Mathematical Research Letters **12**, 10001–10012 (2005)

AN INVERSE SCATTERING PROBLEM FOR SHORT-RANGE SYSTEMS IN A TIME-PERIODIC ELECTRIC FIELD.

François Nicoleau

Abstract. We consider a time-dependent Hamiltonian $H(t) = \frac{1}{2}p^2 - E(t) \cdot x + V(t,x)$ on $L^2(\mathbb{R}^n)$, where the external electric field $E(t)$ and the short-range electric potential $V(t,x)$ are time-periodic with the same period. It is well-known that the short-range notion depends on the mean value E_0 of the external electric field. When $E_0 = 0$, we show that the high energy limit of the scattering operators determines uniquely $V(t,x)$. When $E_0 \neq 0$, the same result holds in dimension $n \geq 3$ for generic short-range potentials. In dimension $n = 2$, one has to assume a stronger decay on the electric potential.

1. Introduction

In this note, we study an inverse scattering problem for a two-body short-range system in the presence of an external time-periodic electric field $E(t)$ and a time-periodic short-range potential $V(t,x)$ (with the same period T). For the sake of simplicity, we assume that the period $T = 1$.

The corresponding Hamiltonian is given on $L^2(\mathbb{R}^n)$ by :

$$H(t) = \frac{1}{2}p^2 - E(t) \cdot x + V(t,x), \tag{1.1}$$

where $p = -i\partial_x$. When $E(t) = 0$, the Hamiltonian $H(t)$ describes the dynamics of the hydrogen atom placed in a linearly polarized monochromatic electric field, or a light particle in the restricted three-body problem in which two other heavy particles are set on prescribed periodic orbits. When $E(t) = \cos(2\pi t)\ E$ with $E \in \mathbb{R}^n$, the Hamiltonian describes the well-known AC-Stark effect in the E-direction [7].

In this paper, we assume that the external electric field $E(t)$ satisfies :

$$(A_1) \qquad t \to E(t) \ \in L^1_{loc}(\mathbb{R};\mathbb{R}^n) \quad , \quad E(t+1) = E(t) \ a.e \ .$$

Moreover, we assume that the potential $V \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$, is time-periodic with period 1, and satisfies the following estimations :

$$(A_2) \qquad \forall \alpha \in \mathbb{N}^n,\ \forall k \in \mathbb{N},\ |\ \partial_t^k \partial_x^\alpha V(t,x)\ | \leq\ C_{k,\alpha}\ < x >^{-\delta-|\alpha|},\ \text{with } \delta > 0,$$

where $< x >=\ (1+x^2)^{\frac{1}{2}}$. Actually, we can accommodate more singular potentials (see [10], [11], [12] for example) and we need (A_2) for only $k,\ \alpha$ with finite order . It is well-known that under assumptions $(A_1) - (A_2)$, $H(t)$ is essentially

Received by the editors June 28, 2005.

self-adjoint on $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space, [16]. We denote $H(t)$ the self-adjoint realization with domain $D(H(t))$.

Now, let us recall some well-known results in scattering theory for time-periodic electric fields. We denote $H_0(t)$ the free Hamiltonian :

$$H_0(t) = \frac{1}{2}p^2 - E(t) \cdot x \ , \tag{1.2}$$

and let $U_0(t,s)$, (resp. $U(t,s)$) be the unitary propagator associated with $H_0(t)$, (resp. $H(t)$) (see section 2 for details).

For short-range potentials, the wave operators are defined for $s \in \mathbb{R}$ and $\Phi \in L^2(\mathbb{R}^n)$ by :

$$W^{\pm}(s)\ \Phi = \lim_{t \to \pm\infty} U(s,t)\ U_0(t,s)\ \Phi. \tag{1.3}$$

We emphasize that the short-range condition depends on the value of the mean of the external electric field :

$$E_0 \ = \ \int_0^1 \ E(t)\ dt\ . \tag{1.4}$$

• **The case** $E_0 = 0$

By virtue of the Avron-Herbst formula (see section 2), this case falls under the category of two-body systems with time-periodic potentials and this case was studied by Kitada and Yajima ([10], [11]), Yokoyama [22].

We recall that for a unitary or self-adjoint operator U, $\mathcal{H}_c(U)$, $\mathcal{H}_{ac}(U)$, $\mathcal{H}_{sc}(U)$ and $\mathcal{H}_p(U)$ are, respectively, continuous, absolutely continuous, singular continuous and point spectral subspace of U.

We have the following result ([10], [11], [21]) :

Theorem 1.
Assume that hypotheses (A_1), (A_2) *are satisfied with* $\delta > 1$ *and with* $E_0 = 0$.
Then : (i) the wave operators $W^{\pm}(s)$ *exist for all* $s \in \mathbb{R}$.
(ii) $W^{\pm}(s+1) = W^{\pm}(s)$ *and* $U(s+1,s)\ W^{\pm}(s) = W^{\pm}(s)\ U_0(s+1,s)$.
(iii) Ran $(W^{\pm}(s)) = \mathcal{H}_{ac}\left(U(s+1,s)\right)$ *and* $\mathcal{H}_{sc}\left(U(s+1,s)\right) = \emptyset$.
(iv) the purely point spectrum $\sigma_p(U(s+1,s))$ *is discrete outside* $\{1\}$.

Comments
1 - The unitary operators $U(s+1,s)$ are called the Floquet operators and they are mutually equivalent. The Floquet operators play a central role in the analysis of time periodic systems.
The eigenvalues of these operators are called Floquet multipliers. In [5], Galtbayar, Jensen and Yajima improve assertion (iv) : for $n = 3$ and $\delta > 2$, $\mathcal{H}_p\left(U(s+1,s)\right)$ is finite dimensional.

2 - For general $\delta > 0$, $W^{\pm}(s)$ do not exist and we have to define other wave operators. In ([10], [11]), Kitada and Yajima have constructed modified wave operators $W^{\pm}_{HJ}$ by solving an Hamilton-Jacobi equation.

• **The case** $E_0 \neq 0$

This case was studied by Moller [12] : using the Avron-Herbst formula, it suffices to examine Hamiltonians with a constant external electric field, (Stark Hamiltonians); the spectral and the scattering theory for Stark Hamiltonians are well established [2]. In particular, a Stark Hamiltonian with a potential V satisfying (A_2) has no eigenvalues [2]. The following theorem, due to Moller [12], is a time-periodic version of these results.

Theorem 2.
Assume that hypotheses (A_1), (A_2) *are satisfied with* $\delta > \frac{1}{2}$ *and with* $E_0 \neq 0$.
Then : (i) the Floquet operators have purely absolutely continuous spectrum.
(ii) the wave operators $W^{\pm}(s)$ *exist for all* $s \in \mathbb{R}$ *and are unitary.*
(iii) $W^{\pm}(s+1) = W^{\pm}(s)$ *and* $U(s+1,s)\ W^{\pm}(s) = W^{\pm}(s)\ U_0(s+1,s)$.

The inverse scattering problem

For $s \in \mathbb{R}$, we define the scattering operators $S(s) = W^{+*}(s)\ W^{-}(s)$. It is clear that the scattering operators $S(s)$ are periodic with period 1.

The inverse scattering problem consists to reconstruct the perturbation $V(s,x)$ from the scattering operators $S(s)$, $s \in [0,1]$.

In this paper, we prove the following result :

Theorem 3.
Assume that $E(t)$ *satisfies* (A_1) *and let* V_j, $j = 1,2$ *be potentials satisfying* (A_2). *We assume that* $\delta > 1$ *(if* $E_0 = 0$*),* $\delta > \frac{1}{2}$ *(if* $E_0 \neq 0$ *and* $n \geq 3$*),* $\delta > \frac{3}{4}$ *(if* $E_0 \neq 0$ *and* $n = 2$*).*
Let $S_j(s)$ *be the corresponding scattering operators.*
Then :

$$\forall s \in [0,1],\ S_1(s) = S_2(s) \implies V_1 = V_2\ .$$

We prove Theorem 3 by studying the high energy limit of $[S(s), p]$, (Enss-Weder's approach [4]). We need $n \geq 3$ in the case $E_0 \neq 0$ in order to use the inversion of the Radon transform [6] on the orthogonal hyperplane to E_0. See also [15] for a similar problem with a Stark Hamiltonian.

We can also remark that if we know the free propagator $U_0(t,s)$, $s,t \in \mathbb{R}$, then by virtue of the following relation :

$$S(t) = U_0(t,s)\ S(s)\ U_0(s,t)\ , \tag{1.5}$$

the potential $V(t,x)$ is uniquely reconstructed from the scattering operator $S(s)$ at only one initial time.

In [21], Yajima proves uniqueness for the case of time-periodic potential with the condition $\delta > \frac{n}{2}+1$ and with $E(t)=0$ by studying the scattering matrices in a high energy regime.

In [20], for a time-periodic potential that decays exponentially at infinity, Weder proves uniqueness at a fixed quasi-energy.

Note also that inverse scattering for long-range time-dependent potentials without external electric fields was studied by Weder [18] with the Enss-Weder time-dependent method, and by Ito for time-dependent electromagnetic potentials for Dirac equations [8].

2. Proof of Theorem 3

2.1. The Avron-Herbst formula. First, let us recall some basic definitions for time-dependent Hamiltonians. Let $\{H(t)\}_{t\in\mathbb{R}}$ be a family of selfadjoint operators on $L^2(\mathbb{R}^n)$ such that $\mathcal{S}(\mathbb{R}^n) \subset D(H(t))$ for all $t\in\mathbb{R}$.

Definition.

We call *propagator* a family of unitary operators on $L^2(\mathbb{R}^n)$, $U(t,s)$, $t,s\in\mathbb{R}$ such that :

1 - $U(t,s)$ is a strongly continuous fonction of $(t,s)\in\mathbb{R}^2$.
2 - $U(t,s)\,U(s,r) = U(t,r)$ for all $t,s,r\in\mathbb{R}$.
3 - $U(t,s)\,(\mathcal{S}(\mathbb{R}^n)) \subset \mathcal{S}(\mathbb{R}^n)$ for all $t,s\in\mathbb{R}$.
4 - If $\Phi\in\mathcal{S}(\mathbb{R}^n)$, $U(t,s)\Phi$ is continuously differentiable in t, s and satisfies :

$$i\,\frac{\partial}{\partial t}U(t,s)\,\Phi = H(t)\,U(t,s)\,\Phi\ ,\quad i\,\frac{\partial}{\partial s}U(t,s)\,\Phi = -U(t,s)\,H(s)\,\Phi\ .$$

To prove the existence and the uniqueness of the propagator for our Hamiltonians $H(t)$, we use a generalization of the Avron-Herbst formula close to the one given in [3].

In [12], the author gives, for $E_0\neq 0$, a different formula which has the advantage to be time-periodic. To study our inverse scattering problem, we use here a different one, which is defined for all E_0. We emphasize that with our choice, $c(t)$ (see below for the definition of $c(t)$) is also periodic with period 1; in particular $c(t)=O(1)$.

The basic idea is to generalize the well-known Avron-Herbst formula for a Stark Hamiltonian with a constant electric field E_0, [2]; if we consider the Hamiltonian B_0 on $L^2(\mathbb{R}^n)$,

$$B_0 = \frac{1}{2}p^2 - E_0\cdot x\ , \tag{2.1}$$

we have the following formula :

$$e^{-itB_0} = e^{-i\frac{E_0^2}{6}t^3}\ e^{itE_0\cdot x}\ e^{-i\frac{t^2}{2}E_0\cdot p}\ e^{-it\frac{p^2}{2}}\ . \tag{2.2}$$

In the next definition, we give a similar formula for time-dependent electric fields.

Definition

We consider the family of unitary operators $T(t)$, for $t \in \mathbb{R}$:

$$T(t) = \ e^{-ia(t)}\ e^{-ib(t)\cdot x}\ e^{-ic(t)\cdot p}\ ,$$

where :

$$b(t) = \ -\int_0^t (E(s)-E_0)\ ds - \int_0^1 \int_0^t (E(s)-E_0)\ ds\ dt\ . \tag{2.3}$$

$$c(t) = \ -\int_0^t b(s)\ ds\ . \tag{2.4}$$

$$a(t) = \int_0^t \left(\frac{1}{2}\ b^2(s) - E_0\cdot c(s)\right)\ ds\ . \tag{2.5}$$

Lemma 4.

The family $\{H_0(t)\}_{t\in\mathbb{R}}$ has an unique propagator $U_0(t,s)$ defined by :

$$U_0(t,s) = \ T(t)\ e^{-i(t-s)B_0}\ T^*(s)\ . \tag{2.6}$$

Proof.

We can always assume $s = \ 0$ and we make the following ansatz :

$$U_0(t,0) = \ e^{-ia(t)}\ e^{-ib(t)\cdot x}\ e^{-ic(t)\cdot p}\ e^{-itB_0}\ . \tag{2.7}$$

Since on the Schwartz space, $U_0(t,0)$ must satisfy :

$$i\ \frac{\partial}{\partial t}\ U_0(t,0) = H_0(t)\ U_0(t,0)\ , \tag{2.8}$$

the functions $a(t),\ b(t), c(t)$ solve :

$$\dot{b}(t) = -E(t) + E_0,\ \dot{c}(t) = -b(t),\ \dot{a}(t) = \frac{1}{2}\ b^2(t) - E_0\cdot c(t). \tag{2.9}$$

We refer to [3] for details and [12] for a different formula. □

In the same way, in order to define the propagator corresponding to the family $\{H(t)\}$, we consider a Stark Hamiltonian with a time-periodic potential : $B(t) = B_0 + V_1(t,x)$ where

$$V_1(t,x) = \ e^{ic(t)\cdot p}\ V(t,x)\ e^{-ic(t)\cdot p} = \ V(t, x+c(t)), \tag{2.10}$$

(we recall that $c(t)$ a is C^1-periodic function). Then, $B(t)$ has an unique propagator $R(t,s)$, (see [16] for the case $E_0 = 0$ and [12] for the case $E_0 \neq 0$). It is easy to see that the propagator $U(t,s)$ for the family $\{H(t)\}$ is defined by :

$$U(t,s) = T(t)\ R(t,s)\ T^*(s). \tag{2.11}$$

Comments

Since the Hamiltonians $H_0(t)$ and $H(t)$ are time-periodic with period 1, one has for all $t, s \in \mathbb{R}$:

$$U_0(t+1, s+1) = U_0(t,s) \quad , \quad U(t+1,s+1) = U(t,s)\ . \tag{2.12}$$

Thus, the wave operators satisfy $W^{\pm}(s+1) = W^{\pm}(s)$.

2.2. The high energy limit of the scattering operators. In this section, we study the high energy limit of the scattering operators by using the well-known Enss-Weder's time-dependent method [4]. This method can be used to study Hamiltonians with electric and magnetic potentials on $L^2(\mathbb{R}^n)$ [1], the Dirac equation [9], the N-body case [4], the Stark effect ([15], [17]), the Aharonov-Bohm effect [18].

In [13], [14] a stationary approach, based on the same ideas, is proposed to solve scattering inverse problems for Schrödinger operators with magnetic fields or with the Aharonov-Bohm effect.
Before giving the main result of this section, we need some notation.

- Φ, Ψ are the Fourier transforms of functions in $C_0^\infty(\mathbb{R}^n)$.
- $\omega \in S^{n-1} \cap \Pi_{E_0}$ is fixed, where Π_{E_0} is the orthogonal hyperplane to E_0.
- $\Phi_{\lambda,\omega} = e^{i\sqrt{\lambda}x.\omega}\Phi,\ \Psi_{\lambda,\omega} = e^{i\sqrt{\lambda}x.\omega}\Psi.$

We have the following high energy asymptotics where $< \, , \, >$ is the usual scalar product in $L^2(\mathbb{R}^n)$:

Proposition 5.
Under the assumptions of Theorem 3, we have for all $s \in [0,1]$,

$$< [S(s), p]\ \Phi_{\lambda,\omega}\ ,\ \Psi_{\lambda,\omega} > = \lambda^{-\frac{1}{2}}\ < \left(\int_{-\infty}^{+\infty} \partial_x V(s, x+t\omega)\ dt \right)\ \Phi\ ,\ \Psi >$$
$$+\ o\ (\lambda^{-\frac{1}{2}})\ .$$

Comments

Actually, for the case $n=2,\ E_0 \neq 0$ and $\delta > \frac{3}{4}$, Proposition 5 is also valid for $\omega \in S^{n-1}$ satisfying $\mid \omega \cdot E_0 \mid < \mid E_0 \mid$, (see ([18], [15]).

Then, Theorem 3 follows from Proposition 5 and the inversion of Radon transform ([6] and [15], Section 2.3).

Proof of Proposition 5

For example, let us prove Proposition 5 for the case $E_0 \neq 0$ and $n \geq 3$, the other cases are similar. More precisely, see [18] for the case $E_0 = 0$, and for the case $n = 2,\ E_0 \neq 0$, see ([17], Theorem 2.4) and ([15], Theorem 4).

- **Step 1**

Since $c(t)$ is periodic, $c(t) = O(1)$. Then, $V_1(t,x)$ is a short-range perturbation of B_0, and we can define the usual wave operators for the pair of Hamiltonians $(B(t), B_0)$:

$$\Omega^{\pm}(s) = \text{s} - \lim_{t\to\pm\infty} R(s,t)\ e^{-i(t-s)B_0}\ . \tag{2.13}$$

Consider also the scattering operators $S_1(s) = \Omega^{+*}(s)\ \Omega^{-}(s)$. By virtue of (2.6) and (2.11), it is clear that :

$$S(s) = T(s)\ S_1(s)\ T^*(s)\ . \tag{2.14}$$

Using the fact that $e^{-ib(s)\cdot x}\ p\ e^{ib(s)\cdot x} = p + b(s)$, we have :

$$[S(s),p] = [S(s), p+b(s)] = T(s)\ [S_1(s),p]\ T^*(s)\ . \tag{2.15}$$

Thus,

$$< [S(s),p]\ \Phi_{\lambda,\omega}\ ,\ \Psi_{\lambda,\omega} >= < [S_1(s),p]\ T^*(s)\ \Phi_{\lambda,\omega}\ ,\ T^*(s)\ \Psi_{\lambda,\omega} > . \tag{2.16}$$

On the other hand,

$$T^*(s)\ \Phi_{\lambda,\omega} = e^{i\sqrt{\lambda}x.\omega}\ e^{ic(s)\cdot(p+\sqrt{\lambda}\omega)}\ e^{ib(s)\cdot x}\ e^{ia(s)}\ \Phi. \tag{2.17}$$

So, we obtain :

$$< [S(s),p]\ \Phi_{\lambda,\omega}\ ,\ \Psi_{\lambda,\omega} > = < [S_1(s),p]\ f_{\lambda,\omega},\ g_{\lambda,\omega}\ >, \tag{2.18}$$

where

$$f = e^{ic(s)\cdot p}\ e^{ib(s)\cdot x}\ \Phi \text{ and } g = e^{ic(s)\cdot p}\ e^{ib(s)\cdot x}\ \Psi\ . \tag{2.19}$$

Clearly, $f,\ g$ are the Fourier transforms of functions in $C_0^{\infty}(\mathbb{R}^n)$.

- **Step 2 : Modified wave operators**

Now, we follow a strategy close to [15] for time-dependent potentials. First, let us define a free-modified dynamic $U_D(t,s)$ by :

$$U_D(t,s) = e^{-i(t-s)B_0}\ e^{-i\int_0^{t-s}\ V_1(u+s,up'+\frac{1}{2}u^2E_0)\ du}\ , \tag{2.20}$$

where p' is the projection of p on the orthogonal hyperplane to E_0.

We define the modified wave operators :

$$\Omega_D^{\pm}(s) = \text{s} - \lim_{t\to\pm\infty} R(s,t)\ U_D(t,s)\ . \tag{2.21}$$

It is clear that :

$$\Omega_D^{\pm}(s) = \Omega^{\pm}(s)\ e^{-ig^{\pm}(s,p')}\ , \tag{2.22}$$

where

$$g^{\pm}(s,p') = \int_0^{\pm\infty} V_1(u+s, up' + \frac{1}{2}u^2 E_0)\ du\ . \tag{2.23}$$

Thus, if we set $S_D(s) = \Omega_D^{+*}(s)\Omega_D^{-}(s)$, one has :

$$S_1(s) = e^{-ig^+(s,p')}\ S_D(s)\ e^{ig^-(s,p')} \tag{2.24}$$

- **Step 3 : High energy estimates**

Denote $\rho = min\ (1,\delta)$. We have the following estimations, (the proof is exactly the same as in ([15], Lemma 3) for time-independent potentials).

Lemma 6.
For $\lambda >> 1$, we have :

(i) $$|| \left(V_1(t,x) - V_1(t,(t-s)p' + \frac{1}{2}(t-s)^2 E_0)\right)\ U_D(t,s)\ e^{ig^{\pm}(s,p')} f_{\lambda,\omega}\ || \leq\ C\ (1+ |\ (t-s)\sqrt{\lambda}\ |)^{-\frac{1}{2}-\rho}\ .$$

(ii) $|| \ \big(R(t,s)\Omega_D^{\pm}(s) - U_D(t,s)\big)\, e^{ig^{\pm}(s,p')} f_{\lambda,\omega}\ || =\ O\ (\lambda^{-\frac{1}{2}})\ ,$ *uniformly for $t,\ s \in \mathbb{R}$.*

- **Step 4**

We denote $F(s,\lambda,\omega) =< [S_1(s),p]\ f_{\lambda,\omega}\ ,\ g_{\lambda,\omega} >$. Using (2.24), we have :

$$\begin{aligned} F(s,\lambda,\omega)\ &= < [e^{-ig^+(s,p')}\ S_D(s)\ e^{ig^-(s,p')}, p]\ f_{\lambda,\omega}\ ,\ g_{\lambda,\omega} > \\ &= < [S_D(s),p]\ e^{ig^-(s,p')} f_{\lambda,\omega}\ ,\ e^{ig^+(s,p')} g_{\lambda,\omega} > \\ &= < [S_D(s)-1, p-\sqrt{\lambda}\omega]\ e^{ig^-(s,p')} f_{\lambda,\omega}\ ,\ e^{ig^+(s,p')} g_{\lambda,\omega} > \\ &= < (S_D(s)-1)\ e^{ig^-(s,p')} (pf)_{\lambda,\omega}\ ,\ e^{ig^+(s,p')} g_{\lambda,\omega} > \\ &\quad -\ < (S_D(s)-1)\ e^{ig^-(s,p')} f_{\lambda,\omega}\ ,\ e^{ig^+(s,p')} (pg)_{\lambda,\omega} > \\ &:= F_1(s,\lambda,\omega) - F_2(s,\lambda,\omega). \end{aligned}$$

First, let us study $F_1(s,\lambda,\omega)$. Writing $S_D(s) - 1 = (\Omega_D^+(s) - \Omega_D^-(s))^*\ \Omega_D^-(s)$ and using

$$\Omega_D^+(s) - \Omega_D^-(s) = i \int_{-\infty}^{+\infty} R(s,t)\ [\ V_1(t,x) - V_1(t,(t-s)p' + \frac{1}{2}(t-s)^2 E_0)\ ]\ U_D(t,s)\ dt\ , \tag{2.25}$$

we obtain :

$$(2.26)\quad S_D(s) - 1 = -i \int_{-\infty}^{+\infty} U_D(t,s)^* \ [\ V_1(t,x) - V_1(t,(t-s)p' + \frac{1}{2}(t-s)^2 E_0)\ ] \ R(t,s)\ \Omega_D^-(s)\ dt\ .$$

Thus,

$$F_1(s,\lambda,\omega) = -i \int_{-\infty}^{+\infty} < R(t,s)\ \Omega_D^-(s)\ e^{ig^-(s,p')}(pf)_{\lambda,\omega}\ ,\ [\ V_1(t,x) - V_1(t,(t-s)p' + \frac{1}{2}(t-s)^2E_0)\ ]\ U_D(t,s)\ e^{ig^+(s,p')} g_{\lambda,\omega} >\ dt$$

$$= -i \int_{-\infty}^{+\infty} < U_D(t,s)\ e^{ig^-(s,p')}(pf)_{\lambda,\omega}\ ,\ [\ V_1(t,x) - V_1(t,(t-s)p' + \frac{1}{2}(t-s)^2E_0)\ ]\ U_D(t,s)\ e^{ig^+(s,p')} g_{\lambda,\omega} >\ dt + R_1(s,\lambda,\omega)\ ,$$

where :

$$(2.27)\quad R_1(s,\lambda,\omega) = -i \int_{-\infty}^{+\infty} <\ [\ R(t,s)\ \Omega_D^-(s) - U_D(t,s)\ ]\ e^{ig^-(s,p')}(pf)_{\lambda,\omega}\ ,\ [\ V_1(t,x) - V_1(t,(t-s)p' + \frac{1}{2}(t-s)^2E_0)\ ]\ U_D(t,s)\ e^{ig^+(s,p')} g_{\lambda,\omega} >\ dt\ .$$

By Lemma 6, it is clear that $R_1(s,\lambda,\omega) = O\ (\lambda^{-1})$. Thus, writing $t = \frac{\tau}{\sqrt{\lambda}} + s$, we obtain :

$$(2.28)\quad F_1(s,\lambda,\omega) = -\frac{i}{\sqrt{\lambda}} \int_{-\infty}^{+\infty} < U_D(\frac{\tau}{\sqrt{\lambda}} + s, s)\ e^{ig^-(s,p')}\ (pf)_{\lambda,\omega}\ ,\ \left(V_1(\frac{\tau}{\sqrt{\lambda}} + s, x) - V_1(\frac{\tau}{\sqrt{\lambda}} + s, \frac{\tau}{\sqrt{\lambda}}p' + \frac{\tau^2}{2\lambda}E_0) \right) U_D(\frac{\tau}{\sqrt{\lambda}} + s, s)\ e^{ig^+(s,p')} g_{\lambda,\omega} >\ d\tau +\ O\ (\lambda^{-1})\ .$$

Denote by $f_1(\tau,s,\lambda,\omega)$ the integrand of the (R.H.S) of (2.28). By Lemma 6 (i),

$$(2.29)\quad |\ f_1(\tau,s,\lambda,\omega)\ | \leq C\ (1+ |\ \tau\ |)^{-\frac{1}{2}-\rho}\ .$$

So, by Lebesgue's theorem, to obtain the asymptotics of $F_1(s,\lambda,\omega)$, it suffices to determine $\lim_{\lambda\to+\infty} f_1(\tau,s,\lambda,\omega)$.

Let us denote :

$$(2.30)\quad U^{\pm}(t,s,p') = e^{i \int_t^{\pm\infty} V_1(u+s,up'+\frac{1}{2}u^2E_0)\ du}.$$

We have :

$$f_1(\tau,s,\lambda,\omega) \ = < e^{-i\frac{\tau}{\sqrt{\lambda}}B_0} \ U^-(\frac{\tau}{\sqrt{\lambda}},s,p') \ (pf)_{\lambda,\omega} \ , \left(V_1(\frac{\tau}{\sqrt{\lambda}}+s,x) - V_1(\frac{\tau}{\sqrt{\lambda}}+s,\frac{\tau}{\sqrt{\lambda}}p' + \frac{\tau^2}{2\lambda} \ E_0) \right) e^{-i\frac{\tau}{\sqrt{\lambda}}B_0} \ U^+(\frac{\tau}{\sqrt{\lambda}},s,p') \ g_{\lambda,\omega} > \ . \tag{2.31}$$

Using the Avron-Herbst formula (2.2), we deduce that :

$$f_1(\tau,s,\lambda,\omega) \ = < e^{-i\frac{\tau}{2\sqrt{\lambda}}p^2} \ U^-(\frac{\tau}{\sqrt{\lambda}},s,p') \ (pf)_{\lambda,\omega} \ , \left(V_1(\frac{\tau}{\sqrt{\lambda}}+s,x+\frac{\tau^2}{2\lambda} \ E_0) - V_1(\frac{\tau}{\sqrt{\lambda}}+s,\frac{\tau}{\sqrt{\lambda}}p' + \frac{\tau^2}{2\lambda} \ E_0) \right) e^{-i\frac{\tau}{2\sqrt{\lambda}}p^2} \ U^+(\frac{\tau}{\sqrt{\lambda}},s,p') \ g_{\lambda,\omega} > \ . \tag{2.32}$$

Then, we obtain :

$$f_1(\tau,s,\lambda,\omega) \ = < e^{-i\frac{\tau}{2\sqrt{\lambda}}(p+\sqrt{\lambda}\omega)^2} \ U^-(\frac{\tau}{\sqrt{\lambda}},s,p'+\sqrt{\lambda}\omega) \ pf \ , \left(V_1(\frac{\tau}{\sqrt{\lambda}}+s,x+\frac{\tau^2}{2\lambda} \ E_0) - V_1(\frac{\tau}{\sqrt{\lambda}}+s,\frac{\tau}{\sqrt{\lambda}}(p'+\sqrt{\lambda}\omega) + \frac{\tau^2}{2\lambda} \ E_0) \right) e^{-i\frac{\tau}{2\sqrt{\lambda}}(p+\sqrt{\lambda}\omega)^2} \ U^+(\frac{\tau}{\sqrt{\lambda}},s,p'+\sqrt{\lambda}\omega) \ g > \ . \tag{2.33}$$

Since

$$e^{-i\frac{\tau}{2\sqrt{\lambda}}(p+\sqrt{\lambda}\omega)^2} \ = \ e^{-i\frac{\tau\sqrt{\lambda}}{2}} \ e^{-i\tau\omega.p} \ e^{-i\frac{\tau}{2\sqrt{\lambda}}p^2} \ , \tag{2.34}$$

we have

$$f_1(\tau,s,\lambda,\omega) \ = < e^{-i\frac{\tau}{2\sqrt{\lambda}}p^2} \ U^-(\frac{\tau}{\sqrt{\lambda}},s,p'+\sqrt{\lambda}\omega) \ \ pf \ , \left(V_1(\frac{\tau}{\sqrt{\lambda}}+s,x+\tau\omega+\frac{\tau^2}{2\lambda} \ E_0) - V_1(\frac{\tau}{\sqrt{\lambda}}+s,\frac{\tau}{\sqrt{\lambda}}(p'+\sqrt{\lambda}\omega) + \frac{\tau^2}{2\lambda} \ E_0) \right) e^{-i\frac{\tau}{2\sqrt{\lambda}}p^2} \ U^+(\frac{\tau}{\sqrt{\lambda}},s,p'+\sqrt{\lambda}\omega) \ g > \ . \tag{2.35}$$

Since $| \ V_1(u+s,u(p'+\sqrt{\lambda}\omega)+\frac{1}{2}u^2E_0)) \ | \leq \ C \ (u^2+1)^{-\delta} \in L^1(\mathbb{R}^+, \ du)$, it is easy to show (using Lebesgue's theorem again) that :

$$s - \lim_{\lambda\to+\infty} \ U^{\pm}(\frac{\tau}{\sqrt{\lambda}},s,p'+\sqrt{\lambda}\omega) \ = 1 \ . \tag{2.38}$$

Then,

$$\lim_{\lambda\to+\infty} \ f_1(\tau,s,\lambda,\omega) \ = < pf \ , \ (V_1(s,x+\tau\omega) - V_1(s,\tau\omega)) \ \ g > \ . \tag{2.39}$$

So, we have obtained :

$$(2.40) \quad F_1(s,\lambda,\omega) \; = \; -\frac{i}{\sqrt{\lambda}} \; < pf, \; \left(\int_{-\infty}^{+\infty} \left(V_1(s, x+\tau\omega) - V_1(s,\tau\omega) \right) \, d\tau \right) \; g > \\ + \, o \, (\frac{1}{\sqrt{\lambda}}).$$

In the same way, we obtain

$$(2.41) \quad F_2(s,\lambda,\omega) \; = \; -\frac{i}{\sqrt{\lambda}} \; < f, \; \left(\int_{-\infty}^{+\infty} \left(V_1(s, x+\tau\omega) - V_1(s,\tau\omega) \right) \, d\tau \right) \; pg > \\ + o \, (\frac{1}{\sqrt{\lambda}}) \; ,$$

so

$$(2.42) \qquad F(s,\lambda,\omega) \; = \; F_1(s,\lambda,\omega) - F_2(s,\lambda,\omega)$$

$$(2.43) \qquad = \; \frac{1}{\sqrt{\lambda}} \; < f, \; \left(\int_{-\infty}^{+\infty} \partial_x V_1(s, x+\tau\omega) \; d\tau \right) \; g > + \; o(\frac{1}{\sqrt{\lambda}}) \; .$$

Using (2.19) and $\partial_x V(s, x+\tau\omega) = e^{-ic(s)\cdot p} \; \partial_x V_1(s, x+\tau\omega) \; e^{ic(s)\cdot p}$, we obtain :

$$(2.44) \quad F(s,\lambda,\omega) \; = \; \frac{1}{\sqrt{\lambda}} \; < \Phi, \; \left(\int_{-\infty}^{+\infty} \partial_x V(s, x+\tau\omega) \; d\tau \right) \; \Psi > \; + o(\frac{1}{\sqrt{\lambda}}) \; . \quad \square$$

References

[1] S. Arians, *Geometric approach to inverse scattering for the Schrödinger equation with magnetic and electric potentials*, J. Math. Phys. **38** (1997), no. 6, 2761–2773.

[2] J. E. Avron and I. W. Herbst, *Spectral and scattering theory for Schrödinger operators related to the Stark effect*, Comm. Math. Phys. **52** (1977) 239–254.

[3] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry*, Texts and Monographs in Physics, Springer Study Edition. Springer Verlag, Berlin, 1987.

[4] V. Enss and R. Weder, *The geometrical approach to multidimensional inverse scattering*, J. Math. Phys. **36** (1995), no. 8, 3902–3921.

[5] A. Galtbayar, A. Jensen, and K. Yajima, *Local time decay of solutions to Schrödinger equations with time-periodic potentials*, J. Statist. Phys. **116** (2004), no. 1-4, 231–282.

[6] S. Helgason, *The Radon Transform*, Progress in Mathematics 5, Birkhäuser, 1980.

[7] J. S. Howland, *Two problems with time-dependent Hamiltonians*, Mathematical Methods and Application of Scattering Theory, ed. J. A. DeSanto, A. W. Saenz and W.W. Zachary, Lecture Notes in Physics, 130, Springer, 1980.

[8] H. T. Ito, *An inverse scattering problem for Dirac equations with time-dependent electromagnetic potential*, Publ. Res. Inst. Math. Sci. **34** (1998), no. 4, 355–381.

[9] W. Jung, *Geometric approach to inverse scattering for Dirac equation*, J. Math. Phys. **36** (1995), no. 8, 3902–3921.

[10] H. Kitada and K. Yajima, *A scattering theory for time-dependent long-range potentials*, Duke Math. J. **49** (1982), no. 2, 341–376.

[11] ———, *Remarks on our paper : A scattering theory for time-dependent long-range potentials*, Duke Math. J. **50**, (1983), no. 4, 1005–1016.

[12] J. S. Moller, *Two-body short-range systems in a time-periodic electric field*, Duke Math. J. **105** (2000), no. 1, 135–166.

[13] F. Nicoleau, *A stationary approach to inverse scattering for Schrödinger operators with first order perturbation*, Comm. Partial Differential Equations **22** (1997), no. 3-4, 527–553.

[14] ______, *An inverse scattering problem with the Aharonov-Bohm effect*, J. Math. Phys. **41** (2000), no. 8, 5223–5237.

[15] ______, *Inverse scattering for Stark Hamiltonians with short-range potentials*, Asymptot. Anal. **35** (2003), no. 3-4, 349–359.

[16] M. Reed and R. Simon, Methods of mathematical physics, Vol. 2, Academic Press, 1978.

[17] R. Weder, *Multidimensional inverse scattering in an electric field*, J. Funct. Anal. **139** (1996), no. 2, 441–465.

[18] ______, *Inverse scattering for N-body systems with time dependent potentials*, Inverse Problems of Wave Propagation and Diffraction, Eds. G. Chavent, P. C. Sabatier, Lecture Notes in Physics 486, Springer Verlag, 1997.

[19] ______, *The Aharonov-Bohm effect and time-dependent inverse scattering theory*, Inverse Problems **18** (2002), no. 4, 1041–1056.

[20] ______, *Inverse scattering at a fixed quasi-energy for potentials periodic in time*, Inverse Problems **20** (2004), no. 3, 893–917.

[21] K. Yajima, Time periodic Schrödinger equations, Topics in the theory of Schrödinger equations, World Scientific Publishing Co. Pte. Ltd, 2004.

[22] K. Yokoyama, *Mourre theory for time-periodic systems*, Nagoya Math. J. **149** (1998) 193–210.

Université de Nantes, Département de Mathématiques, Laboratoire Jean Leray, UMR CNRS-UN 6629, 2, rue de la Houssinière BP 92208 , F-44322 Nantes cedex 03, FRANCE

E-mail address: nicoleau@math.univ-nantes.fr