Mathematical Research Letters **12**, 713–718 (2005)

ON THE NON-EXISTENCE OF A CODIMENSION ONE HOLOMORPHIC FOLIATION TRANSVERSE TO A SPHERE

Toshikazu Ito and Bruno Scárdua

Dedicated to the memory of Professor Haruo Kitahara

1. Introduction

In this paper we address the following question:

Question 1. *Is there any codimension one holomorphic foliation $\mathcal{F}$ in a neighborhood of the closed unit ball $B[0;1] \subset \mathbb{C}^n$ such that $\mathcal{F}$ is transverse to the boundary sphere $S^{2n-1}(0;1)$ for $n \geq 3$?*

We point-out that for $n = 2$ there are linear examples and the situation is well-understood ([1],[5]). We conjecture that, for dimension $n \geq 3$, Question 1 has a negative answer. In this direction we state our main result as:

Theorem 1. *Let $\mathcal{F}$ be a codimension one foliation in a neighborhood U of the closed unit ball $B[0;1] \subset \mathbb{C}^n$, $n \geq 2$ and transverse to the boundary sphere $S^{2n-1}(0;1)$. If $\mathcal{F}$ has some leaf L_0 with $0 \in \overline{L}_0$ and which is closed in $U \setminus \operatorname{sing}(\mathcal{F})$ and transverse to every sphere $S^{2n-1}(0;R)$, $0 < R \leq 1$ then $n = 2$.*

A natural situation happens when $\mathcal{F}$ has a global separatrix: according to [7] if a codimension one foliation $\mathcal{F}$ as above is transverse to $S^{2n-1}(0;1)$ then $\mathcal{F}$ has a single singularity p_0 in the open ball $B^{2n}(0;1)$. If $n \geq 3$ then by Malgrange's Theorem ([8]) the foliation $\mathcal{F}$ admits a local holomorphic first integral $f\colon V \to \mathbb{C}$ in a neighborhood V of p_0 in $\mathbb{C}^n$ with $f(p_0) = 0$. The germ of hypersurface $\Lambda = f^{-1}(0) \subset V$ is called a *separatrix* of $\mathcal{F}$, the existence of a separatrix for dimension 2 is proved in [3]. We shall say that $\mathcal{F}$ has a *global separatrix* $\tilde{\Lambda}$ if the leaf L_0 of $\mathcal{F}$ that contains $\Lambda \setminus \{p_0\}$, is closed in U for $U \supset B(0;1)$ small enough. In this case we put $\tilde{\Lambda} = L_0 \cup \Lambda = L_0 \cup \{p_0\}$. An immediate consequence of our main result is:

Corollary 1. *Let $\mathcal{F}$ be a foliation as above, transverse to $S^{2n-1}(0;1)$ and admitting a global separatrix $\tilde{\Lambda}$ transverse to $S^{2n-1}(0;R)$, $\forall\, R \in (0,1]$. Then $n = 2$.*

A holomorphic function $f\colon U \to \mathbb{C}$ defined in an open subset $0 \in U \subset \mathbb{C}^n$ is *quasi-homogeneous* if there exists a holomorphic vector field $\overrightarrow{\xi} = \sum\limits_{j=1}^{n} \alpha_j z_j \frac{\partial}{\partial z_j}$, with $0 \leq \alpha_j \in \mathbb{Q}$, $\forall\, j$, such that $df(\overrightarrow{\xi}) = \alpha \cdot f$ for some constant $\alpha \in \mathbb{C}$. A

Received by the editors May 6, 2005.

codimension one analytic subset $\Lambda \subset U$ is *quasi-homogeneous* if $\Lambda = f^{-1}(0)$ for some quasi-homogeneous function f as above. As a particular case of the above results we have

Corollary 2. *Let $\mathcal{F}$ be a holomorphic foliation of codimension one on a neighborhood U of the closed ball $B[0;1] \subset \mathbb{C}^n$, $n \geq 2$. Suppose that $\mathcal{F}$ is transverse to the boundary sphere $S^{2n-1}(0;1)$ and has a quasi-homogeneous invariant hypersurface $\Lambda \subset U$. Then $n = 2$.*

Essentially, we reduce Question 1 of transversality for foliations to a question of transversality for a closed leaf of the foliation. Another interesting consequence we obtain is:

Corollary 3. *Let ω be a closed meromorphic one-form in a neighborhood U of $B[0;1] \subset \mathbb{C}^n$, $n \geq 2$ and such that the corresponding holomorphic foliation $\mathcal{F}_\omega$ is transverse to $S^{2n-1}(0;1)$. Suppose that the polar set of ω is transverse to $S^{2n-1}(0;R)$, $\forall\, R \in (0,1]$. Then $n = 2$. Moreover, there is a holomorphic mapping Φ from a neighborhood of $B[0;1]$ to a neighborhood of the origin $0 \in \mathbb{C}^2$ such that either $\mathcal{F} = \Phi^*(\mathcal{L}_\lambda)$ or $\mathcal{F} = \Phi^*(\mathcal{L}_{a,m})$ where $\mathcal{L}_\lambda$ is the linear foliation given by $xdy - \lambda ydx = 0$, $\lambda \in \mathbb{C}\backslash\mathbb{R}_-$ and $\mathcal{L}_{a,m}$ is the Poncaré-Dulac normal form foliation given by $xdy - (my + ax^m)dx = 0$, $\quad m \in \mathbb{N}\backslash\{0,1\}$, $\quad a \in \mathbb{C}\backslash\{0\}$.*

2. Preliminaries

Let $f\colon U \to \mathbb{C}$ be a holomorphic function defined in a neighborhood U of the ball $B[0;1] \subset \mathbb{C}^n$ with $f(0) = 0$. We fix the standard metric on $\mathbb{C}^n$ corresponding to the norm $||z||^2 = \sum_{j=1}^{n} |z_j|^2 = \sum_{j=1}^{n} z_j \cdot \bar{z}_j$ where $z = (z_1, \ldots, z_n)$. We assume that either f is a submersion at each point of $f^{-1}(0)$ or that the origin is the only singularity of f in $f^{-1}(0)$ and this singularity is isolated. According to Milnor [9] we have the following: For any $\varepsilon > 0$ small enough $f^{-1}(0)$ is (smooth and) transverse to the sphere $S^{2n-1}(0;\varepsilon)$ and the topology of the *link* $K(f;R) = f^{-1}(0) \cap S^{2n-1}(0;R)$ is the same for all $R \in (0,\varepsilon]$. We shall use the following remark:

Lemma 1. *Let $f\colon U \to \mathbb{C}$ be a holomorphic function as above with $f(0) = 0$ and U a neighborhood of $B[0;1]$ in $\mathbb{C}^n$. Assume that $f^{-1}(0) - \{0\}$ is transverse to the spheres $S^{2n-1}(0;R)$ for every $R \in (0,1]$. Then the links $K(f;R)$ and $K(f;1)$ are diffeomorphic for every $R \in (0,1]$.*

Proof: Denote by $\rho\colon \mathbb{C}^n \simeq \mathbb{R}^{2n} \to [0,+\infty)$ the C^∞-function $\rho(z_1,\ldots,z_n) = \sum_{j=1}^{n} |z_j|^2$. Then we may consider the smooth manifold $M := f^{-1}(0) - \{0\} \subset U$ with its natural C^∞ differentiable structure. Denote by $\varphi\colon M \to (0,+\infty)$ the restriction $\rho\big|_M$. Then φ is of class C^∞ and we have $\varphi^{-1}(R) = K(f;R)$ for every $R \in (0,1]$. Moreover an easy computation shows that a point $p \in M$ is a critical point of φ if and only if $T_p(M) \subset d\rho(p)^{-1}(0)$ if and only if

$T_p(M) \subset T_p\big(S^{2n-1}(0;||p||)\big)$. Since by hypothesis M is transverse to every sphere $S^{2n-1}(0;R)$ it follows that φ has no critical points on M. Now clearly φ is proper in $M\backslash[B[0;\varepsilon]\cap M]$ for every $\varepsilon > 0$. Therefore by standard arguments of Morse Theory the flow of the gradient of φ gives a diffeomorphism from $K(f;1)$ with $K(f;R)$ for every $0 < \varepsilon \leq R < 1$. $\square$

Now we investigate some examples of the situation in Lemma 1. Let $f\colon U \to \mathbb{C}$ be a quasi-homogeneous holomorphic function with respect to the vector field $\vec{\xi} = \sum\limits_{j=1}^{n} \alpha_j z_j \frac{\partial}{\partial z_j}$ as in the introduction. In this case $df(\vec{\xi})$ vanishes identically on $\{f = 0\}$ and therefore $f^{-1}(0)$ is a union of orbits of $\vec{\xi}$. On the other hand the hermitian product of $\vec{\xi}$ with the radial vector field $\vec{\mathcal{R}} = \sum\limits_{j=1}^{n} z_j \frac{\partial}{\partial z_j}$ is $\langle\vec{\xi},\vec{\mathcal{R}}\rangle = \sum\limits_{j=1}^{n} \alpha_j |z_j|^2 \geq 0$. This product vanishes only at the origin and therefore $\vec{\xi}$ is transverse to the spheres $S^{2n-1}(0;R)$, $\forall\, R \in (0,1]$, showing that $f^{-1}(0)$ is transverse to $S^{2n-1}(0;R)$, $\forall\, R > 0$ provided that the origin is an isolated singularity of f. For instance we can take $f = \sum\limits_{j=1}^{n} z_j^{m_j}$, $m_j \in \mathbb{N}$ and $\vec{\xi} = \sum\limits_{j=1}^{n} \frac{1}{m_j} z_j \frac{\partial}{\partial z_j}\cdot$ Since we only ask for the transversality of the level $f^{-1}(0)$ with the spheres $S^{2n-1}(0;R)$, $0 < R \leq 1$ we may obtain other examples of functions f with $f^{-1}(0) - \{0\}$ transverse to small spheres centered at the origin, by considering functions of the form $f = f_0 + P$, where f_0 is quasi-homogeneous and P is a small perturbation.

Lemma 2. *If $\Lambda \subset U$ is quasi-homogeneous and has an isolated singularity at the origin then Λ is transverse to the spheres $S^{2n-1}(0;R), \forall R > 0$.*

3. Proof of the results

Let $\mathcal{F}$ be a holomorphic foliation of codimension one in U; $B[0;1] \subset U \subset \mathbb{C}^n$, $n \geq 2$ and transverse to $S^{2n-1}(0;1)$. We may assume that $n \geq 3$. According to [7] we must have n even and $\operatorname{sing}(\mathcal{F}) \cap B(0;1) = \{p_0\}$ is a single simple-singularity. In particular $n \geq 4$. We can assume that either $\mathcal{F}$ has a global separatrix in the situation of Corollary 1 or, more generally, that $\mathcal{F}$ has a closed leaf L_0 in $U\backslash \operatorname{sing}(\mathcal{F}) = U\backslash\{p_0\}$ with $0 \in \overline{L}_0$. By Remmert-Stein Theorem [10] the closure $\overline{L}_0 \subseteq L_0 \cup \{p_0\}$ is an analytic subvariety of U, of pure codimension one and, since U is a neighborhood of $B[0;1]$, by a classical Theorem of Cartan, there exists a holomorphic function $f\colon U \to \mathbb{C}$ such that $f(0) = 0$ and $\overline{L}_0 = f^{-1}(0)$. Now, according to Milnor since f has an isolated singularity at the origin (or even f is non-singular) and $n \geq 4$, the link $f^{-1}(0) \cap S^{2n-1}(0;\varepsilon) = K(f,\varepsilon)$ is simply-connected ([4],[9]) for any $\varepsilon > 0$ small enough. Lemma 1 then implies that the link $K(f;1)$ is also simply-connected. Let us use the transversality $\mathcal{F} \pitchfork S^{2n-1}(0;1)$. Denote by $\mathcal{F}_1$ the restriction $\mathcal{F}\big|_{S^{2n-1}(0;1)}$ then $\mathcal{F}_1$ is a codimension two real foliation with a natural transversely holomorphic structure. Also the link $K(f,1)$ corresponds to a simply-connected compact leaf of $\mathcal{F}_1$. From now on we proceed as in [6] in order to obtain a contradiction. First we apply the Global Stability Theorem of [2] to conclude that every leaf of $\mathcal{F}_1$ is compact with trivial fundamental group. This implies that the leaf space $S^{2n-1}(0;1)\big/\mathcal{F}_1$ of $\mathcal{F}_1$ is a compact Riemann surface and admits therefore some non-constant meromorphic mapping $S^{2n-1}(0;1)\big/\mathcal{F}_1 \to \overline{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$ onto the Riemann Sphere (it is possible to prove directly that $S^{2n-1}(0;1)\big/\mathcal{F}_1$ is simply-connected and therefore isomorphic to $\overline{\mathbb{C}}$). Using this we obtain a transversely holomorphic first integral $F_1\colon S^{2n-1}(0;1) \to \overline{\mathbb{C}}$ for $\mathcal{F}_1$. By transversality of $\mathcal{F}$ with $S^{2n-1}(0;1)$ the map F_1 admits an extension to a holomorphic map $F\colon W \to \overline{\mathbb{C}}$ in a neighborhood W_0 of $S^{2n-1}(0;1)$ in U and constant along the leaves of the restriction $\mathcal{F}\big|_{W_0}$. By standard Hartogs' Extension results we can extend F as a holomorphic mapping $F\colon W_0 \cup B[0;1] \to \overline{\mathbb{C}}$ constant along the leaves of $\mathcal{F}$. By Stein's Factorization Theorem we may assume that F has connected fibers. We may write $F = \frac{\alpha}{\beta}$ for some holomorphic functions α, β in a neighborhood W of $B[0;1]$ and without non-trivial common factors in $\mathcal{O}(W)$. Since the only singularity of $\mathcal{F}$ in $B(0;1)$ admits a local first integral of holomorphic type into $\mathbb{C}$ it follows that $\alpha^{-1}(0)\cap\beta^{-1}(0) = \emptyset$ and F has no indeterminacy points in $B[0;1]$. In particular the restriction $F_1 = F\big|_{S^{2n-1}(0;1)}$ defines a locally trivial C^∞ fibration of $S^{2n-1}(0;1)$ over the sphere $S^2 \simeq \mathbb{C}\cup\{\infty\}$ with simply-connected fibers. By the homotopy sequence of a fibration [11] we must have $2n-1 \leq 3$, contradiction. This proves Theorem 1. □

Proof of Corollary 1. Let ω be a closed meromorphic one form in $U \supset B(0;1)$. Write $(\omega)_\infty = \bigcup\limits_{j=1}^{r}\{f_j = 0\}$ for suitable (reduced) holomorphic functions $f_j\colon U \to \mathbb{C}$, $j = 1,\dots,r$. Since we can take U simply-connected ω can be written

$$(*)\ \ \omega = \sum_{j=1}^{r}\lambda_j\,(df_j/f_j) + d\big(g/\prod_{j=1}^{r} f_j^{n_j-1}\big)$$

for some $\lambda_j \in \mathbb{C}$ and $n_j \in \mathbb{N}$ and some holomorphic function $g\colon U \to \mathbb{C}$. In particular either ω is holomorphic in U or $\{f_1 = 0\} \subset (\omega)_\infty$ gives a closed leaf of $\mathcal{F}_\omega$. In this last case we apply Theorem 1 to obtain $n = 2$. In the first case $\omega = dg$ and $\mathcal{F}_\omega$ admits a holomorphic first integral in U. By the Maximum modulus principle we conclude that $\mathcal{F}_\omega$ cannot be transverse to $S^{2n-1}(0;1)$ even for $n = 2$. Thus $n = 2$ and we have a unique simple singularity for $\mathcal{F}_\omega$ in the ball $B(0;1)$ which is necessarily a singularity in the Poincaré-domain (cf. [5]). By Poincaré-Dulac theorem we know that either $\mathcal{F}_\omega$ is linearizable as $\mathcal{L}_\lambda$ with $\lambda \in \mathbb{C}\setminus\mathbb{R}_-$ in a neighborhood of the singularity or it is analytically conjugate in a neighborhood of the singularity to a Poincaré-Dulac normal form $\mathcal{L}_{a,m}$. Comparing these local models for ω with the global writing $(*)$ above we conclude. □

Corollary 2 follows immediately from Lemma 2 and Corollary 1.

Remark 1. If we do not assume that $0 \in \overline{L}_0$ in Theorem 1 then we cannot apply Lemma 1 in its present formulation. Nevertheless, we can proceed as follows. Suppose that $p_0 \neq 0$ and let T be an automorphism of the closed ball $B^{2n}[0;1]$ which maps p_0 to 0 and such that $T^2 = Id$. Then both $\mathcal{F}$ and the pull-back foliation $T^*(\mathcal{F})$ are transverse to the sphere $S^{2n-1}(0;1)$. The foliation $T^*(\mathcal{F})$ has a leaf $T^{-1}(L_0)$ which is closed in $T^{-1}(U) \setminus \text{sing}(T^*(\mathcal{F})) = T^{-1}(U) \setminus \{0\}$ and transverse to all hyperbolic balls of hyperbolic center $T^{-1}(0) = p_0$. Lemma 1 can be stated accordingly to this situation with essentially the same proof and also the notion of quasi-homogeneity. This suggests that Theorem 1 might hold for codimension one holomorphic foliations transverse to the boundary of a strongly convex domain and having a global separatrix transverse to the boundary of all Caratheodory or Kobayashi balls centered at some singularity. We want to thank the referee for this and other valuable remarks.

References

[1] M. Brunella, *On holomorphic foliations in complex surfaces transverse to a sphere*, Bol. Soc. Brasil. Mat. (N.S.) 26 (1995), no. 2, 117–128.

[2] ———, *A Global Stability Theorem for transversely holomorphic foliations*, Ann. Global Anal. Geom. **15** (1997), no. 2, 179–186.

[3] C. Camacho, P. Sad, *Invariant varieties through singularities of holomorphic vector fields*, Ann. of Math. (2) **115** (1982), no. 3, 579–595.

[4] A. Dimca, Singularities and Topology of Hypersurfaces. Universitext, Springer Verlag, 1992.

[5] T. Ito, A. Poincaré-Bendixson type theorem for holomorphic vector fields, RIMS Pub. 878 (June 1994), 1–9.

[6] T. Ito, B. Scárdua, On holomorphic foliations transverse to spheres. To appear in Moscow Mathematical Journal, 2005.

[7] ———, A Poincaré-Hopf Index Theorem for holomorphic one-forms. Topology **44** (2005) no.1, 73–84.

[8] B. Malgrange, Frobenius avec singularités [d'aprés B. Malgrange, J. F. Mattei et R. Moussu]. (French) Séminaire Bourbaki, 30e année (1977/78), Exp. No. 523, pp. 290–299, Lecture Notes in Math., 710, Springer, Berlin, 1979.

[9] J. Milnor, Singular points of complex hypersurfaces. Ann. Math. Studies, Vol. 61, Princeton Univ. Press, 1968.

[10] R. C. Gunning, Introduction to holomorphic functions of several variables. Vol. II. Local theory. The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA, 1990.

[11] N. Steenrod, The topology of fibre bundles. Reprint of the 1957 edition. Princeton University Press, Princeton, NJ, 1999.

Department of Natural Science, Ryukoku University, Fushimi-ku, Kyoto 612-8577, JAPAN

Instituto de Matemática C.P. 68530, Universidade Federal do Rio de Janeiro, 21.945-970 Rio de Janeiro-RJ, BRAZIL
E-mail address: scardua@impa.br