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ON THE MOTIVE OF THE HILBERT SCHEME OF POINTS ON A SURFACE

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1. Introduction

Let S be a smooth quasiprojective variety over an algebraically closed field k of characterstic 0. In this note we determine the class $[S^{[n]}]$ of the Hilbert scheme $S^{[n]}$ of subschemes of length n on S in the Grothendieck ring $K_0(V_k)$ of k-varieties. The result expresses $[S^{[n]}]$ in terms of the classes of the symmetric powers $S^{(l)} = S^l/\mathfrak{S}_l$. Here $\mathfrak{S}_l$ is the symmetric group acting by permutation of the factors of S^l.

Theorem 1.1. *In the Grothendieck ring of k-varieties we have the equality*

$$[S^{[n]}] = \sum_{\alpha \in P(n)} [S^{(\alpha)} \times \mathbb{A}^{n-|\alpha|}].$$

Here $P(n)$ is the set of partitions of n. We write a partition $\alpha \in P(n)$ as $(1^{a_1}, 2^{a_2}, \dots, n^{a_n})$, where a_i is the number of occurences of i in α. Then the length $|\alpha|$ of α is the sum of the a_i and $S^{(\alpha)} = S^{(a_1)} \times \ldots \times S^{(a_n)}$.

In the case $k = \mathbb{C}$ the cohomology of $S^{[n]}$ has been studied by a number of authors [E-S1],[Gö1],[G-S],[Ch1],[N],[Gr],[L],[dC-M1]. In particular the Betti numbers and the Hodge structure have been determined. The class $[X]$ of a smooth projective variety over $\mathbb{C}$ (or more generally of a projective variety with finite quotient singularities) determines its Hodge structure; so Theorem 1.1 gives a new and elementary proof of the corresponding formulas.

By a result of [Gi-Sou] and [Gu-Na] our result implies that the same formula holds in the Grothendieck ring of effective Chow motives.

Similar arguments apply to the incidence variety $S^{[n,n+1]} \subset S^{[n]} \times S^{[n+1]}$. At the end we give some applications to moduli spaces of rank two sheaves on surfaces.

While I was finishing this paper, the preprint [dC-M2] appeared, in which the Chow groups and the Chow motive of $S^{[n]}$ are determined over any field using different methods.

Our approach is mostly motivated by [Ch1] and by [dB]. Lemma 4.4 plays an important role in this paper. I am very thankful to B. Totaro who proved it for

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me. B. Totaro also pointed out a mistake in an earlier version of the paper and explained to me how to deduce Conjecture 2.5 from a conjecture of Beilinson and Murre. I thank M.S. Narasimhan and K. Paranjape for very useful discussions.

2. Grothendieck rings of varieties and motives and Chow groups

In this paper let k be an algebraically closed field of characteristic 0. Let $K_0(V_k)$ be the Grothendieck ring of k-varieties. This is the abelian group generated by the isomorphism classes of k-varieties with the relation that $[X \setminus Y] = [X] - [Y]$, when Y is a closed subvariety of X. The addition and multiplication in this ring are given by the disjoint union and the product of varieties.

Let M_k be the category of effective Chow motives over k. For the precise definitions and some results about motives see e.g. [Ma],[Kl],[Sch]. Let $A^l(X)$ be the l-th Chow group of the variety X with $\mathbb{Q}$-coefficients. Let X, Y be smooth projective varieties. Assume X has dimension d. The group $\mathrm{Hom}_{\mathrm{C}}(X,Y) := A^d(X \times Y)$ is the group of correspondences from X to Y of degree 0. An object in M_k is a pair (X,p) where X is a smooth projective variety over k and $p \in \mathrm{Hom}_{\mathrm{C}}(X,X)$ with $p^2 = p$. The morphisms are $\mathrm{Hom}((X,p),(Y,q)) = q\,\mathrm{Hom}_{\mathrm{C}}(X,Y)p$. There is a contravariant functor h from the category of smooth projective varieties to M_k by sending X to $(X, [\Delta_X])$ (where $\Delta_X \subset X \times X$ is the diagonal) and $f : X \to Y$ to the class of the transpose of its graph $[\Gamma'_f] \in A^*(Y \times X)$. We define $(X,p) \oplus (Y,q) := (X \sqcup Y, p \oplus q)$ $(X,p) \otimes (Y,q) := (X \times Y, p \times q)$.

The Grothendieck ring $K_0(M_k)$ is the quotient of the free abelian group on the isomorphism classes $[M]$ of effective Chow motives by the subgroup generated by elements $[M] - [M'] - [M'']$ whenever $M \simeq M' \oplus M''$. We denote by $[N]$ the class of a motive in $K_0(M_k)$.

Theorem 2.1. [Gi-Sou],[Gu-Na]. *Let k be a field of characteristic zero. There exists a unique ring homomorphism $\overline{h} : K_0(V_k) \to K_0(M_k)$ with $\overline{h}([X]) = [h(X)]$ for X smooth projective.*

The Lefschetz motive L is defined by $h(\mathbb{P}^1) = 1 \oplus L$, where $1 := h(pt)$ for $pt = \operatorname{Spec} k$. By $[\mathbb{P}^1] = [pt] + [\mathbb{A}^1]$ we see that $[L] = \overline{h}([\mathbb{A}^1])$. So Theorem 1.1 immediately implies the identity

$$[h(S^{[n]})] = \sum_{\alpha \in P(n)} \overline{h}([S^{(\alpha)}])[L^{\otimes(n-|\alpha|)}].$$

in $K_0(M_k)$.

If a finite group G acts on a k-variety X, the motive $h(X/G)$ of the quotient can be defined as $(X, \frac{1}{|G|}\sum_{g \in G}[g])$ where $[g]$ is the graph of the action by g. Therefore we can associate two a priory different elements of $K_0(M_k)$ to the quotient X/G, namely $[h(X/G)]$ and $\overline{h}([X/G])$.

Theorem 2.2. [dB], *Chapter 2,* [dB-Na]. $[h(X/G)] = \overline{h}([X/G])$.

Therefore we obtain

Corollary 2.3.

$$[h(S^{[n]})] = \bigoplus_{\alpha \in P(n)} [h(S^{(\alpha)}) \otimes L^{\otimes(n-|\alpha|)}].$$

The Chow groups of a Chow motive N are defined by $A^l(N) = \mathrm{Hom}(L^{\otimes l}, N)$ and for a smooth projective variety X we have $A^l(h(X)) = A^l(X)$.

Remark 2.4. If N and M are motives with $[N] = [M]$, then there exists a motive P with $N \oplus P = M \oplus P$. By the definition of $\oplus$ and of Chow groups of motives is is evident that as graded vector spaces $A^*(N \oplus P) = A^*(N) \oplus A^*(P)$. Therefore $A^*(N) \oplus A^*(P) = A^*(M) \oplus A^*(P)$. If the Chow groups of P are finite dimensional, it follows that $A^*(N) = A^*(M)$.

We expect that this result holds without the restriction of finite dimensionality.

Conjecture 2.5. *If N and M are effective Chow motives with $[N] = [M]$ in $K_0(M_k)$, then M and N are isomorphic. In particular they have the same Chow groups with rational coefficients.*

In a previous version of this paper the result of Remark 2.4 was claimed without the restriction of finite dimensionality. The mistake was pointed out to me by B. Totaro. He also explained to me the following argument how Conjecture 2.5 follows (over any field k) from the following conjecture of Beilinson and Murre.

Conjecture 2.6. *(see* [Ja] *Conj. 2.1.). Let H^* be a Weil cohomology theory. For each smooth projective variety X over k and all $j \geq 0$, there exists a descending filtration $F^\bullet$ on $A^j(X)$ such that*

1. *$F^0 A^j(X) = A^j(X)$ and $F^1 A^j(X)$ is the kernel of the cyle map $A^j(X) \to H^{2j}(X)$,*
2. *$F^r A^i(X) \cdot F^s A^j(X) \subset F^{r+s} A^{i+j}(X)$ for the intersection product,*
3. *$F^\bullet$ is respected by f^*, f_* for morphisms $f : X \to Y$,*
4. *$F^l A^j(X) = 0$ for $l \gg 0$.*

Now we assume Conjecture 2.6 and show Conjecture 2.5. Let $M = (X, p)$ be an effective Chow motive. Let $R := \mathrm{End}\,(M) \subset A^*(X \times X)$. The cycle map induces a homomorphism $R \to \mathrm{End}\,(H^*(X))$. Let I be the kernel. By the definition of the composition of correspondences and parts 2. and 3. of Conjecture 2.6, we see that for $f \in I$, $f^n \in F^n A^*(X \times X)$. So, by part 4., I is nilpotent.

Our aim is to show that this implies that effective Chow motives satisfy the Krull-Schmidt Theorem: Every effective Chow motive is the direct sum of finitely many indecomposable motives, whose isomorphism classes are uniquely defined. This immediately implies Conjecture 2.5: If $M, N \in M_k$ with $[M] = [N]$, then $M \oplus P \simeq N \oplus P$ for $P \in M_k$. By the Krull-Schmidt Theorem it follows that $M \simeq N$.

In the theorem in Section 3.3 in [Ga-Ro] it is shown that an additive category $\mathcal{C}$ whose isomorphism classes form a set satisfies the Krull Schmidt Theorem if the following holds: Every idempotent in $\mathcal{C}$ splits and for each object A in $\mathcal{C}$, if we write $R := \mathrm{End}\,(A)$, then $R/\mathrm{rad}\,(R)$ is semisimple and all idempotents in $R/\mathrm{rad}\,(R)$ are the images of idempotents in R. Here $\mathrm{rad}\,(R)$ is the Jacobson radical of R. We check these conditions for the category M_k of effective Chow motives. Idempotents split because M_k is pseudoabelian. For a motive $M = (X, p)$ let as above $R := \mathrm{End}\,(M)$ and $I = ker(R \to \mathrm{End}\,(H^*(X))$. Then I is nilpotent and R/I is a finite dimensional $\mathbb{Q}$-algebra. Since I is nilpotent, $I \subset \mathrm{rad}\,(R)$. So $R/\mathrm{rad}\,(R)$ is a finite dimensional $\mathbb{Q}$-algebra with radical 0. So it is semisimple. Furthermore we see that $\mathrm{rad}\,(R)$ is nilpotent: $\mathrm{rad}\,(R/I)$ is nilpotent because R/I is finite dimensional. The result follows for $\mathrm{rad}\,(R)$ because I is nilpotent. Then Theorem 1.7.3 in [Be] implies that all idempotents of $R/\mathrm{rad}\,(R)$ lift to idempotents of R.

Corollary 2.3 and Conjecture 2.5 imply the formulas

$$h(S^{[n]}) = \bigoplus_{\alpha \in P(n)} h(S^{(\alpha)}) \otimes L^{\otimes(n-|\alpha|)},$$

$$A^i(S^{[n]}) = \bigoplus_{\alpha \in P(n)} A^{i+|\alpha|-n}(S^{(\alpha)}).$$

These formulas have been shown in [dC-M2] over an arbitrary field.

3. The stratification of $S^{(n)}$ and $S^{[n]}$

Let $\omega_n : S^{[n]} \to S^{(n)}$ be the Hilbert-Chow morphism, which associates to each subscheme Z its support with multiplicities. $S^{[n]}$ and $S^{(n)}$ have a natural stratification parameterized by the partitions of n, which has been used before [Gö1],[G-S],[Ch1],[dC-M1] to compute the cohomology of $S^{[n]}$. Let $P(n)$ be the set of all partitions of n. A partition $\alpha = (n_1, \ldots, n_r)$ is also written as $\alpha = (1^{a_1}, \ldots, n^{a_n})$, where a_i is the number of occurences of a_i in α. We write $|\alpha| = r = \sum_i a_i$. The corresponding locally closed stratum $S^{(n)}_\alpha$ is the set of all zero-cycles $\xi = n_1x_1 + \ldots n_rx_r$ with $x_1, \ldots, x_r$ distinct points of S. We put $S^{[n]}_\alpha := \omega_n^{-1}(S^{[n]}_\alpha)_{red}$. The strata $S^{(n)}_\alpha$ are smooth, but the $S^{[n]}_\alpha$ and the closures $\overline{S^{(n)}_\alpha}$ usually are singular. There is a natural map $h_\alpha : S^{(\alpha)} \to S^{(n)}$; $(\xi_1, \ldots, \xi_n) \mapsto \sum_{i=1}^n i\xi_i$ whose image is the closure $\overline{S^{(n)}_\alpha}$; in fact it is easy to see that it is the normalization of $\overline{S^{(n)}_\alpha}$. Let $g_\alpha : S^{(\alpha)} \times \mathbb{A}^{n-|\alpha|} \to S^{(n)}$ be the composition of the projection to $S^{(\alpha)}$ with h_α, and let $g : \coprod_{\alpha \in P(n)} S^{(\alpha)} \times \mathbb{A}^{n-|\alpha|} \to S^{(n)}$ be the map induced by the g_α. A more precise version of Theorem 1.1 is the following.

Proposition 3.1. *$[g^{-1}(S^{(n)}_\beta)] = [S^{[n]}_\beta]$ in $K_0(V_k)$ for all $\beta \in P(n)$.*

Theorem 1.1 follows from Proposition 3.1 by summing over all $\beta \in P(n)$.

4. Proof of the main result

We will determine both sides of the equality in Proposition 3.1. We need some preliminaries.

Remark 4.1. In the Grothendieck ring of k-varieties we have:

1. If $f : X \to Y$ is a Zariski locally trivial fibre bundle with fibre F, then $[X] = [Y][F]$ (stratify Y such that f is trivial over the strata).
2. If $f : X \to Y$ is a bijective morphism, then $[X] = [Y]$. (There is a dense open subset $U \subset X$, on which f is an isomorphism (here we use $\mathrm{char}\,(k) = 0$), replacing X by $X \smallsetminus U$ and Y by $Y \smallsetminus f(U)$, we can argue by induction over the dimension.)

Remark 4.2. If a quasiprojective variety X has a decomposition $X = X_1 \sqcup \ldots \sqcup X_l$ into locally closed subvarieties, then it is immediate to see that $\coprod_{n_1+\ldots+n_l=n} \prod_{i=1}^{l} X_i^{(n_i)}$ is a decomposition of $X^{(n)}$ into locally closed subvarieties. Therefore we can define $[X]^{(n)} = [X^{(n)}]$ for X a variety, and $([X_1] + \ldots + [X_l])^{(n)} = \sum_{n_1+\ldots+n_l=n} \prod_{i=1}^{l} [X_i]^{(n_i)}$.

Notation 4.3. Let X be a k-variety. If Y is a variety with a natural morphism to $X^{(m)}$ for some $m > 0$, we write Y_* for the preimage of the open subvariety of all zero cycles $\xi \in X^{(m)}$ whose support consists of m distinct points.

The following lemma was proved for me by Burt Totaro.

Lemma 4.4. *Let X be a variety over k. Let $p : (X \times \mathbb{A}^l)^{(n)} \to X^{(n)}$ be the obvious projection. Then $[(X \times \mathbb{A}^l)^{(n)}] = [X^{(n)} \times \mathbb{A}^{nl}]$, and $[p^{-1}(X_\alpha^{(n)})] = [X_\alpha^{(n)} \times \mathbb{A}^{nl}]$ for all $\alpha \in P(n)$.*

Proof. The first statement follows from the second. It is enough to treat the case $l = 1$; the general case follows by trivial induction. There is a cartesian diagram

$$\begin{array}{ccc} X_*^{|\alpha|} \times \prod_{i=1}^{n} ((\mathbb{A}^1)^{(i)})^{a_i} & \xrightarrow{\overline{p}} & X_*^{|\alpha|} \\ \downarrow & & \downarrow \\ p^{-1}(X_\alpha^{(n)}) & \xrightarrow{p} & X_\alpha^{(n)} \end{array}$$

By the fundamental theorem of symmetric functions the fibre product is $X^{|\alpha|} \times \prod_i (\mathbb{A}^i)^{a_i} = X_*^{|\alpha|} \times \mathbb{A}^n$. So we get an étale trivialization of p. Any two trivializations are related by the action of the group $\mathfrak{S}_{a_1} \times \ldots \times \mathfrak{S}_{a_n}$ by reordering the factors in the X^{a_i}. This acts on the fibres of $\overline{p}$ by reordering the factors $(\mathbb{A}^1)^{(i)}$ in the $((\mathbb{A}^1)^{(i)})^{a_i}$. Choosing an origin in $\mathbb{A}^1$ determines an origin in $\mathbb{A}^n$, and the action becomes linear on k^n, so $p^{-1}(X_\alpha^{(n)})$ is an étale locally trivial vector bundle over $X_\alpha^{(n)}$. Therefore, by Hilbert Theorem 90 [Se] p. 1.24, it is locally trivial in the Zariski topology. Thus we get $[p^{-1}(X_\alpha^{(n)})] = [X_\alpha^{(n)} \times \mathbb{A}^n]$. □

Now we determine the right hand side in Proposition 3.1. Let $R_n = \mathrm{Hilb}^n(\mathbb{A}^2, 0)$ be the punctual Hilbert scheme of subschemes of length n of $\mathbb{A}^2$

concentrated in 0. Then by [E-S1] R_n has a cell decomposition and

$$[R_n] = \sum_{\beta\in P(n)} [\mathbb{A}^{n-|\beta|}].$$

Lemma 4.5. $[S^{[n]}_\alpha] = \big[\big(\prod_{i=1}^n (S\times R_i)^{(a_i)}\big)_*\big]$.

Proof. By Lemma 2.1.4 of [Gö2] $S^{[l]}_{(l)}$ is a locally trivial fibre bundle over S with fibre R_l, thus $[S^{[l]}_{(l)}] = [S\times R_l]$. There is a natural morphism $f : \big(\prod_{i=1}^n (S^{[i]}_{(i)})^{(a_i)}\big)_* \to S^{[n]}_\alpha$ defined on T-valued points by sending $(Z_1,\dots,Z_n)$ to $\coprod_{i=1}^n Z_i$. f is obviously invariant under the action of $\mathfrak{S}_{a_1}\times\ldots\times\mathfrak{S}_{a_n}$ by permuting the factors in the $(S^{[i]}_{(i)})^{(a_i)}$, and the induced morphism from the quotient to $S^{[n]}_\alpha$ induces a bijection on k-valued points. This implies $[S^{[n]}_\alpha] = \big[\big(\prod_{i=1}^n (S^{[i]}_{(i)})^{(a_i)}\big)_*\big]$. □

Notation 4.6.

1. For any $x\in S$ and any $\xi\in S^{(n)}$ we call $m_x(\xi)$ the multiplicity with which x occurs in ξ.
2. We denote $P := \bigcup_{n>0} P(n)$. For $\alpha = (1^{a_1}, 2^{a_2},\dots)\in P(n)$, $\beta := (1^{b_1}, 2^{b_2},\dots)\in P(m)$ and $l\in\mathbb{Z}_{\ge 0}$ we denote $l\alpha := (1^{la_1}, 2^{la_2}\dots)\in P(nl)$, $\alpha+\beta := (1^{a_1+b_1}, 2^{a_2+b_2}\dots)\in P(n+m)$.

Lemma 4.7. $[S^{[n]}_\alpha] = \big[\coprod_f \big(\prod_{\beta\in P} S^{(f(\beta))}\big)_* \times \mathbb{A}^{n-\sum_{\beta\in P} f(\beta)|\beta|}\big]$.
Here f runs through the $f : P\to\mathbb{Z}_{\ge 0}$ with $\sum_{\beta\in P(i)} f(\beta) = a_i$ for all i.

Proof. By Lemma 4.5 we get $[S^{[n]}_\alpha] = \big[\big(\prod_{i=1}^n \big(\coprod_{\beta_i\in P(i)} S\times\mathbb{A}^{i-|\beta_i|}\big)^{(a_i)}\big)_*\big]$. By Remark 4.2 this implies $[S^{[n]}_\alpha] = \big[\big(\prod_{i=1}^n \coprod_{f_i}\prod_{\beta_i\in P(i)} (S\times\mathbb{A}^{i-|\beta_i|})^{(f_i(\beta_i))}\big)_*\big]$, where the f_i run through the $f_i : P(i)\to\mathbb{Z}_{\ge 0}$ with $\sum_{\beta\in P(i)} f_i(\beta) = a_i$. The result follows by Lemma 4.4. □

Now we determine the left hand side in Proposition 3.1. Let $\beta = (1^{b_1}, 2^{b_2},\dots) = (n_1,\dots,n_r)\in P(n)$.

Lemma 4.8. $h_\alpha^{-1} S^{(n)}_\beta \simeq \coprod_f \big(\prod_{\gamma\in P} S^{(f(\gamma))}\big)_*$.
Here the sum is over all functions $f : P\to\mathbb{Z}_{\ge 0}$ with $\sum_{\gamma\in P(i)} f(\gamma) = b_i$ and $\sum_{\gamma\in P} f(\gamma)\gamma = \alpha$.

Using that $g_\alpha^{-1}(S^{(n)}_\beta) = h_\alpha^{-1}(S^{(n)}_\beta)\times\mathbb{A}^{n-|\alpha|}$, Proposition 3.1 follows immediately from Lemma 4.7 and Lemma 4.8 by summing over all β.

Proof. Any $\xi = (\xi_1,\dots,\xi_n)\in h_\alpha^{-1} S^{(n)}_\beta$ induces a map $f_\xi : P\to\mathbb{Z}_{\ge 0}$ as follows. Let $h_\alpha(\xi) = \sum_{i=1}^r n_i x_i$. For all $x\in S$ let $\gamma_x(\xi) := \sum_{j=1}^n (j^{m_x(\xi_j)})$. For each $i = 1,\dots r$ we have $\sum_{j=1}^n j m_{x_i}(\xi_j) = n_i$, so $\gamma_{x_i}(\xi)\in P(n_i)$; furthermore $\sum_i \gamma_{x_i}(\xi) = \alpha$. We define $f_\xi : P\to\mathbb{Z}_{\ge 0}$ by $f_\xi(\gamma) := \#\{x\in S \mid \gamma_x(\xi) = \gamma\}$. Then $\sum_{\gamma\in P(j)} f_\xi(\gamma) = b_j$ and $\sum_{\gamma\in P} f_\xi(\gamma)\gamma = \gamma_{x_1}(\xi)+\ldots+\gamma_{x_r}(\xi) = \alpha$.

Now fix $f : P \to \mathbb{Z}_{\geq 0}$ with the above properties. Let $S^{(\alpha)}_f := \{\xi \in S^{(\alpha)} \mid f_\xi = f\}$. We claim that $S^{(\alpha)}_f \simeq \left(\prod_{\gamma \in P} S^{(f(\gamma))}\right)_*$.

For $\xi \in S^{(\alpha)}_\Gamma$ define $\phi(\xi) = (\phi(\xi)_\gamma)_{\gamma \in P}$ by letting $\phi(\xi)_\gamma \in S^{(f(\gamma))}$ be the sum over all $x \in S$ with $\gamma_x(\xi) = \gamma$. For $\zeta = (\zeta_\gamma)_{\gamma \in P}$ with $\zeta_\gamma \in S^{(f(\gamma))}$ let $\psi(\zeta) := (\xi_1, \dots, \xi_n)$ with $\xi_i = \sum_{\gamma \in P} c_i \zeta_\gamma \in S^{(a_i)}$, where we write $\gamma = (1^{c_1}, 2^{c_2}, \dots)$. It is straightforward from the definitions that ϕ and ψ are inverse to each other. □

Example 4.9. 1. Let S be a projective rational surface. Then $[S] = [\mathbb{A}^0] + b[\mathbb{A}^1] + [\mathbb{A}^2]$ for suitable $b > 0$, and

$$\sum_{n \geq 0} [S^{[n]}] t^n = \prod_{l > 0} \frac{1}{(1 - [\mathbb{A}^{l-1}] t^l)(1 - [\mathbb{A}^l] t^l)^b (1 - [\mathbb{A}^{l+1}] t^l)}.$$

2. Let S be a ruled surface over a curve C. Then

$$\sum_{n \geq 0} [S^{[n]}] t^n = \prod_{l > 0} \Big(\sum_{m \geq 0} [C^{(m)} \times \mathbb{A}^{m(l-1)}] t^{ml} \Big) \Big(\sum_{m \geq 0} [C^{(m)} \times \mathbb{A}^{ml}] t^{ml} \Big).$$

3. If $\widehat{S}$ is the blowup of S in a point then

$$\sum_{n \geq 0} [\widehat{S}^{[n]}] t^n = \frac{\sum_{n \geq 0} [S^{[n]}] t^n}{\prod_{l > 0} (1 - [\mathbb{A}^l] t^l)}.$$

This follows from Theorem 1.1 and Lemma 4.4.

5. The incidence variety

Similar but simpler arguments to those for $S^{[n]}$ can be used for the incidence variety $S^{[n,n+1]} := \{(Z, W) \in S^{[n]} \times S^{[n+1]} \mid Z \subset W\}$, which plays a role in inductive arguments for $S^{[n]}$ [E-S2],[E-G-L]. The Hodge numbers of $S^{[n,n+1]}$ were computed in [Ch1].

Theorem 5.1. $[S^{[n,n+1]}] = \displaystyle\sum_{l=0}^{n} [S \times S^{[l]} \times \mathbb{A}^{n-l}].$

By Theorem 2.1 this immediately implies

Corollary 5.2. $[h(S^{[n,n+1]})] = \displaystyle\bigoplus_{l=0}^{n} [h(S \times S^{[l]}) \otimes L^{\otimes (n-l)}].$

For the proof we introduce a stratification of $S^{[n,n+1]}$. Let

$$\overline{\omega} : S^{[n,n+1]} \to S \times S^{(n)}, (Z, W) \mapsto (\omega_{n+1}(W) - \omega_n(Z), \omega_n(Z)).$$

For $0 \leq m \leq n$ let $(S \times S^{(n)})_m := \{(x, \xi) \in S \times S^{(n)} \mid m_x(\xi) = m\}$, and let $S^{[n,n+1]}_m := \overline{\omega}^{-1}((S \times S^{(n)})_m)_{red}$. The $(S \times S^{(n)})_m$ and the $S^{[n,n+1]}_m$ form stratifications of $S \times S^{(n)}$ and $S^{[n,n+1]}$ respectively into locally closed subvarieties.

Let

$$\overline{g}:\coprod_{m=0}^{n} S\times S^{[m]}\times \mathbb{A}^{n-m}\to S\times S^{(n)};(x,Z,a)\mapsto (x,(n-m)x+\omega_m(Z)).$$

Then Theorem 5.1 follows from the following.

Proposition 5.3. $[S_m^{[n,n+1]}]=[\overline{g}^{-1}((S\times S^{(n)})_m).]$

Proof. If X is a variety with a natural map to $S\times S^{(m)}$ for some $m\geq 0$, we will write X_0 for the preimage of the locus of (x,ξ) with $x\notin \operatorname{supp}(\xi)$. Let $(\mathbb{A}^2,0)^{[n,n+1]}:=\big\{(Z,W)\in(\mathbb{A}^2)^{[n,n+1]}\bigm| Z\subset W, \operatorname{supp}(W)=\{0\}\big\}$ with the reduced structure. In [Ch2] it is shown that $(\mathbb{A}^2,0)^{[n,n+1]}$ has a cell decomposition. Her formula for the numbers of cells of different dimensions implies that $[(\mathbb{A}^2,0)^{[n,n+1]}]=\sum_{l=0}^n[R_l\times\mathbb{A}^{n-l}]$. We now determine $[S_m^{[n,n+1]}]$. First it is easy to see analoguously to the case of $S_{(n)}^{[n]}$ in [Gö2] that $S_n^{[n,n+1]}$ is a locally trivial fibre bundle over S with fibre $(\mathbb{A}^2,0)^{[n,n+1]}$. Therefore

$$[S_n^{[n,n+1]}]=\sum_{l=0}^{n}[S\times R_l\times\mathbb{A}^{n-l}]=\sum_{l=0}^{n}[S_{(l)}^{[l]}\times\mathbb{A}^{n-l}].$$

There is a natural morphism $\sigma:(S_m^{[m,m+1]}\times S^{[n-m]})_0\to S_m^{[n,n+1]}$ given on T-valued points by $((Z,W),X)\mapsto(Z\sqcup X,W\sqcup X)$. σ is obviously a bijection on k valued points. Thus we get

$$[S_m^{[n,n+1]}]=[(S_m^{[m,m+1]}\times S^{[m-n]})_0]=\sum_{l=0}^{m}[(S_{(l)}^{[l]}\times S^{[n-m]})_0\times\mathbb{A}^{m-l}].$$

Now we determine $\overline{g}^{-1}(S\times S^{(n)})_m)$. Let $(S\times S^{[m]})_l=\big\{(x,Z)\in S\times S^{[m]}\bigm| len_x(Z)=l\big\}$. Then $\overline{g}^{-1}((S\times S^{(n)})_m)=\coprod_{l=0}^m(S\times S^{[n-m+l]})_l\times\mathbb{A}^{m-l}$. Furthermore we have a morphism $\phi:(S_{(l)}^{[l]}\times S^{[m-l]})_0\to(S\times S^{[m]})_l$, sending (Y,Z) to $(\operatorname{supp}(Y),Y\sqcup Z)$, which is bijective on k-valued points. Thus $[g^{-1}((S\times S^{(n)})_m)]=\sum_{l=0}^m[(S_{(l)}^{[l]}\times S^{[n-m]})_0\times\mathbb{A}^{m-l}]$.

□

6. Moduli of stable sheaves

Let S be a projective surface over k. Fix $C\in\operatorname{Pic}(S)$ and let H be an ample line bundle on S. We denote by $NS(S)$ the Picard group of S modulo numerical equivalence. The moduli space $M_S^H(C,d)$ of H-semistable rank 2 torsion-free sheaves E with $\det(E)=C$ and $c_2-C^2/4=d$ depends on H via a system of walls and chambers. This dependence has been studied and used by various authors (e.g. [Q1],[F-Q],[E-G],[Gö3]). A class $\xi\in NS(S)+C/2$ is of type (C,d) if $0<-\xi^2\leq d$ and there exists an ample divisor H with $\xi\cdot H=0$. In this case we say that H lies on the corresponding wall. Ample divisors $H,L\in\operatorname{Pic}(S)$ are separated by ξ if $(\xi\cdot H)(\xi\cdot L)<0$. Assume that H and L do not lie on a wall of type (C,d). If they are not separated by a class of type (C,d) we say that they lie in the same chamber of type (C,d). In this case $M_S^L(C,d) = M_S^H(C,d)$. More generally let $F \in \operatorname{Pic}(S)$ be nef with $F^2 \geq 0$ and FC odd. Then $F + nH$ is ample for any ample divisor H, and for n sufficiently large the chamber of $F + nH$ does not depend on H. We will write $M_S^F(C,d) := M_S^{F+nH}(C,d)$ for $n \gg 0$. Let L, H be ample divisors not on a wall of type (C,d). If $2\xi + K_S$ is not effective for all ξ separating L and H (we say that L and H are separated only by good walls), then $M_S^L(C,d)$ is obtained from $M_S^H(C,d)$ by successively blowing up along projective space bundles over products $S^{[l]} \times S^{[n]}$ of Hilbert schemes of points followed by blowdowns of the exceptional divisor to another projective space bundle over $S^{[l]} \times S^{[n]}$. Therefore the proof of Theorem 3.4 in [Gö3] shows:

Proposition 6.1. *Let L, H be ample divisors not on a wall of type (C,d) separated only by good walls. Then*

$$[M_S^L(C,d)] - [M_S^H(C,d)] = [\operatorname{Pic}^0(S)] \cdot \sum_{\xi} [(S \sqcup S)^{[d+\xi^2]}]\big([\mathbb{P}^{d-\xi^2+\xi K_S-\chi(\mathcal{O}_S)-1}] - [\mathbb{P}^{d-\xi^2-\xi K_S-\chi(\mathcal{O}_S)-1}]\big).$$

The sum is over all ξ of type (C,d) with $\xi L > 0 > \xi H$, and we use the convention $\mathbb{P}^{-1} = \emptyset$.

Corollary 6.2. *Under the assumptions of Proposition 6.1, if S is a rational surface, then $[M_S^L(C,d)] - [M_S^H(C,d)]$ is a $\mathbb{Z}$-linear combination of the $[\mathbb{A}^l]$ with $l \leq 4d-3$.*

This follows from Proposition 6.1, and Example 4.9.

Corollary 6.3. *If K_S is numerically trivial, then $[M_S^H(C,d)]$ does not depend on H as long as H does not lie on a wall.*

In [G-Z] and [Gö4] Theta functions for indefinite lattices were introduced to study the wallcrossing. Let Γ be $NS(S)$ with the negative of the intersection form as quadratic form, which we denote by $\langle \ , \ \rangle$. Then for $F, G \in \operatorname{Pic}(S)$ with $F^2 \geq 0$, $G^2 \geq 0$, $F \cdot G > 0$, we define

$$\Theta_{\Gamma,C}^{F,G}(\tau,x) := \sum_{\xi \in \Gamma + C/2} (\mu(\langle \xi, F\rangle - \mu(\langle \xi, G\rangle)q^{\langle \xi,\xi\rangle/2} e^{2\pi i \langle \xi, x\rangle}.$$

Here $\mu(t) = 1$ of $t \geq 0$, and $\mu(t) = 0$ otherwise, and $q = e^{2\pi i\tau}$ for τ in the complex upper half plane $\mathfrak{H}$ and $x \in \Gamma \otimes_{\mathbb{Z}} \mathbb{C}$. This function is defined on a suitable open subset of $\mathfrak{H} \times \Gamma \otimes_{\mathbb{Z}} \mathbb{C}$ and has a meromorphic extension to the whole of $\mathfrak{H} \times \Gamma \otimes_{\mathbb{Z}} \mathbb{C}$. For a meromorphic function $f : \mathfrak{H} \times \Gamma \otimes_{\mathbb{Z}} \mathbb{C}$ and $v \in \Gamma \otimes_Z \mathbb{Q}$ we write

$$f|_v(\tau,x) := q^{\langle v,v\rangle/2} e^{2\pi i\langle v,x\rangle} f(\tau, x+v\tau).$$

Then $\Theta_{\Gamma,C}^{F,G}(\tau,x) = \Theta_{\Gamma}^{F,G}|_{C/2}(\tau,x)$, where we have written $\Theta_{\Gamma}^{F,G} := \Theta_{\Gamma,0}^{F,G}$.

The reason for introducing these theta functions was that they can be expressed in terms of standard theta functions in case $F^2 = G^2 = 0$. In the rest of this section we write $y := e^{2\pi i z}$ for z a complex variable. Recall the standard theta functions

$$\Theta_{\mu,\nu}(\tau,z) := \sum_{n\in Z}(-1)^{n\nu}q^{(n+\mu/2)^2/2}y^{n+\mu/2} \quad (\mu,\nu\in\{0,1\}).$$

If $F^2>0$ and $G^2>0$ or $F\cdot C$ and $G\cdot C$ are odd, then for every $L\in\Gamma$, $\Theta^{F,G}_{\Gamma,C}(\tau,Lz)$ is a power series in $q^{1/8}$ with coefficients Laurent polynomials in $y^{1/2}$. We will write $\Theta^{F,G}_{\Gamma,C}(2\tau,K_Sz)^*$ for the power series in $t^{1/4}$ with coefficients Laurent polynomials in $[\mathbb{A}^1]$, which we obtain by replacing y by $[\mathbb{A}^1]$ and q by $[\mathbb{A}^2]t$ in $\Theta^{F,G}_{\Gamma,C}(2\tau,K_Sz)$. There are no half integer powers of $[\mathbb{A}^1]$, because K_S is characteristic.

The following follows from Theorem 3.4 in [Gö3] in the same way as Theorem 4.1 in [Gö4].

Corollary 6.4. *Assume H,L do not lie on a wall of type (C,d) for any d, and are separated only by good walls. Then, in $K_0(V_k)$*

$$\sum_{d\geq 0}([M^H_S(C,d)]-[M^L_S(C,d)])t^d$$
$$=[\mathrm{Pic}^0(S)]\Big(\sum_{n\geq 0}[S^{[n]}][\mathbb{A}^n]t^n\Big)^2\frac{\Theta^{L,H}_{\Gamma,C}(2\tau,K_Sz)^*}{[\mathbb{A}^1]^{-\chi(\mathcal{O}_S)}([\mathbb{A}^1]-1)}.$$

On the r.h.s. we mean that, after multiplying out, all negative powers of $[\mathbb{A}^1]$ vanish.

In [L-Q1], [L-Q2] a blowup formula was proven for the Euler numbers and the virtual Hodge numbers of moduli spaces of stable rank 2 sheaves on surfaces. In [Ka] a blowup formula is proven for principal bundles. We can show that the formula of [L-Q1], [L-Q2] holds in $K_0(V_k)$. Let H be ample on S and assume that $C\cdot H$ is odd. Let $\widehat{S}$ be the blowup of S in a point and denote by E the exceptional divisor, we denote by H also the pullback of H to $\widehat{S}$.

Theorem 6.5. *Assume $k=\mathbb{C}$. Let $a\in\{0,1\}$ then*

$$\frac{\sum_{d\geq 0}[M^H_S(C+aE,d)]t^d}{\sum_{d\geq 0}[M^H_S(C,d)]t^d}=\frac{\sum_{n\in\mathbb{Z}}[\mathbb{A}^{\binom{2n+a+1}{2}}]t^{(n+\frac{a}{2})^2}}{\prod_{l>0}(1-[\mathbb{A}^{2l}]t^l)}.$$

Proof. In [L-Q1] the authors use virtual Hodge polynomials $e(X:x,y)$ in order to show that there exists a universal power series $Z_a(x,y,t)$ such that

$$\sum_{d\geq 0}e(M^H_{\hat{S}}(C+aE,d),:x,y)t^d=Z_a(x,y,z)\Big(\sum_{d\geq 0}e(M^H_S(C,d):x,y)t^d\Big).$$

To do this they only use the basic property of virtual Hodge polynomials that $e(X\setminus Y:x,y)=e(X:x,y)-e(Y:x,y)$ for Y a closed subvariety of X. So their proof shows that there is a universal power series $Y_a(t)$ in $K_0(V_k)[[t]]$, such that

$$\sum_{d\geq 0}[M^H_{\hat{S}}(C+aE,d)]t^d = Y_a(t)\Big(\sum_{d\geq 0}[M^H_S(C,d)]t^d\Big).$$

In the paper [L-Q2] they compare the wallcrossing on a rational ruled surface and its blowup in a point to determine $Z_a(x,y,t)$. We can translate their argument into our language, where it proves the theorem: Let H, L be ample on S with $C\cdot H$ and $C\cdot L$ odd. Write $M_d := [M^H_S(C,d)] - [M^L_S(C,d)]$ and $M_{a,d} := [M^H_{\hat{S}}(C+aE,d)] - [M^L_{\hat{S}}(C+aE,d)]$. Write $\Gamma = H^2(S,\mathbb{Z})$. Then, as $Y_a(t)$ is universal, we get, using Corollary 6.4,

$$Y_a(t) = \frac{\sum_{d\geq 0} M_{a,d}t^d}{\sum_{d\geq 0} M_d t^d} = \frac{\Theta^{L,H}_{\Gamma\oplus\langle E\rangle, C+aE}(2\tau, K_S z)^*\Big(\sum_{n\geq 0}[\widehat{S}^{[n]}]t^n\Big)^2}{\Theta^{L,H}_{\Gamma,C}(2\tau, K_S z)^*\Big(\sum_{n\geq 0}[S^{[n]}]t^n\Big)^2}$$

By definition it is obvious that $\Theta^{L,H}_{\Gamma\oplus\langle E\rangle,C+aE}(\tau,K_S z) = \theta_{a,0}(\tau,z)\Theta^{L,H}_{\Gamma,C}(\tau,K_S z)$. The result follows by Example 4.9.(3) and the identity

$$\theta_{a,0}(2\tau,z)^* = \sum_{n\in\mathbb{Z}}[\mathbb{A}^{\binom{2n+a+1}{2}}]t^{(n+\frac{a}{2})^2}.$$

□

Corollary 6.4 and Theorem 6.5 imply in particular that the computations of [Gö4] hold in the Grothendieck group of varieties (replacing y in the formulas there by $[\mathbb{A}^1]$). As there, we use the following elementary fact.

Remark 6.6. Let S be the blowup of a ruled surface in finitely many points and F the pullback of a fibre of the ruling. Assume $F\cdot C$ is odd. Then $M^F_S(C,d)=\emptyset$ for all d.

In particular we get the following results.

Corollary 6.7. *Let S be a rational surface.*

1. *Let $C\in \operatorname{Pic}(S)\setminus\{0\}$. Let H be an ample divisor not on a wall of type (C,d) and assume that $K_S\cdot H\leq 0$, then $[M^H_S(C,d)]$ is a $\mathbb{Z}$-linear combination of $[\mathbb{A}^l]$ with $l\leq 4d-3$. (*[Gö4]*, Prop.4.9),*
2. *Let S be the blowup of $\mathbb{P}^2$ in 9 points, with exceptional divisors $E_1,\dots,E_9$ and let H be the pullback of the hyperplane class. Let $F := 3H - E_1 - \dots - E_9$, and let $C\in H^2(S,\mathbb{Z})$ with C^2 odd.*
   Then $[M^F_S(C,d)] = [S^{[2d-3/2]}]$. ([Gö4]*, Thm.7.3).*

Remark 6.8. For S a rational surface it follows from [dC-M2] that the rational Chow groups of the $S^{[n]}$ are finite dimensional. By Remark 2.4 and the discussion before Proposition 6.1 it follows that under the conditions of Corollary 6.7.1. the calculations in [Gö4] hold also in the Chow ring of $M^H_S(C,d)$. In particular in 2. we get that $M^F_S(C,d)$ and $S^{[2d-3/2]}$ have the same Chow groups with rational coefficients.

As a final example we determine the classes of some moduli spaces over a ruled surface S over an elliptic curve E with a section σ with minimal self-intersection $\sigma^2 = 1$.

Proposition 6.9. *Let F be the class of a fibre of the ruling. We write $G := 2\sigma - F$. Let $C \in NS(S)$ with C^2 odd. Then $[M_S^G(C,d)]$ is the coefficient of $t^{2d-1/2}$ in*

$$[E]\prod_{m>0}\prod_{i=0,1}\left(1+[E]\left(\sum_{l\geq 1}[\mathbb{P}^{l-1}][\mathbb{A}^{m(2l-i)}]t^{2lm}\right)\right)^2 \cdot \prod_{n>0}((1-[\mathbb{A}^{n-1}]t^n)(1-[\mathbb{A}^n]t^n)^2(1-[\mathbb{A}^{n+1}]t^n))^{(-1)^n}$$

Proof. C^2 odd implies $C\cdot F$ odd and $C\cdot G$ odd, therefore Remark 6.6 implies that $M_S^F(C,d) = \emptyset$. By Proposition 6.1 $[M_S^G(C,d)]$ depends only on the numerical equivalence class of C. Therefore we can assume that $C = \sigma = \frac{G+F}{2}$ or $C = \sigma - F = \frac{G-F}{2}$. We write $\Gamma = NS(S)$ with the negative of the intersection form. We note that $K_S = -G$. By Corollary 6.4 $[M_S^G(C,d)]$ is the coefficient of t^d in

$$[E]\Big(\sum_{n\geq 0}[S^{[n]}][\mathbb{A}^n]t^n\Big)^2 \frac{\Theta^{F,G}_{\Gamma,\frac{F+G}{2}}(2\tau,-Gx)^* + \Theta^{F,G}_{\Gamma,\frac{F-G}{2}}(2\tau,-Gx)^*}{[\mathbb{A}^1]-1}.$$

For $n \geq 1$ $E^{(n)}$ is a locally trivial bundle over E with fibre $\mathbb{P}^{n-1}$, thus $[E^{(n)}] = [E\times\mathbb{P}^{n-1}]$. Using Example 4.9, we only need to show that

$$\begin{aligned}\Theta^{F,G}_{\Gamma,\frac{F+G}{2}}&(2\tau,-Gz) + \Theta^{F,G}_{\Gamma,\frac{F-G}{2}}(2\tau,-Gz)\\ &= (y^{1/2}-y^{-1/2})q^{1/4}\prod_{n>0}\left((1-q^{n/2}y^{-1})(1-q^{n/2})^2(1-q^{n/2}y)\right)^{(-1)^n}.\end{aligned}$$

Let L be the lattice generated by $F/2, G/2$. 0 and $G/2$ are a basis of L modulo Γ. Therefore $\Theta^{F,G}_{\Gamma} + \Theta^{F,G}_{\Gamma,G} = \Theta^{F,G}_{L}$ and $\Theta^{F,G}_{\Gamma,\frac{F+G}{2}} + \Theta^{F,G}_{\Gamma,\frac{F-G}{2}} = \Theta^{F,G}_{L,\frac{F+G}{2}}$. By $F^2 = G^2 = 0$ and $F\cdot G = 2$, formula (2.14) in [Gö4] gives that

$$\Theta^{F,G}_{L}(2\tau,x) = \frac{\eta(\tau)^3\theta_{1,1}(\tau,\langle(-F+G)/2,x\rangle)}{\theta_{1,1}(\tau,\langle -F/2,x\rangle)\theta_{1,1}(\tau,\langle G/2,x\rangle)}.$$

Easy computations give

$$\Theta^{F,G}_{L,\frac{F+G}{2}}(2\tau,x) = \Theta^{F,G}_{L}|_{\frac{F+G}{2}}(2\tau,x) = \frac{\eta(\tau)^3\theta_{1,1}(\tau,\langle(-F+G)/2,x\rangle)}{\theta_{0,1}(\tau,\langle -F/2,x\rangle)\theta_{0,1}(\tau,\langle G/2,x\rangle)},$$

$$\Theta^{F,G}_{L,\frac{F+G}{2}}(2\tau,-Gz) = \frac{\eta(\tau)^3\theta_{1,1}(\tau,z)}{\theta_{0,1}(\tau,z)\theta_{0,1}(\tau,0)}.$$

The result now follows from the product formulas

$$\theta_{1,1}(\tau,z) = q^{\frac{1}{8}}(y^{\frac{1}{2}} - y^{-\frac{1}{2}}) \prod_{n>0}(1-q^n)(1-q^n y)((1-q^n y^{-1}),$$
$$\theta_{0,1}(\tau,z) = \prod_{n>0}(1-q^n)(1-q^{n-\frac{1}{2}}y)((1-q^{n-\frac{1}{2}}y^{-1}).$$

□

7. Remarks and Speculations

Recently there has been a lot of interest in motivic integration [Ko], [D-L], [Lo]. This is a method to determine the class of a variety in a ring $\widehat{M}_k$ which is obtained from $K_0(V_k)$ via localization at the class of $[\mathbb{A}^1]$ and a suitable completion. In many cases the result of motivic integration is an identity between the classes of smooth projective varieties or projective varieties with finite quotient singularities in $\widehat{M}_k$, and one would expect that the identity does indeed hold in $K_0(V_k)$. Using Conjecture 2.5 this conjecturally gives an isomorphism of motives and of the Chow groups with rational coefficients.

Two important instances of this are the following:

(1) Let X, Y be a smooth projective birational k-varieties with $K_X = K_Y = 0$. Then motivic integration is used to show that X and Y have the same Hodge numbers ([Ko],[D-L]) (that they have the same Betti numbers was shown before in [Ba] via p-adic integration). In fact they have the same class in $\widehat{M}_k$. It should be true that $[X] = [Y]$ in $K_0(V_k)$.

This happens e.g. in some cases for moduli spaces of K3-surfaces: Let S be a K3 surface with Pic $(S) = \mathbb{Z}L$ for L an ample divisor. Then $[M_S^L(L, L^2/2+3)] = [S^{[L^2/2+3]}]$. This follows from the proof of Proposition 1.9 in [G-H] We write $M := M_S^L(L, L^2/2+3)$, $X := S^{[L^2/2+3]}$. There is a diagram of birational maps $M \xleftarrow{\phi} N \xrightarrow{\psi} X$ together with stratifications $M = \coprod M_l$, $N = \coprod N_l$, $X = \coprod X_l$ with $N_l = \phi^{-1} M_k = \psi^{-1} X_l$, such that $N_l \to M_l$ and $N_l \to X_l$ are $\mathbb{P}^{l-1}$-bundles.

(2) In a similar way the results of motivic integration on the McKay correspondence [D-L],[R] make it seem likely that the following holds in the Grothendieck ring of varieties. Let X be a smooth projective variety acted upon by a finite group $G \subset Sl(n, \mathbb{C})$ such that the action preserves the canonical divisor. Let Y be a crepant resolution of X/G. Choosing an eigenbasis, we can write $g = \text{diag}\,(\epsilon^{a_1}, \dots, \epsilon^{a_n})$, where ϵ is a primitive r^{th} root of unity for r the order of g. Write $a(g) = \frac{1}{r}\sum a_i$ and let $C(g)$ be the centralizer of $g \in G$. Then one should have in the Grothendieck ring of varieties $[Y] = \sum_{[g]}[X^g/C(g)][\mathbb{A}^{a(g)}]$. Here $[g]$ runs through the conjugacy classes elements of G. In particular we should have $A^i(Y) = \sum_{[g]} A^{i-a(g)}(X^g)^{C(g)}$.

Using [Gö5], Theorem 1.1 and the main result of [dC-M2] say that this is true for the resolution $\omega_n : S^{[n]} \to S^{(n)}$ of the symmetric power $S^{(n)} = S^n/\mathfrak{S}_n$.

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