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# A COMPARISON OF ZEROS OF $L$–FUNCTIONS

Ravi Raghunathan

## 1. Introduction

In this paper we examine the following question: Given two different Dirichlet series $D_1(s)$ and $D_2(s)$ which extend to meromorphic functions $L_1(s)$ and $L_2(s)$ on the complex plane $\mathbb{C}$ and which satisfy suitable functional equations, are there infinitely many zeros of $L_2(s)$ which are not zeros of $L_1(s)$? More precisely, let $S_1$ and $S_2$ denote the sets of (non-trivial) zeros (counted with multiplicity) of $L_1(s)$ and $L_2(s)$ respectively. Then, is $|S_2 \setminus S_1| = \infty$?

In a number of classical examples we are able to give an affirmative answer to the above question. The simplest case occurs in Theorem 3, when $D(\chi_1, s) = \sum_{n=1}^{\infty} \frac{\chi_1(n)}{n^s}$ and $D(\chi_2, s) = \sum_{n=1}^{\infty} \frac{\chi_2(n)}{n^s}$ are both Dirichlet $L$-series, i.e., Dirichlet series attached to distinct primitive Dirichlet characters $\chi_1$ and $\chi_2$. Theorem 3' generalises this result to pairs of cuspidal automorphic $L$-functions over any number field with the same archimedean factors. In Theorem 4, we are able to show the analogue of Theorem 3 for the $L$-functions of two holomorphic cuspidal eigenforms of the same weight or the same level. In Theorem 5, we obtain similar results for the pair $L(\mathrm{Sym}^2(\pi_f) \otimes \chi, s)$ and $L(\chi, s)$, where $f$ denotes a holomorphic cusp form, $\pi_f$ its associated cuspidal representation and $L(\chi, s)$ is a Dirichlet $L$-function (there are some restrictions on $f$ and $\chi$). We also deal with the case when we have the two $L$-functions $L(\mathrm{Sym}^3(\pi_\Delta), s)$ and $L(\pi_\Delta, s)$, where $\pi_\Delta$ denotes the cuspidal representation associated with the Ramanujan cusp form $\Delta$.

We establish results of the above kind using a *non-standard* version of the usual *Converse Theorems* for Dirichlet series [R1,R2]. While Theorems 3 and $3'$ require only the use of Hamburger's Theorem [H1,H2,H3], Theorems 4 and 5 require an extension of Hecke's Converse Theorem for Dirichlet series with a finite number of poles at arbitrary locations. This extension of Hecke's result (which was only for Dirichlet series with at most two poles at the edges of the critical strip) is not trivial and is proved in [R1](see also [R2]). We also emphasise that this extended version is necessary for our theorems. Using the original result of Hecke in Theorems 4 and 5 will, at best, yield only $|S_2 \setminus S_1| = 3$. The other key

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ingredient comes from the *non-vanishing* theorems for $L$-functions on the line $\mathrm{Re}(s) = 1$, generalising the results of Dirichlet for the classical Dirichlet $L$-series. These theorems are due to Jacquet-Shalika [J-S] and Shahidi [Sh1,Sh3].

We briefly indicate our methods in the simplest case. For instance, let $\chi_1$ and $\chi_2$ be two primitive Dirichlet characters of the same parity and conductor. If we assume that $D(\chi_1, s)$ and $D(\chi_2, s)$ have all but finitely many zeros in common, $D(s) = \frac{D(\chi_1,s)}{D(\chi_2,s)}\zeta(s)$ (where $\zeta(s)$ is the Riemann zeta-function) has at most finitely many poles. One then checks easily that $D(s)$ satisfies the same functional equation as $\zeta(s)$, so Hamburger's theorem asserts that, in fact, $D(s) = c\zeta(s)$, for some constant $c$. This is absurd, since it says that $D(\chi_1, s) = cD(\chi_2, s)$. When the level and parity of the characters are not the same the situation is more complicated.

The problem for two Dirichlet characters has been considered by A. Fujii in [F] where a result stronger than Theorem 3 is obtained using explicit formulae for the distribution of zeros of the $L$-functions. The argument cannnot be generalised to get Theorem 3' or any of the other results of this paper. All stronger results of which we are aware depend on the truth of the Generalised Riemann Hypothesis or similar conjectures. This is true of the results in [C-G-G1] and [C-G-G2] for the classical Dirichlet characters. In [B-P] Bombieri and Perelli work in the Selberg Class under similar assumptions as well as under the very strong hypotheses of Selberg's Conjecture B [S]. We stress that the results of the present paper are *unconditional*, and, as far as we know, do not follow from known facts about the distribution of zeros of automorphic $L$-functions.

While most of the examples of Dirichlet series treated in this paper arise from automorphic forms, we note that our methods use only the fact that the *quotients* of the two Dirichlet series being compared satisfy certain analytic criteria (continuation to meromorphic functions, finite growth) and certain functional equations (i.e., they are *nice* Dirichlet series in the terminology of Piatetski-Shapiro). In particular, in Theorem 5, we know only that the $L$-function $L(\mathrm{Sym}^3(\pi_\Delta), s)$ is nice- its automorphy has not yet been established. Similarly, Artin $L$-functions and Dirichlet series in the Selberg Class also fit into our framework. The precise analytic criteria for *niceness* are outlined in Section 2. We note also that our methods frequently work over arbitrary number fields (as in Theorem 3').

More generally, one can ask if results similar to Theorems 3-5 are true for $L$-functions of arbitrary automorphic forms on $GL_n/K$, where $K$ is a number field. In fact, one expects that the $L$-functions $L(\phi_1, s)$ and $L(\phi_2, s)$ corresponding to distinct cuspidal automorphic forms $\phi_1$ on $GL_{r_1}/K$ and $\phi_2$ on $GL_{r_2}/K$, which are eigenforms of the Hecke operators, have at most finitely many common non-trivial zeros (i.e., $|S_1 \cap S_2| < \infty$).

In Section 2 we discuss Converse Theorems and state the main results to be used later in the paper. In Section 3 we discuss the Dirichlet $L$-functions and prove Theorems 3 and 3'. In sections 4 and 5 we imitate the arguments of Theorem 3, but use Theorem 2 in lieu of Hamburger's Theorem. We also use more sophisticated "non-vanishing" results.

In what follows, by "zeros" of an $L$-function we will always mean only its *non-trivial* zeros *counted with multiplicity.* Let $L(s)$ be an $L$-function, satisfying a functional equation, normalised so that the critical strip is between 0 and 1. Then by its set of zeros we shall mean the set

$$S = \{(\rho, m_\rho) |\, L(\rho) = 0, m_\rho = ord_{s=\rho}\, L(s),\ 0 < \mathrm{Re}(\rho) < 1\}.$$

We shall follow this convention in the rest of the paper without further comment.

## 2. Converse theorems — A brief discussion

A *Converse Theorem* is a theorem which establishes criteria for a Dirichlet series to be automorphic. Traditionally, the criteria have required (among other things) that the Dirichlet series, defined a priori in some half-plane, should extend to an entire function on the whole complex plane $\mathbb{C}$. This is true of the celebrated theorems of Hecke [He], Weil [W] and Jacquet-Langlands [J-L] as well as the more recent results in [J-PS-S1] and [C-PS] (The theorems of Hecke and Weil allowed the meromorphic continuation of the Dirichlet series to have at most two poles.). For our purposes, however, it is necesary to relax this requirement and assume only that the Dirichlet series has a continuation to a meromorphic function on $\mathbb{C}$ with at most a finite number of poles, without any restrictions on their locations or orders. The other criteria (such as finite growth in vertical strips and the functional equation) remain unchanged. The first such theorem, proved by H. Hamburger in 1921-22 [H1,H2,H3], uses only this weaker hypothesis. In Theorem 2 (stated below-see also [R1,R2]) we have obtained the analogue of Hecke's theorem under this weaker hypothesis. Extending the Weil-Jacquet- Langlands result for Dirichlet series with a finite (but arbitrary) number of poles at arbitrary locations is work in progress of the author. We do not expect this extension to be trivial.

If $s$ is in the complex plane $\mathbb{C}$, we let $\sigma = \mathrm{Re}(s)$ and $t = \mathrm{Im}(s)$ be the real and imaginary parts of $s$ respectively. If $\chi$ is a Dirichlet character, we let $D(\chi, s)$ be the associated Dirchlet series $\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ and

$$L(\chi, s) = \pi^{\frac{-(s+\epsilon)}{2}} \Gamma\left(\frac{s+\epsilon}{2}\right) D(\chi, s)$$

be the associated $L$-function, where $\epsilon \in \{0, 1\}$ is determined by the equation $\chi(-1) = (-1)^\epsilon$. Let $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ be a Dirichlet series satisfying the following conditions:

(1) $D(s)$ converges in some right half-plane $\sigma > c_1 > 0$ and has a meromorphic continuation (also called $D(s)$) to all of $\mathbb{C}$ of the form $\frac{E(s)}{P(s)}$, where $E(s)$ is an entire function and $P(s)$ is a polynomial in $s$.

(2) For every $c \in \mathbb{R}^+$ there exist $t_0$, $K$, $\rho \geq 0$ such that

$$|D(\sigma + it)| < Ke^{|t|^\rho},$$

for $-c < \sigma < c$ and for all $|t| > t_0$.

(3a) $D(s)$ satisfies one of the following functional equations

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)D(s) = \varepsilon(s)\pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)F(1-s), \tag{1.1}$$

$$\pi^{-(s+1)/2}\Gamma\left(\frac{s+1}{2}\right)D(s) = \varepsilon(s)\pi^{-\left(\frac{2-s}{2}\right)}\Gamma\left(\frac{2-s}{2}\right)F(1-s), \tag{1.2}$$

where $\varepsilon(s) = k^{-s}$, $k > 0$, $k \in \mathbb{R}$, $\Gamma(s)$ denotes the usual gamma function and $F(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$ is an arbitrary Dirichlet series converging uniformly and absolutely for all $s$ such that $\mathrm{Re}(s) > c_2 > 0$.

We can now state Hamburger's Theorem: If $D(s)$ is a Dirichlet series satisfying conditions (1), (2) and (3a) then we have

**Theorem 1.**

*(i) If $k \notin \mathbb{Z}^+$ then* (1.1) *and* (1.2) *have no non-trivial solutions.*

*(ii) If $k \in \mathbb{Z}^+$ let $n^+$ (resp. $n^-$) be the dimension of the space of solutions $W^+$ of equation* (1.1) *(resp. $W^-$ of equation* (1.2)*). Then*

$$n^+ + n^- = k.$$

*Further, the L-functions $L(\chi, s)$, where $\chi$ denotes a Dirichlet character, form a basis for the space $W^+ \oplus W^-$.*

*(iii) If $L(s)$ satisfies* (1.1) *then $L(s)$ has at most two poles-at the points $s = 0$ or $s = 1$. If $L(s)$ satisfies* (1.2) *it will be entire.*

**Remarks.** 1. The smallest $k$ for which $L(s)$ satisfies one of the functional equations (3a) is called the *conductor* of $L(s)$. For a given $L(s)$, we will always consider the functional equation with this minimal $k$.

2. If $D(s)$ satisfies one of the functional equations (3a) with $\varepsilon(s) = k^{-s}$, then $k$ will be an integer multiple of the conductor (see the Corollary of Proposition 1 in Section 3 of [PS-R]).

Hamburger's Theorem is a Converse Theorem for $GL(1)/Q$, that is, the functional equations (1.1) and (1.2) are those satisfied by the $L$-functions of automorphic forms on $GL(1)/Q$ (these are just Dirichlet characters). The first analogue for $GL(2)/Q$ was proved by Hecke for Dirichlet series which extend to meromorphic functions on $\mathbb{C}$ which are entire or have at most two poles at

$s = 0$ and $s = k$. Theorem 2 stated below is the analogue of Hecke's theorem for Dirichlet series whose meromorphic continuations may have a finite number of poles at arbitrary locations satisfying the functional equation

(3b)

$$(2\pi)^{-s}\Gamma(s)D(s) - \frac{a_0}{s} = (2\pi)^{-(k-s)}\Gamma(k-s)F(k-s) - \frac{a_0}{k-s}. \tag{2.1}$$

More precisely, let $D(s)$ be a Dirichlet series satisfying conditions (1), (2) and (3b). Then we have

**Theorem 2.** *If $k > 2$, $\frac{k}{2} \in \mathbb{N}$, $D(s)$ is modular, i.e., $f(z) = \sum_{n=0}^{\infty} a_n q^n$ is a modular form of weight $k$ associated to $SL_2(\mathbb{Z})$, where $q = e^{2\pi i z}$. If $k = 2$, $D(s)$ arises from the non-holomorphic Eisenstein series of weight 2.*

**Corollary.** *If $k > 2$, the only possible pole of $D(s)$ is simple and lies at $s = k$. If $k = 2$, the poles of $D(s)$ are simple and lie at $s = 1$ and $s = 2$.*

**Remarks.** 1. In [R1,R2] Theorem 2 was stated (and proved) for Dirichlet series of *self-dual* type, i.e., $F(s) = D(s)$ in the functional equation (2.1). However, there is no difficulty in modifying the arguments to obtain the version of Theorem 2 stated above. Of course, once Theorem 2 is proved we see that $F(s) = D(s)$ is forced, since modular forms of level 1 give rise to self-dual Dirichlet series.
2. The functional equation (2.1) is the one satisfied by the $L$-functions of level 1 modular forms. After making the transformation $s \to s + \frac{k-1}{2}$ and normalising $D(s)$, (2.1) is equivalent to

$$G(s)D(s) - \frac{a_0}{s+m} = G(1-s)F(1-s) - \frac{a_0}{1-s+m}, \tag{2.2}$$

where $m = \frac{k-1}{2}$ and $G(s) = (2\pi)^{-(s+m)}\Gamma(s+m)$. It will sometimes be more convenient to use this form of the functional equation. Equation (2.2) is satisfied by the $L$-functions of automorphic representations of $GL_2/Q$ that are unramified at all the finite places and are holomorphic discrete series at infinity. Of course, these are precisely the representations that correspond to holomorphic cuspidal eigenforms of level 1.
3. In all the applications of Theorem 2 in this paper we will typically consider a quotient of $L$-functions of the form $L(s) = L_1(s)/L_2(s)$, where both $L_1(s)$ and $L_2(s)$ are known to be entire of finite order. Since we will be assuming that $L(s)$ has at most a finite number of poles in the whole complex plane, we see that $L(s)$ must satisfy condition (2) of the theorems above. Establishing the finite order of the individual $L$-functions in the quotient is not, however, trivial. In sections 3 and 4 the results for the relevant $L$-functions can be found in [Go-J], while the

finiteness of the orders of the symmetric power $L$-functions in Theorem 5 has been proved by Gelbart and Shahidi [G-Sh]. In the rest of this paper we will assume that the relevant Dirichlet series satisfy condition (2) without further comment.

4. The assumption that $D(s)$ has at most a finite number of poles is equivalent to the assumption that $L(s)$ has a finite number of poles, once it is known that one of the functional equations above is satisfied [R1,R2]. Hence, we may make either assumption without any loss of generality.

We call Dirichlet series $D(s)$ that satisfy conditions (1) and (2), as well as suitable functional equations (invariance under $s \mapsto 1-s$, like those in (3a) or (3b)), *nice* Dirichlet series, and their corresponding $L$-functions $L(s)$ *nice* $L$-functions. Of course, one expects all automorphic $L$-functions to be nice, while Converse Theorems assert that certain nice $L$-functions are automorphic.

## 3. Zeros of the classical Dirichlet $L$-series

**Theorem 3.** *Let $\chi_1$ and $\chi_2$ be distinct primitive Dirichlet characters and let $S_1$ (resp. $S_2$) be the set of zeros of $L(\chi_1, s)$ (resp. $L(\chi_2, s)$). Then $|S_1 \setminus S_2| = \infty$ and $|S_2 \setminus S_1| = \infty$, i.e., there are infinitely many zeros of $L(\chi_1, s)$ (resp. $L(\chi_2, s)$) which are not zeros of $L(\chi_2, s)$ (resp. $L(\chi_1, s)$).*

*Proof.* It is sufficient to show that if $\chi_1$ and $\chi_2$ are distinct primitive Dirichlet characters then $\frac{L(\chi_1,s)}{L(\chi_2,s)}$ has infinitely many (non-integer) poles. This would show that $|S_2 \setminus S_1| = \infty$. Reversing the role of $\chi_1$ and $\chi_2$ will then prove the result.

Let $N_1$ denote the conductor of $L(\chi_1, s)$. The $L$-function $L(\chi_2, s)$ satisfies one of the functional equations (1.1) or (1.2) for some conductor $N_2$. Choose a primitive Dirichlet character $\chi \neq \chi_2$) such that $L(\chi, s)$ satisfies the same functional equation as $L(\chi_2, s)$ (with some conductor $N$ not necessarily the same as $N_2$). Consider the Dirichlet series

$$D(s) = \frac{D(\chi_1, s)}{D(\chi_2, s)} D(\chi, s).$$

Because of the analytic properties of Dirichlet $L$-functions, D(s) extends to a meromorphic function $L(s)$ on $\mathbb{C}$ satisfying either functional equation (1.1) or (1.2) (according to whether $L(\chi_1, s)$ satisfies (1.1) or (1.2)) with $\varepsilon(s) = l^{-s}$, where $l = \frac{N_1}{N_2} N$. If we now suppose that $L(s)$ has only finitely many poles, it is easy to see that $L(s)$ is nice and satisfies (1.1) or (1.2).

If $N_2 \dagger N_1$, we can choose $\chi$ such that $N_2 \dagger N_1 N$, so $L(s)$ will satisfy functional equation (1.1) or (1.2) with conductor $l = \frac{N_1}{N_2} N \notin \mathbb{Z}^+$. By the first part of Theorem 1 we must have $D(s) = 0$, which is absurd. This shows that $L(s)$ cannot have only finitely many poles as assumed above.

If $N_2|N_1$, then $D(s)$ satisfies the functional equation (1.1) or (1.2) with $l = \frac{N_1}{N_2}N \in \mathbb{Z}^+$. Using Theorem 1 (iii), we can conclude that

$$L(s) = \frac{L(\chi_1, s)}{L(\chi_2, s)} L(\chi, s) = \sum_i c_i L(\chi_i, s), \tag{3.1}$$

where $\chi_i$ runs over a set of distinct Dirichlet characters with conductors $d_i$ dividing $l$.

**Proposition 3.1.** *If $\chi$ is suitably chosen, then*

$$L(s) \neq c_1 L(\chi_1, s) + cL(\chi, s),$$

*i.e., $\chi$ and $\chi_1$ are not the only characters appearing in the sum on the right-hand side of* (3.1).

*Proof.* Suppose

$$L(s) = c_1 L(\chi_1, s) + cL(\chi, s).$$

Multiplying both sides by $L(\chi_2, s)$ we get

$$L(\chi_1, s)L(\chi, s) = c_1 L(\chi_1, s)L(\chi_2, s) + cL(\chi, s))L(\chi_2, s). \tag{3.2}$$

Comparing the first coefficients of the Dirichlet series on both sides of (3.2) we see immediately that $c + c_1 = 1$. Comparing the coefficients of $p^{-s}$, where $p$ is any prime, we obtain

$$\chi_1(p) + \chi(p) = c_1(\chi_1(p) + \chi_2(p)) + c(\chi(p) + \chi_2(p)).$$

We now choose $\chi$ by imposing the following conditions: We first choose $p$ so that $\chi_1(p) = \chi_2(p) = 1$ and choose a $\chi \neq \chi_2$ such that $\chi(p) = -1$. This yields $c_1 = 0$, which shows that $c = 1$. Now, choose a prime $q$ such that $\chi_(q) \neq \chi_2(q)$. This shows $c \neq 1$, which is absurd and hence the proposition is proved. □

By Proposition 3.1 there is a Dirichlet character $\chi_3$ which appears in the right-hand side of (3.1) and $\chi_3 \neq \chi, \chi_1$. Twisting both sides of (3.1) by $\chi_3^{-1}$ we get

$$\frac{L(\chi_1\chi_3^{-1}, s)}{L(\chi_2\chi_3^{-1}, s)} L(\chi\chi_3^{-1}, s) = \sum_{\chi_i} c_i L(\chi_i\chi_3^{-1}, s). \tag{3.4}$$

Now, by a classical result of Dirichlet we know that $L(\chi_2\chi_3^{-1}, 1) \neq 0$. We also know that $L(\chi\chi_3^{-1}, 1)$ and $L(\chi_1\chi_3^{-1}, 1)$ are both finite. Hence the left-hand side of (3.4) is finite at $s = 1$. On the other hand, the term $L(\chi_3\chi_3^{-1}, 1) = \infty$ while $L(\chi_i\chi_3^{-1}, 1)$ is finite whenever $\chi_i \neq \chi_3$. Hence, the right-hand side of (3.4) is infinite at $s = 1$, which is absurd. This concludes the proof of Theorem 3. □

We denote by $\mathbb{A}_K$ the ring of adèles over a number field $K$. Using the same ideas as in the proof of Theorem 3, we can prove

**Theorem 3′.** *Let $\pi_1$ and $\pi_2$ be cuspidal automorphic representations of $GL_n(\mathbb{A}_K)$ $(n \geq 2)$. Let $L(\pi_1, s)$ and $L(\pi_2, s)$ be their associated $L$-functions and $D_1(s)$ and $D_2(s)$ their associated Dirichlet series respectively. Let $S_1$ and $S_2$ be their corresponding sets of sets of zeros. If $L(\pi_1, s)$ and $L(\pi_2, s)$ have the same gamma factors at infinity (i.e. the local archimedean $L$-functions are the same) then $|S_2 \setminus S_1| = \infty$.*

*Proof.* We may assume that $n > 1$ or that $K$ is not $Q$. Otherwise, Theorem 3' reduces to Theorem 3.

We form the Dirichlet series

$$D(s) = \frac{D_1(s)}{D_2(s)} D(\chi, s),$$

where $\chi$ is a Dirichlet character as before. We let $L(s)$ denote its meromorphic continuation to the whole plane. By the work of Godement-Jacquet [Go-J] we know that $L(\pi_i, s)$ $(i = 1, 2)$ satisfy a functional equation of the form

$$L(\pi_i, s) = \varepsilon(s) L(\tilde{\pi}_i, 1 - s),$$

where $\varepsilon(s) = AB^s$ ($A \in \mathbb{C}$ and $B > 0$) and $\tilde{\pi}$ is the representation of $GL_n(\mathbb{A}_K)$ contragredient to $\pi$. Since the gamma factors of $\pi_1$ and $\pi_2$ at infinity are the same we see immediately that $L(s)$ satisfies one of the equations (1.1) or (1.2). For a suitably chosen $\chi$ we may assume that the conductor of $D(s)$ in the functional equation is an integer $l$ (this is the essential case). Hence, using the third part of Theorem 1, we can conclude that

$$L(s) = \sum_i c_i L(\chi_i, s), \tag{3.5}$$

where $\chi_i$ runs over a set of distinct Dirichlet characters with conductors $d_i$ dividing $l$. We now twist both sides of (3.5) by any $\chi_j^{-1} \neq \chi^{-1}$ and compare the values at $s = 1$. The right-hand side contains the terms $L(\chi_i \chi_j^{-1}, s)$ which are finite at $s = 1$ unless $i = j$. Hence, the right-hand side is infinite at $s = 1$. On the left-hand side, the functions $L(\pi_1 \otimes \chi_j^{-1}, s)$ and $L(\chi \chi_j^{-1}, s)$ in the numerator are entire (Go-J). In the denominator the function $L(\pi_2 \otimes \chi_j^{-1}, s)$ does not vanish on the line $\mathrm{Re}(s) = 1$ ([J-S] or [Sh1]). Hence, the left-hand side is finite at $s = 1$, which gives us a contradiction and proves Theorem 3′. □

## 4. Comparing zeros of $L$-functions of cusp forms

Let $f_i$ $(i = 1, 2)$ denote a holomorphic cuspidal eigenform, and $L(f_i, s)$ denote its associated $L$-function. We denote by $\tilde{f}_i$ the holomorphic cusp form given by $\tilde{f}_i(z) = N_i^{\frac{k}{2}} z^{-k_i} f(\frac{-1}{Nz})$, where $N_i$ denotes the level and $k_i$ denotes the weight of

$f_i$. Let $S_i$ denote the set of zeros of $L(f_i,s)$. We will assume that the forms are normalised so that the first coefficient $a_1=1$. Our aim is to prove analogues of Theorem 3 for the $L$-functions $L(f_1,s)$ and $L(f_2,s)$.

We apply Theorem 3′ to the case when $f_1$ and $f_2$ have the same weight. We will show that $|S_2 \setminus S_1| = \infty$. Indeed, comparing the zeros of $L(f_1,s)$ and $L(f_2,s)$ is the same as comparing the zeros of $L(\pi_1,s)$ and $L(\pi_2,s)$, where $\pi_i$ denotes the cuspidal automorphic representation of $GL_2(\mathbb{A}_Q)$ corresponding to $f_i$ (i=1,2). Since $f_1$ and $f_2$ have the same weight, $L(\pi_1,s)$ and $L(\pi_2,s)$ have the same gamma factors at infinity and the result follows from Theorem 3′.

We now suppose instead that $f_1$ and $f_2$ have the same level $N$ and that $f_2$ has even weight $k_2 \geq 12$. Let $f_3$ ($\neq cf_2$, for any constant $c$) be a modular eigenform of level 1 of the same weight as $f_2$ and normalised so that its first coefficient $a_1 = 1$ (since $f_2$ has even weight this is always possible). We consider the $L$-function

$$L(s) = \frac{L(f_1,s)}{L(f_2,s)} L(f_3,s).$$

Since $f_2$ and $f_3$ have the same weight, the $L$-functions $L(f_2,s)$ and $L(f_3,s)$ have the same gamma factors at infinity. Hence, when dividing one by the other these factors cancel. Since $f_1$ and $f_2$ have the same level and $f_3$ has level 1, we see immediately that $L(s)$ satisfies the functional equation (2.1) with weight $k_1$. If we now assume that $L(s)$ has only finitely many poles, then $L(s)$ satisfies all the hypothesis of Theorem 2. Thus we may write

$$L(s) = \frac{L(f_1,s)}{L(f_2,s)} L(f_3,s) = L(f_4,s), \tag{4.1}$$

for some level 1 modular form $f_4$. We may express $L(f_4,s)$ in terms of the basis of level 1 eigenforms $g_j$ of weight $k_1$. Hence, we can write

$$L(f_4,s) = \sum_{j=1}^{l} c_l L(g_j,s). \tag{4.2}$$

We now formulate the analogue of Proposition 3.1 as

**Proposition 4.1.** *Let $f_1$ and $f_2$ be cusp forms as above of different weights. Then*

$$L(f_4,s) \neq c_1 L(f_1,s) + c_3 L(f_3,s),$$

*for any constants $c_1$ and $c_3$.*

*Proof of Proposition.* We already know that $f_4$ is a modular form of weight $k_1$. Now $f_3$ cannot have weight $k_1$ since that would imply that $f_2$ also has weight

$k_1$ ($f_3$ was chosen to have the same weight as $f_2$!). Hence, we must have $c_3 = 0$. But this implies that $\frac{L(f_3,s)}{L(f_2,s)} = c_1$ which is clearly absurd. □

We see by the above proposition that there exists at least one $j = j_0$ such that $g_{j_0} = g$ is not in the space spanned by $f_1$ and $f_3$. We let $T_i$ denote the set of finite ramified places of $L(f_i, s)$ for $1 \le i \le 3$ and $T_j$ be the set of finite ramified primes of $L(g_j, s)$ for $1 \le j \le l$. Let $S = \cup T_i \cup T_j$ and $L_S(f_i, s)$ and $L_S(g_j, s)$ be the corresponding incomplete $L$-functions. Twisting both sides of (4.2) by $\tilde{g}$, we get (see Theorem 9.5 of [J-PS-S2] and [Sh4])

$$\frac{L_S(f_1 \times \tilde{g}, s) L_S(f_3 \times \tilde{g}, s)}{L_S(f_2 \times \tilde{g}, s)} = \sum_{j=1}^{l} c_j L_S(g_j \times \tilde{g}, s). \tag{4.3}$$

The functions $L_S(f_1 \times \tilde{g}, s)$ and $L_S(f_3 \times \tilde{g}, s)$ and $L_S(g_j \times \tilde{g}, s)$, $j \ne j_0$, are holomorphic at $s = 1$ by [J], while the function $L_S(g \times \tilde{g}, s)$ has a simple pole at $s = 1$ [J]. Hence, the right-hand side of (4.3) has a pole of order 1 at $s = 1$. On the other hand, by [Sh1] the function $L_S(f_2 \times \tilde{g}, s)$ does not vanish on the line $\mathrm{Re}(s) = 1$ which gives us a contradiction.

We summarise our discussion above as

**Theorem 4.** *Let $f_1$ and $f_2$ be holomorphic cuspidal eigenforms satisfying either of the conditions* (1) *or* (2) *below:*

(1) *$f_1$ and $f_2$ have the same weight.*
(2) *$f_1$ and $f_2$ are forms of the same level and $f_2$ has even weight $k \ge 12$*

*Then $|S_2 \setminus S_1| = \infty$.*

It is not hard to formulate an analogue of Theorem 4 for two Maass cusp forms or for a holomorphic cusp form and a Maass cusp form. A "level $N$" version of Theorem 2 (that is, the analogue of Weil's Converse Theorem for Dirichlet series with poles) would allow us to remove the conditions in Theorem 4.

## 5. More variations on the theme: Symmetric power $L$-functions

We now apply Theorem 2 to pairs of symmetric power $L$-functions. Let $\pi = \pi_f$ denote a cuspidal automorphic representation of $GL_2(\mathbb{A}_Q)$ associated to a holomorphic cuspidal eigenform $f$ of weight $k$ and nebentypus $\omega$. We will assume that $\pi$ is not monomial. Let $\chi$ be a primitive Dirichlet character. We denote by $L(\mathrm{Sym}^n(\pi), s)$ (resp. $L(\mathrm{Sym}^n(\pi) \otimes \chi, s)$) the nth symmetric power $L$-function of $\pi$ (resp. the nth symmetric power $L$-function twisted by $\chi$) and by $D(\mathrm{Sym}^n(\pi), s)$ (resp. $D(\mathrm{Sym}^n(\pi) \otimes \chi, s)$) its associated Dirichlet series. We note that both the quotients we consider below have the additional feature of an Euler product.

We first specialise to the following case: We suppose that $\omega(-1) = -1$ and that the level of $f$ is $p$ for for some prime $p$. We also suppose that $\chi$ has conductor $p$. Consider now the quotient $D(s) = \frac{D(\mathrm{Sym}^2(\pi)\otimes\chi,s)}{L(\chi,s)}$. Let $L(s)$ be its meromorphic continuation to $\mathbb{C}$. If we assume that $L(s)$ has a finite number of poles, then it is easy to see that $L(s)$ satisfies all the hypotheses of Theorem 2 and also has an Euler product. In particular, it satisfies the functional equation (2.2)- this is because the local archimedean $L$-factor of $L(\mathrm{Sym}^2(\pi)\otimes\chi, s)$ is (upto an exponential function) of the form $\Gamma(\frac{s+\epsilon}{2})\Gamma(s+k-1)$, where $\epsilon$ is 0 or 1 as determined by the equation $\chi(-1) = (-1)^\epsilon$. Hence $L(s)$ is the $L$-function of a *cuspidal* automorphic representation $\rho$ (not just of an automorphic representation- this is because $L(\rho, s)$ has an Euler product!). As before, if we twist by $\chi^{-1}$, we see that $L(\chi\chi^{-1}, s)$ has a pole at $s = 1$ while $L(\mathrm{Sym}^2(\pi), s)$ is entire by [G-J]. This shows that that $L(\rho\otimes\chi, s)$ vanishes at $s = 1$, contradicting [J-S] or [Sh1]. Hence, $D(s)$ must have infinitely many poles.

This example is of particular interest because the Euler product for $L(\mathrm{Sym}^2(\pi)\otimes\chi, s)$ contains the local factors $(1-\frac{\chi(p)}{p^s})^{-1}$ at every prime $p$, and these are precisely the local Euler factors for $L(\chi, s)$. Hence, the quotient by $L(\chi, s)$ is an Euler product of degree 2, absolutely convergent for $\mathrm{Re}(s) > 1$, which extends to a meromorphic function with infinitely many poles.

The next example was suggested to us by Professor Ramakrishnan. By the work of Shahidi [Sh2, Sh3] we know that $L(\mathrm{Sym}^3(\pi), s)$ is entire in many cases (in fact, in all cases when $\pi$ is not monomial by the recent result of [K-Sh]). In particular, this is true for $\pi = \pi_\Delta$, where $\Delta$ is the Ramanujan cusp form [M-Sh]. For this $\pi$ we now consider the quotient $L(s) = \frac{L(\mathrm{Sym}^3(\pi),s)}{L(\pi,s)}$. Arguing as before, we see that $L(s)$ is automorphic and hence we see that

$$\frac{L(\mathrm{Sym}^3(\pi), s)}{L(\pi, s)} = L(\rho, s), \tag{5.1}$$

for some cuspidal automorphic representation $\rho$ of $GL_2(\mathbb{A}_Q)$ (note that $L(\rho, s)$ has an Euler product again!). Twisting both sides of (5.1) by $\tilde{\pi}$, the contragredient representation of $\pi$ we obtain by [Sh4]

$$\frac{L(\mathrm{Sym}^3(\pi)\times\tilde{\pi}, s)}{L(\pi\times\tilde{\pi}, s)} = L(\rho\times\tilde{\pi}, s),$$

which gives (recall that $\pi_\Delta$ is self-dual)

$$\frac{L(\mathrm{Sym}^4(\pi), s)L(\mathrm{Sym}^2(\pi), s)}{L(\pi\times\pi, s)} = L(\rho\times\pi, s). \tag{5.2}$$

Now, it is known that $L(\mathrm{Sym}^4(\pi), s)$ is finite at $s = 1$ [Sh3], and we have already seen that $L(\mathrm{Sym}^2(\pi), 1)$ is finite (since it is entire) at $s = 1$. We know that

$L(\pi \times \pi, s)$ has a pole at $s = 1$, and hence, the left-hand side of (5.2) vanishes at $s = 1$. But the right-hand side does not vanish at $s = 1$ (again, see [Sh1]), and hence, we get a contradiction. This shows that the quotient $L(s) = \frac{L(\mathrm{Sym}^3(\pi),s)}{L(\pi,s)}$ has infinitely many poles.

We summarise our results as

**Theorem 5.** *Let $f$, $\Delta$ and $\chi$ be as above. The quotients $\frac{L(\mathrm{Sym}^2(\pi_f)\otimes\chi,s)}{L(\chi,s)}$ and $\frac{L(\mathrm{Sym}^3(\pi_\Delta),s)}{L(\pi_\Delta,s)}$ have infinitely many poles.*

Using the same circle of ideas (but with Theorem 1 instead of Theorem 2) we can prove:

**Theorem 5′.** *Let $f$ be as above with $k = 1$. The quotients $\frac{L(\mathrm{Sym}^n(\pi_f),s)}{L(\mathrm{Sym}^{n-1}(\pi_f),s)}$ $(n = 1, 2, 3)$ have infinitely many poles.*

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## References

[B-P] E. Bombieri and A. Perelli, *Distinct zeros of L-functions*, Acta Arith. **83** (1998), 271–281.

[C-G-G1] J.B. Conrey, A. Ghosh, and S.M. Gonek, *Simple zeros of the zeta-function of a quadratic number field, I*, Invent. Math. **86** (1986), 563–576.

[C-G-G2] ———, *Simple zeros of the zeta-function of a quadratic number field, II*, Analytic Number Theory and Diophantine Problems (Stillwater, OK, 1984), Progr. Math., vol. 70, 1987, pp. 87–144.

[C-PS] J.W. Cogdell and I.I. Piatetski-Shapiro, *Converse theorems for $GL_n$*, Inst. Hautes Études Sci. Publ. Math. **79** (1994), 157–214.

[F] A. Fujii, *On the zeros of Dirichlet L-functions. V.*, Acta Arithmetica **28** (1975), 395–403.

[G-J] S. Gelbart and H. Jacquet, *A relation between automorphic representations of $GL(2)$ and $GL(3)$*, Ann. Sci. Ecole Norm. Sup. (4) **11** (1978), 471–552.

[G-Sh] S. Gelbart and F. Shahidi, *On the boundedness of automorphic L-functions*, in preparation (1999).

[Go-J] R. Godement and H. Jacquet, *Zeta-functions of simple algebras*, Lecture Notes in Mathematics, vol. 260, Springer Verlag, Berlin-New York, 1972.

[H1] H. Hamburger, *Über die Funktionalgleichung der $\zeta$-Funktion*, Math. Z. **10** (1921), 240–258.

[H2] ———, *Über die Funktionalgleichung der $\zeta$-Funktion*, Math. Z. **11** (1921), 224–245.

[H3] ———, *Über die Funktionalgleichung der $\zeta$-Funktion*, Math. Z. **13** (1922), 283–311.

[He1] E. Hecke, *Über die Bestimmung Dirichletscher Reihen durch ihre Funktionalglechung*, Math. Ann. (Mathematische Werke, No. 33) **112** (1936), 664–699.

[He2] ———, *Theorie der Eisenstein Reihen höhre Stuffe und ihre Anwendung auf Funkiontheorie und Arithmetic*, Abh. Math. Sem. Univ. Hamburg **3** (1927), 199–224.

[J] H. Jacquet, *Automorphic forms on $GL(2)$ Part II*, Lecture Notes in Mathematics, vol. 278, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[J-L] H. Jacquet and R.P. Langlands, *Automorphic forms on $GL(2)$*, Lecture Notes in Mathematics, vol. 114, Springer Verlag, Berlin-Heidelberg-New York, 1970.

[J-PS-S1] H. Jacquet, I.I. Piatetski-Shapiro and J. Shalika, *Rankin-Selberg Convolutions*, Amer. J. Math. **105** (1983), 367-464.

[J-PS-S2] ———, *Automorphic forms on $GL_3$, I and II*, Ann. of Math. (2) **109** (1979), 169-258.

[J-S] H. Jacquet and J. Shalika, *A non-vanishing theorem for zeta-functions of $GL_n$*, Invent. Math. **38** (1976), 1–16.

[K-Sh] H. Kim and F. Shahidi, *Symmetric cube L-functions are entire*, preprint (1997).

[M-Sh] C.J. Moreno and F. Shahidi, *The L-function $L_3(s, \pi_\Delta)$ is entire*, Invent. Math. **79** (1985), 247–251.

[PS-R] I.I. Piatetski-Shapiro and R. Raghunathan, *On Hamburger's Theorem*, Amer. Math. Soc. Transl. **169 (2)** (1995), 109–120.

[R1] R. Raghunathan, *Converse theorems for Dirichlet series with poles*, Doctoral Dissertation, Yale University (1996).

[R2] ———, *A converse theorem for Dirichlet series with poles*, C. R. Acad. Sci. Paris Sér. I Math. **327** (1998), 231–235.

[S] A. Selberg, *Old and new conjectures and results about a class of Dirichlet series*, Proceedings of the Amalfi Conference on Analytic Number Theory (Maiori, 1989), Univ. Salerno, Salerno, 1992, pp. 367–385.

[Sh1] F. Shahidi, *On certain L-functions*, Amer. J. Math. **103** (1981), 297–356.

[Sh2] ———, *Third symmetric power L-functions for $GL(2)$*, Compositio Math. **70** (1989), 245–273.

[Sh3] ———, *Symmetric power L-functions for $GL(2)$*, CRM Proc. Lecture Notes 4, Amer. Math. Soc., Providence, RI, 1994, pp. 159–192.

[Sh4] ———, *Local coefficients as Artin factors for real groups*, Duke Math. J. **52** (1985), 279–289.

[W] A. Weil, *Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen*, Math. Ann. **168** (1967), 149–156.



Department of Mathematics, California Institute of Technology, Pasadena, CA 91125

*E-mail address*: ravi@cco.caltech.edu