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COORDINATES FOR THE MODULI SPACE OF FLAT $PSL(2,\mathbb{R})$-CONNECTIONS

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Abstract. Let $\mathcal{M}$ be the moduli space of irreducible flat $PSL(2,\mathbb{R})$ connections on a punctured surface of finite type with parabolic holonomies around punctures. By using a notion of *admissibility* of an ideal arc, $\mathcal{M}$ is covered by dense open subsets associated to ideal triangulations of the surface. A principal bundle over $\mathcal{M}$ is constructed which, when restricted to the Teichmüller component of $\mathcal{M}$, is isomorphic to the decorated Teichmüller space of Penner. The construction gives a generalization to $\mathcal{M}$ of Penner's coordinates for the Teichmüller space.

1. Introduction

In this paper we consider a punctured surface of finite type $\Sigma = \Sigma_{g,s}$ which is the complement of a finite set of points $V \subset \overline{\Sigma}$, $|V| = s$, called *punctures*, in a closed oriented surface $\overline{\Sigma}$ of genus g:

$$\Sigma = \overline{\Sigma} \setminus V$$

We assume the following restrictions on g and s:

$$s > 0, \quad \kappa \equiv -\chi(\Sigma) = 2g - 2 + s > 0$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ.

Denote by G the group $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm 1\}$. Let $\mathcal{M}$ be the moduli space of irreducible flat G-connections on Σ with parabolic holonomies around the punctures. Connections, representing points of $\mathcal{M}$ will be called *flat connections*. There is a one to one correspondence between elements of $\mathcal{M}$ and equivalence classes of irreducible representations in G of the fundamental group of the surface with parabolicity conditions at punctures. In other words, the moduli space $\mathcal{M}$ can be identified with the open subset of the quotient space $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ where the group G acts by conjugations freely and properly so that $\mathcal{M}$ has the structure of a smooth manifold as the base of a principal G-bundle.

In this paper we extend the Penner's construction of the decorated Teichmüller space [8] to the whole moduli space $\mathcal{M}$. Motivation for this work comes from nice properties of the decorated Teichmüller space (such as simple and explicit realization of the mapping class group action, simple form of the Weil–Petersson symplectic structure, etc.) which are very useful for quantizing the Teichmüller

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space [1, 6]. The main results of this paper are formulated below in Theorems 1 and 2.

With any path a in Σ, starting at some $x \in \Sigma$ and ending at some puncture P, we associate the homotopy class $\ell(a)$ of loops based at x which go along a towards P, then go around P along a small circle, and then return back to x along a^{-1}.

Let a and b be two paths starting from one and the same point in Σ and ending at some punctures. Let $m \in \mathcal{M}$ be represented by flat connection h.

Definition 1. *The homotopy class of the path $a^{-1}b$ is called m-*admissible *if h-holonomies along loops $\ell(a)$ and $\ell(b)$ belong to distinct unipotent subgroups (of parabolic elements) of $PSL(2,\mathbb{R})$.*

It is clear that this notion depends on only the homotopy class $[a^{-1}b]$ and moduli $m = [h]$, but not on the choice of their representatives.

Recall that an *ideal arc* on Σ is a nontrivial isotopy class of a simple path running between punctures, and an *ideal triangulation* of Σ is a maximal family of pairwise nonintersecting ideal arcs. The sets $\mathcal{A}_\Sigma$ and Δ_Σ of ideal arcs and ideal triangulations, respectively, are countable infinite.

With each ideal triangulation τ we can now associate an open dense subset $\mathcal{M}_\tau$ of $\mathcal{M}$ defined as those moduli of flat connections for which τ is an admissible ideal triangulation, i.e. with all ideal arcs admissible.

Theorem 1. *Let $\Sigma, \mathcal{M}, \{\mathcal{M}_\tau\}_{\tau\in\Delta_\Sigma}$ be as above. Then*

(i): *the collection $\{\mathcal{M}_\tau\}_{\tau\in\Delta_\Sigma}$ is a covering for $\mathcal{M}$;*

(ii): *there exists a finite subcovering $\{\mathcal{M}_\tau\}_{\tau\in\Pi}$, $\Pi \subset \Delta_\Sigma$, $|\Pi| < \infty$.*

Theorem 2. *Let $\Sigma, \mathcal{M}, \{\mathcal{M}_\tau\}_{\tau\in\Delta_\Sigma}$ be as above. Then there exists a principal $\mathbb{R}^s_{>0}$-bundle $\pi\colon \tilde{\mathcal{M}} \to \mathcal{M}$ such that*

(i): *for each $\tau \in \Delta_\Sigma$ one has $\pi^{-1}(\mathcal{M}_\tau) = \coprod_{\epsilon\in\{\pm1\}^{2\kappa}} \mathcal{R}_\epsilon(\tau)$ with each $\mathcal{R}_\epsilon(\tau)$ being a principal $\mathbb{R}^s_{>0}$-bundle homeomorphic to the complement of $\psi^{-1}_{\tau,\epsilon}(0)$ for certain rational mapping $\psi_{\tau,\epsilon}\colon \mathbb{R}^{3\kappa}_{>0} \to \mathbb{R}$;*

(ii): *if $\sigma\colon \{\pm1\}^{2\kappa} \to \mathbb{Z}$ is defined by $\sigma(\epsilon) = \frac{1}{2}\sum_{i=1}^{2\kappa}\epsilon_i$, then for $-\kappa \le k \le \kappa$, the sets $\mathcal{R}_k = \bigcup_{\tau\in\Delta_\Sigma}\coprod_{\epsilon\in\sigma^{-1}(k)}\mathcal{R}_\epsilon(\tau)$ are principal $\mathbb{R}^s_{>0}$-bundles disjoint for different k;*

(iii): *there exist principal bundle isomorphisms between $\mathcal{R}_{\pm\kappa}$ and the decorated Teichmüller space $\tilde{\mathcal{T}}$ of Penner.*

Theorem 1(i) and Theorem 2(ii) imply that

$$\mathcal{M} = \coprod_{-\kappa\le k\le\kappa} \mathcal{M}_k, \quad \mathcal{M}_k = \mathcal{R}_k/\mathbb{R}^s_{>0}$$

Part (iii) of Theorem 2 implies that $\mathcal{M}_{\pm\kappa}$, being homeomorphic to the Teichmüller space, are open cells of dimension $3\kappa - s$. The total number of these components is $2\kappa + 1$ which formally at $s = 0$ coincides with the number of connected components of the moduli space of flat $PSL(2,\mathbb{R})$ connections on a closed surface [4]. In our case, however, the components $\mathcal{M}_k$ can be empty or not connected. For example, in the case $\kappa = 1$, $\mathcal{M}_0 = \emptyset$. The non-connectedness of $\mathcal{M}_k$ originates from the fact that there are two distinct conjugacy classes of parabolic elements in $PSL(2,\mathbb{R})$ distinguished by the sign of the off diagonal matrix element when represented by an upper triangular matrix with trace equal to $+2$. For example, consider a sphere with four punctures. One can show that there are two points $m, m' \in \mathcal{M}_0$ such that the holonomies around two fixed punctures are in one and the same conjugacy class for m and in distinct conjugacy classes for m'. Thus, it is clear that m and m' cannot be in one and the same connected component of the moduli space $\mathcal{M}$. However, taking into account the formal agreement of the invariant k with Goldman's relative Euler class, it is natural to expect that for fixed conjugacy types of the holonomies around the punctures nonempty $\mathcal{M}_k$'s are connected.

Part (i) of Theorem 2 extends to $\mathcal{M}$ Penner's coordinate charts for the Teichmüller space. Coordinatization is given by assigning positive real numbers to edges and signs to faces of an ideal triangulation. Part (iii) implies that the Teichmüller component of the moduli space corresponds to putting all signs to one and the same value. In this parameterization one still has an explicit description of both the action of the mapping class group in $\mathcal{M}$ and Goldman's symplectic structure [3] which extends the Weil–Petersson symplectic structure on the Teichmüller component. The pullback of Goldman's symplectic form to $\tilde{\mathcal{M}}$ is given by the same formula as that of Penner in [9] for the pullback of the Weil–Petersson form.

Restriction to parabolic holonomies around the punctures is essential for our construction. Nevertheless, any other case, including the case of closed surfaces, can also be considered in our framework by fixing holonomies around some essential loops. The fact, that we are considering the whole moduli space ensures that holonomies along essential loops can be of any type: hyperbolic, parabolic, elliptic or trivial.

The rest of this paper is organized as follows. In Section 2 we collect the material to be used in proving the theorems. Then, Sections 3 and 4 contain the proofs of two parts of Theorem 1. In the last Section 5 we prove Theorem 2.

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2. Preliminaries

2.1. Subgroups of $PSL(2,\mathbb{R})$. Fix a one-parameter subgroup $U \subset G = PSL(2,\mathbb{R})$ of parabolic elements corresponding to upper triangular unipotent matrices, i.e. matrices with trace equal to ± 2. To avoid a confusion with the term "parabolic subgroup" in the algebraic-group context, in this paper subgroups conjugated to U will be called *unipotent subgroups*.

The normalizer of U, $B = N(U)$, is a Borel subgroup corresponding to upper triangular matrices. In the exact sequence of group homomorphisms

$$1 \to U \to B \to T \to 1$$

the group $T = B/U$ is identified with the Cartan subgroup represented by diagonal matrices. The Bruhat decomposition of G with respect to B consists of only two cells

$$G = B\theta B \sqcup B$$

where $\theta \in N(T) \subset G$ is a fixed representative of the only nontrivial element of the Weyl group $N(T)/T$. We choose it in a unique way by the condition $\theta^2 = 1$ so that it is represented by the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

In the standard action of $PSL(2, \mathbb{R})$ by linear fractional transformations in the upper half of the complex plane, the Cartan subgroup is characterized by the condition that it fixes 0 and ∞, while the Borel subgroup is the stabilizer of ∞. Element θ is given by the inversion with respect to the unit circle, and elements in the two Bruhat cells are distinguished depending on whether ∞ is moved or not.

2.2. Ideal triangulations. For any $x, y \in \overline{\Sigma}$ denote by $\Omega(x, y)$ the set of paths in Σ starting at x and ending at y. The subset $\Omega_{\mathrm{s}}(x, y) \subset \Omega(x, y)$ consists of simple paths. Denote also $\Omega(x) = \Omega(x, x)$ and $\Omega_{\mathrm{s}}(x) = \Omega_{\mathrm{s}}(x, x)$. Two paths are called *not intersecting* if their interior parts do not intersect.

Ideal arcs are nontrivial isotopy classes in $\Omega_{\mathrm{s}}(P, Q)$, for all $P, Q \in V$. Associated to each ideal arc e there are two vertices $P_e, Q_e \in V$, possibly coinciding, such that e is an isotopy class in $\Omega_{\mathrm{s}}(P_e, Q_e)$. The set of all ideal arcs is denoted $\mathcal{A}_\Sigma$.

An *ideal triangulation* of Σ is a maximal set of pairwise nonintersecting ideal arcs. The set of all ideal triangulations of Σ is denoted by Δ_Σ.

Let an ideal triangulation be represented by a system of simple pairwise nonintersecting curves. It is clear that the complement of Σ to such system is a disjoint union of cells homeomorphic to triangles. The isotopy classes of such triangles will be called *ideal triangles*. Each ideal triangulation τ is thus associated with a cell complex homeomorphic to $\overline{\Sigma}$ with the set of punctures V as vertices, the ideal arcs $E(\tau) = \tau$ as (oriented) edges, and ideal triangles $F(\tau) = \bar{\tau}$ as faces. In addition, we denote by $C(\tau) = \coprod_{P \in V} C(\tau, P)$ the set of oriented corners of ideal triangles, i.e. the connected components of intersections of ideal triangles with small disks centered at punctures, where $C(\tau, P)$ is the set of corners containing vertex P. In fact, one has s vertices, 3κ edges, 2κ faces, and 6κ corners in any ideal triangulation of Σ. Each corner $c \in C(\tau, P)$ belongs to unique triangle T_c with the side $O_c \in \tau$ opposite to P. Denote two other sides of T_c by A_c and B_c. For an ideal polygon p in Σ we denote by $E(p)$ and $C(p)$ the sets of its sides and corners, respectively.

If $a \in \Omega(x, P)$, where $x \in \overline{\Sigma}$ and $P \in V$, we denote by $\ell(a)$ the homotopy class of the loop based at x and obtained by going along a towards P, going around P along a small circle in the direction induced from the orientation of Σ, and returning back to x along a^{-1}. In the case, when $x = Q \in V$, the class $\ell(a)$, when it corresponds to a simple loop, is naturally associated with an ideal arc based at Q.

2.3. Graph connections.

Definition 2. *A* flat graph G-connection *on a graph* $\Gamma \subset \Sigma$ *is an isomorphism class of flat* G*-connections on* Σ *with fixed parallel transport operators along edges of* Γ*. A* graph gauge transformation *of a flat graph connection is a usual gauge transformation modulo gauge transformations relating different representatives of graph connections.*

Equivalently, one can define a flat graph G-connection as a representation in G of the edge-path groupoid of Γ, graph gauge transformations being equivalence transformations of representations. It is clear that if graph Γ contains a subgraph homotopically equivalent to Σ, then there is a one to one correspondence between equivalence classes of flat graph connections on Γ (with respect to graph gauge transformations) and moduli of flat connections on Σ, see, for example, [2]. A typical example of a flat graph connection is given by a representation of the fundamental group $\pi_1(\Sigma, x)$ in G, where the graph Γ is given by homotopy classes in $\Omega(x)$ as edges and a single vertex x. Graph gauge transformations in this case correspond to simultaneous conjugations by elements of G.

We shall confine ourselves to special graphs on Σ associated with arbitrary collections of ideal arcs. For this define a pairing

$$I\colon \mathcal{A}_\Sigma \times \mathcal{A}_\Sigma \to \mathbb{Z}_{\geq 0}, \quad I(e, f) = \min(|a \cap b|\ a \in e, b \in f)$$

Let $\hat{\Sigma} \subset \Sigma$ be the complement of a disjoint union of small open disks centered at the punctures. The boundary of $\hat{\Sigma}$ is a disjoint union of s circles $\{L(P)\}_{P \in V}$ associated with punctures

$$\partial\hat{\Sigma} = \coprod_{P \in V} L(P)$$

For each $\alpha \subset \mathcal{A}_\Sigma$ we associate a set $\Gamma(\alpha) = \{(e, p(e), q(e))\}_{e \in \alpha}$, where $p(e) \in L(P_e)$ and $q(e) \in L(Q_e)$ are chosen so that for any ideal arcs e and f there exist such $a \in e$ and $b \in f$, that $a \cap \hat{\Sigma} \in \Omega_{\mathrm{s}}(p(e), q(e))$, $b \cap \hat{\Sigma} \in \Omega_{\mathrm{s}}(p(f), q(f))$, and $|a \cap b| = I(e, f)$. We shall think of $\Gamma(\alpha)$ as a graph in Σ with points $\{p(e), q(e)\}_{e \in \alpha}$ as vertices and two types of edges: "long" edges corresponding to ideal arcs, and "short" edges corresponding to segments of the boundary components of $\hat{\Sigma}$ between vertices. If $\alpha \subset \beta$, we identify $\Gamma(\alpha)$ with the corresponding subgraph of $\Gamma(\beta)$. Thus, for any $\alpha \subset \mathcal{A}_\Sigma$, $\Gamma(\alpha)$ is a subgraph of $\Gamma(\mathcal{A}_\Sigma)$.

Any polygon p in $\Gamma(\mathcal{A}_\Sigma)$ is uniquely associated with an ideal polygon p' in Σ, the long and short sides of p corresponding to sides and corners of p', respectively.

3. Proof of Theorem 1(i)

3.1. Preparation. We shall find the following lemmas useful. We fix an irreducible flat connection.

Lemma 1. *There exists an admissible ideal arc starting and ending at any given puncture.*

Proof. Suppose that there are no such admissible ideal arcs. Fix a base point $x \in \Sigma$ and choose a puncture P together with a curve $b \in \Omega_{\mathrm{s}}(x, P)$. Denote by h_0 the associated to $\ell(b)$ holonomy. Let $\delta \in \Omega_{\mathrm{s}}(x)$ be such that it does not intersect b. Such loops generate the fundamental group $\pi_1(\Sigma, x)$. We associate to δ the loop $b^{-1}\delta b \in \Omega(P)$, the homotopy class of which uniquely defines an ideal arc e based at P. Choose a representative $a^{-1}b$ for e, where $a \in \Omega(x, P)$. Then the holonomies corresponding to loops $\ell(b)$ and $\ell(a)$ are given by elements h_0 and $h_\delta^{-1}h_0h_\delta$, respectively, where h_δ is the holonomy associated to δ. The nonadmissibility of e implies that the elements h_0 and $h_\delta^{-1}h_0h_\delta$ belong to one and the same unipotent subgroup, and thus the element h_δ belongs to this subgroup's normalizer. Conjugating the element h_0 to an upper triangular form, one simultaneously brings to an upper triangular form also element h_δ. Taking into account the fact that element h_0 remains the same independently of the choice of the loop δ, we conclude that the corresponding to the flat connection representation of the fundamental group $\pi_1(\Sigma, x)$ is brought to an upper triangular form. This contradicts the irreducibility of the flat connection. □

Lemma 2. *Let $P, Q, R \in V$ (possibly coinciding) and $a \in \Omega(P, Q)$, $b \in \Omega(Q, R)$, $c \in \Omega(R, P)$ be such that abc is homotopic to a trivial loop. Let the homotopy class $[a]$ be admissible. Then at least one of $[b]$ and $[c]$ is also admissible. In particular, an ideal triangle, having at least one admissible side, has at least two admissible sides.*

Proof. Let $x \in \Sigma$. Choose $p \in \Omega(x, P)$, $q \in \Omega(x, Q)$, $r \in \Omega(x, R)$ so that $a = p^{-1}q$, $b = q^{-1}r$, $c = r^{-1}p$. Admissibility of $[a]$ means that the holonomies $h_{\ell(p)}$ and $h_{\ell(q)}$ belong to distinct unipotent subgroups of G. Then the holonomy $h_{\ell(r)}$ can belong to only one of these subgroups. □

Lemma 3. *For any two distinct punctures there exists an admissible ideal arc connecting them.*

Proof. Let P and Q be two distinct punctures, and let e be an admissible ideal arc based at P (which exists by Lemma 1). Fix $x \in \Sigma$ and $c \in \Omega_{\mathrm{s}}(x, Q)$ so that e is represented by $a^{-1}b$, where $a, b \in \Omega_{\mathrm{s}}(x, P)$ do not intersect c. The homotopy classes $[a^{-1}c]$ and $[b^{-1}c]$ define two (possibly coinciding) ideal arcs, connecting P and Q. By Lemma 2 (applied to e, $[b^{-1}c]$ and $[c^{-1}a]$), at least one of them is admissible. □

Lemma 4. *Let $P, Q \in V$ (possibly coinciding) and $d \in \Omega(P, Q)$. Then $[d]$ is admissible if and only if $\ell(d)$ is admissible.*

Proof. Let $x \in \Sigma$, $a, b \in \Omega(x, P)$, $c \in \Omega(x, Q)$ be such that $\ell(d) = [a^{-1}b]$ and $[d] = [a^{-1}c] = [b^{-1}c]$. We have the identity

$$\ell(c)\ell(a) = \ell(b)\ell(c) = \ell(cd^{-1}) \tag{1}$$

Suppose $\ell(d)$ is nonadmissible. Then holonomies $h_{\ell(a)}$ and $h_{\ell(b)}$ belong to one and the same unipotent subgroup $U' \subset G$. Then equation (1) implies that $h_{\ell(c)}$ is in the normalizer of U', so $h_{\ell(c)} \in U'$ since it is itself a parabolic element. Thus, $[d]$ is nonadmissible. Conversely, if $[d]$ is nonadmissible, then the holonomies around $\ell(a)$, $\ell(b)$ and $\ell(c)$ are in one and the same unipotent subgroup, i.e. $\ell(d)$ is nonadmissible too. □

Lemma 5. *On Σ there exists a tree with the set of punctures as vertices and $s-1$ pairwise nonintersecting admissible ideal arcs as edges.*

Proof. Enumerate the punctures $P_1, \dots, P_s$. By Lemma 3, there exists an admissible ideal arc connecting P_1 and P_2. Denote by Γ_2 the tree with two vertices P_1 and P_2 and the admissible ideal arc as its edge. Assume that the punctures $P_1, \dots, P_k$ are vertices of a tree Γ_k with $k-1$ edges being pairwise nonintersecting admissible ideal arcs, where $1 < k < s$. It is clear that one can find such an edge e of Γ_k, connecting vertices P_i and P_j, that there exists an ideal triangle t with vertices P_i, P_j, P_{k+1} and whose edges do not intersect the edges of Γ_k. By Lemma 2 one of the admissible edges of t connects P_{k+1} either to P_i or P_j. By adding this edge and P_{k+1} to Γ_k we obtain a tree Γ_{k+1} with vertices $P_1, \dots, P_{k+1}$ and k pairwise nonintersecting admissible ideal arcs as edges. Thus, Γ_s is a tree with the required properties. □

Definition 3. *A* short diagonal *of an ideal n-gon p is an ideal arc in p whose complement in p is an ideal triangle and an ideal $(n-1)$-gon.*

Lemma 6. *There exists such a system of $\kappa + 1$ pairwise nonintersecting admissible ideal arcs, containing all punctures, that the complement in Σ to this system is an ideal $(2\kappa+2)$-gon. For any such system the corresponding polygon has at least one admissible short diagonal.*

Proof. Let Γ_s be the tree of Lemma 5, and let P be a puncture. There exists such a system Γ' of $2g$ pairwise nonintersecting ideal arcs (not necessarily admissible), based at P and not intersecting the edges of the tree Γ_s, that the complement of Γ' in Σ is an $4g$-gon with Γ_s inside of it and attached to one of its vertices. This means that the complement of the graph $\Gamma^0 = \Gamma_s \cup \Gamma'$ is an ideal polygon p_0 with $(2\kappa + 2)$ sides. In fact, Γ^0 contains at least one admissible ideal arc. This is clear for $s > 1$ as the $s-1$ edges of the subgraph Γ_s are admissible. In the case, when $s = 1$, one can choose Γ' with at least one admissible loop (this is possible due to Lemma 1). Thus, we can assume that the polygon p_0 has nonempty set of admissible sides. Suppose p_0 has $2n$ nonadmissible sides, where $1 \le n < \kappa + 1$. Let a be an admissible side next to a nonadmissible side b. Let c be the short diagonal of p_0 which together with a and b bounds an ideal triangle. By Lemma 2, c is admissible. Replacing b by c in Γ^0, we obtain another graph Γ^1 corresponding to ideal polygon p_1 with $2n-2$ nonadmissible sides. By repeating this procedure $n-1$ more times we obtain a graph Γ^n and the corresponding polygon p_n whose all $2\kappa+2$ sides are admissible.

Let now Γ be any such graph with the corresponding polygon p. Assume that p does not have admissible short diagonals. Let $P \in V$ and $e_0, e_1, \dots, e_{k-1}$ be the edges of Γ (possibly with repetitions) which intersect a small circle around P with the cyclic order induced from the orientation of Σ. Let $e_i = [d_i]$ for some $d_i \in \Omega(P_i, P)$, $0 \le i < k$. Fix $x \in \Sigma$ and $a_i \in \Omega(x, P_i)$, $0 \le i \le k$, $P_k = P_0$, so that $\{a_i^{-1}a_{i+1}\}_{0\le i<k}$ represent k shorts diagonals (possibly with repetitions) of p. Since all short diagonals of p are nonadmissible, then the holonomies around $\ell(a_i)$, $0 \le i \le k$, belong to one and the same unipotent subgroup of G. This contradicts Lemma 4 according to which $\ell(d_0) = [a_0^{-1}a_k]$ is admissible. ☐

Lemma 7. *An ideal n-gon p, where $n > 4$, with all sides admissible and at least one admissible short diagonal has such a pair of nonintersecting admissible diagonals, that its complement in p is a union of an ideal $(n-2)$-gon and two ideal triangles.*

Proof. As a pair of admissible nonintersecting short diagonals has the required property, we assume that there are no such pairs. Let d be an admissible short diagonal with $p \setminus d = q \sqcup t$, where q is an ideal $(n-1)$-gon and t an ideal triangle. Then q has at least one admissible short diagonal, since otherwise, by Lemma 2, the two nonintersecting short diagonals of p containing the puncture opposite to d in t are admissible. ☐

Lemma 8. *An ideal n-gon, where $n > 3$, with all sides admissible and at least one admissible short diagonal, has $n-3$ pairwise nonintersecting admissible diagonals.*

Proof. The case $n = 4$ is automatic. The case $n = 5$ is equivalent to Lemma 7. Assume that $n = 5+k$, where $k > 0$. Denote the initial n-gon by p_0. Of the two diagonals of Lemma 7, at least one is short. Let it be d_1 in p_0. Then $p_0 \setminus d_1 = p_1 \sqcup t_1$, where t_1 is an ideal triangle and p_1 an ideal $(n-1)$-gon. Polygon p_1 itself satisfies the conditions of Lemma 7. Iterating, we obtain a sequence of pairwise nonintersecting admissible diagonals $\{d_i\}_{1\le i\le k}$, ideal triangles $\{t_i\}_{1\le i\le k}$, and a pentagon p_k, satisfying the conditions of Lemma 7. ☐

3.2. Proof of part (i). This part of the theorem is equivalent to the following

Proposition 1. *For each flat connection there exists an admissible ideal triangulation.*

Proof. Given a flat connection. Let Γ be the system of $\kappa+1$ ideal arcs of Lemma 6. By Lemma 8 we can complete Γ to a system $\tilde\Gamma$ of $\kappa+1+2(\kappa+1)-3 = 3\kappa$ pairwise nonintersecting admissible ideal arcs. $\tilde\Gamma$ is evidently an admissible ideal triangulation. ☐

4. Proof of Theorem 1(ii)

4.1. Preparation. Let $\tau \in \Delta_\Sigma$. Consider an ideal quadrilateral q in τ with diagonal $e \in \tau$. Replacing e by another diagonal e' of q (such operation is called a *flip on* e), we obtain another ideal triangulation τ'.

In the rest of this subsection we assume that a flat connection is given.

Lemma 9. *Any ideal triangulation of* Σ *has at least one admissible edge.*

Proof. Let $\tau \in \Delta_\Sigma$. Assume that all edges of τ are nonadmissible. Then, due to Lemma 2, for any ideal quadrilateral in τ both its diagonals are nonadmissible. Thus, any flip in τ leads to an ideal triangulation with all edges nonadmissible. As any two ideal triangulations can be related to each other by a finite sequence of flips [5, 7], there are no admissible ideal triangulations on Σ. This contradicts Theorem 1(i). □

Definition 4. *An* almost admissible *ideal triangle is an ideal triangle with at most one nonadmissible side.*

Lemma 10. *Any ideal triangulation of* Σ *can be replaced by an ideal triangulation with all ideal triangles almost admissible.*

Proof. Let $\tau_0 \in \Delta_\Sigma$, $A(\tau_0)$ be the subset of almost admissible ideal triangles, and $\bar{A}(\tau_0) = F(\tau_0) \setminus A(\tau_0)$. By Lemma 9, $A(\tau_0) \neq \emptyset$. Assume $k = |\bar{A}(\tau_0)| > 0$. Let $e \in \tau_0$ be such that the quadrilateral q, containing e as a diagonal, is composed of triangles $t_1 \in \bar{A}(\tau_0)$ and $t_2 \in A(\tau_0)$. Then, by Lemma 2, t_2 has exactly two admissible edges, and the second diagonal e' of q is admissible. Replacing e by e', we obtain ideal triangulation τ_1 with $|\bar{A}(\tau_1)| = k - 1$. Repeating such operation $k-1$ more times we obtain ideal triangulation τ_k with $\bar{A}(\tau_k) = \emptyset$. □

4.2. Proof of part (ii). Fix $\tau \in \Delta_\Sigma$. For any $f \colon \tau \to \{\pm 1\}$ associate a subset $\mathcal{M}(f) \subset \mathcal{M}$ given by the collection of those $m \in \mathcal{M}$, for which edge $e \in \tau$ is admissible if $f(e) = 1$ and nonadmissible if $f(e) = -1$. It is clear that

$$\mathcal{M} = \coprod_{f \in S(\tau)} \mathcal{M}(f)$$

where $S(\tau)$ is the collection of those f for which $\mathcal{M}(f) \neq \emptyset$. Evidently, $|S(\tau)| < 2^{3\kappa}$. Now, Theorem 1(ii) directly follows from

Proposition 2. *For each* $f \in S(\tau)$ *one has a finite collection of ideal triangulations*

$$\rho(f) \subset \Delta_\Sigma, \quad |\rho(f)| = \frac{1}{2\kappa+1}\binom{4\kappa}{2\kappa}$$

such that $\mathcal{M}(f) \subset \bigcup_{\tau \in \rho(f)} \mathcal{M}_\tau$.

Proof. Given $f \in S(\tau)$. By using Lemma 10, we replace τ by $\tau_f \in \Delta_\Sigma$ such that for each $m \in \mathcal{M}(f)$ all faces of τ_f are almost admissible. It is easily seen that there exists such a subset $A \subset \tau_f$ of $\kappa + 1$ edges that the complement of

A in Σ is an ideal $(2\kappa+2)$-gon p_A with all sides admissible for any $m \in \mathcal{M}(f)$. To each ideal triangulation of p_A there corresponds a unique ideal triangulation of Σ with A as a subset of ideal arcs. We define $\rho(f)$ as the collection of all such ideal triangulations. The size of the set $\rho(f)$ is equal to the number of triangulations of a $(2\kappa+2)$-gon, i.e. the Catalan number

$$\frac{1}{2\kappa+1}\binom{4\kappa}{2\kappa}$$

Let $m \in \mathcal{M}(f)$. By construction, all sides of p_A are m-admissible, and, according to Lemma 6, p_A has at least one m-admissible short diagonal. Then, by Lemma 8, there exists an m-admissible triangulation of p_A, i.e. there exists such $\tau \in \rho(f)$ that $m \in \mathcal{M}_\tau$. □

It remains only to note that in Theorem 1(ii) one can take $\Pi = \bigcup_{f\in S(\tau)} \rho(f)$. It is clear that

$$|\Pi| < \frac{2^{3\kappa}}{2\kappa+1}\binom{4\kappa}{2\kappa}$$

This is very rough estimation. In fact, for example, in the case $\Sigma = \Sigma_{0,4}$ one can show that there exists $\Pi_{\min}$ of size $|\Pi_{\min}| = 7$.

5. Proof of Theorem 2

For $\alpha \subset \mathcal{A}_\Sigma$ define an open subset $\mathcal{M}(\alpha) \subset \mathcal{M}$ by the condition that for any $m \in \mathcal{M}(\alpha)$ the set α is the maximal family of m-admissible ideal arcs. To avoid trivial cases with $\mathcal{M}(\alpha) = \emptyset$, we define also

$$\mathcal{W} = \{\alpha \subset \mathcal{A}_\Sigma \,|\, \mathcal{M}(\alpha) \neq \emptyset\}$$

It is clear that

$$\mathcal{M} = \coprod_{\alpha\in\mathcal{W}} \mathcal{M}(\alpha)$$

and the Teichmüller component $\mathcal{T} \subset \mathcal{M}(\mathcal{A}_\Sigma)$. Theorem 1 partially characterizes elements of $\mathcal{W}$, namely each of them contains at least one ideal triangulation. It is thus clear that

$$\mathcal{M}_\tau = \coprod_{\alpha\in\mathcal{W}(\tau)} \mathcal{M}(\alpha), \quad \mathcal{W}(\tau) = \{\alpha \in \mathcal{W} | \, \tau \subset \alpha\}$$

We construct a fiber bundle $\pi\colon \tilde{\mathcal{M}} \to \mathcal{M}$ as the disjoint union of fiber bundles $\pi\colon \tilde{\mathcal{M}}(\alpha) \to \mathcal{M}(\alpha)$ for all $\alpha \in \mathcal{W}$. The fiber $\pi^{-1}(m)$ over a point $m \in \mathcal{M}(\alpha)$ consists of flat graph connections on $\Gamma(\alpha)$ which represent m and whose parallel transport operators belong to the unipotent subgroup U for all short edges, and to the subset θT for all long edges. Let us show that $\tilde{\mathcal{M}}$ is in fact a principal T^s-bundle over $\mathcal{M}$.

Given $\alpha \in \mathcal{W}$ and $m \in \mathcal{M}(\alpha)$. Our strategy is to show that any flat connection h, representing m, is equivalent to any flat graph connection in $\pi^{-1}(m)$.

As every vertex of $\Gamma(\alpha)$ belongs to a boundary component of $\hat{\Sigma}$, the h-holonomies around boundary components, based at vertices of $\Gamma(\alpha)$, are parabolic. Thus, one always can replace h by such an equivalent connection that all associated holonomies around the boundary components, based at vertices, belong to one and the same unipotent subgroup $U \subset G$. This automatically makes the parallel transport operators along short edges to belong to $B = N(U)$. The parallel transport operators along long edges belong to the big Bruhat cell $B\theta B$. Thus, making an appropriate graph gauge transformation with values in B, we can make the parallel transport operators along all long edges to belong to the subset θT. The remaining freedom in graph gauge transformations consists of arbitrary T valued functions on vertices of $\Gamma(\alpha)$, which can be used to eliminate the T-parts of parallel transport operators along all short edges, thus obtaining elements of $\pi^{-1}(m)$. This still leaves unfixed the graph gauge transformations, given by T valued functions taking one and the same value on all vertices associated with one boundary component. This residual gauge group is isomorphic to T^s, which freely and transitively acts in the space $\pi^{-1}(m)$. Thus, the union $\tilde{\mathcal{M}}(\alpha) = \coprod_{m\in\mathcal{M}(\alpha)} \pi^{-1}(m)$ is a principal T^s-bundle over $\mathcal{M}(\alpha)$.

5.1. Proof of part (i).

Lemma 11. *Given $\alpha \in \mathcal{W}$ and $m \in \mathcal{M}(\alpha)$. Let f be a hexagonal face in $\Gamma(\alpha)$ corresponding to ideal triangle t with sides in α. Then, there exists $\epsilon = \pm 1$ such that the parallel transport operators along short edges of f are uniquely defined in terms of ϵ and the parallel transport operators along long edges of f.*

Proof. Let us identify B with the set $\mathbb{R}_{>0} \times \mathbb{R}$ where the Lie group structure is that of upper triangular two-by-two matrices of unit determinant

$$(u_1, v_1)(u_2, v_2) = (u_1u_2, u_1v_2 + v_1u_2^{-1}), \quad (u_i, v_i) \in \mathbb{R}_{>0} \times \mathbb{R}, \quad i = 1, 2$$

Then, the subgroups T and U are identified with the components $\mathbb{R}_{>0}$ and $\mathbb{R}$ respectively. In this way the parallel transport operators are associated with mapping $u\colon C(t) \to \mathbb{R}$ for short edges, and $v\colon E(t) \to \mathbb{R}_{>0}$ for long edges. Now, explicit calculation of the condition that the holonomy around ∂f is trivial shows that the real number $u(c)$, associated with corner c of t, is given by the formula

$$u(c) = \epsilon \frac{v(O_c)}{v(A_c)v(B_c)} \tag{2}$$

with $\epsilon = \pm 1$ independent of c. □

Lemma 12. *Given $\alpha \in \mathcal{W}$ and $m \in \mathcal{M}(\alpha)$. Let f be a hexagonal face in $\Gamma(\alpha)$ corresponding to ideal triangle t with sides in α. Then, the parallel transport operators along edges of f are uniquely determined in terms of parallel transport operators along any two long edges and any one short edge of f.*

Proof. This is a direct consequence of Lemma 11. Indeed, let a, b, c be the corners of t, a and b being opposite to A_c and B_c, respectively. Solving equations (2), for example, with respect to $v(O_c)$, $u(a)$, and $u(b)$, we obtain explicitly

$$v(O_c) = |u(c)|v(A_c)v(B_c), \quad u(a) = \frac{1}{u(c)(v(B_c))^2}, \quad u(b) = \frac{1}{u(c)(v(A_c))^2}$$

It is clear that instead of $u(c)$ one can take also either $u(a)$ or $u(b)$. □

Let $\Delta_\Sigma \ni \tau \subset \alpha \in \mathcal{W}$. The parallel transport operators along edges of $\Gamma(\tau)$ uniquely determine all parallel transport operators along edges of $\Gamma(\alpha)$. This is a consequence of Lemma 12. Indeed, for any $e \in \alpha \setminus \tau$ there exists such a finite sequence of ideal arcs $\{e_i\}_{1\le i\le n} \subset \alpha$ that $e_n = e$ and e_i is a side of an ideal triangle whose other two sides are contained in $\tau \cup \{e_j\}_{1\le j<i}$. Using recurrently Lemma 12, one can thus calculate the parallel transport operators along all edges of $\Gamma(\alpha)$ in terms of those of $\Gamma(\tau)$. Besides, by Lemma 11, each point of $\pi^{-1}(\mathcal{M}_\tau)$ is uniquely associated with an element of the set $\mathbb{R}_{>0}^{\tau} \times \{\pm 1\}^{\bar\tau}$, where we denote by X^Y the set of mappings from set Y to set X, $\tau = E(\tau)$, $\bar\tau = F(\tau)$.

Proposition 3. *Let $\tau \in \Delta_\Sigma$ and mapping $\phi_\tau \colon \mathbb{R}_{>0}^{\tau} \times \{\pm 1\}^{\bar\tau} \to \mathbb{R}$ be defined by the formula*

$$\phi_\tau(f,\epsilon) = \prod_{P\in V} \phi_{\tau,P}(f,\epsilon), \quad \phi_{\tau,P}(f,\epsilon) = \sum_{c\in C(\tau,P)} \epsilon(T_c)\frac{f(O_c)}{f(A_c)f(B_c)},$$

Then, there is a one to one correspondence between $\pi^{-1}(\mathcal{M}_\tau)$ and the complement of $\phi_\tau^{-1}(0)$, the action of the structure group T^s being realized by the action of the group $\mathbb{R}_{>0}^{V} \simeq T^s$:

$$\mathbb{R}_{>0}^{\tau} \times \{\pm 1\}^{\bar\tau} \times \mathbb{R}_{>0}^{V} \to \mathbb{R}_{>0}^{\tau} \times \{\pm 1\}^{\bar\tau}$$

$$(f,\epsilon,h) \mapsto (f^h,\epsilon), \quad f^h(e) = f(e)h(P_e)h(Q_e) \tag{3}$$

Proof. One only needs to satisfy the s conditions for the holonomies around the punctures to be parabolic, which, by using Lemma 11, take the form $\phi_\tau(f,\epsilon) \neq 0$.

Verification of the action of the structure group is straightforward. □

To complete the proof of part (i), it remains to notice that the component ϵ in formula (3) is not affected under the action of the structure group, so that for each fixed ϵ there corresponds a subbundle $\mathcal{R}_\epsilon(\tau) \subset \pi^{-1}(\mathcal{M}_\tau)$.

5.2. Proof of part (ii).

Proposition 4. *Consider two ideal triangulations τ and τ' related by a single flip on $e \in \tau$ so that ideal triangles $t_1, t_2 \in \bar\tau$ with sides e, a, b and e, c, d, respectively (the cyclic order of the edges being induced from the orientation of Σ), are replaced by ideal triangles $t_1', t_2' \in \bar\tau'$ with sides e', d, a and e', b, c, respectively. Then two pairs*

$$(f,\epsilon) \in \mathbb{R}_{>0}^{\tau} \times \{\pm 1\}^{\bar\tau} \setminus \phi_\tau^{-1}(0), \quad (f',\epsilon') \in \mathbb{R}_{>0}^{\tau'} \times \{\pm 1\}^{\bar\tau'} \setminus \phi_{\tau'}^{-1}(0)$$

correspond to one and the same element in $\pi^{-1}(\mathcal{M}_\tau \cap \mathcal{M}_{\tau'})$ if and only if

$$f'(x) = f(x) \quad \forall\, x \in \tau \cap \tau' \tag{4}$$

$$\epsilon'(t) = \epsilon(t) \quad \forall\, t \in \bar{\tau} \cap \bar{\tau}' \tag{5}$$

$$\epsilon'(t_1')\epsilon'(t_2') = \epsilon(t_1)\epsilon(t_2) \tag{6}$$

$$\epsilon'(t_1')f(e)f'(e') = \epsilon(t_2)f(a)f(c) + \epsilon(t_1)f(b)f(d) \tag{7}$$

In particular, if pair (f, ϵ)*, representing* $m \in \mathcal{M}_\tau$*, is such that*

$$\frac{f(a)f(c)}{f(b)f(d)} + \epsilon(t_1)\epsilon(t_2) = 0$$

then $m \notin \mathcal{M}_{\tau'}$*.*

Proof. This is a straightforward calculation by using Lemma 11. Indeed, let $\alpha \in \mathcal{W}$ be such that $\tau \cup \tau' \subset \alpha$ and $m \in \mathcal{M}(\alpha)$. Then, it is immediate to see that the equality between the parallel transport operators along edges of $\Gamma(\tau) \cap \Gamma(\tau')$, computed in terms of coordinates associated with τ and τ', is equivalent to equations (4), (5). Let q be the ideal quadrilateral with sides a, b, c, d and diagonals e, e'. Then, one easily sees that equations (4)–(7) imply that the parallel transport operators along edges of q are equal in both systems of coordinates.

On the other hand, by Lemma 11, the parallel transport operator along the short edge of $\Gamma(\tau')$, corresponding to the corner of t_1' opposite to e', is described by the number

$$\epsilon'(t_1')\frac{f'(e')}{f'(a)f'(d)} = \epsilon'(t_1')\frac{f'(e')}{f(a)f(d)}$$

while the same path in $\Gamma(\tau)$ is represented by composition of two short edges, corresponding to the corner of t_1 opposite to b and the corner of t_2 opposite to c. Thus, the corresponding number is given by the sum

$$\epsilon(t_1)\frac{f(b)}{f(a)f(e)} + \epsilon(t_2)\frac{f(c)}{f(d)f(e)}$$

Equating the two numbers we obtain equation (7). Similar calculation for the corner of t_2' opposite to e', leads to another equation similar to (7). Thus obtained equations in particular imply equation (6). □

To complete the proof of part (ii), we note that equation (6) implies that the quantity $k = \frac{1}{2}\sum_{t\in\bar{\tau}} \epsilon(t)$ is independent of τ and thus the sets $\mathcal{R}_k(\tau) = \coprod_{\epsilon\in\sigma^{-1}(k)} \mathcal{R}_\epsilon(\tau)$ are such that $\mathcal{R}_k(\tau) \cap \mathcal{R}_l(\tau') = \emptyset$ if $k \neq l$ and any $\tau, \tau' \in \Delta_\Sigma$.

5.3. Proof of part (iii). This is a direct consequence of Propositions 3 and 4, which in the two cases when all signs, associated with ideal triangles, coincide give the formulas identical to those of Penner for the decorated Teichmüller space.

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