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GEOMETRIC RATIONALITY OF EQUAL-RANK SATAKE COMPACTIFICATIONS

Leslie Saper

Dedicated to the memory of Armand Borel, with respect and admiration

0. Introduction

Let G be the group of real points of a semisimple algebraic group defined over $\mathbb{Q}$, let $K \subseteq G$ be a maximal compact subgroup, and let $\Gamma \subset G$ be an arithmetic subgroup. For every irreducible representation (σ, U) of G, Satake [21] has constructed a compactification of the corresponding symmetric space $D = G/K$. The procedure is to pass from (σ, U) to a representation (π, V) having a nonzero K-fixed vector v (a *spherical representation*); the compactification is then the closure of D under the embedding $D \hookrightarrow \mathbb{P}(V)$, $gK \mapsto [\pi(g)v]$. We denote the compactification by $\overline{D_\sigma}$ or D^*_π; it is a disjoint union of *real boundary components*, one of which is D itself and the others are symmetric spaces of lower rank. Under certain conditions, Satake [22] creates a corresponding compactification X^*_π of the locally symmetric space $X = \Gamma\backslash D$. These conditions were reformulated by Borel [4] and christened *geometric rationality* by Casselman [8].

Although the classification of Satake compactifications D^*_π depends only on G as an $\mathbb{R}$-group, geometric rationality depends *a priori* on the $\mathbb{Q}$-structure. Borel [4, Théorème 4.3] proved geometric rationality in the case where (σ, U) is strongly rational[1] over $\mathbb{Q}$ in the sense of [7]. Baily and Borel [3] considered the case where $D \subseteq \mathbb{C}^N$ is a a bounded symmetric domain and D^* is the closure of D. This natural compactification is topologically equivalent to a certain Satake compactification [21, §5.2], [17, Theorem 4] and all real boundary components are Hermitian symmetric spaces. Baily and Borel prove geometric rationality, without any rationality condition on (σ, U), by a careful consideration of root systems; another proof was given in [2]. Zucker [25] was the first to raise the

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[1]This condition is stated in [4, §1.6] but only implicitly; the term was actually coined later in [7]. Unfortunately the adverb "strongly" is sometimes omitted in the literature, e.g. see [8].

general question of which (σ, U) lead to geometrically rational compactifications. In [8] Casselman gives a criterion for geometric rationality in terms of π and the $\mathbb{Q}$-index of G [24].

In this paper we establish geometric rationality in two broad situations. The first main result, Theorem 8 in §5, is that $\overline{D_\sigma}$ is geometrically rational if (σ, U) is rational over $\mathbb{Q}$. This generalizes Borel's result [4] above which required that (σ, U) be strongly rational over $\mathbb{Q}$. The second main result concerns *equal-rank symmetric spaces*, those that can be written as G/K where $\mathbb{C}\text{-rank}\, G = \operatorname{rank} K$. Define a *real equal-rank Satake compactification* to be one for which each real boundary component is an equal-rank symmetric space; the possibilities have been listed by Zucker [26, (A.2)]. We prove in Theorem 20 of §6 that any real equal-rank Satake compactification is geometrically rational aside from certain $\mathbb{Q}$-rank 1 and 2 exceptions. No rationality assumption on (σ, U) or (π, V) is made. Every Hermitian symmetric space is an equal-rank symmetric space, so this generalizes the geometric rationality result of Baily and Borel [3]. The exceptional cases are described and treated in Theorem 22 of §7.

The equal-rank result is important since in [20] the author proved that the intersection cohomology of a real equal-rank Satake compactification X^*_π agrees with both the intersection cohomology and the weighted cohomology of the reductive Borel-Serre compactification $\widehat{X}$. In the Hermitian case this result on intersection cohomology was conjectured independently by Rapoport [18], [19] and by Goresky and MacPherson [10], while the result on weighted cohomology in the Hermitian case was proved previously by Goresky, Harder, and MacPherson [9].

We have chosen to avoid classification theory as much as possible in order to emphasize the role played by the equal-rank condition; in particular we do not rely on Zucker's list. In fact classification theory is only used in Proposition 10 (which uses the classification of semisimple Lie algebras over $\mathbb{C}$), in Corollary 15 (which uses the classification of real forms of F_4), and in the treatment of the exceptional cases in Theorem 22.

Our main results answer several questions raised by Armand Borel [5] in a letter to the author. It is a pleasure to thank him and Bill Casselman for helpful correspondence and discussions concerning this work. Much of this work took place at the 2002 IAS/Park City Mathematics Institute on Automorphic Forms; I would like to thank the organizers of the Institute for providing a stimulating environment. I would also like to thank the referee for helpful suggestions regarding the exposition.

1. Basic Notions

In this paper G will be the group of real points of a semisimple algebraic $\mathbb{Q}$-group, K will be a maximal compact subgroup of G, and $D = G/K$ the associated symmetric space. For a subfield k of $\mathbb{C}$ we will find it convenient to speak of a parabolic k-subgroup of G when we strictly mean the group of real points of a parabolic k-subgroup of the algebraic group underlying G; similar liberties will be taken for other subgroups, such as tori or unipotent radicals, and concepts such as roots. By a representation (π, V) of G we will mean a finite-dimensional $\mathbb{C}$-vector space V and a homomorphism $\pi\colon G \to \mathrm{GL}(V)$ induced by a $\mathbb{C}$-morphism of the underlying algebraic varieties.

Fix a maximal $\mathbb{Q}$-split torus ${}_{\mathbb{Q}}S$ contained in a maximal $\mathbb{R}$-split torus ${}_{\mathbb{R}}S$ which itself is contained in a maximal torus ${}_{\mathbb{C}}S$ of G defined over $\mathbb{Q}$.[2] Assume compatible orderings on the corresponding root systems have been chosen. For $k = \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$, let ${}_k\Delta$ denote the simple k-roots and let ${}_kW$ be the Weyl group of the k-root system. For $\alpha \in {}_k\Delta$, let $s_\alpha \in {}_kW$ denote the corresponding simple reflection. The *Coxeter graph* of the system of k-roots is a labeled graph with ${}_k\Delta$ for its vertex set and an edge labeled r between $\alpha \neq \beta$ if the product of simple reflections $s_\alpha s_\beta$ in the Weyl group has order $r > 2$. If we add an arrow on an edge labeled greater than 3 which points to the shorter root we obtain the *Dynkin diagram*. As customary, we transfer topological terminology from the graph to ${}_k\Delta$. Thus we may speak of a connected subset of ${}_k\Delta$ or a path in ${}_k\Delta$.

The Galois group $\mathrm{Gal}(\mathbb{C}/k)$ acts on ${}_{\mathbb{C}}\Delta$ (and hence on $\mathbb{C}$-weights) via the $*$-action [7]. Namely, $g^*\colon {}_{\mathbb{C}}\Delta \to {}_{\mathbb{C}}\Delta$ for $g \in \mathrm{Gal}(\mathbb{C}/k)$ is defined to be $w_g \circ g$, where $w_g \in {}_{\mathbb{C}}W$ is uniquely determined by $w_g(g\,{}_{\mathbb{C}}\Delta) = {}_{\mathbb{C}}\Delta$. In particular, if $k \subseteq \mathbb{R}$ we have c^* where c denotes complex conjugation. The *opposition involution* $\iota\colon {}_{\mathbb{C}}\Delta \to {}_{\mathbb{C}}\Delta$ is defined similarly by replacing g with negation. The $*$-action of $\mathrm{Gal}(\mathbb{C}/k)$ and ι commute and are automorphisms of the Dynkin diagram.

Restriction of roots defines

$$\rho_{\mathbb{C}/k}\colon {}_{\mathbb{C}}\Delta \to {}_k\Delta \cup \{0\}.$$

The fibers $\rho_{\mathbb{C}/k}^{-1}(\alpha)$ for $\alpha \in {}_k\Delta$ are nonempty Galois orbits; the fiber $\Delta^0_{\mathbb{C}/k} = \rho_{\mathbb{C}/k}^{-1}(0)$ is Galois invariant and its elements are the *k-anisotropic roots*. For $\theta \subseteq {}_k\Delta$ define $\varepsilon_{\mathbb{C}/k}(\theta) = \rho_{\mathbb{C}/k}^{-1}(\theta \cup \{0\})$. These are the types of the parabolic k-subgroups of G.

The *k-index* of G [24] consists of the Dynkin diagram for ${}_{\mathbb{C}}\Delta$, the subset $\Delta^0_{\mathbb{C}/k}$, and the $*$-action of $\mathrm{Gal}(\mathbb{C}/k)$. This can be represented diagrammatically; in the case $k = \mathbb{R}$ one recovers the *Satake diagram*. We will often by abuse of notation refer to the index simply as ${}_{\mathbb{C}}\Delta$.

We will use the following repeatedly [7, Proposition 6.15]:

Proposition 1. *For α, $\beta \in {}_k\Delta$, $\{\alpha, \beta\}$ is connected if and only if for every $\tilde\alpha \in \rho_{\mathbb{C}/k}^{-1}(\alpha)$, there exists $\tilde\beta \in \rho_{\mathbb{C}/k}^{-1}(\beta)$ and $\psi \subseteq \Delta^0_{\mathbb{C}/k}$ such that $\psi \cup \{\tilde\alpha, \tilde\beta\}$ is connected.*

Let χ be the highest weight of an irreducible representation (π, V) of G and set $\delta = \{\, \alpha \in {}_{\mathbb{C}}\Delta \mid s_\alpha\chi \neq \chi \,\}$. (If necessary we will denote this δ_π or δ_χ.) Let ${}_k\chi = \chi|_{{}_kS}$ and define ${}_k\delta \subseteq {}_k\Delta$ analogously. An analogue of Proposition 1 shows that $\beta \in {}_k\delta$ if and only if there exists $\tilde\beta \in \rho_{\mathbb{C}/k}^{-1}(\beta)$ and $\psi \subseteq \Delta^0_{\mathbb{C}/k}$ such

[2] Much of this paper would become simpler if we could assume that ${}_{\mathbb{R}}S$ is defined over $\mathbb{Q}$. As Borel pointed out to me, this is not always possible; there are counterexamples due to Serre.

that $\psi \cup \{\tilde{\beta}\}$ is connected and contains an element of δ [8, Corollary 7.2], [25, Remark in (2.4)].

For a linear combination of simple k-roots $\sum d_\alpha \alpha$, the *support* $\operatorname{supp}_k(\sum d_\alpha \alpha)$ is defined as $\{\alpha \in {}_k\Delta \mid d_\alpha \neq 0\}$; the linear combination is called *codominant* if all $d_\alpha \geq 0$. A subset $\theta \subseteq {}_k\Delta$ is called *δ-connected* if every connected component of θ contains an element of ${}_k\delta$. For every k-weight λ of (π, V), the difference ${}_k\chi - \lambda$ is codominant with δ-connected support (and integral coefficients). In fact every δ-connected subset of ${}_k\Delta$ arises in this way [7, 12.16].

For $\theta \subseteq {}_k\Delta$, let $\kappa(\theta)$ denote the largest δ-connected subset of θ and let $\omega(\theta)$ denote the largest subset Υ of ${}_k\Delta$ with $\kappa(\Upsilon) = \kappa(\theta)$. Clearly

$$\kappa(\theta) \subseteq \theta \subseteq \omega(\theta).$$

Let $\zeta(\theta)$ denote the complement of $\kappa(\theta)$ in $\omega(\theta)$. Equivalently, $\zeta(\theta)$ consists of those roots that are not in ${}_k\delta$, not in $\kappa(\theta)$ and not joined by an edge to a root in $\kappa(\theta)$. It will sometimes be useful, especially in §6, to use the notation

$$\theta^+ = \theta \cup \{\, \alpha \in {}_k\Delta \mid \alpha \text{ is connected by an edge to a root in } \theta \,\}.$$

We need to recall the various notions of rationality for a representation [7, §12]. Assume that V has a k-structure. An irreducible representation (π, V) is called *projectively rational over k* if the associated projective representation $\pi' \colon G \to \mathrm{PGL}(V)$ is defined over k. It is called *rational over k* if $\pi \colon G \to \mathrm{GL}(V)$ is itself defined over k; it is called *strongly rational over k* if furthermore the parabolic subgroup P_π stabilizing the line spanned by the highest weight vector is defined over k. In terms of the highest weight χ of (π, V), the representation is projectively rational over k if and only if χ is $\mathrm{Gal}(\mathbb{C}/k)$-invariant under the $*$-action and it is strongly rational over k if and only if in addition $s_\alpha \chi = \chi$ for all $\alpha \in \Delta^0_{\mathbb{C}/k}$.

An irreducible representation (π, V) is called *spherical* if there exists a nonzero K-fixed vector $v \in V$; such a vector v is unique up to scalar multiplication. By a theorem of Helgason [12], [14, Theorem 8.49], (π, V) is spherical if and only if its highest weight χ is trivial on the maximal $\mathbb{R}$-anisotropic subtorus of ${}_{\mathbb{C}}S$ and $(\chi, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$ for all $\alpha \in {}_{\mathbb{R}}\Delta$. In particular, a spherical representation (π, V) is strongly rational over $\mathbb{R}$. In this case δ is $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariant and disjoint from $\Delta^0_{\mathbb{C}/\mathbb{R}}$; it follows that $\epsilon_{\mathbb{C}/\mathbb{R}}(\omega(\theta)) = \omega(\epsilon_{\mathbb{C}/\mathbb{R}}(\theta))$ for all $\theta \subseteq {}_{\mathbb{R}}\Delta$. It also follows from [14, proof of Theorem 8.49] that a K-fixed vector v has a nonzero component along any highest weight vector.

2. Satake Compactifications

As in [8] we take as our starting point an irreducible spherical representation (π, V) of G, nontrivial on each noncompact $\mathbb{R}$-simple factor of G, with K-fixed vector v; for the relation with Satake's original construction, see §3. The Satake compactification D^*_π associated to (π, V) is defined to be the closure of the image of $D = G/K$ under the embedding $D \hookrightarrow \mathbb{P}(V)$, $gK \mapsto [\pi(g)v]$. The action of G on D extends to an action on D^*_π. For every parabolic $\mathbb{R}$-subgroup P, the subset $D_{P,h} \subseteq D_\pi^*$ of points fixed by N_P, the unipotent radical, is called a *real boundary component*. The Satake compactification is the disjoint union of the real boundary components, however different P may yield the same real boundary component. By associating to each real boundary component its normalizer we obtain a one-to-one correspondence.

A parabolic $\mathbb{R}$-subgroup P is *δ-saturated* if it is conjugate to a standard parabolic $\mathbb{R}$-subgroup with type $\epsilon_{\mathbb{C}/\mathbb{R}}(\omega(\theta))$ for some $\theta \subseteq {}_{\mathbb{R}}\Delta$. The subgroups that arise as normalizers of real boundary components are the precisely the δ-saturated parabolic $\mathbb{R}$-subgroups. The action of P on $D_{P,h}$ descends to an action of its Levi quotient $L_P = P/N_P$ and the subgroup $L_{P,\ell} \subseteq L_P$ which fixes $D_{P,h}$ pointwise will be called the *centralizer group*. It has as its identity component (in the Zariski topology) the maximal normal connected $\mathbb{R}$-subgroup with simple $\mathbb{R}$-roots $\zeta(\theta)$; the simple $\mathbb{C}$-roots of $L_{P,\ell}$ are $\zeta(\varepsilon_{\mathbb{C}/\mathbb{R}}(\theta))$. Thus $D_{P,h}$ is the symmetric space corresponding to $L_{P,h} = L_P/L_{P,\ell}$, a semisimple $\mathbb{R}$-group with simple $\mathbb{R}$-roots $\kappa(\theta)$; the simple $\mathbb{C}$-roots of $L_{P,h}$ are $\kappa(\varepsilon_{\mathbb{C}/\mathbb{R}}(\theta))$, the connected components of $\varepsilon_{\mathbb{C}/\mathbb{R}}(\kappa(\theta))$ which are not wholly contained in $\Delta^0_{\mathbb{C}/\mathbb{R}}$.

If $D_{P,h}$ and $D_{P',h}$ are two standard real boundary components with normalizers having type $\epsilon_{\mathbb{C}/\mathbb{R}}(\omega(\theta))$ and $\epsilon_{\mathbb{C}/\mathbb{R}}(\omega(\theta'))$ respectively, define $D_{P,h} \leq D_{P',h}$ if $\kappa(\theta) \subseteq \kappa(\theta')$; this is a partial order on the standard real boundary components.

More generally the same procedure allows one to associate a Satake compactification D_π^* to a triple (π, V, v) consisting of a (not necessarily irreducible) representation (π, V) which is nontrivial on each noncompact $\mathbb{R}$-simple factor of G and a K-fixed vector $v \in V$ whose G-orbit spans V. Note that every irreducible component of (π, V) is automatically spherical. We will only consider (π, V, v) satisfying a very restrictive condition (although Casselman points out in [8] that the general case is worthy of further study):

(R) Assume that (π, V) may be decomposed into the direct sum of submodules (π_0, V_0) and (π', V') such that (π_0, V_0) is irreducible with highest weight χ_0 and the difference $\chi_0 - \chi'$, where χ' is the highest weight of any irreducible component of (π', V'), is codominant with δ_{χ_0}-connected support.

Lemma 2. *If (π, V, v) satisfies condition* (R)*, then $D_\pi^* \cong D_{\pi_0}^*$ as G-spaces.*

Proof. Let $y \in D_\pi^* - D$ and let $y_i = [\pi(g_i)v] \in D$ be a sequence converging to y. Write $g_iK = k_ia_iK \in K\overline{A^+}K$ by the Cartan decomposition [13, IX, Theorem 1.1]; here A is the identity component (in the classical topology) of ${}_{\mathbb{R}}S$ and $\overline{A^+} = \{\, a \in A \mid a^\alpha \geq 1,\ \alpha \in {}_{\mathbb{R}}\Delta \,\}$. By passing to a subsequence and conjugating ${}_{\mathbb{R}}S$ if necessary, we may assume that $k_i \to e$ in which case we may assume simply that $y_i = [\pi(a_i)v]$. Choose homogeneous coordinates x_λ on $\mathbb{P}(V)$ corresponding to a basis of $\mathbb{C}$-weight vectors of V. We know there exists a $\mathbb{C}$-weight λ of V such that y is not in the zero set of x_λ; let χ' be the highest weight of an irreducible component of V containing the $\mathbb{C}$-weight λ. Then $(x_{\chi_0}/x_\lambda)(y_i) \propto a_i^{\chi_0 - \chi' + \chi' - \lambda} \geq 1$ since the exponent is codominant; consequently y is not in the zero set of x_{χ_0}.

Since $D_\pi^* \cap \mathbb{P}(V')$ is therefore empty, the projection $V \to V/V' \cong V_0$ induces a continuous map $D_\pi^* \to D_{\pi_0}^* \subseteq \mathbb{P}(V_0)$ of G-spaces. Write $v = v_0 + v'$. In order to see this map is bijective, we need to show that if a sequence $y_{0,i} = [\pi_0(a_i)v_0]$ converges to $y_0 \in D_{\pi_0}^*$, then the sequence $y_i = [\pi(a_i)v]$ converges in D_π^*. By hypothesis, $(x_{\lambda_0}/x_{\chi_0})(y_{0,i}) \propto a_i^{-(\chi_0-\lambda_0)}$ converges for every $\mathbb{C}$-weight λ_0 of V_0. Consequently for any δ_{χ_0}-connected subset $\theta_0 \subseteq {}_{\mathbb{C}}\Delta$, either $a_i^{-\alpha}$ converges for all $\alpha \in \theta_0$ or there exists $\alpha \in \theta_0$ with $a_i^{-\alpha} \to 0$; the proof is by induction on $\#\theta_0$. Since $a_i^{-\alpha} \le 1$ for all α, this implies

(1) $\quad a_i^{-\nu}$ converges for all ν codominant with δ_{χ_0}-connected support.

We need to show for any $\mathbb{C}$-weight λ of an irreducible component with highest weight χ' that $(x_\lambda/x_{\chi_0})(y_i) \propto a_i^{-(\chi_0-\chi'+\chi'-\lambda)}$ converges. Since $\operatorname{supp}(\chi'-\lambda)$ is $\delta_{\chi'}$-connected and $\delta_{\chi'} \subseteq \delta_{\chi_0} \cup \operatorname{supp}(\chi_0-\chi')$, it follows from (R) that $\operatorname{supp}(\chi_0 - \chi') \cup \operatorname{supp}(\chi'-\lambda)$ is δ_{χ_0}-connected. Convergence then follows from (1). ☐

3. Satake's Original Construction

In this section we relate Satake's original construction [21] of compactifications of D to those considered in §2. Let (σ, U) be an irreducible representation of G (not necessarily spherical) which is nontrivial on each $\mathbb{R}$-simple factor. Fix a positive definite Hermitian inner product on U which is *admissible* in the sense that $\sigma(g)^* = \sigma(\theta g)^{-1}$ (here θ denotes the Cartan involution associated to K); such an inner product always exists [16] and is unique up to rescaling. Let $S(U)$ denote the corresponding real vector space of self-adjoint endomorphisms of U. Satake defines the compactification $\overline{D_\sigma}$ associated to (σ, U) to be the closure of the image of D under the embedding $D \hookrightarrow \mathbb{P}S(U)$, $gK \mapsto \sigma(g)\sigma(g)^*$.

Let $g \in G$ act on $S(U)$ by $T \mapsto \sigma(g) \circ T \circ \sigma(g)^* = \sigma(g) \circ T \circ \sigma(\theta g)^{-1}$ and extend to a representation on $S(U)_{\mathbb{C}}$, the space of all endomorphisms of U. This representation is $U \otimes {}^\theta U^*$, where ${}^\theta U^*$ is the twist by θ of the usual contragredient representation. The identity endomorphism $I \in U \otimes {}^\theta U^*$ is a K-fixed vector and we let (π, V) denote the smallest subrepresentation containing I. It is clear that $\overline{D_\sigma}$ is isomorphic with the Satake compactification D_π^* associated to (π, V, I) as in §2.

Proposition 3. *The triple* (π, V, I) *satisfies condition* (R) *with* $\chi_0 = 2\operatorname{Re}\mu$, *where* μ *is the highest weight of* (σ, U).

Proof. The $\mathbb{C}$-weights of the K-fixed vector I include the highest weight χ' of any irreducible component of (π, V). However $I = \sum_\lambda u_\lambda \otimes u_\lambda^*$, where u_λ ranges over a basis of $\mathbb{C}$-weight vectors of U and u_λ^* ranges over the corresponding dual basis. Thus $\chi' = \lambda - \theta\lambda = 2\operatorname{Re}\lambda$, where λ is a $\mathbb{C}$-weight of U. Since $\operatorname{Re}\lambda$ is an $\mathbb{R}$-weight of U, the difference $\operatorname{Re}\mu - \operatorname{Re}\lambda$ is $\mathbb{R}$-codominant and $\operatorname{supp}_{\mathbb{R}}(\operatorname{Re}\mu - \operatorname{Re}\lambda)$ is ${}_{\mathbb{R}}\delta_\mu$-connected. If $\alpha \in {}_{\mathbb{R}}\Delta$, then $\operatorname{supp}_{\mathbb{C}}(2\alpha)$ has at most 2 connected components, each containing an element of $\rho_{\mathbb{C}/\mathbb{R}}^{-1}(\alpha)$. We see that $\operatorname{supp}_{\mathbb{C}}(2\operatorname{Re}\mu - 2\operatorname{Re}\lambda)$ is $\delta_{2\operatorname{Re}\mu}$-connected since $\delta_{2\operatorname{Re}\mu} = \rho_{\mathbb{C}/\mathbb{R}}^{-1}({}_{\mathbb{R}}\delta_\mu)$ and since if $(\alpha, \beta) \neq 0$ then each connected component of $\operatorname{supp}_{\mathbb{C}}(2\alpha)$ is connected to a connected component of $\operatorname{supp}_{\mathbb{C}}(2\beta)$ [25, (1.7)]. Finally a component (π_0, V_0) with highest weight $2\operatorname{Re}\mu$ does in fact occur, since one may check that $\sum_{\lambda|_{\mathbb{R}S}=\mu|_{\mathbb{R}S}} u_\lambda \otimes u_\lambda^*$ is a highest weight vector.[3] □

Corollary 4. *If (σ, U) has highest weight μ and (π_0, V_0) is the irreducible spherical representation with highest weight $2\operatorname{Re}\mu$, then $\overline{D_\mu} \cong D^*_{\pi_0}$ as G-spaces.*

4. Geometric Rationality

A real boundary component $D_{P,h}$ of D^*_π is called *rational* [3, §§3.5, 3.6] if

(i) its normalizer P is defined over $\mathbb{Q}$, and
(ii) the centralizer group $L_{P,\ell}$ contains a normal subgroup $\tilde{L}_{P,\ell}$ of L_P defined over $\mathbb{Q}$ such that $L_{P,\ell}/\tilde{L}_{R,\ell}$ is compact.

A Satake compactification D^*_π is called *geometrically rational* if every real boundary component $D_{P,h}$ whose normalizer has type $\epsilon_{\mathbb{C}/\mathbb{R}}(\omega(\epsilon_{\mathbb{R}/\mathbb{Q}}(\Upsilon)))$ for some δ-connected subset $\Upsilon \subseteq {}_{\mathbb{Q}}\Delta$ is rational. If $\mathbb{Q}$-rank $G = 0$ (and hence $X = \Gamma\backslash D$ is already compact) any Satake compactification D^*_π is geometrically rational, so our main interest is when $\mathbb{Q}$-rank $G > 0$.

Casselman [8, Theorems 8.2, 8.4] proves the following criterion:

Theorem 5. *The Satake compactification associated to (π, V) is geometrically rational if and only if*

(i) *$\omega(\Delta^0_{\mathbb{C}/\mathbb{Q}})$ is Galois invariant, and*
(ii) *$\kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}})$ is Galois invariant modulo $\mathbb{R}$-anisotropic roots, that is, the $\operatorname{Gal}(\mathbb{C}/\mathbb{Q})$-orbit of $\kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}})$ is contained in $\kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}}) \cup \Delta^0_{\mathbb{C}/\mathbb{R}}$.*

5. Representations Rational over $\mathbb{Q}$

Theorem 6. *The Satake compactification associated to (π, V) is geometrically rational if δ is Galois invariant.*

Proof. We need to verify (i) and (ii) from Theorem 5. Now $\kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}})$ is the union of those connected components of $\Delta^0_{\mathbb{C}/\mathbb{Q}}$ which contain an element of δ. Since both δ and $\Delta^0_{\mathbb{C}/\mathbb{Q}}$ are Galois invariant, clearly $\kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}})$ is Galois invariant. This proves (ii) (in fact $\mathbb{R}$-anisotropic roots are not needed). Now $\zeta(\Delta^0_{\mathbb{C}/\mathbb{Q}})$ consists of roots not in δ and not in $\kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}})^+$, hence it is Galois invariant. Since $\omega(\Delta^0_{\mathbb{C}/\mathbb{Q}}) = \kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}}) \cup \zeta(\Delta^0_{\mathbb{C}/\mathbb{Q}})$ this proves (i). □

Corollary 7. *The Satake compactification associated to (π, V) is geometrically rational if (π, V) is projectively rational over $\mathbb{Q}$.*

[3]A more direct proof that $2\operatorname{Re}\mu$ is the highest weight of an irreducible spherical representation is given by Harish-Chandra [11, Lemma 2].

Proof. By [7, §12.6], the associated projective representation $\pi'\colon G \to \mathrm{PGL}(V)$ is defined over $\mathbb{Q}$ if and only if the highest weight of V is Galois invariant; thus δ is Galois invariant. $\square$

Note that in Satake's original construction, the passage from (σ, U) to (π_0, V_0) does not necessarily preserve rationality. Thus Corollary 7 does not settle the question of geometric rationality if (σ, U) is assumed to be rational over $\mathbb{Q}$. Nonetheless we can prove:[4]

Theorem 8. *The Satake compactification associated to (σ, U) by Satake's original construction is geometrically rational if (σ, U) is projectively rational over $\mathbb{Q}$.*

Proof. By Corollary 4, we need to verify (i) and (ii) from Theorem 5 for $\delta = \delta_{2\,\mathrm{Re}\,\mu}$ under the hypothesis that δ_μ is Galois invariant. By the remarks following Proposition 1 and the fact that δ_μ is c^*-invariant, we can write

$$\delta = \{\alpha \in {}_{\mathbb{C}}\Delta - \Delta^0_{\mathbb{C}/\mathbb{R}} \mid \text{there exists } \psi \subseteq \Delta^0_{\mathbb{C}/\mathbb{R}} \text{ such that } \psi \cup \{\alpha\} \text{ is } \delta_\mu\text{-connected}\}.$$

Let $\kappa = \kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}})$ is the union of those connected components of $\Delta^0_{\mathbb{C}/\mathbb{Q}}$ which contain an element of δ; similarly let $\kappa_\mu = \kappa_\mu(\Delta^0_{\mathbb{C}/\mathbb{Q}})$ is the union of those connected components of $\Delta^0_{\mathbb{C}/\mathbb{Q}}$ which contain an element of δ_μ. Clearly $\kappa \subseteq \kappa_\mu$ while $\kappa_\mu - \kappa$ is the union of those connected components of $\Delta^0_{\mathbb{C}/\mathbb{Q}}$ which lie within $\Delta^0_{\mathbb{C}/\mathbb{R}}$ and contain an element of δ_μ. Since κ_μ is Galois invariant, (ii) is satisfied.

For (i) we calculate ${}_{\mathbb{C}}\Delta - \omega(\Delta^0_{\mathbb{C}/\mathbb{Q}}) = ({}_{\mathbb{C}}\Delta - \kappa) \cap (\delta \cup \kappa^+) = ({}_{\mathbb{C}}\Delta - \kappa_\mu) \cap (\delta \cup \kappa_\mu^+)$, where the second equality holds since $\kappa_\mu - \kappa \subseteq \Delta^0_{\mathbb{C}/\mathbb{R}}$ is disjoint from $\delta \cup \kappa^+$ and since $(\kappa_\mu^+ - \kappa^+) - \kappa_\mu \subseteq \delta$. Since $\delta_\mu \cap ({}_{\mathbb{C}}\Delta - \Delta^0_{\mathbb{C}/\mathbb{Q}}) \subseteq \delta \subseteq (\delta_\mu \cap ({}_{\mathbb{C}}\Delta - \Delta^0_{\mathbb{C}/\mathbb{Q}})) \cup \kappa_\mu^+$, we can thus write ${}_{\mathbb{C}}\Delta - \omega(\Delta^0_{\mathbb{C}/\mathbb{Q}}) = ({}_{\mathbb{C}}\Delta - \kappa_\mu) \cap ((\delta_\mu \cap ({}_{\mathbb{C}}\Delta - \Delta^0_{\mathbb{C}/\mathbb{Q}})) \cup \kappa_\mu^+)$, which is clearly Galois invariant. $\square$

6. Equal-rank Satake Compactifications

We begin with some generalities about the structure of root systems, here applied to the simple $\mathbb{C}$-roots ${}_{\mathbb{C}}\Delta$ of G. Let ι be the opposition involution, that is, the negative of the longest element of the Weyl group. For a connected subset $\psi \subseteq {}_{\mathbb{C}}\Delta$ which is invariant under ι, let $\iota|_\psi$ denote the restriction, while ι_ψ denotes the opposition involution of the subroot system with simple roots ψ. Recall

$$\psi^+ = \psi \cup \{\, \alpha \in {}_{\mathbb{C}}\Delta \mid \alpha \text{ is connected by an edge to a root in } \psi \,\}$$

Definition 9. Let $\widetilde{\mathcal{F}}$ denote the family of nonempty connected ι-invariant subsets $\psi \subseteq {}_{\mathbb{C}}\Delta$ for which $\iota|_\psi = \iota_\psi$. Let $\mathcal{F}$ consist of those $\psi \in \widetilde{\mathcal{F}}$ such that

(i) $\psi^+ - \psi$ modulo ι has cardinality ≤ 1, and

[4] The result of Theorem 8 is asserted in [25, (3.3), (3.4)], however in [8, §9] it is noted that [25, Prop. (3.3)(ii)] is incorrect. Also the proof of [25, Prop. (3.4)] seems to implicitly assume that (σ, U) is strongly rational over $\mathbb{R}$.

(ii) there exists $\psi' \supseteq \psi^+$, $\psi' \in \widetilde{\mathcal{F}}$ satisfying (i), such that all components of $\psi' - \psi^+$ are in $\widetilde{\mathcal{F}}$.

Let $\widetilde{\mathcal{F}}^* \subseteq \widetilde{\mathcal{F}}$ exclude sets of cardinality 1 which are not components of ${}_{\mathbb{C}}\Delta$ and for each component $C \subseteq {}_{\mathbb{C}}\Delta$, let $\widetilde{\mathcal{F}}_C = \{\psi \in \widetilde{\mathcal{F}} \mid \psi \subseteq C\}$; similarly define $\mathcal{F}^*$ and $\mathcal{F}_C$.

We view $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ as partially ordered sets by inclusion. For any partially ordered set $\mathcal{P}$, recall that $\psi' \in \mathcal{P}$ *covers* $\psi \in \mathcal{P}$ if $\psi < \psi'$ and there does not exists $\psi'' \in \mathcal{P}$ with $\psi < \psi'' < \psi'$. The *Hasse diagram* of the partially ordered set $\mathcal{P}$ has nodes $\mathcal{P}$ and edges the cover relations; if ψ' covers ψ we draw ψ' to the right of ψ. (For the basic terminology of partially ordered sets, see [23, Chapter 3], but note that in [23] ψ' is drawn *above* ψ.)

Proposition 10. *The Hasse diagrams of $\mathcal{F}_C$ and $\widetilde{\mathcal{F}}_C$ are given in Figure* 1. *In particular if $\psi < \psi' \in \widetilde{\mathcal{F}}$ and $\psi \in \mathcal{F}$ then $\psi' \in \mathcal{F}$. Also if C is not type F_4, then*:

(i) *$\widetilde{\mathcal{F}}^*_C$ is totally ordered.*

(ii) *Suppose ψ, $\psi' \in \mathcal{F}_C$ are incomparable. If C is not type B_n, C_n, or G_2, then one of ψ and ψ' is type A_1 and $\psi \cup \psi'$ is disconnected. If C is type B_n, C_n, or G_2, one of ψ and ψ' is type A_1 and is covered by C.*

Proof. Recall [24] that $\iota|_C$ is the unique nontrivial involution in the cases A_n $(n > 1)$, D_n $(n > 4$, odd), and E_6, and is trivial otherwise. If $\operatorname{rank} C > 1$, let $C_0 \subseteq C$ be defined as follows: if $\iota|_C$ is nontrivial, C_0 is the unique subdiagram of type A_2 or A_3 which is nontrivially stabilized under $\iota|_C$; if $\iota|_C$ is trivial, C_0 is the unique subdiagram of type B_2, D_4, or G_2. The subdiagram C_0 "determines" $\iota|_C$. In particular if $\psi \subseteq C$ is a connected ι-invariant subset with $\operatorname{rank} \psi > 1$, then $\iota|_\psi = \iota_\psi$ is equivalent to $C_0 = \psi_0$. Such ψ are easy to enumerate. For example, if C is type B_n, then ψ can be any connected segment containing the double bond at one end. Together with the cardinality 1 subsets, such calculations yield $\widetilde{\mathcal{F}}_C$. To determine $\mathcal{F}_C$, one then checks which of these ψ satisfy the additional conditions of Definition 9. The results are pictured in Figure 1 and the proposition follows. $\square$

Definition 11. Given a Satake compactification D^*_π, let $\mathcal{B}$ denote the family of subsets $\kappa(\varepsilon_{\mathbb{C}/\mathbb{R}}(\theta)) \subseteq {}_{\mathbb{C}}\Delta$, where $\theta \subseteq {}_{\mathbb{R}}\Delta$ is a nonempty connected and δ-connected subset. Let $\mathcal{B}^* \subseteq \mathcal{B}$ exclude those subsets $\kappa(\varepsilon_{\mathbb{C}/\mathbb{R}}(\theta))$ with cardinality 1 which are not components of ${}_{\mathbb{C}}\Delta$. For every component C of ${}_{\mathbb{C}}\Delta$, set $\mathcal{B}_{C\cup c^*C} = \{\psi \in \mathcal{B} \mid \psi \subseteq C \cup c^*C\}$ and $\mathcal{B}^*_{C\cup c^*C} = \mathcal{B}_{C\cup c^*C} \cap \mathcal{B}^*$.

An element of $\mathcal{B}$ is simply the set of simple roots for the automorphism group of a nontrivial standard irreducible boundary component. The correspondence $\theta \leftrightarrow \kappa(\varepsilon_{\mathbb{C}/\mathbb{R}}(\theta))$ above is an isomorphism of partially ordered sets for the inclusion ordering. Note that in the situation to be considered below, $C \cup c^*C = C$.

A semisimple algebraic $\mathbb{R}$-group is *equal-rank* if $\mathbb{C}\text{-rank}\, G = \operatorname{rank} K$. A symmetric space D is *equal-rank* if $D = G/K$ for an equal-rank group G. A *real equal-rank Satake compactification* is a Satake compactification D^*_π for which all

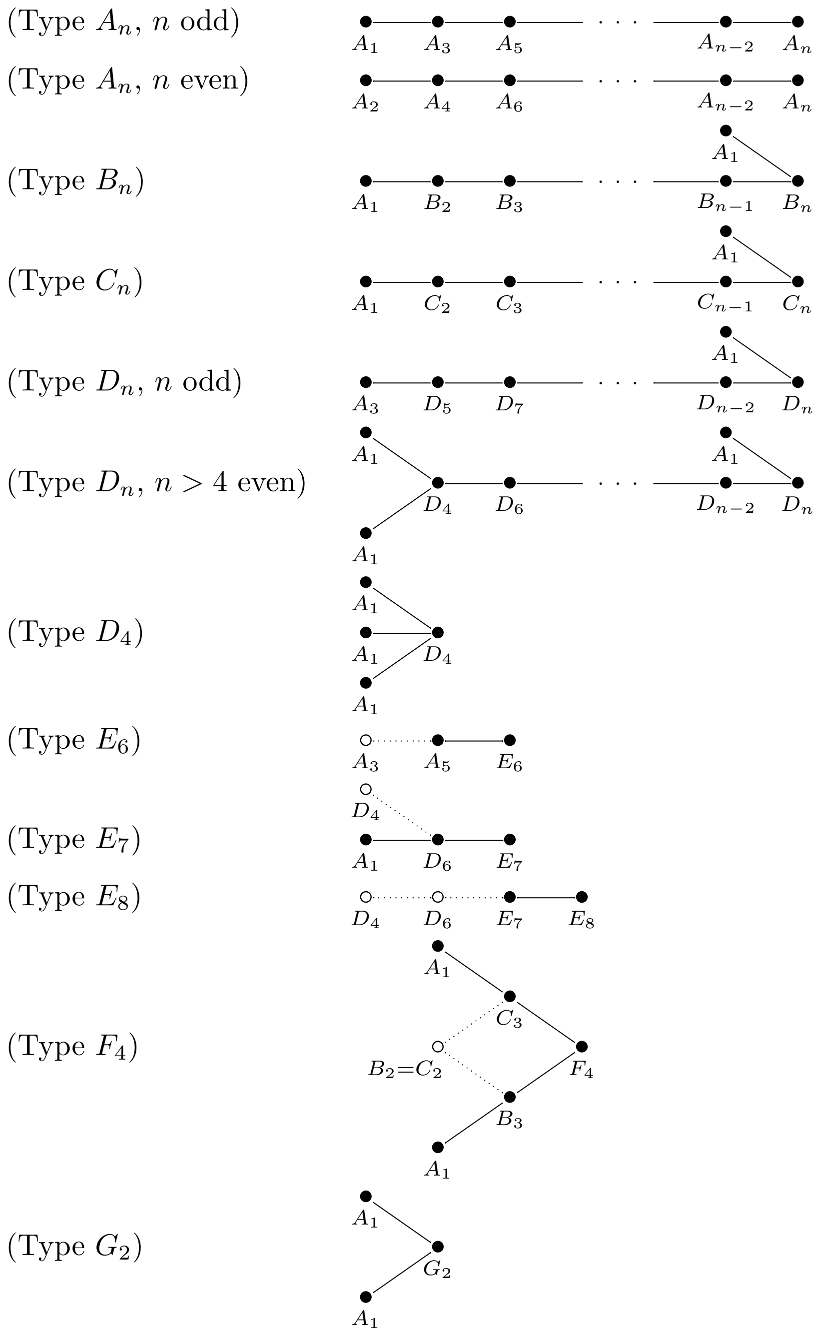


FIGURE 1. Hasse Diagrams of $\widetilde{\mathfrak{F}}_C$ ($\bullet$ = node of $\mathfrak{F}_C$; $\circ$ = node of $\widetilde{\mathfrak{F}}_C - \mathfrak{F}_C$). Each node ψ is labeled by the type of the root system generated by ψ; for simplicity, A_1 nodes in $\widetilde{\mathfrak{F}}_C - \mathfrak{F}_C$ are omitted.

real boundary components $D_{P,h}$ are equal-rank. In the case of a real equal-rank Satake compactification we wish to relate $\mathcal{B}$ (which depends on the $\mathbb{R}$-structure of G and the given Satake compactification) with $\mathcal{F}$ (which only depends on G as a $\mathbb{C}$-group). We begin with a basic lemma.

Lemma 12 (Borel and Casselman)**.** *A semisimple $\mathbb{R}$-group G is equal-rank if and only if $c^* = \iota$, where $c \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is complex conjugation.*

Proof. See [6, §1.2(1) and Corollary 1.6(b)]. $\square$

Corollary 13. *If a semisimple $\mathbb{Q}$-group G is equal-rank, then c^* commutes with g^* for all $g \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$.*

Lemma 14. *Let D^*_π be a real equal-rank Satake compactification. Then $\mathcal{B} \subseteq \widetilde{\mathcal{F}}$, every component of $\Delta^0_{\mathbb{C}/\mathbb{R}}$ belongs to $\widetilde{\mathcal{F}}$, and every component C of ${}_{\mathbb{C}}\Delta$ is c^*-stable.*

Proof. Let c^*_ψ denote the $*$-action of c for the subroot system with simple roots $\psi = \kappa(\varepsilon_{\mathbb{C}/\mathbb{R}}(\theta))$; one may check that $c^*_\psi = c^*|_\psi$. By our equal-rank assumption and Lemma 12, $c^*_\psi = \iota_\psi$ and $c^* = \iota$. It follows that ψ is ι-stable and that $\iota|_\psi = \iota_\psi$. It also follows that ψ is connected, since otherwise c^*_ψ would interchange the components and ι_ψ would preserve them. The same argument applies to a component ψ of $\Delta^0_{\mathbb{C}/\mathbb{R}}$ since $c^*_\psi = c^*|_\psi$ and ψ is the index of an equal-rank (even $\mathbb{R}$-anisotropic) group. $\square$

Corollary 15. *Let D^*_π be a real equal-rank Satake compactification. If a component C of ${}_{\mathbb{C}}\Delta$ has type F_4, then* $\mathbb{R}\text{-rank}\, C = 0$ *or* 1*.*

Proof. If $\mathbb{R}\text{-rank}\, C = 4$, then $\mathcal{B}_C \subseteq \widetilde{\mathcal{F}}_C$ contains a chain of length 4. By Figure 1, the chain must begin with one of the roots in the B_2 subdiagram; this root must therefore belong to δ. But then $\mathcal{B}_C$ would also contain an element of type A_2 which is excluded by the figure. Thus $\mathbb{R}\text{-rank}\, C \leq 3$. However F_4 only has real forms with $\mathbb{R}$-rank $= 0$, 1, and 4 [1, §5.9], [13, Chapter X, Table V]. $\square$

Corollary 16. *Let D^*_π be a real equal-rank Satake compactification. If $\theta \subseteq {}_{\mathbb{R}}C$ is a δ-connected subset of a connected component ${}_{\mathbb{R}}C$ of ${}_{\mathbb{R}}\Delta$, then θ is connected. Consequently $\kappa(\varepsilon_{\mathbb{C}/\mathbb{R}}(\theta)) \subseteq {}_{\mathbb{C}}\Delta$ is connected and* (*if nonempty*) *belongs to $\mathcal{B}$.*

Proof. Let $C = \varepsilon_{\mathbb{C}/\mathbb{R}}({}_{\mathbb{R}}C)$ be the corresponding component of ${}_{\mathbb{C}}\Delta$; it is connected by Lemma 14. We will prove that any two elements of ${}_{\mathbb{R}}\delta \cap {}_{\mathbb{R}}C$ are connected by an edge, from which the corollary follows. To see this, assume otherwise. Then there are clearly two distinct connected and δ-connected subsets of ${}_{\mathbb{R}}C$ with cardinality 2. This is impossible if the type of C is not F_4, since $\mathcal{B}^*_C \subseteq \widetilde{\mathcal{F}}^*_C$ is totally ordered by Proposition 10(i) and Lemma 14. Type F_4 is excluded by Corollary 15. $\square$

We introduce the following convenient notational convention: for $i \geq 1$, ψ_i will always denote some element of $\mathcal{B}$ with $\mathbb{R}\text{-rank}\,\psi_i = i$; let $\psi_0 = \emptyset$. The following observation will be used repeatedly.

Lemma 17. *Let D^*_π be a real equal-rank Satake compactification. For any $\psi_{i-1} < \psi_i$ we may decompose*

$$\psi_i - \psi_{i-1} = \rho^{-1}_{\mathbb{C}/\mathbb{R}}(\alpha) \cup \eta \qquad \text{for some } \alpha \in {}_{\mathbb{R}}\Delta \text{ and } \eta \subseteq \Delta^0_{\mathbb{C}/\mathbb{R}}. \tag{2}$$

The subset η is a union of components of $\Delta^0_{\mathbb{C}/\mathbb{R}}$ and

$$\psi_{i-1} \cup \{\beta\} \text{ is } \begin{cases} \text{connected} & \text{for } \beta \in \rho^{-1}_{\mathbb{C}/\mathbb{R}}(\alpha), \\ \text{disconnected} & \text{for } \beta \in \eta \text{ if } i > 1. \end{cases} \tag{3}$$

Proof. This follows from Definition 11 except for the assertion of "connected" in (3) as opposed to "δ-connected"; for that use (the proof of) Corollary 16. ☐

Proposition 18. *Let D^*_π be a real equal-rank Satake compactification. Then $\mathcal{B} \subseteq \mathcal{F}$.*

Proof. By Lemma 14 it suffices to verify that any $\psi = \psi_i \in \mathcal{B}_C$ satisfies conditions (i) and (ii) from Definition 9. If $\psi_i^+ - \psi_i$ had two elements modulo ι, there would exist two incomparable elements ψ_{i+1} and ψ'_{i+1} of $\mathcal{B}^*_C$ strictly containing ψ_i. For C not type F_4, this is impossible by Lemma 14 and Proposition 10(i) since $\mathcal{B}^*_C \subseteq \widetilde{\mathcal{F}}^*_C$ is totally ordered. Type F_4 is excluded by Corollary 15. Condition (i) follows.

As for condition (ii), let $\psi_{i+1} \in \mathcal{B}_C$ contain ψ_i; (i) implies that $\psi_{i+1} \supseteq \psi_i^+$. Then $\psi_{i+1} - \psi_i^+ = \eta$ is a union of components of $\Delta^0_{\mathbb{C}/\mathbb{R}}$ by Lemma 17; these components belong to $\widetilde{\mathcal{F}}_C$ by Lemma 14. ☐

Corollary 19. *Let D^*_π be a real equal-rank Satake compactification. Consider $\psi_{i-1} < \psi_i \in \mathcal{B}$ and let $\alpha \in {}_{\mathbb{R}}\Delta$ be as in (2). If $\psi = \psi_{i-1}$ or if $\psi \in \mathcal{F}$ is a component of $(\psi_i - \psi_{i-1}) \cap \Delta^0_{\mathbb{C}/\mathbb{R}}$ (the* noncompact *case and the* compact *case respectively), then $\psi^+ - \psi = \rho^{-1}_{\mathbb{C}/\mathbb{R}}(\alpha)$.*

Proof. In the noncompact case, (3) implies $\rho^{-1}_{\mathbb{C}/\mathbb{R}}(\alpha) \subseteq \psi^+ - \psi$; the same inclusion holds in the compact case since ψ_i is connected. However $\psi \in \mathcal{F}$ by the proposition (or by hypothesis in the compact case). Thus Definition 9(i) and Lemma 12 imply that $\psi^+ - \psi$ is a single c^*-orbit. The corollary follows. ☐

Theorem 20. *Let G be an almost $\mathbb{Q}$-simple semisimple group and let D^*_π be a real equal-rank Satake compactification. If G has an $\mathbb{R}$-simple factor H with* $\mathbb{R}$-rank $H = 2$ *and $\mathbb{C}$-type B_n, C_n, or G_2, assume that the Satake compactification associated to $\pi|_H$ does not have a real boundary component of type A_1. Then D^*_π is geometrically rational.*

Remark 21. In particular, this implies Baily and Borel's result [3] on geometric rationality for the natural compactification of a Hermitian symmetric space except possibly when G is the restriction of scalars of a group with $\mathbb{C}$-type B_n, C_n, or G_2. But the Hermitian condition excludes G_2 and implies for B_n or C_n that all simple factors of G have the same $\mathbb{R}$-type, so δ is Galois invariant and Theorem 6 applies.

Proof. For every component C of ${}_{\mathbb{C}}\Delta$, define

$$\mathcal{F}_C^\circ = \mathcal{F}_C - \{\psi \in \mathcal{F}_C | \psi \text{ is type } A_1 \text{ and is covered by } C \text{ which is type } B_n,\, C_n, \text{ or } G_2.\}$$

and set $\mathcal{F}^\circ = \coprod_C \mathcal{F}_C^\circ$. If $\psi \in \mathcal{F}_C - \mathcal{F}_C^\circ$ belonged to $\mathcal{B}$, this would imply that $\mathbb{R}\text{-rank}\, C = 2$. Since such a component is excluded by our hypotheses, Proposition 18 may be strengthened to $\mathcal{B} \subseteq \mathcal{F}^\circ$. Proposition 10(ii) implies that

(4) if $\psi \in \mathcal{B}_C$ and $\psi' \in \mathcal{F}_C^\circ$ are incomparable, then $\psi \cup \psi'$ is disconnected.

(If C has type F_4, $\mathbb{R}\text{-rank}\, C = 1$ by Corollary 15 which implies $\mathcal{B}_C = \{C\}$ and the above equation is vacuous.) It follows that

(5) $$\mathcal{B}_C \text{ is totally ordered.}$$

For if ψ, $\psi' \in \mathcal{B}_C$ were incomparable, the union would be both disconnected by (4) and connected by Corollary 16.

Define

$$\mathcal{K}_{nc} = \{\, \psi \mid \psi \text{ is a component of } \kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}})\,\},$$

$$\mathcal{K}_c = \left\{ \psi \in \mathcal{F}^\circ \,\middle|\, \begin{array}{c} \psi \text{ is a component of } (\psi_i - \psi_{i-1}) \cap \Delta^0_{\mathbb{C}/\mathbb{R}} \text{ for} \\ \text{some } \psi_{i-1} < \psi_i \in \mathcal{B} \text{ with } \psi_i \not\subseteq \kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}}) \end{array} \right\},$$

and set $\mathcal{K} = \mathcal{K}_{nc} \coprod \mathcal{K}_c$; we call the elements of $\mathcal{K}_{nc}$ *noncompact* and the elements of $\mathcal{K}_c$ *compact*. For a component C of ${}_{\mathbb{C}}\Delta$, we define $\mathcal{K}_C$, $\mathcal{K}_{C,nc}$, and $\mathcal{K}_{C,c}$ as usual.

Note that $\kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}}) = \kappa(\varepsilon_{\mathbb{C}/\mathbb{R}}(\Delta^0_{\mathbb{R}/\mathbb{Q}})) = \kappa(\varepsilon_{\mathbb{C}/\mathbb{R}}(\kappa(\Delta^0_{\mathbb{R}/\mathbb{Q}})))$; thus by Corollary 16

(6) $$\mathcal{K}_{nc} = \{\, \kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}}) \cap C \mid \delta \cap \Delta^0_{\mathbb{C}/\mathbb{Q}} \cap C \neq \emptyset \,\} \subseteq \mathcal{B} \subseteq \mathcal{F}^\circ.$$

We first show that if $\psi \in \mathcal{K}$ and $g \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, then

(7) $g^*\psi \in \mathcal{F}^\circ$, $g^*\psi \subseteq \Delta^0_{\mathbb{C}/\mathbb{Q}}$, and $g^*\psi$ is maximal among such sets.

It suffices to prove this for g^* the identity since the definition of $\mathcal{F}^\circ$ depends only the $\mathbb{C}$-root system and since $\Delta^0_{\mathbb{C}/\mathbb{Q}}$ is $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$-invariant. Then the first assertion is part of the definition if ψ is compact and follows from (6) if ψ is noncompact. The second assertion, that $\psi \subseteq \Delta^0_{\mathbb{C}/\mathbb{Q}}$, is clear. For the final assertion, assume $\psi \in \mathcal{K}_C$ and suppose that there exists $\psi' \in \mathcal{F}^\circ$ such that $\psi \subset \psi' \subseteq \Delta^0_{\mathbb{C}/\mathbb{Q}} \cap C$. If ψ is noncompact then it is δ-connected and hence ψ' (being connected) must also be δ-connected; this contradicts the fact that $\psi = \kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}}) \cap C$ is the largest δ-connected subset of $\Delta^0_{\mathbb{C}/\mathbb{Q}} \cap C$. If instead $\psi \subseteq \psi_i - \psi_{i-1}$ is compact, let $\alpha \in {}_{\mathbb{R}}\Delta$ be as in (2). Since $\psi^+ - \psi = \rho^{-1}_{\mathbb{C}/\mathbb{R}}(\alpha)$ by Corollary 19, any connected set strictly containing ψ, such as ψ', must contain an element of $\rho^{-1}_{\mathbb{C}/\mathbb{R}}(\alpha)$. Thus (a) $\psi_i - \psi_{i-1} \subseteq \Delta^0_{\mathbb{C}/\mathbb{Q}}$ by (2) and (b) $\psi_{i-1} \cup \psi'$ is connected by (3). If $i = 1$, (a) implies $\psi_1 \subseteq \kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}})$ which contradicts the definition of $\mathcal{K}_c$. If $i > 1$, the same argument shows that $\psi_{i-1} \not\subseteq \psi'$ and hence ψ_{i-1} and ψ' are incomparable; this contradicts (4) and (b).

We now prove that

$$\mathcal{K} \text{ is } \mathrm{Gal}(\mathbb{C}/\mathbb{Q})\text{-invariant}, \tag{8}$$

that is, given $\psi \in \mathcal{K}_C$ and $g \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ we will show that $g^*\psi \in \mathcal{K}$. Since we can assume G is not $\mathbb{Q}$-anisotropic, $g^*\psi \neq g^*C$. Let i be minimal such that $g^*\psi < \psi_i$ for some $\psi_i \in \mathcal{B}$ and consider some $\psi_{i-1} < \psi_i$. We know that $\psi_i \not\subseteq \kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}})$, else the maximality in (7) would be contradicted. There are three cases.

Case 1: $i = 1$. Let α and η be as in Lemma 17. We have $\rho^{-1}_{\mathbb{C}/\mathbb{R}}(\alpha) \cap \Delta^0_{\mathbb{C}/\mathbb{Q}} = \emptyset$ else $\psi_1 \subseteq \kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}})$. Thus $g^*\psi \in \mathcal{F}^\circ$ is contained in a component of $\eta = \psi_1 \cap \Delta^0_{\mathbb{C}/\mathbb{R}}$; since Proposition 10 and Lemmas 14 and 17 imply such a component is in $\mathcal{F}^\circ$, and hence $\mathcal{K}_c$, the maximality in (7) implies $g^*\psi$ equals this component.

Case 2: $i > 1$ and $\psi_{i-1} \leq g^*\psi$. In this case, $\psi_{i-1} \subseteq g^*\psi \subseteq \Delta^0_{\mathbb{C}/\mathbb{Q}}$ where ψ_{i-1} is nonempty and δ-connected and $g^*\psi$ is connected. This implies $g^*\psi$ is δ-connected and hence equals a component of $\kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}})$ by the maximality in (7). So $g^*\psi \in \mathcal{K}_{nc}$.

Case 3: $i > 1$ and $\psi_{i-1} \not\leq g^*\psi$. Since $\psi_{i-1} \cup g^*\psi$ is disconnected by (4), $g^*\psi$ is contained in a connected component of $(\psi_i - \psi_{i-1}) \cap \Delta^0_{\mathbb{C}/\mathbb{R}}$ by (3); as in Case 1, this component must be in $\mathcal{K}_c$ and $g^*\psi$ equals it.

This finishes the proof that $\mathcal{K}$ is $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$-invariant.

Let $\widehat{\mathcal{K}} = \mathrm{Gal}(\mathbb{C}/\mathbb{Q}) \cdot \mathcal{K}_{nc}$. We claim that if $\psi \in \widehat{\mathcal{K}}_C$ is compact, say $\psi \subseteq (\psi_i - \psi_{i-1}) \cap \Delta^0_{\mathbb{C}/\mathbb{R}}$, then $\psi_{i-1} = \kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}}) \cap C$. In the case $i = 1$, the claim asserts that $\kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}}) \cap C = \emptyset$; this holds since otherwise $\kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}}) \cap C \supseteq \psi_1$ (by (5)) which contradicts the definition of $\mathcal{K}_c$. As for the case $i > 1$, let $g \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ be such that $g^*\psi$ is noncompact. Then $g^*\psi \in \mathcal{B}$ and $g^*\psi_{i-1}$ are incomparable; let $\tilde{\psi} \in \mathcal{B}$ contain $g^*\psi$ and be maximal such that $\tilde{\psi}$ and $g^*\psi_{i-1}$ are incomparable. It follows from (4) and Lemma 17 that $g^*\psi_{i-1} \subseteq \Delta^0_{\mathbb{C}/\mathbb{R}}$ and hence $\psi_{i-1} \subseteq \Delta^0_{\mathbb{C}/\mathbb{Q}}$. Thus $\psi_{i-1} \subseteq \kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}}) \cap C \in \mathcal{B}_C$, while $\psi_i \not\subseteq \kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}})$. The claim now follows from (5).

Assume now that $\widehat{\mathcal{K}}_C \neq \emptyset$ for one and hence all components C. For a fixed C, it follows from the above claim and Corollary 19 that $\psi^+ - \psi$ is independent of the choice of $\psi \in \widehat{\mathcal{K}}_C$ and, if there does not exist a noncompact element in $\widehat{\mathcal{K}}_C$, then $\delta \cap C = \psi^+ - \psi$. It follows that

$$\beta \in \omega(\Delta^0_{\mathbb{C}/\mathbb{Q}}) \cap C \quad \Longleftrightarrow \quad \beta \notin \psi^+ - \psi \text{ for some (and hence any) } \psi \in \widehat{\mathcal{K}}_C. \tag{9}$$

For in the case that there exists $\psi \in \widehat{\mathcal{K}}_C$ noncompact, equation (9) holds since $\psi = \kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}}) \cap C$, while in the case that every $\psi \in \widehat{\mathcal{K}}_C$ is compact, $\omega(\Delta^0_{\mathbb{C}/\mathbb{Q}}) \cap C$ is simply the complement of $\delta \cap C = \psi^+ - \psi$ in C. Since the right hand side of (9) is $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$-invariant, condition (i) of Theorem 5 holds. Also if $g \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ and

$\psi \in \mathcal{K}$ is a component of $\kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}})$, then $g^*\psi \in \mathcal{K}$ and thus either $g^*\psi \subseteq \kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}})$ or $g^*\psi \subseteq \Delta^0_{\mathbb{C}/\mathbb{R}}$; this verifies condition (ii) of Theorem 5.

The remaining case is where $\widehat{\mathcal{K}}_C = \emptyset$ for all C. Then $\bigcup_{g\in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})} g^*\mathcal{B} \cap C$ is totally ordered for any C; this may be proved similarly to (8) but more easily. Let $\psi_1 \in \mathcal{B}_C$ and compare $g^*\psi_1$ with $\psi_1' \in \mathcal{B}_{g^*C}$. Say $g^*\psi_1 \leq \psi_1'$. We must have $g^*(\delta \cap C) \subseteq \delta \cap g^*C$, since otherwise $\kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}})$ would not be empty. Since g^* commutes with c^* (Corollary 13) and $\delta \cap C$ has only one element modulo c^* (by equation (5)), we have equality. Thus δ is Galois invariant and the theorem follows from Theorem 6. $\square$

7. Exceptional cases

It remains to handle the cases excluded from Theorem 20. In contrast to the situation of that theorem, where geometric rationality was automatic with no rationality assumption on (π, V), these exceptional cases are only geometrically rational under a certain condition which depends on the $\mathbb{Q}$-rank.

Theorem 22. *Let G be an almost $\mathbb{Q}$-simple semisimple group which is the restriction of scalars of a group G' with $\mathbb{C}$-type B_n, C_n, or G_2. Let D^*_π be a real equal-rank Satake compactification. Assume G has an $\mathbb{R}$-simple factor H with* $\mathbb{R}$-rank $H = 2$ *for which the Satake compactification associated to $\pi|_H$ has a real boundary component of type A_1.*

(i) *In the case* $\mathbb{Q}$-rank $G = 2$, *D^*_π is geometrically rational if and only if δ is Galois invariant.*

(ii) *In the case* $\mathbb{Q}$-rank $G = 1$, *D^*_π is geometrically rational if and only if $\delta \cap \Delta^0_{\mathbb{C}/\mathbb{Q}}$ is empty or meets every component of ${}_{\mathbb{C}}\Delta$ with* $\mathbb{R}$-rank ≥ 2.

Proof. Let C be the component of ${}_{\mathbb{C}}\Delta$ corresponding to H and let $\alpha_1 \in \delta \cap C$ correspond to the real boundary component of type A_1. By Lemma 17 and our hypotheses, α_1 is at an end of C and (if we denote its unique neighbor by α_2) we have $C - \{\alpha_1, \alpha_2\} = \Delta^0_{\mathbb{C}/\mathbb{R}} \cap C \subseteq \Delta^0_{\mathbb{C}/\mathbb{Q}} \cap C$. If H has $\mathbb{C}$-type C_n, the classification of semisimple Lie algebras over $\mathbb{R}$ [1], [13, Chapter X, Table VI] shows that the two adjacent non-$\mathbb{R}$-anisotropic roots imply H is $\mathbb{R}$-split; thus C_n only occurs for $n = 2$ and we can absorb this case into that of B_n. Classification theory over $\mathbb{R}$ also shows that if $n > 2$, the simple root α_1 is long.

For (i) we only need to prove that δ is Galois invariant under the assumption that (i) and (ii) of Theorem 5 hold; the opposite direction was already proved in Theorem 6. Thus assume that $\mathbb{Q}$-rank $G = 2$. Then for all $g \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ we have $\kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}}) \cap g^*C \subseteq \Delta^0_{\mathbb{C}/\mathbb{Q}} \cap g^*C = g^*(C - \{\alpha_1, \alpha_2\}) \subseteq \omega(\Delta^0_{\mathbb{C}/\mathbb{Q}}) \cap g^*C$. Since $g^*\alpha_1$ is orthogonal to $g^*(C - \{\alpha_1, \alpha_2\})$, it follows that $g^*\alpha_1 \notin \omega(\Delta^0_{\mathbb{C}/\mathbb{Q}}) \cap g^*C$ if and only if $g^*\alpha_1 \in \delta \cap g^*C$. But the first condition is independent of g by condition (i) of Theorem 5 while the second condition holds for $g = e$, so we must have $g^*\alpha_1 \in \delta \cap g^*C$ for all $g \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. If $\#(\delta \cap g_0^*C) > 1$ for some $g_0 \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ then $\mathbb{R}$-rank $g_0^*C = 2$ by Propositions 10(ii) and 18, in which case $g_0^*(C - \{\alpha_1, \alpha_2\}) = \Delta^0_{\mathbb{C}/\mathbb{R}} \cap g_0^*C$; thus the additional element of $\delta \cap g_0^*C$

must be $g_0^*\alpha_2$. In this case however $\kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}}) = \emptyset$ so that for all $g \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, $g^*\alpha_2 \notin \omega(\Delta^0_{\mathbb{C}/\mathbb{Q}}) \cap g^*C$ if and only if $g^*\alpha_2 \in \delta \cap g^*C$. Another application of condition (i) from Theorem 5 concludes the proof that δ is Galois invariant in this case.

For (ii) assume that $\mathbb{Q}$-rank $G = 1$. Write $G = R_{k/\mathbb{Q}}G'$ where k is a finite extension of $\mathbb{Q}$ (totally real by Lemma 14) and G' is an almost k-simple group with k-rank $= 1$ and $\mathbb{C}$-type B_n or G_2. The classification of semisimple groups over $\mathbb{Q}$ [24] shows the case G_2 cannot occur and that in the case B_n the unique simple root which is not k-anisotropic is the long root at the end of the Dynkin diagram. Since $\Delta^0_{\mathbb{C}/\mathbb{Q}} \cap C$ is thus connected for every component C of ${}_{\mathbb{C}}\Delta$, it follows that

$$\kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}}) \cap C \text{ is nonempty} \quad \Longleftrightarrow \quad \delta \cap \Delta^0_{\mathbb{C}/\mathbb{Q}} \text{ meets } C. \tag{10}$$

If this condition holds, then $\kappa(\Delta^0_{\mathbb{C}/\mathbb{Q}}) \cap C = \Delta^0_{\mathbb{C}/\mathbb{Q}} \cap C$, while if the condition does not hold, then $\delta \cap C = C - (\Delta^0_{\mathbb{C}/\mathbb{Q}} \cap C)$. In either case, $\omega(\Delta^0_{\mathbb{C}/\mathbb{Q}}) \cap C = \Delta^0_{\mathbb{C}/\mathbb{Q}} \cap C$. Furthermore

$$\Delta^0_{\mathbb{C}/\mathbb{Q}} \cap C \not\subseteq \Delta^0_{\mathbb{C}/\mathbb{R}} \quad \Longleftrightarrow \quad \mathbb{R}\text{-rank}\, C \geq 2. \tag{11}$$

Part (ii) now easily follows from Theorem 5. □

Remarks 23. Assume G satisfies the hypotheses of Theorem 22.

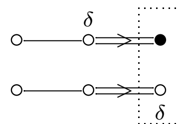
(i) In the $\mathbb{Q}$-rank 2 case, if D^*_π is geometrically rational then all $\mathbb{R}$-simple factors of G have $\mathbb{R}$-rank $= 2$. For the proof shows that every component g^*C has an A_1 boundary component corresponding to $g^*\alpha_1 \in \delta \cap g^*C$ and that if $n > 2$ then α_1 is long. However if $\mathbb{R}$-rank $g^*C > 2$ then the entry for B_n in Figure 1 implies that $\delta \cap g^*C$ is a singleton short root.

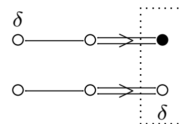
(ii) In the $\mathbb{Q}$-rank 1 case, $\delta \cap \Delta^0_{\mathbb{C}/\mathbb{Q}}$ automatically meets any component with $\mathbb{R}$-rank > 2; this follows from the proof and the entry for B_n in Figure 1. Thus if there is any component with $\mathbb{R}$-rank > 2 , the condition for geometric rationality is that $\delta \cap \Delta^0_{\mathbb{C}/\mathbb{Q}}$ meets every component with $\mathbb{R}$-rank $= 2$.

(iii) Also in the $\mathbb{Q}$-rank 1 case, if $\mathbb{C}$-rank $G' > 2$ then G has a component with $\mathbb{R}$-rank 1. For by the proof above, G' (up to strict k-isogeny) is the special orthogonal group of a quadratic form q in at least 7 variables with k-index 1. Thus we can decompose q into a hyperbolic plane and a k-anisotropic form q' in at least 5 variables. Since every quadratic form in at least 5 variables is isotropic at every non-archimedean place [15, VI.2.12], q' must be anisotropic at some real place by the Hasse-Minkowski principle [15, VI.3.5].

Examples 24. Let k be a real quadratic extension of $\mathbb{Q}$. In the Satake diagrams below, the roots in $\Delta^0_{\mathbb{C}/\mathbb{R}}$ are colored black, while roots in $\Delta^0_{\mathbb{C}/\mathbb{Q}}$ are enclosed in a dotted box. Nodes placed vertically above each other form a $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$-orbit. The elements of δ are so labeled.

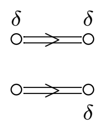
(i) Let q be a k-anisotropic form in three variables, $\mathbb{R}$-anisotropic at one real place and of signature $(2,1)$ at the other. Let G' be the orthogonal group of $h \oplus h \oplus q$, where h is hyperbolic space, and let $G = R_{k/\mathbb{Q}}G'$. Then both of the Satake compactifications



and


are real equal-rank. The first is geometrically rational (Theorem 20 applies) and the second is not (Theorem 22(i) applies). More specifically, condition (i) from Theorem 5 fails in the second example; this example also illustrates Remark 23(i).

If q is instead nonsingular of dimension 1, then of three real equal-rank Satake compactifications



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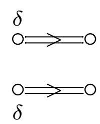
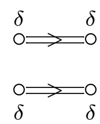


, and


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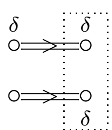
Theorem 22(i) shows that the first is not geometrically rational and the last two are. Similar examples can be constructed with q totally $\mathbb{R}$-anisotropic in any odd number of variables or where G' is the split form of G_2.

(ii) Let G' be the orthogonal group of $h \oplus q$, where q is a k-anisotropic form in three variables which has signature $(2,1)$ at both real places, and let $G = R_{k/\mathbb{Q}}G'$. Among the real equal-rank Satake compactifications



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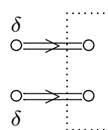
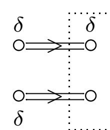


, and


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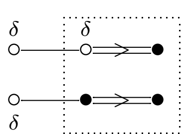
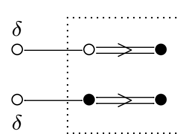
the first two are geometrically rational (even though in the first δ is not Galois invariant) and the last is not; this illustrates Theorem 22(ii).

(iii) Let G' be the orthogonal group of $h \oplus q$, where q is a k-anisotropic form in five variables which has signature $(4,1)$ at one real place and is $\mathbb{R}$-anisotropic at the other (see Remark 23(iii)) and let $G = R_{k/\mathbb{Q}}G'$. There are two real equal-rank Satake compactifications covered by Theorem 22(ii),



and


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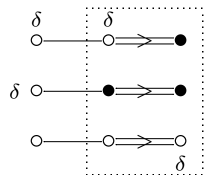
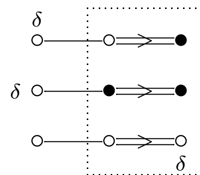
both of which are geometrically rational. If, however, k is a totally real degree 3 extension of $\mathbb{Q}$ and q has signature $(4,1)$, $(5,0)$, and $(3,2)$ at the three real places, the two real equal-rank Satake compactifications covered by Theorem 22(ii) are



and



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Here the first is geometrically rational and the second is not; this illustrates Remark 23(ii).

References

[1] S. Araki, *On root systems and an infinitesimal classification of irreducible symmetric spaces*, J. Math. Osaka City Univ. **13** (1962), 1–34.

[2] A. Ash, D. Mumford, M. Rapoport, and Y. Tai, *Smooth compactification of locally symmetric varieties*, Math Sci Press, Brookline, 1975.

[3] W. Baily and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. **84** (1966), 442–528.

[4] A. Borel, *Ensembles fondamentaux pour les groupes arithmétiques*, Colloque sur la Théorie des Groupes Algébriques, Tenu à Bruxelles du 5 au 7 juin 1962, Centre Belge de Recherches Mathématiques, 1962, 23–40.

[5] ———, March 19, 2002, letter to L. Saper.

[6] A. Borel and W. Casselman, *L^2-cohomology of locally symmetric manifolds of finite volume*, Duke Math. J. **50** (1983), 625–647.

[7] A. Borel and J. Tits, *Groupes réductifs*, I.H.E.S. Publ. Math. **27** (1965), 55–151.

[8] W. Casselman, *Geometric rationality of Satake compactifications*, Algebraic groups and Lie groups, Austral. Math. Soc. Lect. Ser., **9**, Cambridge Univ. Press, Cambridge, 1997, 81–103.

[9] M. Goresky, G. Harder, and R. MacPherson, *Weighted cohomology*, Invent. Math. **116** (1994), 139–213.

[10] M. Goresky and R. MacPherson, *Weighted cohomology of Satake compactifications*, Centre de recherches mathématiques, preprint #1593, 1988.

[11] Harish-Chandra, *Spherical functions on a semisimple Lie group. I*, Amer. J. Math. **80** (1958), 241–310.

[12] S. Helgason, *A duality for symmetric spaces with applications to group representations*, Adv. Math. **5** (1970), 1–154.

[13] ———, *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics, **80**, Academic Press, New York, 1978.

[14] A. Knapp, *Lie groups, beyond an introduction*, Birkhäuser, Boston, 1996.

[15] T. Y. Lam, *The algebraic theorem of quadratic forms*, Benjamin, Reading, Mass., 1973.

[16] Y. Matsushima and S. Murakami, *On vector bundle valued harmonic forms and automorphic forms on symmetric Riemannian manifolds*, Ann. of Math. **78** (1963), 365–416.

[17] C. Moore, *Compactifications of symmetric spaces II: The Cartan domains*, Amer. J. Math. **86** (1964), 358–378.

[18] M. Rapoport, 1986, letter to A. Borel.

[19] ———, *On the shape of the contribution of a fixed point on the boundary: The case of $\mathbb{Q}$-rank 1 (with an appendix by L. Saper and M. Stern)*, The Zeta functions of Picard modular surfaces, Univ. Montréal, Montréal, QC, 1992, 479–488.

[20] L. Saper, *$\mathcal{L}$-modules and micro-support*, preprint, `math.RT/0112251`, 2001.

[21] I. Satake, *On representations and compactifications of symmetric Riemannian spaces*, Ann. of Math. **71** (1960), 77–110.

[22] ———, *On compactifications of the quotient spaces for arithmetically defined discontinuous groups*, Ann. of Math. **72** (1960), 555–580.

[23] R. P. Stanley, *Enumerative combinatorics*, **1**, Cambridge Univ. Press, Cambridge, 1997.

[24] J. Tits, *Classification of algebraic semisimple groups*, Algebraic Groups and Discontinuous Subgroups, Proc. Sympos. Pure Math., **9**, Amer. Math. Soc., Providence, 1966, 33–62.

[25] S. Zucker, *Satake compactifications*, Comment. Math. Helv. **58** (1983), 312–343.

[26] ———, *L_2-cohomology and intersection homology of locally symmetric varieties, II*, Compositio Math. **59** (1986), 339–398.

Department of Mathematics, Duke University, Box 90320, Durham, NC 27708, U.S.A.

E-mail address: `saper@math.duke.edu`

URL: `http://www.math.duke.edu/faculty/saper`