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A NOTE ON THE UNIFORMIZATION OF GRADIENT KÄHLER RICCI SOLITONS

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Abstract. Applying a well known result for attracting fixed points of biholomorphisms [5, 7], we observe that one immediately obtains the following result: if (M^n, g) is a complete non-compact gradient Kähler-Ricci soliton which is either steady with positive Ricci curvature so that the scalar curvature attains its maximum at some point, or expanding with non-negative Ricci curvature, then M is biholomorphic to $\mathbb{C}^n$.

In this note we will prove the following:

Theorem 1. *If (M^n, g) is a complete non-compact gradient Kähler-Ricci soliton which is either steady with positive Ricci curvature so that the scalar curvature attains its maximum at some point, or expanding with non-negative Ricci curvature, then M is biholomorphic to $\mathbb{C}^n$.*

In [2], Cao constructed examples of complete rotationally symmetric gradient Kähler-Ricci solitons with positive holomorphic bisectional curvature on $\mathbb{C}^n$. Theorem 1 implies that in fact all complete gradient Kähler-Ricci solitons with positive holomorphic bisectional curvature, satsifying the assumptions in the theorem, are defined on $\mathbb{C}^n$. The theorem is related to the uniformization conjecture of Yau which states that all complete non-compact Kähler manifolds with positive holomorphic bisectional curvature are biholomorphic to $\mathbb{C}^n$ (see [3] for more details).

Recall that a Kähler-Ricci soliton is a solution to the un-normalized Kähler-Ricci flow

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}} \tag{0.1}$$

which evolves only by dilation and pull back along a one parameter family of biholomorphisms. More specifically, $(M, g_{i\bar{j}}(x))$ is said to be a Kähler-Ricci soliton if there is a family of biholomorphisms ϕ_t on M, given by a holomorphic vector field V, such that $g_{ij}(x,t) = \phi_t^*(g_{ij}(x))$ is a solution of the Kähler-Ricci

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flow:

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}} - 2\rho g_{i\bar{j}} \qquad (0.2)$$
$$g_{i\bar{j}}(x,0) = g_{i\bar{j}}(x)$$

for $0 \le t < \infty$, where $R_{i\bar{j}}$ denotes the Ricci tensor at time t and ρ is a constant. If $\rho = 0$, then the Kähler-Ricci soliton is said to be of *steady type* and if $\rho > 0$ then the Kähler-Ricci soliton is said to be of *expanding type*. We always assume that g is complete and M is non-compact. If in addition, the holomorphic vector field is given by the gradient of a real valued function f, then it is called a gradient Kähler-Ricci soliton. Note that in this case, we have that

$$f_{i\bar{j}} = R_{i\bar{j}} + 2\rho g_{i\bar{j}} \qquad (0.3)$$
$$f_{ij} = 0.$$

If (M, g) is a gradient Kähler-Ricci soliton which is either steady with positive Ricci curvature so that the scalar curvature attains its maximum at some point, or expanding with non-negative Ricci curvature, then one can show that ϕ_t, the flow on M along the vector field $-\nabla f$, satisfies:

(i) ϕ_t is a biholomorphism from M to M for all $t \ge 0$,
(ii) ϕ_t has a unique fixed point p, i.e. $\phi_t(p) = p$ for all $t \ge 0$,
(iii) M is attracted to p under ϕ_t in the sense that for any open neighborhood U of p and for any compact subset W of M, there exists $T > 0$ such that $\phi_t(W) \subset U$ for all $t \ge T$.

Condition (i) is clear. Condition (ii) is shown in [3, 4]. To see that condition (iii) holds, we consider any $R > 0$ and let $B(R)$ be the geodesic ball of radius R with center at p with respect to the metric $g(0)$. From the proof of Lemma 3.2 in [3], there exists $C_R > 0$ such that for any $q \in B(R)$ and for any $v \in T^{1,0}(M)$ at q,

$$||v||_{\phi_t^*(g)} \le \exp(-C_R t)||v||_g.$$

Since $\phi_t(p) = p$, it is easy to see that given any open set $U \subset M$ containing p, we have $\phi_t(B(R)) \subset U$ provided t is large, and thus condition (iii) is satisfied.

We now observe a general result on the biholomorphic structure of a basin of attraction for any biholomprphism on a complex manifold. The following theorem was proved for the case $M = \mathbb{C}^n$ in [5], and was later observed to be true on a general complex manifold M in [7].

Theorem 2. *Let F be a biholomorphism from a complex manifold M^n to itself and let $p \in M^n$ be a fixed point for F. Fix a complete Riemannian metric g on M and define*

$$\Omega := \{x \in M : \lim_{k\to\infty} dist_g(F^k(x), p) = 0\}$$

where $F^k = F \circ F^{k-1}, F^1 = F$. Then Ω is biholomorphic to $\mathbb{C}^n$ provided Ω contains an open neighborhood around p.

Proof of Theorem 1. By conditions (i)-(iii) we may apply Theorem 2 to the biholomorphism $\phi_1 : M \to M$ to conclude that M is biholomorphic to $\mathbb{C}^n$. □

Remark 1. In the first version of this article we proved Theorem 2 in a special case. We would like to thank Dror Varolin for pointing out to us that what we proved had been known earlier [5, 7].

Remark 2. After posting the first version of this article we learned that Theorem 1 in the case of a steady gradient Kähler Ricci soliton had been known independently to Robert Bryant [1].

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