Mathematical Research Letters **12**, 817–820 (2005)

ADDENDUM TO THE PAPER AFFINELY INFINITELY DIVISIBLE DISTRIBUTIONS AND THE EMBEDDING PROBLEM

S.G. Dani and Klaus Schmidt

Abstract. In our paper [5], in proving the general case of our theorem, a result from [3] on embedding of infinitely divisible measures on certain Lie groups with compact center was used. An error has been found in the proof in [3]. In this context we show in this note that the proof of the theorem in [5] can be completed without recourse to the result from [3].

1. Introduction

Let A be a locally compact abelian group and let $P(A)$ denote the semigroup of probability measures on A, with the convolution product. Given $\mu \in P(A)$, a $\lambda \in P(A)$ is said to be an *affine k-th root* of μ (where k is any natural number) if there exists a continuous automorphism ρ of A such that $\rho^k = I$ (the identity transformation) and $\lambda * \rho(\lambda) * \rho^2(\lambda) * \cdots * \rho^{k-1}(\lambda) = \mu$, and μ is said to be *affinely infinitely divisible* (on A) if it has affine k-th roots for all k. We recall also that $\mu \in P(A)$ is said to be *infinitely divisible* if, for every natural number k, μ admits a k-th (convolution) root. The following is the main theorem from [5]:

Theorem 1.1. *Every affinely infinitely divisible probability measure on a connected abelian Lie group A is infinitely divisible on A.*

In [5], after various preparatory results, the theorem is first proved for $A = \mathbb{R}^n$ for any n, and then for a general A as above, namely $A = \mathbb{T}^m \times \mathbb{R}^n$ for some m and n. In the proof of the general case a theorem from [3] on the embeddability of infinitely divisible probability measures on a class of Lie groups with compact (nontrivial) center is used. It turns out that the proof in [3] has an error; see [4] for details. In this context we describe here a modified proof of Theorem 1.1 as above.

2. Proof of the theorem

As in [5] let S be the maximal torus in A, B the subgroup of A containing S and such that B/S is the vector subspace of A/S spanned by $(\mathrm{supp}\mu)S/S$, and V a vector subgroup of B such that B is the direct product of S and V.

Received by the editors June 17, 2005.

Let Γ be the group of automorphisms of B acting trivially on S, Θ the subgroup of Γ consisting of automorphisms whose factor action on B/S is trivial, and Δ the subgroup of Γ consisting of automorphisms leaving V invariant. Then the arguments in [5], until the penultimate paragraph of the proof show that there exists a compact subgroup K of Δ such that μ is infinitely divisible on $B\Theta K$. From this point the next step is to prove that there exists a periodic one-parameter subgroup ϕ of ΘK such that μ is infinitely divisible on $B\phi$; this would enable, together with Corollary 4.2 of [5] to conclude that μ is infinitely divisible on B. To achieve this, in [5] we had appealed to a result from [3] on the embeddability of infinitely divisible measures on groups of the form $B\Theta K$ as above, but the proof of that result is found to have an error.

We shall therefore now proceed as follows. Let M be a minimal closed subgroup of ΘK of the form UC with U a vector subspace of Θ, and C a compact subgroup of ΘK (not necessarily contained in K), such that μ is infinitely divisible on $BM = BUC$; such a subgroup exists, by considerations of dimension and the number of connected components. If M^0 is the connected component of the identity in M then BM^0 is a subgroup of finite index in BM, and an argument as in the penultimate paragraph of [5] shows that μ, which is infinitely divisible on BM, would also be infinitely divisible on BM^0. The minimality condition on M therefore shows that $M^0 = M$, namely M is connected. Hence C is also connected.

Now let H be the subgroup of M consisting of all elements whose action on B leaves μ invariant. Then H is a closed subgroup, and by Lemma 2.2 of [5] μ is infinitely divisible on BH. A priori one does not know at this stage whether H is a semidirect product of a subspace of Θ with a compact subgroup, so one can not conclude immediately that $H = M$. We shall however show that H is compact, and hence $H = M = C$.

We note firstly that if Q is a closed subgroup of M such that every element x of M which is of finite order can be expressed as hyh^{-1} for some $h \in H$ and $y \in Q$, then μ is infinitely divisible on BQ. This may be seen as follows: Let k be any natural number, and ν be a k-th root of μ on BM. Then it has the form $\nu = \lambda x$ where λ is a probability measure on B and $x \in M$ is such that $x^k = e$, the identity element; see [5]. Now let x be expressed as hyh^{-1}, with $h \in H$ and $y \in Q$ as above. Then $(h^{-1}\nu h)^k = h^{-1}\mu h = \mu$, since μ is h-invariant. Thus $h^{-1}\nu h$ is a k-th root of μ. On the other hand, $h^{-1}\nu h = (h^{-1}\lambda h)(h^{-1}xh) = (h^{-1}\lambda h)y$, so its support is contained in BQ. This shows that μ is infinitely divisible on BQ, as claimed.

We now return to the subgroup $M = UC$ as above. The vector subspace U can be decomposed under the conjugation action of C as $U_0 \oplus U_1$ such that U_0 is pointwise fixed and U_1 contains no nonzero fixed points. Then M is a direct product of $M_1 = U_1C$ and U_0. Since $\mathrm{supp}\mu \subset B \subset BM_1$ and BM/BM_1 is a vector group, it follows that for every root λ of μ on BM, $\mathrm{supp}\lambda$ is contained in BM_1, and hence that μ is infinitely divisible on BM_1. By the minimality of M

we get therefore that $M_1 = M$; thus U_0 is trivial and the action of C on U has no nonzero fixed point.

Consider now the subgroup $\overline{UH}$ (the closure of UH). It is of the form UC' for some compact subgroup C' of C, and so by the minimality condition on M we get that $M = \overline{UH}$. We note that $H \cap U$ is normalised by H and U, and hence the preceding conclusion implies that it is a normal subgroup of M. Let $W = H \cap U$. Then W can be expressed as a direct product of its identity component W^0 with a discrete subgroup D which is invariant under the action of C. Since C is connected and its action on U has no nontrivial fixed point, it follows that D is trivial, and hence W is a vector subspace of U. We can now express U as $U = W \oplus W'$ where W' is a C-invariant subspace of U. It can be verified, using elementary linear algebra, that if τ is an affine automorphism of W of the form $w \mapsto \sigma(w) + w_0$ for all $w \in W$, where σ is an automorphism of W and $w_0 \in W$, and if τ is of finite order then τ and σ are conjugate as affine automorphisms, by a translation from W. Using this we see that every element x of UC which has finite order can be expressed as hyh^{-1}, with $h \in W \subset H$ and $y \in W'C$. Therefore by the remark above μ is infinitely divisible on $BW'C$, and hence by the minimality condition on M we have $M = W'C$. Thus, in the notation as above, $H \cap U$ is trivial.

Let R be the (solvable) radical of (the connected Lie group) M and H^0 be the connected component of the identity in H. Since R contains U, H^0R is normalised by U. It is also normalised by H, and since UH is dense in M it follows that H^0R is a normal Lie subgroup of M. Since M/R is a semisimple Lie group this implies that H^0R/R is closed, and furthermore M/R can be expressed as $M_1(H^0R/R)$, where M_1 is a compact connected normal subgroup of M/R such that $M_1 \cap (H^0R/R)$ is finite. Let T be a maximal torus in the compact group H^0R/R and let M' be the closed subgroup of M containing R and such that $M'/R = M_1T$. By the conjugacy of maximal tori (see [6], Chapter 5, Theorem 15) in H^0R/R we get that every x in M can be expressed as hyh^{-1} for some $h \in H^0$, and $y \in M'$. Therefore, by our observation above, μ is infinitely divisible on BM', and hence by the minimality condition on M we have $M' = M$. Thus $M/R = M'/R = M_1T$, and since M/R is semisimple we see that T must be trivial. Therefore H^0 is a solvable Lie group.

Let P be the connected component of the identity in $\overline{HR}$. Since H^0 is solvable, by a theorem of L. Auslander (see [7], Theorem 8.2.4) P is solvable. As the subgroup P is normalised by UH and as the latter is dense in M, it follows that P is normal in M. As M/R is a semisimple Lie group and P/R is a connected solvable normal subgroup, it follows that $P = R$. This implies that HR is closed and R is open in HR. Also, as R contains U, HR has the form UC' for some compact subgroup C' of C. Since μ is infinitely divisible on $BH \subset BHR$, the minimality condition on M now implies that $M = HR$. Also, since M is connected and R is open in HR we further get that $M = R$. Thus M is solvable, and hence the compact connected subgroup C is abelian. Since $H \cap U$ is trivial this further implies that H is abelian.

Now let $p : M \to C$ be the canonical projection homomorphism, and $H' = p(H)$. Then H' is a dense subgroup of C. Since C is an abelian group and its action on U has no nonzero fixed point, it follows that the set of elements of C whose action on U admits a nonzero fixed point is a proper closed subset of C. Therefore there exists $h' \in H'$ whose action on U has no nonzero fixed point. Let $h \in H$ be such that $p(h) = h'$. Then there exists a $u \in U$ such that $uhu^{-1} = h'$. The centraliser of h' in M is compact and hence the preceding conclusion implies that the centraliser of h in M is compact. As H is abelian this shows that H is compact. As μ is infinitely divisible on BH the minimality condition on M now implies that $H = M = C$.

Since C is compact there exists a vector subgroup V of B such that V is invariant under the action of C and $B = SV$, a direct product. Hence BC is a direct product of S and VC, which shows in particular that it is a linear Lie group, namely a Lie group with a faithful finite-dimensional representation. Therefore by the general embedding theorem in [2] we get that μ, which is infinitely divisible on BC, is embeddable on BC; the group involved here being a direct product of a group of rigid motions and a compact abelian group, embeddability in this case can also be obtained along the lines of the (simpler) proof in [1] for measures on the group of affine automorphisms of $\mathbb{R}^n$, $n \geq 1$.

As in the argument in [5] for the vector group case we now deduce, from the embeddability of μ on BC, that there exists a periodic one-parameter subgroup ϕ of C such that μ is infinitely divisible (in fact embeddable) on $B\phi$. Then by Corollary 4.2 of [5] μ is infinitely divisible on B; this proves the theorem.

References

[1] S.G. Dani and M. McCrudden, *Embedding infinitely divisible probability measures on the affine group*, in: Probability Measures on Groups IX (Ed: H. Heyer), (Proceedings of a conf.: Oberwolfach, 1988), Lect. Notes in Math. 1379, Springer Verlag, 1989, pp. 36–49.

[2] ———, *Embeddability of infinitely divisible distributions on linear Lie groups*, Invent. Math. **110** (1992), 237–261.

[3] S.G. Dani, M. McCrudden and S. Walker, *On the embedding problem for infinitely divisible distributions on certain Lie groups with toral center*, Math. Zeits. **245** (2003), 781–790.

[4] ———, *Erratum to our paper "On the embedding problem for infinitely divisible distributions on certain Lie groups with toral center*, Math. Zeits. **245** (2003), 781–790.", Math. Zeits., to appear.

[5] S.G. Dani and Klaus Schmidt, *Affinely infinitely divisible distributions and the embedding problem*, Math. Res. Lett. **9** (2002), 607–620.

[6] A.V. Onishchik and E.B. Vinberg, Lie Groups and Algebraic Groups, Springer, 1990.

[7] M.S. Raghunathan, Discrete Subgroups of Lie Groups, Springer, 1972.

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Mumbai 400 005, India

E-mail address: dani@math.tifr.res.in

Mathematics Institute, University of Vienna, Nordbergstrasse 15, A-1090 Vienna, Austria and Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, A-1090 Vienna, Austria

E-mail address: klaus.schmidt@univie.ac.at