

NORMALIZATION OF IDEALS AND BRIANÇON–SKODA NUMBERS

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ABSTRACT. We establish bounds for the coefficient $\bar{e}_1(I)$ of the Hilbert function of the integral closure filtration of equimultiple ideals. These values are shown to help control all algorithmic processes of normalization that make use of extensions satisfying the condition S_2 of Serre.

1. Introduction

Let R be a Noetherian ring and let I be an R -ideal. The *integral closure* of I is the ideal \bar{I} consisting of all $z \in R$ which are solutions of equations of the form

$$z^n + a_1 z^{n-1} + \dots + a_n = 0, \quad a_i \in I^i.$$

The authors are not aware of any direct algorithm that builds \bar{I} from I , a situation that is aggravated by the lack of numerical measures to distinguish between the two ideals. A *non direct* construction of the integral closure of an ideal passes through the integral closure $\overline{R[It]}$ in $R[t]$ of the Rees algebra $R[It]$. In fact \bar{I} is the degree one component of $\overline{R[It]}$,

$$I \rightsquigarrow \overline{R[It]} = R \oplus \overline{It} \oplus \overline{I^2 t^2} \oplus \dots \rightsquigarrow \bar{I}.$$

This begs the issue since the construction of $\overline{R[It]}$ takes place in a much larger setting, while in a *direct* construction $I \rightsquigarrow \bar{I}$ the steps of the algorithm would take place entirely in R .

We refer to $\overline{R[It]}$ as the *normalization* of I . Its construction is a standard step in the theory of desingularization. It is very significant that there are numerical measures to tell the two algebras $R[It]$ and $\overline{R[It]}$ apart for classes of ideals of interest. We are going to identify one such measure, establish bounds for its value on $\overline{R[It]}$ by data on I , and show how it bounds the number of iterations of any algorithm that builds $\overline{R[It]}$ by a succession of graded extensions

$$R[It] \rightarrow A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_n = \overline{R[It]}$$

satisfying the condition S_2 of Serre. Recall that if one chooses A_1 to be the S_2 -ification of $R[It]$, then the algorithm of [25] indeed produces such chains of

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S_2 algebras, provided R is a S_2 domain of dimension ≥ 2 and is essentially of finite type over a field of characteristic zero.

We now describe the results of the paper. Let (R, \mathfrak{m}) be an analytically unramified local Cohen–Macaulay ring of dimension $d \geq 2$ and type t with infinite residue field, I an \mathfrak{m} –primary ideal, and $R[It] \subset A \subset B \subset \overline{R[It]}$ inclusions of graded R –algebras. With $e_1(A)$ denoting the first Hilbert coefficient of the ideal filtration $\{A_n\}$ arising from A , we prove in Theorem 2.2 that $e_1(A) < e_1(B)$, provided A satisfies S_2 and is properly contained in B . The monotonicity of the function $e_1(-)$ on these algebras yields the crucial role of $\bar{e}_1(I) = e_1(\overline{R[It]})$ in a numerical criterion of normality. When R has a canonical module the S_2 –ification of $R[It]$ is given by $\text{End}_{R[It]}(\omega_{R[It]})$, which is relatively easy to compute. Now Corollary 2.5 says that $\overline{R[It]}$ can be obtained as the S_2 –ification of $R[It]$ if and only if $\bar{e}_1(I) = e_1(I)$; in particular I is normal if and only if $R[It]$ satisfies S_2 and $\bar{e}_1(I) = e_1(I)$. In general $\bar{e}_1(I)$ can be used to bound the lengths of chains of graded S_2 algebras lying between $R[It]$ and $\overline{R[It]}$, see Corollary 2.4.

Thus one is led to search for effective upper bounds on $\bar{e}_1(I)$. Notice that any such inequality also bounds the first Hilbert coefficient $e_1(I)$, an issue that has been addressed in [12, 6, 7, 23] for instance. The bounds we are looking for should estimate $\bar{e}_1(I)$ in terms of the multiplicity $e_0(I)$ of the ideal I . The link between these two Hilbert coefficients is provided by the *Briançon–Skoda number* $b(I)$ of I , which is the smallest integer b such that $\overline{I^{n+b}} \subset J^n$ for every n and every reduction J of I . Indeed, in Theorem 3.2(c) we prove that

$$\bar{e}_1(I) \leq b(I) \min \left\{ \frac{t}{t+1} e_0(I), e_0(I) - \lambda(R/\bar{I}) \right\},$$

where $\lambda(-)$ denotes length. Furthermore, motivated by this result we estimate the Briançon–Skoda number of I in Proposition 3.7. In a regular local ring the above inequality reads

$$\bar{e}_1(I) \leq (d-1) \min \left\{ \frac{e_0(I)}{2}, e_0(I) - \lambda(R/\bar{I}) \right\},$$

since in this case $b(I) \leq d-1$ by the classical Briançon–Skoda theorem. If in addition $d=2$ and I is integrally closed we obtain the equalities $\bar{e}_1(I) = e_1(I) = e_0(I) - \lambda(R/I)$, which in turn imply the well-known facts that I has reduction number at most one and $R[It]$ is Cohen–Macaulay and normal.

In Section 3 we also establish bounds for $\bar{e}_1(I)$ that avoid any reference to the Briançon–Skoda number and instead only involve the multiplicities of I and of I modulo an element in the Jacobian of R . Our proofs are based on a general Briançon–Skoda type theorem due to Hochster and Huneke that applies to non regular rings as well. In Theorem 3.2(a),(b) we show that if R is an algebra essentially of finite type over a perfect field k and δ is a non zerodivisor in $\text{Jac}_k(R)$, then

$$\bar{e}_1(I) \leq \frac{t}{t+1} \left[(d-1)e_0(I) + e_0(I + \delta R/\delta R) \right]$$

and

$$\bar{e}_1(I) \leq (d - 1)[e_0(I) - \lambda(R/\bar{I})] + e_0(I + \delta R/\delta R).$$

In Section 4 we extend these results to arbitrary equimultiple ideals.

2. Normalization of Rees algebras

The computation (and its control) of the integral closure of a standard graded algebra over a field benefits greatly from Noether normalizations and of the structures built upon them. If $A = R[It]$ is the Rees algebra of an ideal I in a Noetherian ring R , it does not allow for many such constructions. We would still like to develop some tracking of the complexity of the task required to build \bar{A} (assumed A -finite) through sequences of graded extensions

$$A \rightarrow A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_n = \bar{A}$$

where A_{i+1} is obtained from a specific procedure applied to A_i . As in [26], if the A_i satisfy the condition S_2 of Serre, we will call such chains *divisorial*. At a minimum, we would want to bound the length of divisorial chains. In this section we show how this can be realized for Rees algebras of ideals.

We now review some definitions and basic facts. For ideals $J \subset I$ in a Noetherian ring one says that J is a *reduction* of I if $\bar{I} = \bar{J}$. If in addition R is local with infinite residue field, we define *minimal* reductions of I to be reductions minimal with respect to inclusion. The minimal number of generators of every minimal reduction of I is the *analytic spread* of I , which is bounded below by the height of the ideal I and above by the dimension of the ring R . Thus if I is an \mathfrak{m} -primary ideal every minimal reduction of I is generated by $\dim R$ elements. Finally, we say that I is *equimultiple* if every minimal reduction of I is generated by $\text{ht } I$ elements.

Let (R, \mathfrak{m}) be a Noetherian local ring of dimension $d > 0$ and let I be an \mathfrak{m} -primary ideal. Let $D = \bigoplus_{n \geq 0} D_n t^n$ be a graded R -subalgebra of $R[t]$ with $R[It] \subset D \subset R[t]$ and assume that D is a finite $R[It]$ -module. For any such algebra we consider the Hilbert-Samuel function $\lambda(R/D_n)$. For $n \gg 0$ this function is given by the Hilbert-Samuel polynomial

$$e_0(D) \binom{n + d - 1}{d} - e_1(D) \binom{n + d - 2}{d - 1} + \text{lower terms}.$$

Notice that $e_i(R[It])$ coincide with the usual *Hilbert coefficients* $e_i(I)$ of I . Furthermore $e_0(D) = e_0(I)$. By $\overline{R[It]}$ we will always denote the integral closure of $R[It]$ in $R[t]$. We write $\bar{e}_i(I)$ for the *normalized Hilbert coefficients* $e_i(\overline{R[It]})$ of I in case $\overline{R[It]}$ is a finite $R[It]$ -module. The coefficient $\bar{e}_1(I)$ will be the main object of interest in this paper.

The condition that $\overline{R[It]}$ be a finite $R[It]$ -module is satisfied for any ideal I in an analytically unramified local ring R , see [22, 1.5]. Under this assumption there exists an integer b such that

$$\overline{I^{n+b}} \subset J^n \quad \text{for every } n \text{ and every reduction } J \text{ of } I,$$

where we use the convention that $I^m = R$ for $m \leq 0$. The smallest such $b \geq 0$ is called the *Briançon–Skoda number* $b(I)$ of I . In a regular local ring R of positive dimension one has $b(I) \leq \dim R - 1$ according to the classical Briançon–Skoda theorem ([16, Theorem 1]).

In order to relate $e_0(I)$, $\bar{e}_1(I)$ and $b(I)$ the next lemma is needed. We use the notation $\deg(-)$ for the multiplicity of a finite module over a Noetherian local ring or of a finite graded module over a Noetherian standard graded algebra over an Artinian local ring.

Lemma 2.1. *Let (R, \mathfrak{m}) be a Noetherian local ring of dimension $d > 0$ and let I be an \mathfrak{m} -primary ideal. Let A and B be graded R -subalgebras of $R[t]$ with*

$$R[It] \subset A \subset B \subset R[t]$$

and assume that B is a finite $R[It]$ -module. Write C for the graded $R[It]$ -module B/A .

- (a) $\dim C \leq d$, and equality holds if R is Cohen–Macaulay, A satisfies the condition S_2 of Serre and $A \neq B$.
- (b) If $\dim C < d$ then $e_1(B) = e_1(A)$.
- (c) If $\dim C = d$ then $e_1(B) - e_1(A) = \deg(C) > 0$.

Proof. Part (a) is obvious. To prove (b) and (c) notice that C is a finite graded module over a Noetherian standard graded algebra over an Artinian ring. Hence it has a Hilbert polynomial whose degree is $\dim C - 1$. On the other hand the exact sequences

$$0 \longrightarrow C_n \longrightarrow R/A_n \longrightarrow R/B_n \longrightarrow 0$$

show that $(e_1(B) - e_1(A))/(d - 1)!$ is the coefficient of the term of degree $d - 1$ in the Hilbert polynomial of C . □

Theorem 2.2. *Let (R, \mathfrak{m}) be an analytically unramified local Cohen–Macaulay ring of positive dimension with infinite residue field and let I be an \mathfrak{m} -primary ideal. Let A and B be distinct graded R -subalgebras of $R[t]$ with*

$$R[It] \subset A \subsetneq B \subset \overline{R[It]}$$

and assume that A satisfies the condition S_2 of Serre. Then

$$0 \leq e_1(I) \leq e_1(A) < e_1(B) \leq \bar{e}_1(I) \leq b(I)e_0(I).$$

Proof. Let J be a minimal reduction of I , and notice that $e_1(J) = 0$ since R is Cohen–Macaulay. Now Lemma 2.1 implies the asserted inequalities except for the last one. To show that $\bar{e}_1(I) \leq b(I)e_0(I)$ write $d = \dim R$ and $b = b(I)$. We

consider the $R[Jt]$ -module $C = \overline{R[It]}/R[Jt]$. Since $R[Jt]$ satisfies the condition S_2 of Serre, Lemma 2.1 shows that

$$\bar{e}_1(I) = \bar{e}_1(I) - e_1(J) = \deg(C).$$

In turn, by the definition of b , C is a submodule of the graded $R[Jt]$ -module D whose n -th components are J^{n-b}/J^n . The inclusions $J^{n-b} \supset J^{n-b+1} \supset \dots \supset J^n$ induce a filtration of D . From their Hilbert functions one sees that the factors in this filtration all have dimension d and multiplicity $e_0(J) = e_0(I)$. Hence $\dim D = d = \dim C$ and $\deg(D) = b e_0(I)$. Now the containment $C \subset D$ gives

$$\deg(C) \leq \deg(D) = b e_0(I),$$

and we obtain $\bar{e}_1(I) = \deg(C) \leq b(I)e_0(I)$, as asserted. □

Remark 2.3. Applying Theorem 2.2 to any minimal reduction J of I one obtains the stronger estimate $\bar{e}_1(I) = \bar{e}_1(J) \leq b(J)e_0(J) = b(J)e_0(I)$.

A measure for the complexity of $\overline{R[It]}$ is the number of steps needed to construct it. We address this issue in the next corollary, which is a direct consequence of Theorem 2.2.

Corollary 2.4. *Let (R, \mathfrak{m}) be an analytically unramified local Cohen–Macaulay ring of positive dimension and let I be an \mathfrak{m} -primary ideal. Then $\bar{e}_1(I)$ bounds the length of any chain of graded R -subalgebras satisfying the condition S_2 of Serre lying strictly between $R[It]$ and $\overline{R[It]}$.*

The following corollary provides a numerical criterion for when the integral closure coincides with the S_2 -ification $\text{End}_{R[It]}(\omega_{R[It]})$ of $R[It]$. Let (R, \mathfrak{m}) be a local Cohen–Macaulay ring of dimension ≥ 2 with a canonical module and infinite residue field, let I be an \mathfrak{m} -primary ideal, and J a minimal reduction of I . Writing $(-)^{\vee} = \text{Hom}_{R[Jt]}(-, R[Jt])$ we consider these embeddings of graded algebras,

$$R[It] \subset \text{End}_{R[It]}(\omega_{R[It]}) \subset \text{End}_{R[It]}(R[It]^{\vee}) = R[It]^{\vee\vee} \subset \overline{R[It]}.$$

Corollary 2.5. *Let (R, \mathfrak{m}) be an analytically unramified local Cohen–Macaulay ring of dimension ≥ 2 with a canonical module and infinite residue field, and let I be an \mathfrak{m} -primary ideal. Then $\overline{R[It]} = \text{End}_{R[It]}(\omega_{R[It]})$ if and only if $\bar{e}_1(I) = e_1(I)$; in this case $\overline{R[It]} = R[It]^{\vee\vee}$. In particular, the ideal I is normal if and only if $R[It]$ satisfies the condition S_2 of Serre and $\bar{e}_1(I) = e_1(I)$.*

Proof. Notice that by Lemma 2.1(b), $\text{End}_{R[It]}(\omega_{R[It]})$ has first Hilbert coefficient $e_1(I)$. Now apply Theorem 2.2. □

3. Bounds on $\bar{e}_1(I)$ and the Briançon–Skoda number

We discuss the role of Briançon–Skoda type theorems (see [1], [16]) in determining some relationships between the coefficients $e_0(I)$ and $\bar{e}_1(I)$. We will use a Briançon–Skoda theorem that works in non-regular rings. We are going to provide a short proof along the lines of [16] for the special case we need: \mathfrak{m} -primary ideals in a local Cohen–Macaulay ring. The general case is treated by Hochster and Huneke in [11, 1.5.5 and 4.1.5]. Let k be a perfect field, let R be a reduced and equidimensional k -algebra essentially of finite type, and assume that R is affine with $d = \dim R$ or (R, \mathfrak{m}) is local with $d = \dim R + \text{trdeg}_k R/\mathfrak{m}$. Recall that the *Jacobian ideal* $\text{Jac}_k(R)$ of R is defined as the d -th Fitting ideal of the module of differentials $\Omega_k(R)$ – it can be computed explicitly from a presentation of the algebra. By varying Noether normalizations one deduces from [16, Theorem 2] that the Jacobian ideal $\text{Jac}_k(R)$ is contained in the conductor $R : \bar{R}$ of R (see also [19], [2, 3.1] and [10, 2.1]); here \bar{R} denotes the integral closure of R in its total ring of fractions.

Theorem 3.1. *Let k be a perfect field, let R be a reduced local Cohen–Macaulay k -algebra essentially of finite type, and let I be an equimultiple ideal of height $g > 0$. Then for every integer n ,*

$$\text{Jac}_k(R) \overline{I^{n+g-1}} \subset I^n.$$

Proof. We may assume that k is infinite. Then, passing to a minimal reduction, we may suppose that I is generated by a regular sequence of length g . Let S be a finitely generated k -subalgebra of R so that $R = S_{\mathfrak{p}}$ for some $\mathfrak{p} \in \text{Spec}(S)$, and write $S = k[x_1, \dots, x_e] = k[X_1, \dots, X_e]/\mathfrak{a}$ with $\mathfrak{a} = (h_1, \dots, h_t)$ an ideal of height c . Notice that S is reduced and equidimensional. Let $K = (f_1, \dots, f_g)$ be an S -ideal with $K_{\mathfrak{p}} = I$, and consider the extended Rees ring $B = S[Kt, t^{-1}]$. Now B is a reduced and equidimensional affine k -algebra of dimension $e - c + 1$.

Let $\varphi: k[X_1, \dots, X_e, T_1, \dots, T_g, U] \rightarrow B$ be the k -epimorphism mapping X_i to x_i , T_i to $f_i t$ and U to t^{-1} . Its kernel has height $c + g$ and contains the ideal \mathfrak{b} generated by $\{h_i, T_j U - f_j \mid 1 \leq i \leq t, 1 \leq j \leq g\}$. Consider the Jacobian matrix of these generators,

$$\Theta = \left(\begin{array}{c|ccc} \frac{\partial h_i}{\partial X_j} & & 0 & \\ \hline & U & & T_1 \\ * & & \ddots & \vdots \\ & & & U \quad T_g \end{array} \right).$$

Notice that $I_{c+g}(\Theta) \supset I_c\left(\left(\frac{\partial h_i}{\partial X_j}\right)\right)U^{g-1}(T_1, \dots, T_g)$. Applying φ we obtain $\text{Jac}_k(B) \supset I_{c+g}(\Theta)B \supset \text{Jac}_k(S)Kt^{-g+2}$. Thus $\text{Jac}_k(S)Kt^{-g+2}$ is contained in the conductor of B . Localizing at \mathfrak{p} we see that $\text{Jac}_k(R)It^{-g+2}$ is in the conductor of the extended Rees ring $R[It, t^{-1}]$. Hence for every n ,

$\text{Jac}_k(R) I \overline{I^{n+g-1}} \subset I^{n+1}$, which yields

$$\text{Jac}_k(R) \overline{I^{n+g-1}} \subset I^{n+1} : I = I^n,$$

as $(\text{gr}_I(R))_+$ has positive grade. □

We now use Theorem 3.1 to sharpen the bound on \bar{e}_1 given in Theorem 2.2.

Theorem 3.2. *Let (R, \mathfrak{m}) be a reduced local Cohen–Macaulay ring of dimension $d > 0$ and let I be an \mathfrak{m} –primary ideal.*

- (a) *If in addition R is an algebra essentially of finite type over a perfect field k with type t , and $\delta \in \text{Jac}_k(R)$ is a non zerodivisor, then*

$$\bar{e}_1(I) \leq \frac{t}{t+1} [(d-1)e_0(I) + e_0(I + \delta R/\delta R)].$$

- (b) *If the assumptions of (a) hold, then*

$$\bar{e}_1(I) \leq (d-1)[e_0(I) - \lambda(R/\bar{I})] + e_0(I + \delta R/\delta R).$$

- (c) *If R is analytically unramified and R/\mathfrak{m} is infinite, then*

$$\bar{e}_1(I) \leq b(I) \min \left\{ \frac{t}{t+1} e_0(I), e_0(I) - \lambda(R/\bar{I}) \right\}.$$

Proof. We may assume that R/\mathfrak{m} is infinite. Then, passing to a minimal reduction we may suppose that I is generated by a regular sequence f_1, \dots, f_d . Notice this can only decrease $b(I)$. Let S be a local ring obtained from R by a purely transcendental residue field extension and by factoring out $d-1$ generic elements a_1, \dots, a_{d-1} of I . To be more precise, $S = R(\{X_{ij}\})/(a_1, \dots, a_{d-1})$ with $\{X_{ij}\}$ a set of $(d-1)d$ indeterminates and $a_i = \sum_{j=1}^d X_{ij} f_j$. Notice that S is also a birational extension of a localization of a polynomial ring over R , and hence is analytically unramified according to [18, 36.8] and [22, 1.6]. Furthermore S is a one–dimensional local ring and the S –ideal IS is generated by a single non zerodivisor, say $IS = fS$. From [14, Theorem 1] one has

$$(1) \quad \bar{I}S = \overline{IS},$$

$$(2) \quad \bar{I}^n S = \overline{(IS)^n} \quad \text{for } n \gg 0.$$

The last fact combined with the genericity of a_1, \dots, a_{d-1} yields $\bar{e}_1(I) = \bar{e}_1(IS)$. Moreover $\bar{e}_1(IS) = \lambda(\bar{S}/S)$ as S is a one–dimensional analytically unramified local ring. Thus

$$(3) \quad \bar{e}_1(I) = \lambda(\bar{S}/S).$$

In the setting of (a) and (b) the element δ is a non zerodivisor on S . Furthermore Theorem 3.1 shows that

$$(4) \quad \delta I^{d-1} \subset \delta \overline{I^{d-1}} \subset I^n : \bar{I}^n \quad \text{for every } n.$$

For $n \gg 0$, by (2), (4) and since $f^n S$ is contained in the conductor $S : \bar{S}$, we obtain

$$\delta f^{d-1} S \subset f^n S : \overline{f^n S} = f^n S : f^n \bar{S} = S : \bar{S}.$$

Hence

$$(5) \quad \delta f^{d-1} \bar{S} \subset S : \bar{S}.$$

We prove (a) by computing lengths along the inclusions

$$(6) \quad \delta f^{d-1} S \subset \delta f^{d-1} \bar{S} \subset S : \bar{S} \subset S.$$

Also recall that

$$(7) \quad \lambda(\bar{S}/S) \leq t \lambda(S/S : \bar{S})$$

by [9, the proof of 3.6] (see also [3, Theorem 1] and [5, 2.1]). We obtain

$$\begin{aligned} \frac{t+1}{t} \bar{e}_1(I) &= \lambda(\bar{S}/S) + \frac{1}{t} \lambda(\bar{S}/S) && \text{by (3)} \\ &\leq \lambda(\delta f^{d-1} \bar{S} / \delta f^{d-1} S) + \lambda(S/S : \bar{S}) && \text{by (7)} \\ &\leq \lambda(S / \delta f^{d-1} S) && \text{by (6)} \\ &= (d-1) \lambda(S/fS) + \lambda(S/\delta S) \\ &= (d-1)e_0(I) + e_0(I + \delta R/\delta R), \end{aligned}$$

where the last equality holds by the genericity of a_1, \dots, a_{d-1} .

Next we prove part (b). The inclusion (5) yields the filtration

$$S = \delta f^{d-1} \bar{S} + S \subset f^{d-1} \bar{S} + S \subset \dots \subset f^2 \bar{S} + S \subset f \bar{S} + S \subset \bar{S},$$

which shows

$$(8) \quad \bar{e}_1(I) = \lambda(\bar{S}/S) = \sum_{i=1}^{d-1} \lambda(f^{i-1} \bar{S} + S / f^i \bar{S} + S) + \lambda(f^{d-1} \bar{S} + S / \delta f^{d-1} \bar{S} + S).$$

Multiplication by f induces epimorphisms of S -modules

$$(9) \quad \frac{f^{i-1} \bar{S} + S}{f^i \bar{S} + S} \twoheadrightarrow \frac{f^i \bar{S} + S}{f^{i+1} \bar{S} + S}.$$

Now (8) and (9) show

$$(10) \quad \bar{e}_1(I) \leq (d-1) \lambda(\bar{S}/f\bar{S} + S) + \lambda(f^{d-1} \bar{S} + S / \delta f^{d-1} \bar{S} + S).$$

Next we claim that

$$(11) \quad \lambda(\bar{S}/f\bar{S} + S) = \lambda(\bar{I}/I).$$

Indeed,

$$\begin{aligned} \lambda(\overline{S}/f\overline{S} + S) &= \lambda(\overline{S}/f\overline{S}) - \lambda(f\overline{S} + S/f\overline{S}) \\ &= \lambda(S/fS) - \lambda(S/S \cap f\overline{S}) \\ &= \lambda(S \cap f\overline{S}/fS) \\ &= \lambda(\overline{f\overline{S}}/fS) \\ &= \lambda(\overline{I}/I) \quad \text{by (1).} \end{aligned}$$

On the other hand,

$$\begin{aligned} \lambda(f^{d-1}\overline{S} + S/\delta f^{d-1}\overline{S} + S) &\leq \lambda(f^{d-1}\overline{S}/\delta f^{d-1}\overline{S}) \\ &= \lambda(\overline{S}/\delta\overline{S}) \\ &= \lambda(S/\delta S) \\ &= e_0(I + \delta R/\delta R), \end{aligned}$$

where the last equality holds by the genericity of a_1, \dots, a_{d-1} . Therefore

$$(12) \quad \lambda(f^{d-1}\overline{S} + S/\delta f^{d-1}\overline{S} + S) \leq e_0(I + \delta R/\delta R).$$

Combining (10), (11) and (12) we deduce

$$\begin{aligned} \overline{e}_1(I) &\leq (d-1) \lambda(\overline{I}/I) + e_0(I + \delta R/\delta R) \\ &= (d-1)[e_0(I) - \lambda(R/\overline{I})] + e_0(I + \delta R/\delta R). \end{aligned}$$

Finally we prove part (c). Write $b = b(I)$. We first claim that

$$(13) \quad f^b\overline{S} \subset S : \overline{S}.$$

Indeed, for $n \gg 0$

$$\begin{aligned} f^n S &\supset \overline{I^{n+b}} S \\ &= \overline{(IS)^{n+b}} \quad \text{by (2)} \\ &= \overline{f^{n+b}\overline{S}} \\ &= f^{n+b}\overline{S} \quad \text{since } n \gg 0. \end{aligned}$$

Therefore $f^b\overline{S} \subset S$, proving (13). Now (13) yields the filtrations

$$(14) \quad f^b S \subset f^b\overline{S} \subset S : \overline{S} \subset S,$$

$$(15) \quad S = f^b\overline{S} + S \subset \dots \subset f^2\overline{S} + S \subset f\overline{S} + S \subset \overline{S}.$$

Filtration (14) implies

$$\begin{aligned} \frac{t+1}{t} \overline{e}_1(I) &= \lambda(\overline{S}/S) + \frac{1}{t} \lambda(\overline{S}/S) \quad \text{by (3)} \\ &\leq \lambda(f^b\overline{S}/f^b S) + \lambda(S/S : \overline{S}) \quad \text{by (7)} \\ &\leq \lambda(S/f^b S) \quad \text{by (14)} \\ &= b \lambda(S/fS) \\ &= b e_0(I), \end{aligned}$$

where the last equality holds by the genericity of a_1, \dots, a_{d-1} . On the other hand filtration (15) yields

$$\begin{aligned} \bar{e}_1(I) &= \lambda(\bar{S}/S) \\ &= \sum_{i=1}^b \lambda(f^{i-1}\bar{S} + S/f^i\bar{S} + S) \\ &\leq b \lambda(\bar{S}/f\bar{S} + S) && \text{by (9)} \\ &= b \lambda(\bar{I}/I) && \text{by (11)} \\ &= b (e_0(I) - \lambda(R/\bar{I})). \end{aligned}$$

□

Remark 3.3. The multiplicity $e_0(I + \delta R/\delta R)$ occurring in Theorem 3.2 can be bounded by

$$e_0(I + \delta R/\delta R) \leq (d - 1)! e_0(I) \deg(R/\delta R),$$

where $\deg(R/\delta R)$ is the multiplicity of the local ring $R/\delta R$. Indeed, [15, Theorem 3] gives

$$e_0(I + \delta R/\delta R) \leq (d - 1)! \lambda(R/I + \delta R) \deg(R/\delta R).$$

Corollary 3.4. *Let (R, \mathfrak{m}) be a regular local ring of dimension $d > 0$ and let I be an \mathfrak{m} -primary ideal. Then*

$$e_1(I) \leq \bar{e}_1(I) \leq (d - 1) \min\left\{\frac{e_0(I)}{2}, e_0(I) - \lambda(R/\bar{I})\right\}.$$

Proof. We may assume that R/\mathfrak{m} is infinite. The classical Briançon–Skoda theorem gives that $b(I) \leq d - 1$, see [16, Theorem 1]. The assertions now follow from Theorems 2.2 and 3.2(c). □

We are now going to use Corollary 3.4 to bound the length of divisorial chains for classes of Rees algebras.

Corollary 3.5. *Let (R, \mathfrak{m}) be a regular local ring of dimension $d > 0$ and let I be an \mathfrak{m} -primary ideal. Then $(d - 1) \min\{\frac{e_0(I)}{2}, e_0(I) - \lambda(R/\bar{I})\}$ bounds the length of any chain of graded R -subalgebras satisfying the condition S_2 of Serre lying strictly between $R[It]$ and $\overline{R[It]}$.*

Proof. The assertion follows from Corollaries 2.4 and 3.4. □

Remark 3.6. (a) Let (R, \mathfrak{m}) be a regular local ring of dimension 2 and let I be an \mathfrak{m} -primary integrally closed ideal. Then $e_1(I) = \bar{e}_1(I) = e_0(I) - \lambda(R/I) \leq \frac{e_0(I)}{2}$. This follows, for instance, from Corollary 3.4 combined with the inequality $e_1(I) \geq e_0(I) - \lambda(R/I)$, see [20, Theorem 1]. Furthermore by [13, 2.1] or [21, 3.3], the equality $e_1(I) = e_0(I) - \lambda(R/I)$ implies that I has reduction number at most one if R/\mathfrak{m} is infinite, a fact

proved in [17, 5.4]. Thus $R[It]$ is Cohen–Macaulay according to [24, 3.1] and [8, 3.10]. Now Corollary 2.5 yields the well-known result that I is normal, see [27, Theorem 2', p. 385].

- (b) Let k be an infinite field, write $R = k[X_1, \dots, X_d]_{(X_1, \dots, X_d)}$, let \mathfrak{m} denote the maximal ideal of R , and let I be an \mathfrak{m} -primary R -ideal generated by homogeneous polynomials in $k[X_1, \dots, X_d]$ of degree s . Then $\bar{e}_1(I) = e_1(\mathfrak{m}^s) = \frac{d-1}{2}e_0(I)(1 - \frac{1}{s}) \approx \frac{d-1}{2}e_0(I)$. This shows that the estimate of Corollary 3.4 is essentially sharp.

Proposition 3.7. *Let k be a perfect field, let (R, \mathfrak{m}) be a reduced local Cohen–Macaulay k -algebra essentially of finite type of dimension $d > 0$, and let $\delta \in \text{Jac}_k(R)$ be a non zerodivisor. Then for any \mathfrak{m} -primary ideal I ,*

$$b(I) \leq d - 1 + e_0(I + \delta R/\delta R).$$

Proof. We may assume that R/\mathfrak{m} is infinite. Then, replacing I by a minimal reduction with the same Briançon–Skoda number we may suppose that I is generated by a regular sequence of length d . As in the proof of Theorem 3.2 let S be a local ring obtained from R by a purely transcendental residue field extension and by factoring out $d - 1$ generic elements a_1, \dots, a_{d-1} of I . Write $IS = fS$ and let b be the smallest non negative integer with $f^b \bar{S} \subset S : \bar{S}$.

We first claim that

$$(16) \quad b(I) \leq b.$$

Indeed, for any integer $n \geq 0$ we have

$$\overline{I^{n+b}S} \subset \overline{I^{n+b}\bar{S}} \subset f^{n+b}\bar{S} \subset f^n S,$$

hence $\overline{I^{n+b}S} \subset I^n S$. As $\text{gr}_I(R)$ is a polynomial ring in d variables over R/I , the generic choice of a_1, \dots, a_{d-1} gives that $\text{gr}_I(R)$ embeds into $\text{gr}_{IS}(S)$. Therefore $\overline{I^{n+b}} \subset I^n$, proving (16).

By (16) it suffices to show that $b \leq d - 1 + e_0(I + \delta R/\delta R)$. To this end we may assume $b \geq d - 1$. The definition of b yields the filtration

$$(17) \quad S = f^b \bar{S} + S \subset \dots \subset f^{d-1} \bar{S} + S.$$

On the other hand (5) implies

$$(18) \quad S = \delta f^{d-1} \bar{S} + S \subset f^{d-1} \bar{S} + S.$$

If $f^i \bar{S} + S = f^{i-1} \bar{S} + S$ for some $b \geq i \geq d$, then multiplication by f^{b-i} yields $S = f^b \bar{S} + S = f^{b-1} \bar{S} + S$, contradicting the minimality of b . Thus (17) gives

$$\lambda(f^{d-1} \bar{S} + S/S) \geq b - d + 1.$$

On the other hand from (12) and (18) we deduce

$$\lambda(f^{d-1} \bar{S} + S/S) \leq e_0(I + \delta R/\delta R).$$

Thus $b - d + 1 \leq e_0(I + \delta R/\delta R)$. □

Remark 3.8. In the setting of the proof of Proposition 3.7, $b(I) = b(IS) = b$. Indeed, (2) implies that for $n \gg 0$,

$$f^{n+b(I)}\overline{S} = \overline{I^{n+b(I)}S} = \overline{I^{n+b(I)}S} \subset I^n S = f^n S,$$

showing that $f^{b(I)}\overline{S} \subset S$. Hence $b \leq b(I)$ and then $b = b(I)$ by (16). Clearly $b(IS) \leq b$. For $n \gg 0$, $f^{n+b(IS)}\overline{S} = \overline{f^{n+b(IS)}S} \subset f^n S$ and therefore $f^{b(IS)}\overline{S} \subset S$, showing that $b \leq b(IS)$.

4. Equimultiple ideals

In this section we extend Theorems 2.2 and 3.2 to arbitrary equimultiple ideals. The technical change involves Hilbert functions. Let R be a Noetherian local ring with infinite residue field and I an ideal of height $g > 0$. Let $D = \bigoplus_{n \geq 0} D_n t^n$ be a graded R -subalgebra of $R[t]$ with $R[It] \subset D \subset R[t]$ and assume that D is a finite $R[It]$ -module. Instead of the length function as in Theorem 2.2, we consider the multiplicity $\text{deg}(R/D_n)$ of the R -module R/D_n . According to the associativity formula for multiplicities one has

$$\text{deg}(R/D_n) = \sum_{\mathfrak{p}} \lambda(R_{\mathfrak{p}}/(D_n)_{\mathfrak{p}}) \text{deg}(R/\mathfrak{p}),$$

where $n \gg 0$ and the sum is taken over the minimal primes \mathfrak{p} of I with $\dim R/\mathfrak{p} = \dim R/I$. It follows that this function behaves as a polynomial of degree g , which we still call the Hilbert–Samuel polynomial of D ,

$$E_0(D) \binom{n+g-1}{g} - E_1(D) \binom{n+g-2}{g-1} + \text{lower terms}.$$

The coefficients $E_i(D)$ can be expressed in terms of the local Hilbert coefficients $e_i(D_{\mathfrak{p}})$,

$$E_i(D) = \sum_{\mathfrak{p}} e_i(D_{\mathfrak{p}}) \text{deg}(R/\mathfrak{p}).$$

If $g = 0$ we set $E_0(D) = \sum_{\mathfrak{p}} \lambda(R_{\mathfrak{p}}) \text{deg}(R/\mathfrak{p})$. We will use the notation $E_i(I)$ when $D = R[It]$ and $\overline{E}_i(I)$ when $D = \overline{R[It]}$, the integral closure of $R[It]$ in $R[t]$.

One glaring difficulty with the above formula lies in the numbers $e_i(D_{\mathfrak{p}})$ or even $e_i(I_{\mathfrak{p}})$, which are hard to get hold of. At least for equimultiple ideals on the other hand, the qualitative behavior of the $E_i(I)$ is that of the usual Hilbert coefficients and $E_0(I)$ can be expressed as a multiplicity:

Proposition 4.1. *Let R be a local Cohen–Macaulay ring with infinite residue field and let I be an equimultiple ideal of positive height.*

- (a) $E_0(I) = E_0(J) = \text{deg}(R/J)$ for every minimal reduction J of I .
- (b) The ideal I is a complete intersection if and only if $E_0(I) = \text{deg}(R/I)$ if and only if $E_1(I) = 0$.

Proof. Notice that J is a complete intersection. Furthermore the minimal primes of I and of J coincide, and hence all have maximal dimension. For any such prime \mathfrak{p} , $e_0(I_{\mathfrak{p}}) = e_0(J_{\mathfrak{p}}) = \lambda(R_{\mathfrak{p}}/J_{\mathfrak{p}})$, proving (a). Moreover $e_0(I_{\mathfrak{p}}) \geq \lambda(R_{\mathfrak{p}}/I_{\mathfrak{p}})$ and $e_1(I_{\mathfrak{p}}) \geq 0$; either inequality is an equality if and only if $I_{\mathfrak{p}}$ is a complete intersection ([20, Theorem 1]). According to [4, Theorem], the last condition holds for every \mathfrak{p} if and only if I is a complete intersection. This proves part (b). \square

The version of Theorem 2.2 for equimultiple ideals can now be stated. In its proof we will only discuss the points that require a new justification.

Theorem 4.2. *Let R be an analytically unramified local Cohen–Macaulay ring with infinite residue field and let I be an equimultiple ideal of positive height. Let A and B be distinct graded R -subalgebras of $R[t]$ with*

$$R[It] \subset A \subsetneq B \subset \overline{R[It]}$$

and assume that A satisfies the condition S_2 of Serre. Then

$$0 \leq E_1(I) \leq E_1(A) < E_1(B) \leq \overline{E}_1(I) \leq b(I)E_0(I).$$

Proof. Let g be the height of I and \mathfrak{p} a minimal prime of I . By Theorem 2.2, $e_1(A_{\mathfrak{p}}) \leq e_1(B_{\mathfrak{p}})$ and $e_1(A_{\mathfrak{p}}) = e_1(B_{\mathfrak{p}})$ only when $A_{\mathfrak{p}} = B_{\mathfrak{p}}$. Now $A_{\mathfrak{p}} = B_{\mathfrak{p}}$ for every minimal prime \mathfrak{p} of I is equivalent to saying that the R -annihilator L of $C = B/A$ is an ideal of height at least $g + 1$. Since I is equimultiple of height g , we conclude that the height of the A -ideal LA is at least 2. As $LAB \subset A$ and A satisfies the condition S_2 , it would follow that $A = B$. This proves the asserted inequalities except for the last one. To see the last inequality notice that $b(J_{\mathfrak{p}}) \leq b(I)$ for every minimal reduction J of I , and apply Remark 2.3. \square

Remark 4.3. The proof of Theorem 4.2 shows that when passing from the algebra A to B , one of the values $e_1(A_{\mathfrak{p}})$ increases. Thus, the integer $\sum_{\mathfrak{p}} e_1(A_{\mathfrak{p}})$ would give tighter control. Padding the summands with the $\deg(R/\mathfrak{p})$ into an ‘Ersatzintegral’ however provides a value that becomes ‘visible’, unlike the $e_1(A_{\mathfrak{p}})$.

It is also possible to derive sharper estimates for equimultiple ideals based on the bounds of Theorem 3.2. We will indicate some of these by making use of a very general inequality for $E_0(I)$ that arises from Lech’s formula ([15, Theorem 3]).

Proposition 4.4. *Let R be an equidimensional and catenary local Nagata ring and let I be an ideal of height g . Then*

$$E_0(I) \leq g! \deg(R/\overline{I}) \deg(R).$$

Proof. We estimate $E_0(I)$ as given above using Lech’s inequality. Indeed, adding over all minimal primes \mathfrak{p} of I of height g we obtain

$$\begin{aligned} E_0(I) &= \sum e_0(I_{\mathfrak{p}}) \deg(R/\mathfrak{p}) \\ &= \sum e_0(\bar{I}_{\mathfrak{p}}) \deg(R/\mathfrak{p}) \\ &\leq \sum g! \lambda(R_{\mathfrak{p}}/\bar{I}_{\mathfrak{p}}) \deg(R_{\mathfrak{p}}) \deg(R/\mathfrak{p}) \\ &\leq g! \left[\sum \lambda(R_{\mathfrak{p}}/\bar{I}_{\mathfrak{p}}) \deg(R/\mathfrak{p}) \right] \deg(R) \\ &= g! \deg(R/\bar{I}) \deg(R), \end{aligned}$$

where we have used the fact that $\deg(R_{\mathfrak{p}}) \leq \deg(R)$ by our assumption on R ([18, 40.1]). □

Theorem 4.5. *Let R be a reduced local Cohen–Macaulay ring and let I be an equimultiple ideal of height $g > 0$.*

(a) *If in addition R is an algebra essentially of finite type over a perfect field k with type t , and $\delta \in \text{Jac}_k(R)$ is a non zerodivisor, then*

$$\begin{aligned} \bar{E}_1(I) &\leq \frac{t}{t+1} [(g-1)E_0(I) + E_0(I + \delta R/\delta R)] \quad \text{and} \\ \bar{E}_1(I) &\leq \frac{t}{t+1} [(g-1)g! \deg(R/\bar{I}) \deg(R) + \\ &\quad (g-1)! \deg(R/(\overline{I + \delta R})) \deg(R/\delta R)]. \end{aligned}$$

(b) *If the assumptions of (a) hold, then*

$$\begin{aligned} \bar{E}_1(I) &\leq (g-1) [E_0(I) - \deg(R/\bar{I})] + E_0(I + \delta R/\delta R) \quad \text{and} \\ \bar{E}_1(I) &\leq (g-1) \deg(R/\bar{I}) [g! \deg(R) - 1] + \\ &\quad (g-1)! \deg(R/(\overline{I + \delta R})) \deg(R/\delta R). \end{aligned}$$

(c) *If R is Nagata and R/\mathfrak{m} is infinite, then*

$$\begin{aligned} \bar{E}_1(I) &\leq b(I) \min \left\{ \frac{t}{t+1} E_0(I), E_0(I) - \deg(R/\bar{I}) \right\} \\ &\leq b(I) \frac{t}{t+1} g! \deg(R/\bar{I}) \deg(R). \end{aligned}$$

Proof. To estimate $\bar{E}_1(I)$ in the equimultiple case we start from

$$\bar{E}_1(I) = \sum_{\mathfrak{p}} \bar{e}_1(I_{\mathfrak{p}}) \deg(R/\mathfrak{p})$$

with the sum taken over the minimal primes \mathfrak{p} of I , and make use of Theorem 3.2 to bound the $\bar{e}_1(I_{\mathfrak{p}})$ in terms of the $e_0(I_{\mathfrak{p}})$ and $e_0((I + \delta R/\delta R)_{\mathfrak{p}})$. Notice that either $\delta \notin \mathfrak{p}$ and then $e_0((I + \delta R/\delta R)_{\mathfrak{p}}) = 0$, or $\delta \in \mathfrak{p}$ and $\mathfrak{p}/\delta R$ is also a minimal prime of $I + \delta R/\delta R$, in which case $I + \delta R/\delta R$ has height $g - 1$. We

now process the summation as in the proof of Proposition 4.4 for the two ideals of two different rings. For example in (a) we have

$$\begin{aligned} \overline{E}_1(I) &= \sum_{\mathfrak{p}} \overline{e}_1(I_{\mathfrak{p}}) \deg(R/\mathfrak{p}) \\ &\leq \sum_{\mathfrak{p}} \frac{\text{type}(R_{\mathfrak{p}})}{\text{type}(R_{\mathfrak{p}}) + 1} [(g-1)e_0(I_{\mathfrak{p}}) + e_0((I + \delta R/\delta R)_{\mathfrak{p}})] \deg(R/\mathfrak{p}) \\ &\leq \frac{t}{t+1} [(g-1) \sum_{\mathfrak{p}} e_0(I_{\mathfrak{p}}) \deg(R/\mathfrak{p}) + \sum_{\mathfrak{p}} e_0((I + \delta R/\delta R)_{\mathfrak{p}}) \deg(R/\mathfrak{p})]. \end{aligned}$$

In the last expression the first sum equals $E_0(I)$ and the second sum is either 0 or else $E_0(I + \delta R/\delta R)$, in which case $I + \delta R/\delta R$ has height $g-1$. We now use Proposition 4.4 to conclude the proof of part (a). \square

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