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# ON THE MORSE INEQUALITIES FOR GEODESICS ON LORENTZIAN MANIFOLDS

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Abstract. We extend the classical Morse inequalities in Riemannian Geometry to the geodesics joining two nonconjugate points on a Lorentzian manifold. The Morse inequalities are obtained developing a Morse Theory for a class of strongly indefinite functionals.

## 1. Introduction

Morse Theory for Riemannian geodesics relates the set of the geodesics joining two nonconjugate points on a complete Riemannian manifold to the topological structure of the manifold. In particular the Morse Inequalities give a lower bound on the number of such geodesics by the Betti numbers of the based loop space. Let $(\mathcal{M}, g)$ be a smooth, connected and complete Riemannian manifold and let $\Omega(\mathcal{M})$ be the based loop space of the manifold $\mathcal{M}$, equipped with the compact–open topology. Moreover, let $H_k(\Omega(\mathcal{M}); \mathcal{K})$ be the $k$–th singular homology group of the space $\Omega(\mathcal{M})$ with respect to the field $\mathcal{K}$. The Betti numbers $\beta_k(\Omega(\mathcal{M}); \mathcal{K})$, $k \in \mathbf{N}$, are defined as the dimension of $H_k(\Omega(\mathcal{M}); \mathcal{K})$. Let $p$ and $q$ two nonconjugate points for the Riemannian metric $g$ on $\mathcal{M}$ and denote by $G(p, q)$ the set of the geodesics joining $p$ and $q$. The Morse Relations state that there exists a formal series $Q(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k$, with $a_k \in \mathbf{N} \cup \{+\infty\}$, such that

$$\sum_{x \in G(p,q)} \lambda^{m(x)} = \sum_{k=0}^{\infty} \beta_k(\mathcal{M}; \mathcal{K}) \lambda^k + (1 + \lambda) Q(\lambda)$$

The series $\sum_{k=0}^{\infty} \beta_k(\mathcal{M}; \mathcal{K}) \lambda^k$ is often called Poincaré polynomial of the manifold $\mathcal{M}$ with coefficients in the field $\mathcal{K}$.

For any $k \in \mathbf{N}$ let $G_k(p, q)$ be the number of geodesics $z$ in $G(p, q)$ having Morse index $m(z)$ equal to $k$. From the Morse Relations one can deduce the Morse Inequalities for the geodesics joining $p$ and $q$, which state that for any $k \in \mathbf{N}$,

$$G_k(p, q) \geq \beta_k(\Omega(\mathcal{M}); \mathcal{K}). \tag{1.1}$$

The proof of the Morse Relations is obtained applying the abstract Morse Theory to the action integral

$$E(x) = \int_0^1 g(x(s))[\dot{x}(s), \dot{x}(s)] \mathrm{d}s \tag{1.2}$$

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defined on the infinite dimensional manifold of the sufficiently smooth curves on $\mathcal{M}$ joining $p$ with $q$, see [Bo,Mo,Mi,Pa]

The extension of Morse Theory to semiriemannian manifolds and in particular to Lorentzian ones immediately presents some difficulty, because the Morse Relations and the Morse Inequalities (1.1) do not make sense. Indeed the Morse index of any geodesic as a critical point of the action integral $E$ is equal to $+\infty$, because of the indefiniteness of the metric. On the other hand one could still require to relate the set of geodesics joining two points on a semiriemannian manifold to the topology of the manifold itself.

Functionals defined on Hilbert manifolds and admitting critical points with Morse index equal to $+\infty$ are often called *strongly indefinite*. There are many examples of strongly indefinite functionals which naturally arise in Nonlinear Analysis and Differential Geometry, as in the study of periodic orbits of Hamiltonian systems, nonlinear wave equations, symplectic geometry and semiriemannian geometry. The study of the critical points of strongly indefinite functionals and the developments of a Morse Theory for them has been the object of several studies in the last 25 years. After the first results on the existence of critical points for strongly indefinite functionals and applications to the study of periodic orbits of Hamiltonian systems [R,AZ], Morse theoretic results for strongly indefinite functionals have been obtained by several authors [CZ,Sz,CLL,Ab,AM], with applications to asymptotically linear Hamiltonian systems. We also mention the deep work of A. Floer to develop a Morse Theory for the action in symplectic geometry. The papers [CZ,Sz,CLL,Ab,AM] all deal with functionals $f(x)$ defined on a Hilbert space $H$ and having the form

$$f(x) = 1/2(Lx, x) + b(x), \tag{1.3}$$

where $L$ is a symmetric Fredholm operator on $H$ and $b(x)$ is a smooth function with compact gradient. In [CZ,Sz,CLL] a Galerkin finite dimensional reduction is developed to get some Morse theoretic information on $f$ as a limit of the Morse properties of the finite dimensional restrictions. A crucial property of the functional (1.3) for the results in the papers [CZ,Sz,CLL] is that the principal part $L$ of the second derivative of the functional $f$ is independent on the critical points of the functional. Indeed it is $f''(x) = L + b''(x)$ and $b''(x)$ is a compact operator at any critical point $x$. This fact allows to develop the Galerkin reduction with a sequence of finite dimensional subspaces $(H_n)_{n\in\mathbf{N}}$ such that $\|P_nL - LP_n\| = o(1)$ in the norm of the convergence of continuous operators, where $P_n$ denotes the orthogonal projector of the Hilbert space $H$ onto $H_n$. On the other hand the second derivative of the action integral at a geodesic $z : [0,1] \to \mathcal{M}$ joining two points on an arbitrary semiriemannian manifold $(\mathcal{M}, g)$ is given by

$$E''(z)[\zeta, \zeta'] = \int_0^1 g(z)[D_s\zeta, D_s\zeta']\mathrm{d}s - \int_0^1 g(z)[R(\dot z, \zeta)\dot z, \zeta']\mathrm{d}s, \tag{1.4}$$

for any couple of smooth vector fields $\zeta$, $\zeta'$ along $z$ such that $\zeta(0) = \zeta'(0) = 0$, $\zeta(1) = \zeta'(1) = 0$, where $R$ denotes the curvature tensor for the metric $g$. By (1.4) it follows that the principal part of $E''(z)$ depends on the geodesic $z$.

An abstract Morse Theory for a class of strongly indefinite functionals defined on Hilbert manifolds, so that the principal part of the second derivative may depend on the critical points, has been recently developed in [ABFM] and applications to Lorentzian Geometry have been obtained. In this note we present the main results of [ABFM].

## 2. Morse Inequalities for strongly indefinite functionals

In this section we state the main results on an abstract Morse Theory for a class of strongly indefinite functionals. First of all we define the relative index for a class of bilinear forms on a Hilbert space. It extends the usual notion of index of a bilinear form to cases in which the index can be equal to $+\infty$ and it allows to define the *relative Morse index* for critical points of functionals whose second derivative is a bilinear form belonging to the class. Let $H$ be a real Hilbert and let $a$, $a_0\colon H \times H \longrightarrow \mathbf{R}$ be two continuous, symmetric, and nondegenerate bilinear forms on $H$ such that $k = a - a_0$, is a bilinear form defining a compact operator. Let $A$ and $A_0$ be the linear isomorphisms on $H$ induced by the forms $a$ and $a_0$ and denote by $V^+(A)$ and $V^-(A)$ the maximal $A$–invariant subspaces on which $A$ is respectively positive definite and negative definite. Analogously, the $A_0$–invariant subspaces $V^+(A_0)$ and $V^-(A_0)$ on which $A_0$ is positive definite and negative definite are defined. Then the index of $a$ relatively to $a_0$, denoted by $j(a, a_0)$ is defined as follows:

$$j(a, a_0) = \dim(V^-(A) \cap V^+(A_0)) - \dim(V^+(A) \cap (V^-(A_0))). \tag{2.1}$$

The relative index $j(a, a_0)$ is an integer number (possibly negative) and it is equal to the index of the form $a$ (i.e. the maximal dimension of a subspace where $a$ is negative definite) if $a_0$ is positive definite.

We introduce now a class of strongly indefinite functionals. We fix a (possibly infinite dimensional) Riemannian manifold $(\Omega, h)$, a Hilbert space $F$, a closed affine submanifold $H = e_0 + H_0$ of $F$, where $H_0$ is an infinite dimensional, closed and separable subspace of $F$ and $e_0 \in F$. Finally we set $\mathcal{Z} = \Omega \times H$. We consider $C^2$ functionals $f\colon \mathcal{Z} \longrightarrow \mathbf{R}$ such that any critical point $z = (x, y)$ is nondegenerate and the second derivative of $f''(z)$ at $z$ satisfies the following assumption:

**(A0)** *for any critical point $z$ of $f$, the second derivative $f''(z)$ is of the form $f''(z) = a_0(z) + k(z)$, where $k(z)$ and $a_0(z)$ are symmetric and continuous bilinear forms on the (Hilbert) tangent space $T_z\mathcal{Z} = T_x\Omega \times H_0$. Moreover, the form $k(z)$ defines a compact operator on $T_z\mathcal{Z}$, the form $a_0(z)$ is nondegenerate and $V^+(a_0(z)) = T_x\Omega$ and $V^-(a_0(z)) = H_0$.*

Since $H_0$ is infinite dimensional, the Morse index $m(z)$ is equal to $+\infty$ for any critical point $z$ of $f$ . On the other hand the *relative Morse index* $j(z) = j(f''(z), a_0(z))$ is well defined.

In order to state the Morse Inequalities for the strongly indefinite functional $f\colon \mathcal{Z} \longrightarrow \mathbf{R}$, we need the $(PS)^*$ compactness condition, which is a version of the classical Palais–Smale condition, very useful in the study of strongly indefinite functionals, see [Ab]. We fix an orthonormal basis $\{e_1, e_2, ...., e_n, ....\}$ of $H_0$ and set for any $n \in \mathbf{N}$, $H_{0,n} = \mathrm{Span}\{e_1, e_2, ....., e_n\}$, $H_n = e_0 + H_{0,n} \subset H$ and $\mathcal{Z}_n = \Omega \times H_n \subset \mathcal{Z}$. Finally, for any $n \in \mathbf{N}$ we shall denote by $f_n$ the restriction of $f$ to $\mathcal{Z}_n$.

**Definition 2.1.** Let $f \colon \mathcal{Z} \longrightarrow \mathbf{R}$ be a $C^1$ functional, we say that $f$ satisfies the $(PS)^*$ condition if any sequence $(z_n)_{n\in\mathbf{N}}$ such that $z_n \in \mathcal{Z}_n$ for any $n \in \mathbf{N}$, $f(z_n)_{n\in\mathbf{N}}$ is bounded and $\|\mathrm{grad} f_n(z_n)\| \to 0$ as $n \to \infty$, contains a converging subsequence, where $\mathrm{grad} f_n(z_n)$ is the gradient of $f_n$ with respect to the restricted Riemannian structure on $\mathcal{Z}_n$.

We state now the *Morse Relations* for the functional $f$ in terms of the singular homology groups of the manifold $\mathcal{Z}$ (or equivalently of the manifold $\Omega$).

**Theorem 2.2.** *Let $f \colon \mathcal{Z} \longrightarrow \mathbf{R}$ be a $C^2$ functional such that all the critical points of $f$ are nondegenerate. Assume that the second derivative at any critical point satisfies* (A0) *and $f$ satisfies $(PS)^*$. Moreover, assume that:*

(A1) *for any $x \in \Omega$, $\sup_{y\in H} f(x, y) < +\infty$;*

(A2) *for any $n \in \mathbf{N}$, $n$ sufficiently large, there exists $R_n > 0$ such that, setting for any $R > 0$,*

$$\mathcal{Z}_n(R) = \{z = (x, y) \in \mathcal{Z}_n : \|y - e_0\|_E \geq R\},$$

*we have:*

(A2i) $\inf_{z\in\mathcal{Z}_n(R_n)} \|\mathrm{grad} f_n(z_n)\| > 0$*;*

(A2ii) *there exists $a_n \in \mathbf{R}$ such that $f_n^{a_n} = \{z \in \mathcal{Z}_n : f_n(z) \leq a_n\} \subset \mathcal{Z}_n(R_n)$.*

*Let $K(f)$ be the set of the critical points of $f$ and set $j(z) = j(f''(z), a_0(z))$ for any $z \in K(f)$. Then for any field $\mathcal{K}$ there exists a Laurent series $Q(\lambda) = \sum_{k=-\infty}^{k=+\infty} a_k \lambda^k$, with $a_k \in \mathbf{N} \cup \{+\infty\}$ such that*

$$\sum_{z\in K(f)} \lambda^{j(z)} = \sum_{k=0}^{\infty} \beta_k(\mathcal{Z}; \mathcal{K})\lambda^k + (1 + \lambda)Q(\lambda).$$

*Moreover the Morse Inequalities hold; let $N_k(f)$ the number of critical points $z$ of $f$ having relative index $j(z) = k$, for any field $\mathcal{K}$ and for any $k \in \mathbf{N}$, we have:*

$$N_k(f) \geq \beta_k(\mathcal{Z}; \mathcal{K}) = \beta_k(\Omega; \mathcal{K}). \tag{2.2}$$

The Morse Inequalities (2.2) can be extended to a class of strongly indefinite functionals not satisfying the $(PS)^*$ condition. Indeed in [ABFM] it is shown that inequalities (2.2) still hold if the functional $f$ does not satisfy the $(PS)^*$ condition, but it is possible to construct a suitable family of perturbing functionals $(f_\delta)_{\delta>0}$ satisfying $(PS)^*$, assumptions (A0)–(A2) and such that $f_\delta \to f$ uniformly on any bounded subset of $\mathcal{Z}$. However in this case it is an open problem if the Morse Relations hold. Actually it is not clear how to completely control the growth of the homology groups of the sublevels of the functionals $f_\delta$ as $\delta \to 0$. However the Morse Inequalities still hold.

The proof of Theorem 2.2 and its generalization to functionals not satisfying $(PS)^*$ are based on a Galerkin reduction argument. We consider the sequence $(H_n)$ of finite dimensional, linear submanifold introduced above and consider the restriction $f_n$ of $f$ to $H_n$. The restricted functionals $f_n$ are unbounded, but the Morse index at any critical point is well defined and the Morse Inequalities can be proved, under assumptions (A1)-(A2). Some delicate arguments on approximation of critical points of the functional $f$ by the critical points of the restricted functionals $f_n$ and on evaluation of the relative index of a critical point of $f$ by the Morse index of the approximating critical points of $f_n$ allow to pass to the limit in the Morse Inequalities for the functional $f$ and to obtain the Morse Inequalities (2.2). In order to obtain these approximation results, the implicit function theorem plays a basic role. Moreover we do not need the assumption that the commutator of the principal part of $f''(z)$ with the orthogonal projections onto $H_n$ is infinitesimal (as $n \to +\infty$), as in [CZ,CLL,Sz]. Indeed the finite dimensional linear submanifold $H_n$ can be arbitrarily chosen, except that $H_n \subset H_{n+1}$ and $\bigcup_{n\in\mathbf{N}} H_n$ is dense in $H$.

## 3. The Morse Inequalities in Lorentzian Geometry

In this section we apply the abstract result of Section 2 to state the Morse Inequalities for the geodesics joining two nonconjugate points on a *standard stationary* or an *orthognal splitting* Lorentzian manifold. We consider a connected semiriemannian manifold $(\mathcal{M}, g)$, the *index* of the metric $g$ is the number $\nu(g)$ of the negative eigenvalues of the bilinear form $g(z)$ on the tangent space $T_z\mathcal{M}$ at $z$ to $\mathcal{M}$. It does not depend on $z \in \mathcal{M}$. The semiriemannian manifold $(\mathcal{M}, g)$ is called *Riemannian* if $\nu(g) = 0$ and it is called *Lorentzian* if $\nu(g) = 1$. The interest to study Lorentzian manifolds comes from General Relativity, where gravitational fields are modeled by four dimensional Lorentzian manifolds, also called *spacetimes*. We refer to the books [BEE,ON] for the basic properties of semiriemannian manifolds. A smooth curve $z(s) : I \longrightarrow \mathcal{M}$ is a *geodesic* if $D_s\dot{z} = 0$, where $D_s$ denotes the covariant derivative along $z$ induced by the Levi-Civita connection of $g$ and $\dot{z}$ is the tangent vector field along $z$. The geodesics on a semiriemannian manifold satisfy a variational principle. The geodesics joining two fixed points $p$ and $q$ are the critical points of the action integral

$$E(z) = \int_0^1 g(z(s))[\dot{z}(s), \dot{z}(s)]\mathrm{d}s \tag{3.1}$$

defined on the infinite dimensional Sobolev manifold $\Omega^{1,2}(p, q; \mathcal{M})$ of the curves $z(s) : [0,1] \longrightarrow \mathcal{M}$ such that $z(0) = p$, $z(1) = q$, $z$ is absolutely continuous and its derivative $\dot{z}$ is square integrable. It is well known that the space $\Omega^{1,2}(p, q; \mathcal{M})$ is equipped with a structure of infinite dimensional manifold modelled on the Sobolev–Hilbert space $H^{1,2}([0,1], \mathbf{R}^n)$. If $z \in \Omega^{1,2}(p, q; \mathcal{M})$, the tangent space $T_z\Omega^{1,2}(p, q; \mathcal{M})$ at $z$ is given by

$$T_z\Omega^{1,2}(p, q; \mathcal{M}) = \{\zeta \in \Omega^{1,2}((p, 0), (q, 0); T\mathcal{M}) : \pi \circ \zeta = z\},$$

where $T\mathcal{M}$ denotes the tangent bundle of $\mathcal{M}$ and $\pi\colon T\mathcal{M} \to \mathcal{M}$ is the bundle projection.

Let $z$ be a geodesic joining $p$ and $q$, the second derivative

$$E''(z) : T_z\Omega^{1,2}(p, q; \mathcal{M}) \times T_z\Omega^{1,2}(p, q; \mathcal{M}) \to \mathbf{R}$$

of $E$ at $z$ is given by (1.4), for any couple of tangent vectors $\zeta, \zeta' \in T_z\Omega^{1,2}(p, q; \mathcal{M})$. Since the curvature term is compact with respect to the $H^{1,2}$ topology, $E''(z)$ is a compact perturbation of the nondegenerate bilinear form $a_0(z)[\zeta, \zeta'] = \int_0^1 g(z)[D_s\zeta, D_s\zeta']ds$. So we define the *relative index* $j(z)$ of a semiriemannian geodesic setting

$$j(z) = j(E''(z), a_0(z)). \tag{3.2}$$

The index $j(z)$ is well defined and $j(z) \in \mathbf{Z}$. Moreover, if the metric $g$ is Riemannian, then $a_0(z)$ is positive definite and $j(z)$ equals to the Morse index $m(z, E)$ of the geodesic.

A geodesic $z \in \Omega^{1,2}(p, q; \mathcal{M})$ is said nondegenerate if it is a nondegenerate critical point of the action integral $E$, i.e. the second derivative defines an invertible linear operator on the tangent space $T_z\Omega^{1,2}(p, q; \mathcal{M})$. Since $E''(z)$ defines a Fredholm operator of index 0, this is equivalent to require that the kernel of $E''(z)$ is trivial and this is equivalent to say that there are no solutions of the Jacobi equations $D_s^2\zeta + R(\zeta, \dot{z})\dot{z} = 0$ such that $\zeta(0) = 0$, $\zeta(1) = 0$. Two points $p$ and $q$ of a semiriemannian manifold $(\mathcal{M}, g)$ are said *nonconjugate* if any geodesic joining $p$ and $q$ is nondegenerate. From a variational point of view, the nonconjugation of the points $p$ and $q$ means that the action integral is a Morse function, i.e. its critical points are nondegenerate. Using the Sard theorem it can be proved that all the couple of points in $\mathcal{M}$, except for a nowhere dense set, are nonconjugate, see [Mi].

The variational properties of the action integral in the Riemannian case are completely known. If a Riemannian manifold is complete, the functional $E$ is bounded from below and satisfies the Palais–Smale condition. Then any complete Riemannian manifold is geodesically connected, i.e. any couple of points of the manifold can be joined by a geodesic. Moreover the Morse Relations hold for any couple of nonconjugate points.

The situation is completely different in the case of a semiriemannian manifold with positive index, in particular a Lorentzian manifold. In this case there are many significant counterexamples to the geodesic connectedeness and then to the Morse Inequalities. Calabi and Markus, see [CM], first gave some example of geodesically complete but not geodesically connected Lorentzian manifolds (we recall that the geodesically completeness is equivalent to the metric completeness for a Riemannian manifold). Moreover there are compact not geodesically connected Lorentzian manifolds, see [W]. We shall present some of these counterexamples at the end of the note.

The variational properties of the action integral in Lorentzian geometry have been studied in the last years and some results have obtained on the geodesic connectedeness (see [Ma] and the references therein). In this note we state the Morse

Inequalities for the geodesics joining two nonconjugate points on two classes of Lorentzian manifolds, the stationary and the orthogonal splitting Lorentzian manifolds.

**Definition 3.1.** A Lorentzian manifold $(\mathcal{M},g)$ is said *splitting* if $\mathcal{M} = \mathcal{M}_0 \times \mathbf{R}$, where $\mathcal{M}_0$ is a smooth connected manifold, and the metric $g$ has the following form: for any $z = (x,t) \in \mathcal{M}$ and for any $\zeta = (\xi,\tau) \in T_z\mathcal{M} = T_x\mathcal{M}_0 \times \mathbf{R}$,

$$g(z)[\zeta,\zeta] = \langle \alpha(x,t)\xi,\xi\rangle + 2\langle \delta(x,t),\xi\rangle\tau - \beta\,(z)\,\tau^2, \tag{3.3}$$

where $\langle\cdot,\cdot\rangle$is a Riemannian metric on $\mathcal{M}_0$, $\alpha(x,t)$ is a positive linear operator on $T_x\mathcal{M}_0$, smoothly depending on $z$, $\delta(x,t)$ is a smooth vector field tangent to $\mathcal{M}_0$ and $\beta\,(z)$ is a smooth scalar field on $\mathcal{M}$. The metric $g$ is said *orthogonal splitting* if $\delta(x,t) \equiv 0$, while the metric $g$ is said *standard stationary* if the linear operator $\alpha$, the vector field $\delta$ and the scalar field $\beta$ do not depend on the variable $t$.

If the metric is stationary we can assume without any loss of generality that the the linear operator $\alpha(x)$ is equal to the identity map and the metric has the following form:

$$g(z)[\zeta,\zeta] = \langle \xi,\xi\rangle + 2\langle \delta(x),\xi\rangle\tau - \beta(x)\tau^2,$$

while an orthogonal splitting metric takes the form

$$g(z)[\zeta,\zeta] = \langle \alpha(x,t)\xi,\xi\rangle - \beta(x,t)\tau^2.$$

**Remark 3.2.** We recall that by a result of Geroch (see [BEE,CBC]), any time-oriented globally hyperbolic Lorentzian manifold is isometric to an orthogonal splitting Lorentzian manifold.

We first consider stationary Lorentzian manifolds. First of all it can be easily proved that the relative index $j(z)$ of any geodesic on a stationary Lorentzian manifold is nonnegative, see [ABFM]. This essentially follows by the fact the second derivative $E''(z)$ is still negative definite on a maximal subspace where the principal part $a_0(z)$ is negative definite. For this class of Lorentzian manifolds, the Morse Relations and the Morse Inequalities hold, under some growth condition on the coefficients of the metric.

**Theorem 3.3.** *Let $(\mathcal{M},g)$ be a stationary Lorentzian manifold and let $z_0 = (x_0,t_0)$, $z_1 = (x_1,t_1)$ be two nonconjugate points of $\mathcal{M}$. Assume that:*

- (S1) *The Riemannian manifold $(\mathcal{M}_0,\langle\cdot,\cdot\rangle)$ is complete;*
- (S2) *There exists two positive constants $0 < \nu \le M$ such that for any $z \in \mathcal{M}$, $\nu \le \beta(z) \le M$;*
- (S3) $\sup\{\langle\delta(x),\delta(x)\rangle_0, x \in \mathcal{M}_0\} < +\infty$.

*Let $G(z_0,z_1)$ the set of the geodesics joining $z_0$ and $z_1$. Then for any field $\mathcal{K}$ there exists a series $Q(\lambda) = \sum_{k=0}^{k=+\infty} a_k\lambda^k$, with $a_k \in \mathbf{N}\cup\{+\infty\}$ such that:*

$$\sum_{z\in G(z_0,z_1)} \lambda^{j(z)} = \sum_{k=0}^{\infty} \beta_k(\mathcal{Z};\mathcal{K})\lambda^k + (1+\lambda)Q(\lambda). \tag{3.4}$$

*Moreover the Morse inequalities hold; let, for any* $k \in \mathbf{N}$, $G_k(z_0, z_1)$ *the number of geodesics joining* $z_0$ *and* $z_1$ *and having index* $k$. *Then*

$$G_k(z_0, z_1) \geq \beta_k(\mathcal{Z}; \mathcal{K}) = \beta_k(\Omega(\mathcal{M}); \mathcal{K}), \tag{3.5}$$

*where* $\beta_k(\Omega(\mathcal{M}), \mathcal{K})$ *is the* $k$*–th Betti number of the based loop space* $\Omega(\mathcal{M})$ *of* $\mathcal{M}$ *with coefficients in the field* $\mathcal{K}$.

**Remark 3.4.** We point out that since the relative index of any geodesic in a stationary spacetime is nonnegative, the series $Q(\lambda)$ in Theorem 3.3 is not a Laurent one as in the abstract Theorem 2.2. For this reason the Morse Relations (3.4) and the Morse Inequalities (3.5) involve all the geodesics joining $z_0$ and $z_1$.

The proof of Theorem 3.3 is obtained by applying the abstract Morse Theory presented in Section 2 to the action integral $E$ defined at (3.1). Assumptions (S1)-(S3) guarantee that $E$ satisfies the $(PS)^*$ condition. If such assumptions do not hold, the Morse Inequalities may be false. A counterexample is provided by the *Anti de Sitter spacetime* (see below). A version of the Morse Relations for a stationary Lorentzian manifold satisfying (S1)–(S3) has been previously proved using a global saddle point reduction (see [Ma]) which allows to define a nonnegative index $m(z)$ to any geodesic $z$ joining $z_0$ and $z_1$. In [ABFM] we show that $j(z) = m(z)$.

The situation is completely different for nonstationary Lorentzian manifolds and in the particular case of orthogonal splitting ones. Roughly speaking we could say that this case is the genuinely strongly indefinite one. First of all it is well known that for nonstationary metrics the $(PS)^*$ condition does not hold for the action integral $E$, see [Ma]. Then the results of Theorem 2.2. can not be applied. However, under some asymptotic conditions on the coefficients of the metric, it can be proved that the action integral can be perturbed as explained in Section 2. So for an orthogonal splitting Lorentzian manifold we state only the Morse Inequalities for the geodesics joining two nonconjugate points.

**Theorem 3.5.** *Let* $(\mathcal{M}, g)$ *be an orthogonal spliting Lorentzian manifold and let* $z_0 = (x_0, t_0)$ *and* $z_1 = (x_1, t_1)$ *be two nonconjugate points of* $\mathcal{M}$. *Assume that:*

(O1) *The Riemannian manifold* $(\mathcal{M}_0, \langle\cdot,\cdot\rangle)$ *is complete;*

(O2) *There exists* $\lambda > 0$, *such that for any* $z = (x, t) \in \mathcal{M}$, *and for any* $\xi \in T_x\mathcal{M}_0$, $\langle\alpha(z)\xi, \xi\rangle \geq \lambda\langle\xi, \xi\rangle$ ;

(O3) *there exists two positive constants* $0 < \nu \leq M$ *such that for any* $z \in \mathcal{M}$, $\nu \leq \beta(z) \leq M$;

(O4) *there exists a positive constant* $L$ *such that for any* $z \in \mathcal{M}$, $|\langle\alpha_t(z)\xi, \xi\rangle| \leq L\langle\xi, \xi\rangle$, $\quad |\beta_t(z)| \leq L$, *where* $\alpha_t$ *and* $\beta_t$ *denote respectively the partial derivative, with respect to* $t$, *of* $\alpha$ *and* $\beta$;

(O5) $\limsup_{t\to+\infty}\langle\alpha_t(x, t)\xi, \xi\rangle \leq 0$ *and* $\liminf_{t\to-\infty}\langle\alpha_t(x, t)\xi, \xi\rangle \geq 0$, *uniformly with respect to* $x \in \mathcal{M}_0$ *and* $\xi \in T_x\mathcal{M}_0$, $\langle\xi, \xi\rangle = 1$.

*Then for any* $k \in \mathbf{N}$ *and for any field* $\mathcal{K}$, *denoting by* $G_k(z_0, z_1)$ *the number of geodesics joining* $z_0$ *and* $z_1$ *and having relative index equal to* $k$, *we have*

$$G_k(z_0, z_1) \geq \beta_k(\Omega(\mathcal{M}), \mathcal{K}). \tag{3.6}$$

The action integral $E$ for a nonstationary metric does not satisfy the $(PS)^*$ condition and Theorem 2.2 can not be directly applied. However, by virtue of assumptions (O1)–(O4), it is possible to find a family $(E_\delta)_{\delta>0}$ of smooth functionals satisfying $(\mathrm{PS})^*$ and such that $E_\delta \to E$, as $\delta \to 0$, on the bounded subsets of the Sobolev manifold $\Omega^{1,2}(z_0, z_1; \mathcal{M})$. Moreover, assumptions (O2)-(O5) permit to prove some apriori estimates on the critical points of $E_\delta$, so we can pass to the limit as $\delta \to 0$ in the Morse Inequalities for the functionals $E_\delta$ obtaining (3.6). We refer to [ABFM] for the details. If assumptions (O1)-(O5) do not hold, the Morse Inequalities may not be hold. A counterexample is given by the *de Sitter spacetime* (see below).

**Remarks 3.6.**

**(1)** The index $j(z)$ of a geodesic for a nonstationary Lorentzian metric can be negative, so in general $j(z) \in \mathbf{Z}$. It can be proved that the index of a causal geodesic is nonnegative, so a geodesic with negative index is spacelike. We mention the paper [U] where the Morse Inequalities on a globally hyperbolic Lorentzian manifold, relating the timelike geodesics joining two causally related points to the topology of the space of timelike curves are proved. It would be interesting to characterize geodesics with negative index with respect to the metric or some of its invariants as the curvature tensor and the Jacobi equation.

**(2)** It is an open problem to prove the Morse Relations (3.4) for an orthogonal splitting manifold $(\mathcal{M}, g)$, as for stationary Lorentzian manifolds. If the Relations (3.4) would be true, then the geodesics with negative index are not due to the topology of the problem (the manifold $\mathcal{M}$), but rather by the geometrical properties of the metric $g$. Moreover the number of the geodesics with negative index is even. Indeed the topological content of (3.4) is given by the Poincaré polynomial $\sum_{k=0}^{\infty} \beta_k(\mathcal{Z}; \mathcal{K})\lambda^k$, which has only nonnegative powers, while the geometric content is in the remainder term $(1+\lambda)Q(\lambda)$. So, if the Laurent series $Q(\lambda)$ contains a term with negative power , then the term $(1+\lambda)$ forces the existence of two critical points with negative index.

**(3)** The Morse Relations in Theorem 3.3 and the Morse Inequalities in Theorem 3.5 are written in terms of the homology groups of the based loop space $\Omega(\mathcal{M})$ rather than the homology groups of the Sobolev manifold $\Omega^{1,2}(z_0, z_1; \mathcal{M})$ on which the abstract Morse Theory is applied . It is well known that the two spaces $\Omega(\mathcal{M})$ and $\Omega^{1,2}(z_0, z_1; \mathcal{M})$ are homotopically equivalent (cf. [Pa]).

**(4)** It is known (see [FH]) that if the manifold $\mathcal{M}$ is noncontractible into itself, there exists infinitely many nonnull Betti numbers $\beta_k(\Omega(\mathcal{M}), \mathbf{R})$. As a consequence we get the following results which is a generalization of the classical Serre Theorem on the existence of infinitely many geodesics joining two points on a compact Riemannian manifold: if the assumptions of Theorems 3.3 and 3.5 are satisfied, there exists a sequence $(z_m)$ of geodesics joining $z_0$ and $z_1$ such that the $j(z_m)_{m\in\mathbf{N}} \to +\infty$.

We present now some counterexample to the Morse Inequalities in Lorentzian Geometry. Indeed, they are counterexamples to the geodesic connectedeness which is a necessary condition in order that the Morse Inequalities hold for any couple of nonconjugate points. For further details see [ABFM].

**1)** The Anti de Sitter spacetime $(\mathcal{M}, g)$, where $\mathcal{M} = ]-\pi/2, \pi/2[\times \mathbf{R}$ and $g$ is the stationary (it is also static, that is $\delta(x) \equiv 0$) metric

$$g(z)[\zeta,\zeta] = g(x,t)[(\xi,\tau),(\xi,\tau)] = \frac{\xi^2}{\cos^2 x} - \frac{\tau^2}{\cos^2 x} \tag{3.7}$$

is an example of a geodesically complete Lorentzian manifold for which the Morse Inequalities do not hold, see [Pe] and also [CM]. Moreover it is a stationary Lorentzian manifold satisfying (S1) and (S3) but not (S2) of Theorem 3.3.

**2)** The de Sitter spacetime $(\mathcal{M}, g)$, where $\mathcal{M} = S^n \times \mathbf{R}$, $S^n$ is the standard $n$-dimensional sphere, $g$ is given by

$$g(z)\,[\zeta,\zeta] = \cosh^2 t \langle \xi, \xi \rangle - \tau^2,$$

and $\langle \cdot, \cdot \rangle$ is the standard Riemannian metric on $S^n$, is an example of orthognal splitting Lorentzian manifold such that the Morse Inequalities do not hold for any couple of points, see [CM,ON,ABFM]. The manifold $(\mathcal{M}, g)$ does not satisfy assumptions (O4)-(O5) of Theorem 3.5. Notice that $(\mathcal{M}, g)$ is geodesically complete and globally hyperbolic, giving a counterxample to the Morse Inequalities in these classes.

**3)** Classical counterexamples to the geodesic connectedeness and the Morse Inequalities for compact Lorentzian manifolds are the two dimensional *Clifton-Pohl* torus, see [ON,W]. For instance consider $\mathbf{R}^2$ with the Lorentzian metric $g(z)[\zeta,\zeta] = g(x,t)[(\xi,\tau),(\xi,\tau)] = (\cos t)\xi^2 + (2\sin t)\xi\tau - (\cos t)\tau^2$. The metric $g$ induces a Lorentzian metric $\tilde{g}$ on the torus $T^2 = \{(x,t) : 0 \le x \le 1, -4\pi \le t \le 4\pi\}$. The Lorentzian manifold $(T^2, \tilde{g})$ is not geodesically connected, because any geodesic with starting point on the circle $t = 0$ remains in the set $\{(x,t) : -5/2\pi \le t \le 7/2\pi\}$.

## References


[Ab] A. Abbondandolo, *Morse theory for Hamiltonian systems*, Chapman & Hall/CRC Research Notes in Mathematics, 425. Chapman & Hall/CRC, Boca Raton, FL, 2001.

[ABFM] A. Abbondandolo, V. Benci, D. Fortunato, A. Masiello, *Morse relations for geodesics on Lorentzian manifolds,* preprint.

[AM] A. Abbondandolo, P. Majer, *Morse homology on Hilbert spaces,* Comm. Pure Appl. Math. **54**, 689–760 (2001).

[AZ] H. Amann, E. Zehnder, *Nontrivial solutions for a class of nonresonant problems and applications to nonlinear differential equations,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **7** (1980), 539–603.

[BEE] J. K. Beem, P. H. Ehrlich, K. L. Easley, *Global Lorentzian geometry*, Second edition. Monographs and Textbooks in Pure and Applied Mathematics, 202. Marcel Dekker, Inc., New York, 1996.

[BO] R. Bott, *Lectures on Morse Theory, old and new,* Bull. Amer. Math. Soc. (N.S.) **7**, 331–358 (1982).

[CM] E. Calabi, L. Markus, *Relativistic space forms,* Ann. of Math. (2) **75** (1962), 63–76.
[CLL] K. C. Chang, J. Q. Liu, M. J. Liu, *Nontrivial periodic solutions for strong resonance Hamiltonian systems,* Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (1997), 103–117.
[CBC] Y. Choquet-Bruhat, S. Cotsakis, *Global hyperbolicity and completeness,* J. Geom. Phys. **43** (2002), 345–350.
[CZ] C. Conley, E. Zehnder, *Morse-type index theory for flows and periodic solutions for Hamiltonian equations,* Comm. Pure Appl. Math. **37** (1984), 207–253.
[FH] E. Fadell, S. Husseini, *Category of loop spaces of open subsets in Euclidean space*, Nonlinear Anal. **17** (1991), 1153–1161.
[Ma] A. Masiello, *Variational methods in Lorentzian geometry*, Pitman Research Notes in Mathematics Series, 309. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1994.
[Mi] J. Milnor, *Morse Theory*, Ann. of Math. Studies **51**, Princeton University Press, Princeton, NJ, 1963.
[Mo] M. Morse, *The Calculus of variations in the large*, Reprint of the 1932 original. American Mathematical Society Colloquium Publications, 18. American Mathematical Society, Providence, RI, 1996.
[ON] B. O'Neill, *Semiriemannian geometry with application to general relativity,* Academic Press, New York, 1983.
[Pa] R. Palais, *Morse theory on Hilbert manifolds,* Topology **2** (1963), 299–340.
[Pe] R. Penrose, *Techniques of differential topology in relativity*, Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 7. Society for Industrial and Applied Mathematics, Philadelphia, PA, 1972.
[R] P. H. Rabinowitz, *Periodic solutions of Hamiltonian systems,* Comm. Pure Appl. Math. **31** (1978), 157–184.
[Sz] A. Szulkin, *Cohomology and Morse theory for strongly indefinite functionals,* Math. Z. **209** (1992), 375–418.
[U] K. Uhlenbeck, *A Morse Theory for geodesics on Lorentz manifolds,* Topology **14** (1975), 69–90.
[W] T. Weinstein, *An Introduction to Lorentz Surfaces*, de Gruyter Expositions in Mathematics, 22. Walter de Gruyter & Co., Berlin, 1996.

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