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AUTOMORPHISM GROUPS OF SOME AFFINE AND FINITE TYPE ARTIN GROUPS

Ruth Charney and John Crisp

Abstract. We observe that, for fixed $n \geq 3$, each of the Artin groups of finite type A_n, $B_n = C_n$, and affine type $\tilde{A}_{n-1}$ and $\tilde{C}_{n-1}$ is a central extension of a finite index subgroup of the mapping class group of the $(n+2)$-punctured sphere. (The centre is trivial in the affine case and infinite cyclic in the finite type cases). Using results of Ivanov and Korkmaz on abstract commensurators of surface mapping class groups we are able to determine the automorphism groups of each member of these four infinite families of Artin groups.

A rank n *Coxeter matrix* is a symmetric $n \times n$ matrix M with integer entries $m_{ij} \in \mathbb{N} \cup \{\infty\}$ where $m_{ij} \geq 2$ for $i \neq j$, and $m_{ii} = 1$ for all $1 \leq i \leq n$. Given any rank n Coxeter matrix M, the *Artin group* of type M is defined by the presentation

$$A(M) \cong \langle\ s_1, \ldots, s_n \ \mid\ \underbrace{s_i s_j s_i \ldots}_{m_{ij}} = \underbrace{s_j s_i s_j \ldots}_{m_{ij}} \ \text{ for all } i \neq j, m_{ij} \neq \infty \rangle\,.$$

Adding the relations $s_i^2 = 1$ to this presentation yields a presentation of the Coxeter group of type M generated by standard reflections s_i and such that the rotation $s_i s_j$ has order m_{ij}, for all $1 \leq i, j \leq n$. A Coxeter matrix M and its Artin group $A(M)$ are said to be of *finite type* if the associated Coxeter group $W(M)$ is finite, and of *affine (or Euclidean) type* if $W(M)$ acts as a proper, cocompact group of isometries on some Euclidean space with the generators $s_1, \ldots, s_n$ acting as affine reflections.

The information contained in the Coxeter matrix M is often displayed in the form of a graph, the *Coxeter graph*, whose vertices are numbered $1, .., n$ and which has an edge labelled m_{ij} between the vertices i and j whenever $m_{ij} \geq 3$ or $m_{ij} = \infty$. With this particular convention, one usually suppresses the labels which are equal to 3 (but not the corresponding edges!). Note that the absence of an edge between two vertices indicates that the corresponding generators of $A(M)$ commute. We say that an Artin group is *irreducible* if its Coxeter graph is connected and observe that every Artin group is isomorphic to a direct product of irreducible Artin groups corresponding to the connected components of its Coxeter graph.

In this paper we concern ourselves with the following four infinite families of Artin groups $A = A(M)$ (see Figure 1): the irreducible finite types $M = A_n$,

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B_n and irreducible affine types $\tilde{A}_{n-1}$ and $\tilde{C}_{n-1}$, of rank $n \geq 3$ in each case. (We refer to [3] for the complete classification of irreducible finite and affine Coxeter systems.) For each Artin group A on this list, we determine its automorphism group $Aut(A)$, its outer automorphism group $Out(A) = Aut(A)/Inn(A)$, and the abstract commensurator group $Comm(A/Z)$ of the group modulo its centre (see Definition 3).

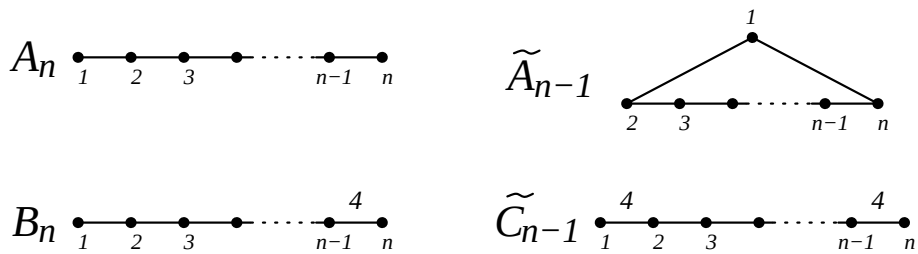

FIGURE 1. Coxeter graphs of finite type A_n, B_n and affine type $\tilde{A}_{n-1}$, $\tilde{C}_{n-1}$, with rank $n \geq 3$. Note that, by convention, the subscript in each case indicates the geometric dimension of the Artin group.

The Artin group $A(A_n)$ is well-known as the braid group on $n+1$ strings, and the automorphism groups of the braid groups were determined by Dyer and Grossman in [7]. In this paper we exploit the fact that the Artin groups of type B, $\tilde{A}$ and $\tilde{C}$ also have descriptions as braid groups (see [1]) in order to study their automorphism groups. Note that the centre of any irreducible Artin group of finite type is an infinite cyclic group, which we generally denote by Z. (On the other hand, the groups $A(\tilde{A}_{n-1})$ and $A(\tilde{C}_{n-1})$ have trivial centres.) As explained in Section 1, each of the groups $A(A_n)/Z$, $A(B_n)/Z$, $A(\tilde{A}_{n-1})$ and $A(\tilde{C}_{n-1})$ is a finite index subgroup of the mapping class group $Mod(S_{n+2})$ of the $(n+2)$-punctured 2-sphere S_{n+2}. Using this fact, and appealing to the work of Ivanov and Korkmaz on the structure of surface mapping class groups [9, 11, 12], we show:

Theorem 1. *Let $C_2 = \mathbb{Z}/2\mathbb{Z}$. For each $n \geq 3$, we have*

(i) $Comm(G) \cong Mod(S_{n+2})$
for each $G = A(A_n)/Z,\ A(B_n)/Z,\ A(\tilde{A}_{n-1}),\ A(\tilde{C}_{n-1})$.

(ii) (a) (Dyer, Grossman [7]) $Out(A(A_n)) \cong Out(A(A_n)/Z) \cong C_2$
(b) $Out(A(B_n)) \cong (\mathbb{Z} \rtimes C_2) \times C_2$ *and* $Out(A(B_n)/Z) \cong C_2 \times C_2$
(c) $Out(A(\tilde{C}_{n-1})) \cong Sym(3) \times C_2$
(d) $Out(A(\tilde{A}_{n-1})) \cong D_{2n} \times C_2$.

In addition, we show that the exact sequence

$$1 \to Inn(A) \to Aut(A) \to Out(A) \to 1$$

splits in each of the above cases with the exception of the case $A = A(\tilde{C}_{n-1})$ with $n \equiv 2 \mod 3$ (see Proposition 6). In the case of $A(\tilde{A}_{n-1})$, for example, the dihedral factor D_{2n} is realised by the group of "graph automorphisms" induced from symmetries of the Coxeter graph, which in this case is an n-cycle. Every Artin group admits an involution $\epsilon : s_i \mapsto s_i^{-1}$, for $i = 1, .., n$, which is never inner (because it reverses the sign of the length function $\ell : A \to \mathbb{Z}$, defined by $s_i \mapsto 1$). This accounts for the central C_2 factor in each of the outer automorphism groups above.

Note that all of the outer automorphism groups in the theorem are finite with the exception of $Out(A(B_n))$. The infinite cyclic factor in $Out(A(B_n))$ is generated by a "transvection" which multiplies each generator s_i by some element of the center Z. In Section 3, we study the group of transvections of a finite type Artin group. Transvection homomorphisms exist in any Artin group with non-trivial center, but in general, they are not automorphisms. In particular, transvection automorphisms do not occur when the Artin group has abelianisation $\mathbb{Z}$, as in the case of the braid groups. However, $A(B_n)$ has abelianisation $\mathbb{Z} \times \mathbb{Z}$ and likewise for Artin groups of dihedral type $I_2(m)$ when m is even. (The groups $Out(A(I_2(m)))$, for $m \geq 3$, were computed in [8] and are discussed in Section 3). By contrast, transvection homomorphisms appear in the abstract commensurator group of *every* finite type Artin group. In Proposition 11, we show that the abstract commensurator group of any finite type Artin group contains an infinitely generated free abelian group generated by transvections.

1. Abstract commensurators and the mapping class group of a punctured sphere

The following is a special case of Theorem 8.5A of Ivanov's survey article [9]. It is a consequence of ideas laid out in the papers of Ivanov on the mapping class groups of higher genus orientable surfaces and the work of Korkmaz [11, 12] on the complex of curves associated to an m-punctured sphere. For simplicity we shall write S_m for the 2-sphere with m points removed. If S is an orientable surface (without boundary), we denote by $Mod(S)$ the group of all diffeomorphisms of S (not necessarily respecting orientation) modulo diffeotopy.

Theorem 2 (Ivanov, Korkmaz)**.** *Let $m \geq 5$. If $\varphi : H \to K$ is an isomorphism between finite index subgroups $H, K < Mod(S_m)$, then φ is the restriction to H of an inner automorphism of $Mod(S_m)$ (conjugation by some element $g \in Mod(S_m)$).*

Definition 3. Let G be a group. We define the *abstract commensurator group of* G to be

$$Comm(G) = \{\varphi : H \xrightarrow{\cong} K \ : \ H, K < G \text{ finite index } \} \ / \ \sim$$

where $\varphi \sim \psi$ if they agree on a finite index subgroup of G. We note that the group structure on $Comm(G)$ is given by composition of isomorphisms after appropriate restriction of their domains to finite index subgroups.

Corollary 4. *Let $\Gamma = Mod(S_m)$, $m \geq 5$, let $H < \Gamma$ denote a finite index subgroup and $N_\Gamma(H)$ the normalizer of H in Γ. Then*

(i) $Comm(H) \cong \Gamma$, *and*
(ii) $Aut(H) \cong N_\Gamma(H)$.

Proof. (i) There is a natural homomorphism $\Gamma \to Comm(H)$ which takes $\gamma \in \Gamma$ to conjugation by γ restricted to the finite index subgroup $H \cap \gamma^{-1} H \gamma$. It follows from Theorem 2 that this map is surjective. To show that it is injective, we observe that Γ has trivial "virtual centre", that is, any element which centralizes a finite index subgroup is necessarily trivial. This is because any mapping class $g \in Mod(S)$ is determined by its action on a finite number of isotopy classes of simple closed curves on the surface S, and therefore by its action on the n-th powers T_i^n of the corresponding Dehn twists for any choice of n. Given a finite index subgroup $K < \Gamma$ centralized by g we can always find a sufficiently large n such that K contains all T_i^n. Then g is determined by its action on K and so must be the identity in Γ.

(ii) There is now an obvious homomorphism $Aut(H) \to \Gamma$ which factors through the inverse of the isomorphism just described, and realizes each automorphism of H as conjugation by an element of Γ. By the preceding argument this map is injective. Its image is precisely $N_\Gamma(H)$. □

2. Some interesting finite index subgroups of $Mod(S_m)$ and their automorphisms

Let $Sym(m)$ denote the group of permutations of the set $\{1, 2, .., m\}$, for $m \geq 1$. If $k < m$ then we consider $Sym(k)$ as a subgroup of $Sym(m)$ consisting of those permutations which fix the subset $\{k+1, .., m\}$. Notice that each mapping class of the m-punctured sphere S_m induces a permutation of the punctures. Moreover, the mapping class group acts on the orientation class of the surface. Thus we have a surjective group homomorphism $\pi : Mod(S_m) \to Sym(m) \times C_2$ with kernel the group of pure orientation preserving mapping classes. Note that, strictly speaking, the definition of π depends on a labelling of the punctures by the numbers 1 through m.

Notation. Fix $n \geq 3$. We shall write $\Gamma = Mod(S_{n+2})$ and Γ_A, Γ_B, $\Gamma_{\tilde{C}}$ respectively, for the finite index subgroup of Γ generated by those orientation preserving diffeomorphisms which fix the last 1, 2, 3 punctures of S_{n+2} respectively. These three subgroups are, in other words, the preimages under π of the subgroups $Sym(k) < Sym(n+2) \times C_2$, with $k = n+1, n, n-1$ respectively. In particular we have $[\Gamma : \Gamma_A] = 2(n+2)$, $[\Gamma : \Gamma_B] = 2(n+1)(n+2)$ and $[\Gamma : \Gamma_{\tilde{C}}] = 2n(n+1)(n+2)$.

It is well known that the $(n+1)$-string braid group, $A(A_n)$, is isomorphic to the orientation preserving mapping class group of the $(n+1)$-punctured disk *relative* to the boundary of the disk. (That is, diffeotopies are required in this case to fix the boundary of the disk *pointwise*). The centre Z of this mapping class group is generated by the Dehn twist about a curve parallel to the boundary, and is precisely the kernel of the natural homomorphism to the mapping class group $\Gamma = Mod(S_{n+2})$ (induced by inclusion of the punctured disk in S_{n+2}). Thus we may realise the group $A(A_n)/Z$ as the finite index subgroup Γ_A of Γ. Likewise, the group $A(B_n)$ may be identified with the subgroup of $A(A_n)$ fixing the last string, so $A(B_n)/Z$ is isomorphic to Γ_B.

It is less well-known that the Artin group $A(\tilde{C}_{n-1})$ is also isomorphic to a subgroup of the braid group. To describe this subgroup, renumber the punctures so that $\Gamma_{\tilde{C}}$ fixes x_1, x_{n+1}, x_{n+2} and let $\widehat{\Gamma}_{\tilde{C}}$ be the inverse image of $\Gamma_{\tilde{C}}$ in $A(A_n)$. (Thus $\widehat{\Gamma}_{\tilde{C}}$ is a "2-pure" braid group.) Forgetting strings 2 through n gives a homomorphism $\widehat{\Gamma}_{\tilde{C}} \to A(A_1) \cong \mathbb{Z}$ which counts the twisting between the first and last string. As shown in [1], the Artin group $A(\tilde{C}_{n-1})$ can be identified with the kernel of this homomorphism. We can picture the kernel as the subgroup of braids in which the first and last strings have been pulled straight so they cannot interact. It is easy to see that this subgroup intersects the center Z (of $A(A_n)$) trivially and together they generate $\widehat{\Gamma}_{\tilde{C}}$. It follows that $A(\tilde{C}_{n-1})$ is isomorphic to $\Gamma_{\tilde{C}}$.

In summary, we have

- $\Gamma_A \cong A(A_n)/Z$,
- $\Gamma_B \cong A(B_n)/Z$, and
- $\Gamma_{\tilde{C}} \cong A(\tilde{C}_{n-1})$.

These isomorphisms may be described explicitly as follows. Distribute the punctures x_i evenly along half the equator in the order $x_{n+2}, x_1, x_2, .., x_n, x_{n+1}$ so that x_{n+1} and x_{n+2} are antipodal (see Figure 2). Then the ith standard generator of $A(A_n)$ maps to the positive braid twist σ_i exchanging x_i with x_{i+1}. The ith standard generator of $A(B_n)$ maps to σ_n^2 if $i = n$ and σ_i otherwise. The ith standard generator of $A(\tilde{C}_{n-1})$ maps to σ_i^2 if $i \in \{1, n\}$ and σ_i if $i \in \{2, .., n-1\}$.

Proposition 5. **(i):** $Comm(\Gamma_A) \cong Comm(\Gamma_B) \cong Comm(\Gamma_{\tilde{C}}) \cong \Gamma$,
(ii): *there exist short exact sequences*

$$\begin{aligned} 1 \to \Gamma_A \to & Aut(\Gamma_A) \to C_2 \to 1 \\ 1 \to \Gamma_B \to & Aut(\Gamma_B) \to C_2 \times C_2 \to 1 \\ 1 \to \Gamma_{\tilde{C}} \to & Aut(\Gamma_{\tilde{C}}) \to Sym(3) \times C_2 \to 1 \end{aligned}$$

Proof. Let H denote one of Γ_A, Γ_B, or $\Gamma_{\tilde{C}}$, and P the set of 1,2 or 3 punctures fixed by all elements of H. It is easily checked that $N_\Gamma(H)$ is exactly the group of all mapping classes (orientation preserving or not) which leave P setwise invariant. It follows that $N_\Gamma(H)/H$ is a direct product of $C_2 = \Gamma/\Gamma^+$ with the group $Sym(P)$ of permutations of the set P. Statements (i) and (ii) now follow immediately from the corresponding statements in Corollary 4. $\square$

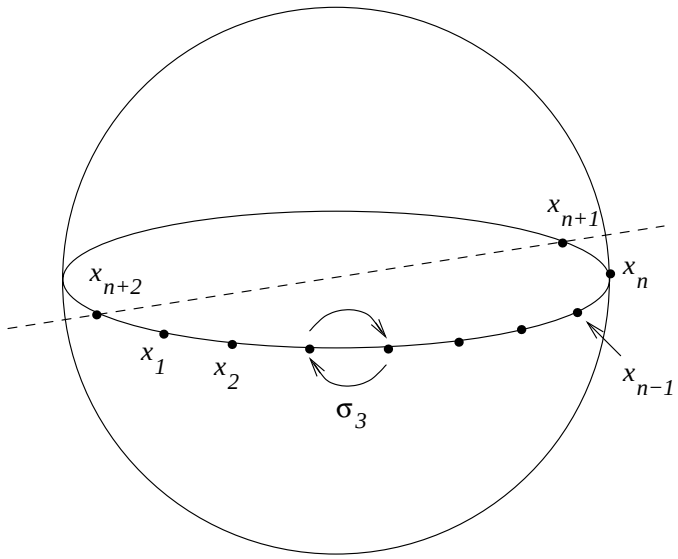

FIGURE 2. Finite index subgroups of $\Gamma = Mod(S_{n+2})$ isomorphic to $A(A_n)/Z$, $A(B_n)/Z$ and $A(\tilde{C}_{n-1})$, $n \geq 3$, fix the subsets $\{x_{n+2}\}, \{x_{n+1}, x_{n+2}\}$ and $\{x_1, x_{n+1}, x_{n+2}\}$ respectively.

It is easy to see, with the aid of Figure 2, that the short exact sequences in the proposition above are split in the cases Γ_A and Γ_B. The group $C_2 \times C_2$ may be generated by a reflection in the equatorial plane and a rotation about an axis in the equatorial plane which exchanges the points x_{n+1} and x_{n+2}. These diffeomorphisms leave Γ_B invariant. Only the reflection leaves Γ_A invariant.

For $\Gamma_{\tilde{C}}$, the exact sequence splits providing $n + 2 \equiv 0, 2 \bmod 3$. To see this, think of S_{n+2} as a sphere with $3k$ punctures arranged symmetrically along the equator and, if $n + 2 \equiv 2(3)$, the remaining two punctures at the north and south poles. Then there is an orientation-preserving action of $Sym(3)$ on S_{n+2} generated by an order three rotation ρ about the vertical axis through the poles and an order two rotation η about a horizontal axis through one (k odd) or two (k even) punctures at the equator. If x is a puncture fixed by η, then the orbit of x under ρ is preserved by $Sym(3)$, hence for an appropriate numbering of the punctures (namely, so that $P = \{x, \rho(x), \rho^2(x)\}$) the action of $Sym(3)$ lies in $Aut(\Gamma_{\tilde{C}})$ and is faithful on P. This splits the $Sym(3)$ factor. The C_2 factor is realized by reflection through the equatorial plane.

Proposition 6. *If $\Gamma_{\tilde{C}} = A(\tilde{C}_{n-1})$ with $n + 2 \equiv 1$ mod 3, then $Aut(\Gamma_{\tilde{C}})$ does not contain a subgroup isomorphic to $Sym(3)$. In this case, the exact sequence for $\Gamma_{\tilde{C}}$ in the previous proposition does not split.*

Proof. Suppose $Aut(\Gamma_{\tilde{C}}) \subset \Gamma$ contains a subgroup K isomorphic to $Sym(3)$ and let H be the subgroup of K generated by a 3-cycle. Then H is a subgroup of $Mod(S_{n+2})$, hence H acts by permutations on the set of punctures $\{x_1, \ldots x_{n+2}\}$.

Every H orbit consists of either 1 or 3 punctures. Since $n+2 \equiv 1(3)$, the number of fixed points of H must also be congruent to 1 mod 3. If H fixes 4 or more punctures, then it lies in a subgroup of Γ isomorphic to $\Gamma_{\tilde{C}}$ which is impossible since $\Gamma_{\tilde{C}}$ is torsion-free (since, by [1], it is a subgroup of the braid group on $n+1$ strings which is known to be torsion free).

Thus H must have a unique fixed puncture. Since H is normal in K, all of K must fix this puncture and hence K lies in a subgroup of Γ isomorphic to Γ_A. By a theorem of Bestvina, [2] Thm 4.5, every finite subgroup of Γ_A is cyclic, so we arrive at a contradiction. □

It has been observed by several authors [1][6][10] that there exists a semidirect product decomposition

$$A(B_n) \cong A(\tilde{A}_{n-1}) \rtimes \mathbb{Z}\,.$$

(See [5] for further discussion of this decomposition.) The Coxeter graph of $A(\tilde{A}_{n-1})$ is an n-cycle and the generator of the cyclic factor in the semi-direct product acts on $A(\tilde{A}_{n-1})$ via an order n rotation of this graph. The centre of $A(B_n)$ is the subgroup $n\mathbb{Z}$ of the cyclic factor. Thus we also have

$$\Gamma_B \cong A(\tilde{A}_{n-1}) \rtimes \mathbb{Z}/n\mathbb{Z}\,.$$

Here we interpret $A(\tilde{A}_{n-1})$ as the subgroup of Γ_B consisting of mapping classes of "zero angular momentum" about the axis through the two fixed punctures. More precisely, suppose that the punctures of S_{n+2} are arranged so that x_{n+1} and x_{n+2} are placed at the north and south pole respectively, and the remaining points $x_1, .., x_n$ are equally spaced (in that cyclic order) around the equator, see figure 3. Define $\Gamma_{\tilde{A}}$ to be the subgroup of Γ generated by the braid twists σ_i which exchange the points x_i and x_{i+1}, for $i = 1, .., n$, indices taken mod n. The isomorphism $A(\tilde{A}_{n-1}) \cong \Gamma_{\tilde{A}}$ is given by sending the ith standard generator of $A(\tilde{A}_{n-1})$ to σ_i. Note that $\Gamma_{\tilde{A}}$ is a finite index subgroup of Γ of index $[\Gamma : \Gamma_{\tilde{A}}] = n[\Gamma : \Gamma_B] = 2n(n+1)(n+2)$. In particular, $Comm(\Gamma_{\tilde{A}}) \cong \Gamma$.

Proposition 7. *Let $\Gamma_{\tilde{A}} \cong A(\tilde{A}_{n-1})$ be as above. Then $N_\Gamma(\Gamma_{\tilde{A}}) = N_\Gamma(\Gamma_B)$. Consequently $Aut(\Gamma_{\tilde{A}}) \cong Aut(\Gamma_B)$ and there is an exact sequence*

$$1 \to \Gamma_{\tilde{A}} \to Aut(\Gamma_{\tilde{A}}) \to D_{2n} \times C_2 \to 1\,,$$

where D_{2n} denotes the dihedral group of order $2n$. Moreover, a splitting of this sequence is given by realizing the group of outer automorphisms as the direct product of the group of graph automorphisms of $A(\tilde{A}_{n-1})$ with the group of order 2 generated by the inversion ϵ which inverts each standard generator.

Proof. We suppose that the punctures of S_{n+2} are arranged as shown in Figure 3. The outer automorphism group of Γ_B may then be generated by the reflection in the equatorial plane and a rotation exchanging the two poles (this gives a splitting of the exact sequence of Proposition 5 slightly different to the

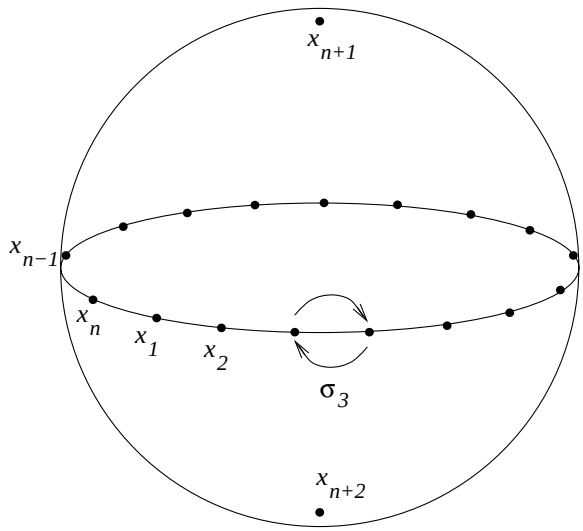

FIGURE 3. The finite index subgroup $\Gamma_{\tilde{A}}$ of $\Gamma = Mod(S_{n+2})$ isomorphic to $A(\tilde{A}_{n-1})$.

one previously mentioned). It follows easily that $\Gamma_{\tilde{A}}$ is invariant by these mapping classes, and hence a characteristic subgroup of Γ_B. In particular, we have $N_\Gamma(\Gamma_B) < N_\Gamma(\Gamma_{\tilde{A}})$.

Let $P = \{x_{n+1}, x_{n+2}\}$. To prove that $N_\Gamma(\Gamma_{\tilde{A}}) = N_\Gamma(\Gamma_B)$ it suffices now to check that every mapping class which normalizes $\Gamma_{\tilde{A}}$ leaves P invariant. But if $g \in N_\Gamma(\Gamma_{\tilde{A}})$ then $\pi(g)$ normalizes $\pi(\Gamma_{\tilde{A}}) = Sym(n) \times C_2$, and this is just to say that g induces a symmetry of the puncture set which leaves P invariant, as required.

The remaining statements are now easily deduced from Corollary 4 and Proposition 5. □

In summary, Propositions 5 and 7 give us explicitly the automorphism groups of $A(A_n)/Z$, $A(B_n)/Z$, $A(\tilde{A}_{n-1})$, and $A(\tilde{C}_{n-1})$, for all $n \geq 3$, as well as the fact that the abstract commensurator group is in each case isomorphic to $Mod(S_{n+2})$. This proves Theorem 1 with the exception of the first isomorphism in each of part (ii)(a) and (b). In the next section we shall recover the group $Aut(A(B_n))$ from the above result on $Aut(A(B_n)/Z)$. In a similar fashion, the result of [7] on automorphisms of the braid group is easily recovered from the computation of $Aut(A(A_n)/Z)$. These steps will complete the proof of Theorem 1.

3. Transvections and automorphisms of $A(B_n)$

Let H be a group with nontrivial centre Z. By a *transvection* of H we mean a homomorphism $T_\lambda : H \to H$ given by $T_\lambda(x) = x\lambda(x)$, where λ denotes a

function $H \to Z$. The fact that T_λ is a homomorphism requires that λ is also a homomorphism, and it is easily checked that the composite of two transvections is again a transvection. We note, however, that transvections are typically not automorphisms. It is easily shown that a transvection is an automorphism (resp. injective, resp. surjective) if and only if its restriction to the centre Z is an automorphism (resp. injective, resp. surjective).

Let $Tv(H)$ denote the group of automorphisms of H which are transvections. Then, almost by definition, we have an exact sequence

$$1 \to Tv(H) \to Aut(H) \to Aut(H/Z)\,.$$

Note that $Inn(H) \cong H/Z$ always maps isomorphically onto $Inn(H/Z)$. In general there is an issue, however, as to whether *every* outer automorphism of H/Z lifts to an automorphism of H, and as to whether the extension splits.

In the case of an irreducible finite type Artin group A, the extension splits at least over a finite index subgroup $Aut^*(A/Z)$ of $Aut(A/Z)$ defined as follows. Let $\ell : A \to \mathbb{Z}$ be the length homomorphism which maps each generator of A to 1. Recall that the centre Z of A is infinite cyclic. Let ζ be a generator of Z and let $d = \ell(\zeta)$. The length homomorphism on A descends to a "length homomorphism" $\bar{\ell} : A/Z \to \mathbb{Z}/d\mathbb{Z}$. Define $Aut^*(A)$, respectively $Aut^*(A/Z)$, to be the group of length preserving or reversing automorphisms, that is, automorphisms α such that $\ell \circ \alpha = \pm\ell$ (respectively $\bar{\ell} \circ \alpha = \pm\bar{\ell}$).

Proposition 8. *Suppose A is an irreducible Artin group of finite type. Then the natural map $\pi : Aut^*(A) \to Aut^*(A/Z)$ is an isomorphism. If, in addition, the abelianization of A is infinite cyclic (i.e., A is not of type $I_2(2m), B_n$, or F_4), then $Tv(A)$ is trivial and $Aut(A) \cong Aut^*(A/Z)$.*

Proof. Note that a non-trivial transvection T_λ of A can never be length preserving, and can only be length reversing if $d = 2$ (and $\lambda(\cdot) = \zeta^{-\ell(\cdot)}$). Since $d > 2$ for all irreducible finite type Artin groups we conclude that $\ker(\pi) = Tv(A) \cap Aut^*(A)$ is trivial.

For any length preserving $\alpha \in Aut^*(A/Z)$, we define a lift $\hat{\alpha} \in Aut^*(A)$ as follows. (The lift for length reversing automorphisms is defined analogously). By definition, for any $a \in A$, $\alpha(aZ) = bZ$ for some $b \in A$ with $\ell(b) \equiv \ell(a) \bmod d$. Since Z is generated by an element of length d, the coset bZ contains a unique representative b' satisfying $\ell(b') = \ell(a)$. Define $\hat{\alpha}(a) = b'$. To verify that $\hat{\alpha}$ is a homomorphism, note that if $\hat{\alpha}(a_1) = b_1$ and $\hat{\alpha}(a_2) = b_2$, then

$$\begin{aligned} \alpha(a_1a_2Z) &= \alpha(a_1Z)\cdot\alpha(a_2Z) = b_1Z\cdot b_2Z = b_1b_2Z, \\ \ell(a_1a_2) &= \ell(a_1)+\ell(a_2) = \ell(b_1)+\ell(b_2) = \ell(b_1b_2), \end{aligned}$$

so it follows that $\hat{\alpha}(a_1a_2) = b_1b_2$. Moreover, it is straightforward to verify that if α_1 and α_2 are two elements of $Aut^*(A/Z)$, then $\alpha_1 \circ \alpha_2$ lifts to $\hat{\alpha}_1 \circ \hat{\alpha}_2$. In particular, the lift of an automorphism is also an automorphism. Thus, $\alpha \mapsto \hat{\alpha}$ defines a section for the projection π, and π is an isomorphism.

If the abelianization of A is infinite cyclic, then the length homomorphism ℓ can be identified with the abelianization map $A \to A^{\mathrm{ab}} \cong \mathbb{Z}$. Hence every

automorphism of A is either length preserving or length reversing. The second statement of the proposition follows. $\square$

In the case of $A = A(A_n)$, it follows from Proposition 5 that $Aut^*(A/Z) = Aut(A/Z) \cong \Gamma_A \rtimes C_2$ which, combined with Proposition 8, yields the main result of [7], that $Out(A(A_n)) = C_2$.

We remark, however, that while $Aut(A) = Aut^*(A) \cong Aut^*(A/Z)$ when A has infinte cyclic abelianization, it is not always the case that $Aut(A) \cong Aut(A/Z)$ (or equivalently that $Out(A) \cong Out(A/Z)$). In particular, for A of type $I_2(m)$ with m odd, $A/Z \cong C_2 * C_m$, and $Out(A/Z)$ is isomorphic to the group of units of $\mathbb{Z}/m\mathbb{Z}$, while $Out(A) = C_2$.

In the case of an Artin group whose abelianization is not infinite cyclic, the situation is more complicated.

Proposition 9. *If A is an irreducible Artin group of type $I_2(2n), B_n$, $n \geq 3$, or F_4, then $\mathrm{Tv}(A) \cong \mathbb{Z}$. If A is of type $I_2(4) \equiv B_2$ then $\mathrm{Tv}(A) \cong D_\infty$. Moreover, in all these cases $Aut(A)$ contains $\mathrm{Tv}(A) \rtimes Aut^*(A)$ as a finite index subgroup.*

Proof. We refer the reader to [3] and [4] for descriptions of the Coxeter graphs, and central elements respectively, for each of the irreducible finite type Artin groups. In each case listed above, the abelianization of A is $\mathbb{Z} \times \mathbb{Z}$, and the generator ζ of Z maps to an element $(r, s) \in \mathbb{Z} \times \mathbb{Z}$, where $(r, s) = (n, n)$ for type $I_2(2n)$, $(12, 12)$ for type F_4, and $(n(n-1), n)$ for type B_n. Any homomorphism $\lambda : A \to Z$ can be obtained by composing the abelianisation homomorphism with a map $\mathbb{Z} \times \mathbb{Z} \to Z$ given by $(r, s) \mapsto \zeta^{pr+qs}$ for a pair of integers $p, q \in \mathbb{Z}$. The associated transvection then satisfies $T_\lambda(\zeta) = \zeta^k$ where $k = 1 + (p+q)n$, or $1 + (p(n-1) + q)n$, depending on the case ($n = 12$ for type F_4). On the other hand T_λ is an automorphism if and only if $k = \pm 1$. When $n \geq 3$ this is only possible if $k = 1$ and $p + q = 0$, or $(p(n-1) + q) = 0$, respectively. This allows exactly an infinite cyclic group of transvection automorphisms (all of which act by the identity on Z).

In the remaining case, type $I_2(4) \equiv B_2$, we have the constraint $k = 1 + 2(p + q) = \pm 1$, and so $p + q = 0$ or -1. We note that $A = A(I_2(4))$ is given by the presentation $\langle a, b \mid abab = baba \rangle$, where $\zeta = abab$. A transvection of the second type ($p = 0, q = -1$) is realized by the automorphism $T_0 : a \mapsto a$ and $b \mapsto (aba)^{-1}$. Since in general we have

$$T_\lambda \circ T_\mu = \begin{cases} T_{\lambda+\mu} \text{ if } T_\lambda(\zeta) = \zeta \\ T_{\lambda-\mu} \text{ if } T_\lambda(\zeta) = \zeta^{-1} \end{cases}$$

it is easily checked that $Tv(A)$ is an infinite dihedral group with T_0 acting as a direction reversing involution.

The subgroup $Aut^*(A/Z)$ is finite index in $Aut(A/Z)$ (since $\bar{\ell}$ takes values in the finite group $\mathbb{Z}/d\mathbb{Z}$ and A/Z, being finitely generated, admits only a finite number of homomorphisms to $\mathbb{Z}/d\mathbb{Z}$). Hence the inverse image $\pi^{-1}(Aut^*(A/Z)) < Aut(A)$ is a finite index subgroup of $Aut(A)$. By the previous proposition, this subgroup splits as a semi-direct product $Tv(A) \rtimes Aut^*(A)$. $\square$

For A of type $I_2(m)$, we have the following presentation:

$$A(I_2(m)) = \langle\ a, b\ \mid\ \underbrace{aba\dots}_{m} = \underbrace{bab\dots}_{m}\ \rangle\,.$$

In this case the outer automorphism groups were computed in [8]:

$$\begin{aligned} &Out(A(I_2(m)) \cong C_2, && m \text{ odd} \\ &Out(A(I_2(m)) \cong (\mathbb{Z} \rtimes \langle\gamma\rangle) \times C_2 \cong D_\infty \times C_2\,, && m \text{ even} \end{aligned}$$

where γ denotes the graph involution $\gamma : a \leftrightarrow b$, and the C_2 factor in each case is generated by the inversion automorphism $\epsilon : a \mapsto a^{-1}, b \mapsto b^{-1}$. In the case m even, the infinite cyclic factor $\mathbb{Z}$ is generated by the automorphism $\eta : a \mapsto aba$ and $b \mapsto a^{-1} = b(ab)^{-1}$, and contains the group of transvections as a subgroup of index $m/2$, when $m \geq 6$. (Note that η fixes both elements ab and ba, and that the centre is generated by $(ab)^{m/2} = (ba)^{m/2}$). It follows that, in the case $A = A(I_2(m))$ with $m \geq 6$ even, the subgroup $Tv(A) \rtimes Aut^*(A)$ has index $m/2$ in the entire automorphism group. This behaviour seems to be repeated in the case $A = A(B_n)$, for $n \geq 3$, where, as we will see below, $\pi^{-1}(Aut^*(A/Z))$ is a subgroup of index 2 in $Aut(A)$. The type $I_2(4) \equiv B_2$ Artin group is however exceptional in this regard. In this case the group of transvections is generated by η^2 and the element $T_0 = \epsilon \circ \eta \circ \gamma$, from which it follows that $Tv(A) \rtimes Aut^*(A)$ is the whole automorphism group. We do not know what the index is (or whether the subgroup is proper) in the case of $A(F_4)$.

Proposition 10. *Let $n \geq 3$ and $A = A(B_n)$. Then*

$$\begin{aligned} Aut(A) &\cong (\Gamma_B \times Tv(A)) \rtimes (C_2 \times C_2) \\ Out(A) &\cong (\mathbb{Z} \rtimes C_2) \times C_2 \cong D_\infty \times C_2 \end{aligned}$$

Proof. We first show that the map $\phi : Aut(A) \to Aut(A/Z)$ is surjective and splits. Recall from Section 2 that $A/Z \cong \Gamma_B \cong \Gamma_{\tilde{A}} \rtimes \mathbb{Z}/n\mathbb{Z}$. It is clear from the proof of Proposition 7 that elements of $Aut(\Gamma_B)$ act on this semidirect product by leaving the characteristic subgroup $\Gamma_{\tilde{A}}$ invariant and by mapping the generator of the cyclic factor $\mathbb{Z}/n\mathbb{Z}$ either to itself or its inverse. Any such automorphism lifts uniquely to an automorphism of $A(B_n) \cong \Gamma_{\tilde{A}} \rtimes \mathbb{Z}$. Thus we have a section to the map ϕ.

By Proposition 5, this proves that

$$Aut(A) \cong Tv(A) \rtimes Aut(\Gamma_B) \cong Tv(A) \rtimes (\Gamma_B \rtimes (C_2 \times C_2)).$$

A direct check shows that inner automorphisms, Γ_B, commute with transvections, and the first statement of the proposition follows.

We claim that the C_2 factor generated by the inversion ϵ also commutes with transvections. This follows from the fact that ϵ restricts to a length reversing automorphism of the center Z and hence takes the generator ζ of Z to ζ^{-1}. For any transvection T and any generator s_i, $T(s_i) = s_i\zeta^k$ for some k, so

$$\epsilon \circ T \circ \epsilon(s_i) = \epsilon \circ T(s_i^{-1}) = \epsilon(s_i^{-1}\zeta^{-k}) = s_i\zeta^k = T(s_i).$$

By Proposition 9, $Tv(A) \cong \mathbb{Z}$ and the second claim of the proposition now follows. $\square$

Remark. The $\mathbb{Z}$ factor in the decomposition $A(B_n) = \Gamma_{\tilde{A}} \rtimes \mathbb{Z}$ is generated by the element $\delta = s_1 s_2 \dots s_n$ of $A(B_n)$ and it acts on $\Gamma_{\tilde{A}}$ via rotation of the $\tilde{A}$ graph. These rotations thus become inner automorphisms in $Aut(A(B_n))$. The remaining graph automorphisms (the reflections) give rise to the first of the two C_2 factors in $Aut(A(B_n))$, the second C_2 factor being generated by the inversion ϵ. A generator of the first C_2 factor is represented by the automorphism τ defined by

$$\begin{aligned} \tau(s_i) &= s_{n-i},\ i = 1, \dots, n-1 \\ \tau(\delta) &= \delta^{-1} \end{aligned}$$

and hence $\tau(s_n) = \tau((s_1 \dots s_{n-1})^{-1}\delta) = (\delta s_{n-1} \dots s_1)^{-1}$. In particular, τ does not lie in $Tv(A) \rtimes Aut^*(A)$ (as long as $n \geq 3$), and this latter group is index 2 in the full automorphism group. (By contrast, when A is of type $B_2 \equiv I_2(4)$, the involution τ just described is exactly the exceptional transvection $T_0 = \epsilon \circ \eta \circ \gamma$ already mentioned in the preceding discussion).

If H is a group whose centre Z is a finite rank free abelian group, the injective transvections which are not automorphisms of H have as image a proper finite index subgroup of H and therefore represent infinite order elements of $Comm(H)$ (*not* equivalent to automorphisms). In the case of a finite type Artin group, this implies that the abstract commensurator group must be large.

Proposition 11. *Let A be any finite type Artin group. Then $Comm(A)$ contains an infinitely generated free abelian subgroup generated by transvections.*

Proof. It suffices to prove the proposition for irreducible A. Let $\ell : A \to \mathbb{Z}$ be the length homomorphism. Let ζ be the generator of the centre Z with positive length $d = \ell(\zeta)$. Then for every $m \in \mathbb{Z}$, there is a homomorphism $\lambda_m : A \to Z$ defined by $\lambda_m(a) = \zeta^{m\ell(a)}$. The associated transvection takes a to $a\zeta^{m\ell(a)}$. In particular, it takes ζ to ζ^{md+1}. We denote this transvection by T_{md+1}.

It is straightforward to verify that $T_{md+1} \circ T_{kd+1} = T_{(md+1)(kd+1)}$. Thus, T_{md+1} and T_{kd+1} are commuting elements of $Comm(A)$ and to find an infinitely generated free abelian subgroup of $Comm(A)$, it suffices to find an infinite sequence of integers $\{n_i\}$ which are relatively prime and satisfy $n_i \equiv 1 \bmod d$. This can be done inductively by setting $n_1 = 1+d$ and $n_{i+1} = 1+d(n_1 n_2 \dots n_i)$. The transvections T_{n_i} are then linearly independent. $\square$

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Department of Mathematics, Brandeis University, Mail Stop 050, Waltham, MA 02454

E-mail address: charney@brandeis.edu

I.M.B.(UMR 5584 du CNRS), Université de Bourgogne, B.P. 47 870, 21078 Dijon, France

E-mail address: jcrisp@u-bourgogne.fr