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CONIFOLD TRANSITIONS AND MORI THEORY

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Abstract. We show there is a symplectic conifold transition of a projective 3-fold which is not deformation equivalent to any Kähler manifold. The key ingredient is Mori's classification of extremal rays on smooth projective 3-folds. It follows that there is a (nullhomologous) Lagrangian sphere in a projective variety which is not the vanishing cycle of any Kähler degeneration, answering a question of Donaldson.

1. A *conifold transition* is a surgery on a (real) six-dimensional manifold X which replaces a three-sphere with trivial normal bundle by a two-sphere with trivial normal bundle, cutting out $\mathbb{S}^3 \times D^3$ and replacing it with $D^4 \times \mathbb{S}^2$. More generally, one can simultaneously replace a collection of disjoint embedded three-spheres. The local geometry is reviewed below, and treated in detail in the Appendix to [STY02]. Such surgeries arise in algebraic geometry when one passes from the smoothing to either small resolution of a 3-fold ordinary double point, coming from the basic diffeomorphism

$$\{z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\} \smallsetminus \{0\} \cong T^*\mathbb{S}^3 \smallsetminus \{0\text{-section}\}$$

given in coordinates by $z \mapsto (\Re(z)/|\Re(z)|, -|\Re(z)|\Im(z))$. The main theorem of [STY02] showed that the surgery can be performed compatibly with respect to a symplectic structure on X whenever the 3-spheres $L_i \subset X$ are Lagrangian and satisfy a homology relation $\sum_i \lambda_i[L_i] = 0 \in H_3(X;\mathbb{Z})$ with $\prod_i \lambda_i \neq 0$. (This result can be viewed as "mirror", in the sense of mirror symmetry, to an older result of Friedman and Tian [Fri86, Tia92] giving sufficient conditions for the smoothing of a complex Calabi-Yau 3-fold with ordinary double points to admit a complex structure. Friedman's results were inspired by earlier work of Clemens [Cle83].)

The aim of this note is to prove the following, which was left open in [STY02]:

Theorem 1. *There is a symplectic conifold transition of a projective 3-fold which is not deformation equivalent to any Kähler 3-fold.*

We will begin with the 3-fold $E \times \mathbb{P}^1$, where E is an Enriques surface, and take the conifold transition along a single nullhomologous Lagrangian sphere. This 3-fold has $h^{2,0} = 0$; using this, an argument in Hodge theory given below in section 8 shows that if the Lagrangian sphere could be degenerated to a node in any degeneration with smooth Kähler total space, the transition would also admit

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Kähler forms. In particular, we obtain the answer to a question of Donaldson ([Don00], Question 4):

Corollary 1. *There is a Lagrangian sphere in a projective algebraic variety which is not the vanishing cycle of any Kähler degeneration.*

The local model for the conifold transition given in section 2 shows that any Lagrangian sphere can be degenerated in a symplectic degeneration. We remark that nullhomologous Lagrangian spheres are sometimes vanishing cycles for projective degenerations: consider a Lefschetz pencil of quadric 3-folds. Other, more striking, examples are given in section 7.

If it were known that Kähler 3-folds either have nef canonical class or admit an extremal ray, our arguments would show the analogous conifold transition of K3 $\times$ $\mathbb{P}^1$ is not Kähler (as it stands the methods only show it is not projective). This example, being simply connected, would necessarily have formal cohomology ring in the sense of rational homotopy theory. This suggests that establishing a Mori-type theory in the Kähler case would have immediate applications in symplectic topology.

2. For the reader's convenience, we give some more background on conifold transitions. Fix once and for all a model of an ordinary double point $W = \{\sum_{j=1}^4 z_j^2 = 0\} \subset \mathbb{C}^4$. This has two small resolutions $W^\pm$, in which the singular point is replaced by a rational curve $\mathbb{P}^1$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, defined by the graphs of the rational maps

$$W^+\colon \quad \frac{z_1 + iz_2}{z_3 + iz_4} = -\frac{z_3 - iz_4}{z_1 - iz_2}; \qquad W^-\colon \quad \frac{z_1 + iz_2}{z_3 - iz_4} = -\frac{z_3 + iz_4}{z_1 - iz_2}.$$

If we take the standard oriented $\mathbb{S}^3 \subset \mathbb{R}^4$ and fix coordinates

$$T^*\mathbb{S}^3 = \{(u,v) \in \mathbb{R}^4 \times \mathbb{R}^4 \mid \ |u| = 1, \langle u, v\rangle = 0\}$$

then there is a symplectomorphism $W \setminus \{0\} \cong W^\pm \setminus \mathbb{P}^1 \cong T^*\mathbb{S}^3 \setminus \{v = 0\}$. The cotangent bundle $T^*\mathbb{S}^3$ is the smoothing $\{\sum z_j^2 = t \in \mathbb{R}_+\}$ of the singularity, with vanishing cycle the real locus $\{\sum z_j^2 = t\} \subset \mathbb{R}^4$. The holomorphic (orientation-preserving) automorphism $z_4 \mapsto -z_4$ of $\mathbb{C}^4$ acts on $\mathbb{R}^4 \subset \mathbb{C}^4$ by a reflection, reversing the orientation of the unit sphere. Hence, the choice of one of the two small resolutions is equivalent to the choice of an orientation of the three-sphere. Passing from one small resolution to the other is an example of a *flop*.

A neighbourhood of a Lagrangian sphere in a symplectic six-manifold is modelled on a neighbourhood of the zero-section in the cotangent bundle [Wei71] (so it is canonically framed up to homotopy, but not canonically oriented). The surgery which replaces a neighbourhood of $L \subset X$ by a neighbourhood of the $\mathbb{P}^1$ in either of $W^\pm$ is unique up to isotopy and the $\mathbb{Z}/2\mathbb{Z}$ choice above [STY02].

If the Lagrangian 3-sphere is homologically trivial, the main theorem of [STY02] implies that the result Y of the conifold transition carries a distinguished deformation equivalence class of symplectic forms. In fact, a bounding four-chain $L = \partial R \subset X$ gives rise, after surgery, to a four-cycle $\hat{R} \subset Y$; the symplectic form ω_X defines a distinguished class $[\omega_X] \in H^2(Y)$ and the symplectic forms on Y live in cohomology classes $[\omega_X] + \varepsilon \mathrm{PD}[\hat{R}]$, where PD denotes the Poincaré dual class, and $0 < |\varepsilon|$ is small. By forcing R to conform to suitable local models and by flopping, one can ensure that $\hat{R}$ meets the exceptional $\mathbb{P}^1$ transversely and positively once, and then ε will be strictly positive. The construction gives existence for symplectic forms in these classes; it says nothing about obstructions to finding forms in other classes, for instance as ε increases.

3. Our main idea is to derive obstructions to symplectic forms being Kähler from Mori's classification of extremal rays on smooth projective 3-folds. This is similar in spirit to the philosophy adopted by Campana and Peternell in [CP94]. More precisely, Mori [Mo82] proves the following remarkable theorem.

Theorem 2 (Mori)**.** *Let X be a smooth projective 3-fold. Then either K_X is nef, or X admits an extremal ray the contraction of which is a morphism $f\colon X \to Y$ of one of the following types:*

Fibring contraction: *$f\colon X \to Y$ is conic fibration over a smooth surface if $\dim Y = 2$; a del Pezzo fibration over a smooth curve if $\dim Y = 1$; or X is a Fano 3-fold if $\dim Y = 0$.*

Divisorial contraction: *f is a birational morphism which contracts a divisor $D \subset X$ to a point or a smooth curve in Y. Moreover, the possible divisors D are:*

- *$D = \mathbb{P}^2$, $\nu_D = \mathcal{O}(-1)$;*
- *$D = \mathbb{P}^2$, $\nu_D = \mathcal{O}(-2)$;*
- *$D = \mathbb{P}^1 \times \mathbb{P}^1$, $\nu_D = \mathcal{O}(-1,-1)$;*
- *$D = Q$ for $Q \subset \mathbb{P}^3$ a quadric cone, $\nu_D = \mathcal{O}(-1)$;*
- *$D \to C$ is a minimal ruled surface over a smooth curve.*

The existence of any such contraction – or of X having nef canonical class – imposes constraints on the Chern classes and the cohomology ring of the underlying almost complex structure. Using this, it is sometimes possible to prove that a homotopy class of almost complex structures on a smooth manifold is not compatible with any projective structure; moreover, the obstructions arising this way have a very different flavour from classical obstructions coming from the Hard Lefschetz theorem, the formality of the cohomology ring or the representation theory of the fundamental group.

An important point is that Mori's work applies to projective, rather than Kähler, 3-folds; however, if one knows *a priori* that for any complex structure $h^{2,0}$ vanishes, the existence of Kähler forms implies the existence of rational Kähler forms, and hence a projective structure. In the example studied below in section 6, the vanishing of $h^{2,0}$ is derived cohomologically.

4. Here is a well-known construction of nullhomologous Lagrangian spheres, taken from [ALP94]. The elementary observation is that for the unit sphere

$\mathbb{S}^3 \subset \mathbb{C}^2$, the restriction of the standard flat Kähler form $\omega_{\mathbb{C}^2}$ coincides with the pullback of the Fubini-Study form $\omega_{\mathbb{P}^1}$ under the Hopf map $\mathbb{S}^3 \to \mathbb{P}^1$

$$(\omega_{\mathbb{C}^2})_{|\mathbb{S}^3} = (\text{hopf})^* \omega_{\mathbb{P}^1}. \tag{1}$$

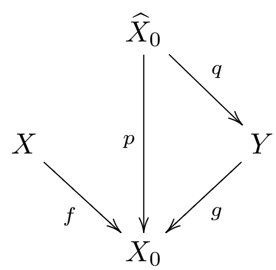
Indeed, following [MS98], this is our definition of the Fubini-Study form on $\mathbb{P}^n$; it is given by quotienting Hopf circles in $\mathbb{S}^{2n+1}$ and pushing forward the flat form from $\mathbb{C}^{n+1}$. (This gives a line in $\mathbb{P}^n$ area π.) It follows that the graph of the Hopf map, inside $(\mathbb{C}^2 \times \mathbb{P}^1, -\omega_{\mathbb{C}^2} \oplus \omega_{\mathbb{P}^1})$, is a Lagrangian sphere; moreover, the sphere has image entirely inside $B^4(r) \times \mathbb{P}^1$ if $r > 1$.

Lemma 1. *Let (S, ω_S) be a complex algebraic surface defined over $\mathbb{R}$ and $\phi\colon B^4(r) \hookrightarrow S$ a symplectic embedding of a ball of radius $r > 1$. Then $S \times \mathbb{P}^1$ contains a nullhomologous Lagrangian 3-sphere.*

Proof. Consider the map $\mathbb{S}^3 \to S \times \mathbb{P}^1$ given by composing the above Hopf embedding $\mathbb{S}^3 \hookrightarrow B^4(r) \times \mathbb{P}^1$ with complex conjugation ι on S (which is an antisymplectic involution: $\iota^* \omega_S = -\omega_S$). The composite map is a Lagrangian embedding. □

The Darboux theorem implies that for any symplectic form of large enough volume on S (equivalently any polarisation of sufficiently large degree), the space of symplectic embeddings of the ball $B^4(r)$ is non-empty and connected.

5. We gather some relevant facts about the topology of a space Y obtained by a conifold transition on a single nullhomologous Lagrangian sphere in X. (In a particular case a similar analysis was carried out in [ST03].) Note that Y dominates the singular space X_0 obtained by collapsing the 3-sphere $\mathbb{S}^3 \subset X$ to a point; denote by $\widehat{X}_0$ the complex blow up of X_0 at the node. The exceptional set of the blow up is diffeomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. We have a diagram:



Lemma 2.

(1) We have exact sequences:

$$0 \to H_4(X;\mathbb{Z}) \to H_4(Y;\mathbb{Z}) \to \mathbb{Z} \to 0$$
$$0 \to H^2(X;\mathbb{Z}) \to H^2(Y;\mathbb{Z}) \to \mathbb{Z} \to 0$$

(2) We have exact sequences:

$$0 \to \mathbb{Z} \to H_2(Y;\mathbb{Z}) \to H_2(X;\mathbb{Z}) \to 0$$
$$0 \to \mathbb{Z} \to H^4(Y;\mathbb{Z}) \to H^4(X;\mathbb{Z}) \to 0$$

The first $\mathbb{Z}$ factor is naturally generated by the class of the exceptional 2-sphere.

(3) The classes in $H_2(\widehat{X}_0;\mathbb{Z})$ of the two rulings of the divisor $D \subset \widehat{X}_0$ contracted by p are linearly independent.

(4) $c_1(Y) = c_1(X)$ under the natural inclusion $H^2(X;\mathbb{Z}) \subset H^2(Y;\mathbb{Z})$.

Proof.

(1) The first sequence is from the exact sequence for the pair (X_0, X) where we view $X_0 = X \cup_{\mathbb{S}^3} B^4$. Excision identifies $H_*(X_0, X;\mathbb{Z}) \cong H_*(B^4, \mathbb{S}^3;\mathbb{Z}) \cong \mathbb{Z}$. We have used the identity $q\colon H_4(Y) \cong H_4(X_0)$ which holds since q has fibres of dimension at most 2. The final zero arises since $[\mathbb{S}^3] = 0 \in H_3(X;\mathbb{Z})$. The second sequence is the Poincaré dual of the first.

(2) Regard $X_0 = Y \cup_{\mathbb{P}^1} B^3$ and write the exact sequence for (X_0, Y). Excision gives $H_*(X_0, Y;\mathbb{Z}) \cong H_*(B^3, \mathbb{S}^2;\mathbb{Z})$; the non-trivial identification $f_*\colon H_2(X_0) \cong H_2(X)$ follows from cellular homology. The second sequence is Poincaré dual to the first.

(3) Since $\widehat{X}_0$ is given by blowing up a curve in Y we have $b_2(\widehat{X}_0) = b_2(Y) + 1 = b_2(X_0) + 2$, which implies the result.

(4) This holds since the surgery is local and arises for Calabi-Yau's; we can interchange parallelisable neighbourhoods of the zero-sections of $T^*\mathbb{S}^3$ and $\mathcal{O}(-1)^{\oplus 2}$ compatibly with a fixed trivialisation of the tangent bundle at the common boundary.

□

The tangent bundle TX has a canonical trivialisation up to homotopy in an open neighbourhood of the Lagrangian 3-sphere L, so the Chern classes $c_i(X)$ lift canonically to relative Chern classes $c_i(X, L) \in H^{2i}_c(X \setminus L)$. The natural map induced by inclusion

$$H^4_c(X \setminus L) \to H^4(Y)$$

is an isomorphism, so we can view $c_2(X, L)$ as an element of $H^4(Y)$.

Lemma 3. *In the notation above,* $c_2(Y) = c_2(X, L) - \mathrm{PD}[\mathbb{P}^1]$.

Proof. It is clear that the surgery changes c_2 by a local contribution which is a multiple of the exceptional sphere. Recall from [Fri91] that under a flop $Y \dashrightarrow Y'$ along a rational curve $\mathbb{P}^1 \subset Y$, and for a divisor $D \subset Y$ with proper transform $D' \subset Y'$, there is an identity

$$c_2(Y') \cdot D' = c_2(Y) \cdot D + 2\mathrm{PD}[\mathbb{P}^1] \cdot D. \tag{2}$$

From the local model of section 2 or [STY02], we can pick a local complex surface $D \subset Y$ which is transverse to the $\mathbb{P}^1$ and intersects it positively at a point and whose proper transform $D' \subset Y'$ contains the flopped $\mathbb{P}^1$ with normal bundle $\mathcal{O}(-1)$. In our case, Y is obtained as the conifold transition of X, and the tangent bundle of X is holomorphically trivial in a neighbourhood of the Lagrangian 3-sphere; so we can pick a cycle Σ representing $c_2(X)$ disjoint from the locus where the surgery takes place, and $c_2(Y) = PD[\Sigma + n\mathbb{P}^1]$ for some $n \in \mathbb{Z}$, where $\mathbb{P}^1$ denotes the exceptional 2-sphere. Symmetry (between Y and Y' relative to X) now implies that the only possibility compatible with Equation 2 is that $n = -1$, and that in our notation $c_2(Y) = c_2(X, L) - \mathrm{PD}[\mathbb{P}^1]$. □

From the existence of the surgery as a local operation on Calabi-Yau's, we immediately get:

Corollary 2. *The following are preserved by the surgery:*

- $c_1^2 = 0$ *in* $H^4(\cdot;\mathbb{Z})$ *and the value* $c_1c_2 \in H^6(\cdot;\mathbb{Z}) = \mathbb{Z}$,
- c_1 *is divisible by* p *in integer cohomology mod torsion.*

The homological independence of the classes of the two rulings in the exceptional divisor $D \subset \widehat{X}_0$ has as standard consequence:

Corollary 3. *If* $\widehat{X}_0$ *is projective then the classes of the two rulings of the exceptional divisor* D *contracted by* p *are distinct extremal rays. Hence, the two small resolutions of* X_0 *are also projective.*

Indeed, the contraction $p\colon \widehat{X}_0 \to X_0$ is projective and has relative Picard rank 2 because the two rulings are different in homology. By Mori theory p can be factored in two different ways to two projective small resolutions of X_0.

Finally, we remark that the *ring* structure in cohomology for the conifold transition depends not just on the homology class of the Lagrangian sphere, but on its embedded isotopy class and its Lagrangian framing. (A Lagrangian 3-sphere always has a distinguished framing up to homotopy, given by choosing a compatible almost complex structure on the ambient manifold and using the exponential map as in the proof of the Lagrangian neighbourhood theorem, cf. [MS98].) This is familiar from classical surgery theory.

6. We now prove Theorem 1. Take a projective Enriques surface E defined over $\mathbb{R}$ and construct the Lagrangian sphere $\mathbb{S}^3 \subset E \times \mathbb{P}^1$ as above. This sphere is nullhomologous, so the symplectic conifold transition Z of $E\times\mathbb{P}^1$ exists [STY02]. We suppose for contradiction that Z admits a Kähler structure. We will need the following facts about Z:

- $\pi_1(Z) = \mathbb{Z}/2\mathbb{Z}$. For clearly the surgery does not affect the fundamental group.
- $c_1^2(Z) = 0$. For (suppressing Poincaré Duality) $c_1(E\times\mathbb{P}^1) = 2E\times\{pt\} + c_1(E)\times\mathbb{P}^1$, which has square zero, using $c_1^2(E) = 0$, $2c_1(E) = 0$. Now use Corollary 2.
- $c_1c_2(Z) = 24$. Indeed this holds for $E \times \mathbb{P}^1$; for example by Riemann-Roch
$$1 = \chi(\mathcal{O}_{E\times\mathbb{P}^1}) = \frac{c_1c_2}{24}$$
or by an explicit calculation of Chern classes. Now use Corollary 2.

That $c_1^2 = 0$ is critical in what follows. The reader can supply a more explicit proof by fixing a 3-chain in E bounding a surface $C_1 \cup C_2$, with each C_i representing $c_1(E)$, and constructing the Lagrangian sphere by starting with a ball in E disjoint from this 3-chain.

Lemma 4. *The Kähler manifold Z is projective.*

Proof. The odd de Rham cohomology groups $H^{2i+1}_{dR}(Z)$ all vanish. Hence, by Hodge theory, $H^1(\mathcal{O}_Z) = H^3(\mathcal{O}_Z) = 0$. Riemann-Roch then gives $h^0(\mathcal{O}_Z) + h^2(\mathcal{O}_Z) = (c_1c_2)/24 = 1$ which implies $h^2(\mathcal{O}_Z) = 0$. Hodge theory now says $H^2_{dR}(Z) = H^{1,1}(Z)$ and hence there are rational Kähler forms. □

Remark 1. For $\mathrm{K3}\times\mathbb{P}^1$, $c_1c_2 = 48$ and the above argument fails and we cannot conclude that the conifold transition is projective. This is the only point at which the analogous proof does not go through. The projectivity of Z is needed in the argument below where we use the theory of extremal rays. If this theory applied to Kähler 3-folds, the argument below would also show that the transition of $\mathrm{K3} \times \mathbb{P}^1$ is not Kähler. For recent results on Mori theory for Kähler 3-folds see [Pe01].

We may now apply Mori's classification of extremal rays, Theorem 2. The space Z cannot be Fano, since $c_1^2 = 0$. A deep theorem of Miyaoka [Miy85] states that $-c_1^2 + 3c_2$ is pseudoeffective. When $K_Z = -c_1$ is nef, this implies that

$$\chi(\mathcal{O}_Z) = \frac{c_1c_2}{24} = K_Z\frac{-c_2}{24} \leq -\frac{K_Z^3}{72} \leq 0.$$

We have $\chi(\mathcal{O}_Z) = 1$, hence there must be some extremal contraction.

(i) For all but the first of the divisorial contractions, there is a rational curve $\mathbb{P}^1 \subset D$ for which $c_1(Z)\cdot\mathbb{P}^1 = 1$. For us $c_1(E\times\mathbb{P}^1)$ is divisible by 2 in integral cohomology mod torsion (since $c_1(E)$ is torsion), hence by Corollary 2 $c_1(Z)$ is also divisible by 2 in integral cohomology mod torsion, therefore no such rational curve can exist. For the first on the list of divisorial contractions, $c_1(Z)^2 \cdot D = 4 \Rightarrow c_1^2(Z) \neq 0$. Hence Z cannot admit any divisorial contraction.

(ii) We now consider the fibring contractions. If $Z \to C$ fibres over a smooth curve with del Pezzo fibres, then either $g(C) = 0 \Rightarrow \pi_1(Z) = 0$ or $g(C) \geq 1 \Rightarrow \pi_1(Z) \twoheadrightarrow \pi_1(C)$, neither of which are compatible with $\pi_1(Z) = \mathbb{Z}/2\mathbb{Z}$. Alternatively, if a del Pezzo fibration $Z \to C$ arises as an extremal contraction, then the Picard rank $\rho(Z) = 2$; in our case as we already remarked $\rho(Z) = h^{1,1}(Z) = b^2(Z) = b^2(E\times\mathbb{P}^1) + 1 = b^2(E) + 2 = 12$, a contradiction. Hence the fibring contraction must be a conic bundle over a surface.

If there are any singular fibres in the conic bundle, there is again a component of some fibre on which c_1 evaluates to give 1, giving a contradiction as above. Hence Z is the total space of a $\mathbb{P}^1$-bundle over a smooth complex surface S which must have $\pi_1(S) = \mathbb{Z}/2\mathbb{Z}$. A general identity for conic bundles asserts that

$$-\pi_* K_Z^2 = 4K_S + \Delta$$

where Δ denotes the locus in S over which the fibres are singular; in our case, this immediately implies $4K_S = 0$. (Alternatively, vanishing of $H^3(Z)$ and hence the topological Brauer class of the $\mathbb{P}^1$-bundle implies it is the projectivisation of a rank two complex topological vector bundle $V \to Z$; the usual presentation of the cohomology ring of $\mathbb{P}(V)$ again shows $4c_1(S) = 0$.) Because $\pi_1(S) = \pi_1(Z) = \mathbb{Z}/2$, and $c_1(S)$ is torsion, the classification of complex surfaces forces S to be an Enriques surface. In that case $b_2(Z) = b_2(S) + 1 = 11$, but in fact $b_2(Z) = b_2(E \times \mathbb{P}^1) + 1 = 12$, which is a contradiction. This excludes all possible extremal rays, and finishes the proof of Theorem 1.

7. Perhaps surprisingly, there are nodal projective degenerations of 3-folds of the form $S \times \mathbb{P}^1$ with nullhomologous vanishing cycles of the form constructed in section 4. The following example was suggested to us by János Kollár.

Proposition 1. *There is a projective degeneration, with smooth total space, of $\mathbb{P}^2 \times \mathbb{P}^1$ to a variety with a single node.*

Proof. We work inside the total space of the scroll $\mathbb{F}$ given by quotienting $\mathbb{C}^2\backslash\{0\} \times \mathbb{C}^4\backslash\{0\}$ by the $\mathbb{C}^* \times \mathbb{C}^*$ action whose weights are:

$$\begin{pmatrix} 1 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

For background and notation on scrolls see [Re97]. This is a $\mathbb{P}^3$-bundle over $\mathbb{P}^1$. Fix coordinates s_0, s_1 on the base $\mathbb{P}^1$ and x_0, x_1, x_2, x_3 on the fibre, and consider the hypersurface

$$\mathcal{X} = \{tx_0 + s_0x_1 + s_1x_2 = 0\}$$

inside $\mathbb{F} \times \Delta$, where $t \in \Delta$ is a coordinate on the disc. One can easily check that the fibre of this is $\mathbb{P}^2 \times \mathbb{P}^1$ for $t \neq 0$ but $\mathbb{P}(F)$ at $t = 0$, where F is the total space of the bundle $\mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2)$ over $\mathbb{P}^1$. The curve $t = 0 = x_1 = x_2 = x_3$ has normal bundle $\mathcal{O}(-1)^{\oplus 3}$ inside $\mathcal{X}$, and can be "antiflipped" (blown up and blown down). As in the Appendix to [BCZ03], the antiflip is effected by changing the linearisation and passing to the scroll $\mathbb{F}'$ defined by quotienting $\mathbb{C}^3\backslash\{0\} \times \mathbb{C}^3\backslash\{0\}$ by the $\mathbb{C}^* \times \mathbb{C}^*$ action with the same weights. In the new scroll $\mathbb{F}'$, the antiflip is given by exactly the same equation, but now there is an affine chart $x_0 = x_3 = 1$ in which the zero-fibre $t = 0$ has equation

$$\{s_0x_1 + s_1x_2 = 0\} \subset \mathbb{C}^3.$$

In particular, the zero-fibre of the new pencil has a single node, as required. □

As an aside, we remark that one can also obtain the total space of the degeneration from $\mathbb{P}^2 \times \mathbb{P}^1 \times \Delta$ by a sequence of birational transformations. First blow up a curve in the zero-fibre with normal bundle $\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$ and then contract the total transform of $\mathbb{P}^2 \times \mathbb{P}^1 \times \{0\}$. This gives a family with generic fibre $\mathbb{P}^2 \times \mathbb{P}^1$ and special fibre $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2))$ and containing a curve with normal bundle $\mathcal{O}(-1)^{\oplus 3}$, and we then antiflip this as before.

From a symplectic perspective, the degeneration above gives rise to a Lagrangian vanishing cycle in $(\mathbb{P}^2 \times \mathbb{P}^1, \Omega_t)$ for any $\Omega_t = \omega_{\mathbb{P}^1} \oplus t\omega_{\mathbb{P}^2}$ with $t \gg 0$.

Corollary 4. *The vanishing cycle of the degeneration above is Lagrangian isotopic to the Lagrangian sphere constructed in section 4.*

Proof. Following an argument of Seidel from [Sei03], we can modify the Kähler potential and hence Kähler form to begin with the restriction of the standard flat form from $\mathbb{C}^3$ near the node. In these co-ordinates, the vanishing cycle is given in a nearby smooth fibre $t = \delta$ by

$$s_0 + x_1 \in \mathbb{R}, s_0 - x_1 \in i\mathbb{R}; \quad s_1 + x_2 \in \mathbb{R}, s_1 - x_2 \in i\mathbb{R}; \quad s_0x_1 + s_1x_2 = \delta \in \mathbb{R}_+.$$

The conditions imply $s_0 = \bar{x}_1$ and $s_1 = \bar{x}_2$. In the original family, before antiflipping, $[s_0; s_1]$ define projective co-ordinates on $\mathbb{P}^1$ and the sphere is given by taking the radius $\sqrt{\delta}$-circles in the complex conjugate of the tautological line bundle over $\mathbb{P}^1$, which co-incides with the construction of section 4. □

The sphere of section 4 can be constructed in $(\mathbb{P}^2 \times \mathbb{P}^1, \Omega_t)$ for any $t > 1$, but does not exist if $t \leq 1$. Correspondingly, for non-trivial $\mathbb{P}^2$-bundles over $\mathbb{P}^1$ (note that $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(a) \oplus \mathcal{O}(b))$ is diffeomorphic to $\mathbb{P}^2 \times \mathbb{P}^1$ if $a + b \equiv 0 \bmod 3$) all polarisations give the $\mathbb{P}^1$-factor larger size, and we expect that there are *no* projective degenerations to a nodal variety. In these examples, symplectic and algebraic geometry seem to match up very closely. By analogy with known results for $\mathbb{P}^1 \times \mathbb{P}^1$ [Hi03], it would be natural to conjecture that *every* Lagrangian sphere in $(\mathbb{P}^2 \times \mathbb{P}^1, \Omega_{t>1})$ is Lagrangian isotopic to the sphere constructed in section 4, even though there are probably many different projective degenerations of $\mathbb{P}^2 \times \mathbb{P}^1$ to a nodal variety.

Rebecca Barlow [Ba85] constructed a simply connected surface of general type S with $p_g = 0$ and $K^2 = 1$. Such a surface is necessarily homeomorphic to a blow up X of $\mathbb{P}^2$ in 8 points, and $S \times \mathbb{P}^1$ and $X \times \mathbb{P}^1$ are in fact diffeomorphic. In the light of the above example, it would be interesting to study conifold transitions of $S \times \mathbb{P}^1$.

8. We now give the derivation of Corollary 1 from Theorem 1. Suppose we have a smooth Kähler manifold $\mathcal{X} \to D$ which fibres over the disc with generic fibre $E \times \mathbb{P}^1$ and with central fibre having a node, in which a nullhomologous Lagrangian 3-sphere has been collapsed. Grauert's theorem [BS76] asserts that the Euler characteristic $\chi(\mathcal{O}_{X_t})$ is constant in t, and a standard result [Ko95] asserts that rational Gorenstein singularities are Du Bois, hence in our case the canonical map $H^i(X, \mathbb{C}) \to H^i(X, \mathcal{O}_X)$ is surjective. Since the odd topological cohomology of X_0 is unchanged from that of $X_t = E \times \mathbb{P}^1$, we deduce that the central fibre has $H^2(X_0, \mathcal{O}_{X_0}) = H^2(X_t, \mathcal{O}_{X_t})$. The central fibre X_0 can be resolved by a single blowup $f \colon \widehat{X}_0 \to X_0$ of the node. Since X_0 has rational singularities, $R^i f_* \mathcal{O} = (0)$ for $i > 0$, therefore $H^2(\mathcal{O}_{\widehat{X}_0}) = (0)$. By Hodge theory $\widehat{X}_0$ has rational Kähler forms, hence it is projective.

Corollary 3 now implies that the small resolutions are projective, contradicting Theorem 1.

Remark 2. The implication that the two small resolutions are projective when the vanishing cycle is nullhomologous was made by Clemens in a slightly different context [Cle83].

9. Because the degeneration of $\mathbb{P}^2 \times \mathbb{P}^1$ constructed in section 7 extends to a family over $\mathbb{P}^1$, the monodromy around the singular fibre – that is, the Dehn twist τ_L in the vanishing cycle L – is symplectically isotopic to the identity. By contrast, the Dehn twist in any essential 3-sphere is differentiably of infinite order (it acts by an infinite order transvection on homology). This in turn has nontrivial consequences for the differential and symplectic topology of the total space: we end with one simple illustration of this, although it is something of a digression from the main theme.

For $t \gg 0$, let $\mathcal{E}_{\Omega_t}(\Sigma, \mathbb{P}^2 \times \mathbb{P}^1)$ denote the path-component of the space of symplectic embeddings of $\Sigma = \mathbb{S}^2 \amalg \mathbb{S}^2$ into $(\mathbb{P}^2 \times \mathbb{P}^1, \Omega_t)$ which contains the inclusion ϕ of $\{0, \infty\} \times \mathbb{P}^1$. Here $\infty \in \mathbb{P}^2$ is any fixed point lying on the line at infinity away from $\mathbb{C}^2 \subset \mathbb{P}^2$. Note that $0, \infty$ are *separated* by the projection $\pi(L)$ of the Lagrangian L to $\mathbb{P}^2$ in the sense that any arc joining $0, \infty$ meets $\pi(L)$ an odd number of times; for, by Corollary 4, the vanishing cycle L can be taken to project to the unit sphere in $\mathbb{C}^2 \subset \mathbb{P}^2$. After generic perturbation one can suppose that $L \cap \mathrm{im}\,(\phi) = \emptyset$ and hence that $\mathrm{supp}(\tau_L) \cap \mathrm{im}\,(\phi) = \emptyset$; then a choice of isotopy $u = (u_t)$ from τ_L to the identity gives a loop $\gamma_u \subset \mathcal{E}_{\Omega_t}(\Sigma, \mathbb{P}^2 \times \mathbb{P}^1)$.

Proposition 2. *The loop γ_u above is homotopically essential. Indeed, it defines a non-trivial element of the kernel of the canonical map induced by inclusion:*

$$\pi_1 \mathcal{E}_{C^\infty}(\Sigma, \mathbb{P}^2 \times \mathbb{P}^1) \to \pi_1 \mathcal{E}_{C^\infty}(\mathbb{S}^2, \mathbb{P}^2 \times \mathbb{P}^1) \times \pi_1 \mathcal{E}_{C^\infty}(\mathbb{S}^2, \mathbb{P}^2 \times \mathbb{P}^1).$$

Proof. If γ_u was inessential, we could find a two-parameter family of embeddings of Σ into $\mathbb{P}^2 \times \mathbb{P}^1$ extending γ_u. In this case, by deforming the given isotopy (u_t) by the flow along the obvious family of vector fields, we could in fact suppose that all the maps u_t fixed $\{0, \infty\} \times \mathbb{P}^1$. At the cost of replacing diffeomorphisms by homotopy self-equivalences, we could in fact find a homotopy from τ_L to the identity through maps which were the identity in a neighbourhood of these two spheres. Gluing such neighbourhoods together, we obtain a homotopy from $\hat{\tau}_L$ to the identity, where $\hat{\tau}_L$ is the Dehn twist along L in the space obtained from the gluing. By the choice of $0, \infty$ as separated by L, however, this space is exactly $W \times P^1$ where W is given by adding a 1-handle to $\mathbb{P}^2$, the 1-handle constructed such that in $W \times \mathbb{P}^1$ the sphere L is homologically essential. This gives a contradiction.

The refined statement about γ viewed as a loop of smooth embeddings relies on some results in surgery theory. First, for any space N and $f : \mathbb{S}^k \to N$, there is an exact sequence [BH82]

$$\pi_2(N) \to \pi_{k+1}(N) \to \pi_1(\mathrm{Maps}(\mathbb{S}^k, N); f) \to \pi_1(N)$$

where the first map is Whitehead product with $[f]$ and the last map is restriction to a base-point. If $N = \mathbb{P}^2 \times \mathbb{P}^1$ and $[f] = \{pt\} \times \mathbb{P}^1$, then the first map in the sequence reduces to the map $\pi_2(\mathbb{S}^2) \to \pi_3(\mathbb{S}^2)$ given by Whitehead product with the generator, which is an isomorphism. Hence $\pi_1(\text{Maps}) = \{0\}$.

Results of Lashof [Las76] and Robinson [Rob73] show that, for mappings of *connected* surfaces into six-manifolds,

- $\pi_2(\text{Maps}, \text{Emb}_{C^0}) = 0$;
- $\pi_i(\text{Emb}_{C^\infty}) \to \pi_i(\text{Emb}_{C^0})$ is an isomorphism for $i = 1$ and is surjective for $i = 2$ (this relies on a homotopy computation due to Milgram [Mil72]).

We have an exact sequence

$$\pi_2(\text{Emb}_{C^0}, \text{Emb}_{C^\infty}) \to \pi_2(\text{Maps}, \text{Emb}_{C^\infty}) \to \pi_2(\text{Maps}, \text{Emb}_{C^0})$$

and the quoted theorems imply that the first and third terms both vanish. Therefore the map $\pi_1(\text{Emb}_{C^\infty}) \to \pi_1(\text{Maps})$ is injective, and the nullhomotopy of the loop $\{u_t(\mathbb{S}^2)\}$ of embeddings of either component $\mathbb{S}^2 \subset \Sigma$ as a family of maps can be realised as a nullhomotopy in the space of embeddings. (The critical role of connectedness in Lashof's theorem is illustrated by the triviality of the isotopy class of τ_L.) □

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