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THE MASLOV INDEX AS A QUADRATIC SPACE

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Abstract. Kashiwara defined the Maslov index (associated to a collection of Lagrangian subspaces of a symplectic vector space over a field F) as a class in the Witt group $W(F)$ of quadratic forms. We construct a canonical quadratic vector space in this class and show how to understand the basic properties of the Maslov index without passing to $W(F)$—that is, more or less, how to upgrade Kashiwara's equalities in $W(F)$ to canonical isomorphisms between quadratic spaces. The quadratic space is defined using elementary linear algebra. On the other hand, it has a nice interpretation in terms of sheaf cohomology, due to A. Beilinson.

1. Introduction.

Let F be a field of characteristic not 2. Let $W(F)$ denote the Witt group of quadratic spaces over F—we will use the term *quadratic space* to mean a finite dimensional F-vector space with a non-degenerate symmetric bilinear form.

1.1. Suppose given a vector space V over F with a symplectic form B and some Lagrangian subspaces $l_1, \dots, l_n$ of V, indexed by $\mathbb{Z}/n\mathbb{Z}$.

To this data Kashiwara associated a class $\tau(l_1, \dots, l_n) \in W(F)$ called *the Maslov index* of the Lagrangians (see the appendices in [LV] and [KS], and section 7 of this article). In this article we answer the following questions:

(i) How can one represent $\tau(l_1, \dots, l_n)$ as the class of a canonically defined quadratic space?
(ii) How can one upgrade the basic equalities satisfied by τ to canonical isomorphisms between quadratic spaces?

By 'basic equalities' we mean dihedral symmetry, i.e.

$$\tau(l_1, l_2, \dots, l_n) = \tau(l_2, l_3, \dots, l_n, l_1) = -\tau(l_n, l_{n-1}, \dots, l_1) \tag{1}$$

and the chain condition, i.e.

$$\tau(l_1, l_2, \dots, l_n) = \tau(l_1, l_2, \dots, l_k) + \tau(l_1, l_k, \dots, l_n) \tag{2}$$

for any $k \in \{3, \dots, n-1\}$. We also mean that when F is a local field (e.g. $F = \mathbb{R}$),

$$\begin{gathered}\tau(l_1, \dots, l_n) \textit{ is locally constant in } l_1, \dots, l_n \\ \textit{if the dimension of } l_i \cap l_{i+1} \textit{ is fixed for each } i \in \mathbb{Z}/n\mathbb{Z}.\end{gathered} \tag{3}$$

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1.2. In section 2 we answer question (i), constructing a quadratic space denoted $T_{1,2,\ldots,n}$ (with bilinear form $q_{1,2,\ldots,n}$). We re-define the Maslov index $\tau(l_1, l_2, \ldots, l_n)$ to be the class of $T_{1,2,\ldots,n}$ in $W(F)$, and in section 7 we verify that our Maslov index is the same as Kashiwara's.

We will give a concrete description of the quadratic space in section 2.2, but abstractly we may say that the vector space $T_{1,2,\ldots,n}$ is the cohomology $H^0(C)$ of a certain complex C (13); the form $q_{1,2,\ldots,n}$ arises from the quasi-isomorphism of C and its dual C^*. This point of view is explained in section 3, where we also give a formula for the dual form on $T^*_{1,2,\ldots,n}$.

A sheaf-theoretic construction of $(T_{1,2,\ldots,n}, q_{1,2,\ldots,n})$ is described in 1.4. The reader chiefly interested in this interpretation can proceed directly to 1.4 and then section 8.

Remark. Having written this paper, we noticed that another answer to question (i) is proposed in [CLM], section 12, using topological methods when $F = \mathbb{R}$. However, one can find a counter-example to their formula for $n = 5$, $\dim V = 2$. Their method rather leads to our formula (10).

1.3. In sections 4, 5, 6 we answer question (ii).

1.3.1. The dihedral symmetry (1) will be realized by canonical identifications

$$(T_{1,2,\ldots,n}, q_{1,2,\ldots,n}) = (T_{2,3,\ldots,n,1}, q_{2,3,\ldots,n,1}) = (T_{n,n-1,\ldots,1}, -q_{n,n-1,\ldots,1})$$

as described in section 4.

1.3.2. For the chain condition (2), suppose first that $l_1 \cap l_k = 0$. In section 5 we describe a canonical isometric isomorphism

$$T_{1,2,\ldots,k} \oplus T_{1,k,\ldots,n} \xrightarrow{\cong} T_{1,2,\ldots,n}.$$

To treat the case $l_1 \cap l_k \neq 0$, we use the notion of *quadratic subquotient*: if T is a quadratic space and I an isotropic subspace, then $I^\perp/I$ is again a quadratic space, called 'the quadratic subquotient of T by I.'

Without assuming $l_1 \cap l_k = 0$, we identify $T_{1,2,\ldots,k} \oplus T_{1,k,\ldots,n}$ with a quadratic subquotient of $T_{1,2,\ldots,n}$, so the following well known lemma shows that (2) holds without any assumptions.

Lemma 1. *If S is a quadratic subquotient of T then S and T have the same class in $W(F)$.*

Proof. Suppose that $S = I^\perp/I$. Choose a linear complement M to $I^\perp$ in T. Then $M + I$ (a direct sum) is a hyperbolic summand of S, and $(M + I)^\perp \subset I^\perp$ maps isometrically onto $I^\perp/I$. □

1.3.3. As for (3), it is not true in general that the isomorphism class of $T_{1,2,\ldots,n}$ is locally constant under the condition described. It would be enough to show that $\dim T_{1,2,\ldots,n}$ is locally constant (see Lemma 9), but in fact (see (14)), $\dim T_{1,2,\ldots,n}$ depends not only on the dimensions of $l_i \cap l_{i+1}$ but also on $\dim(l_1 \cap \cdots \cap l_n)$.

In section 6 we represent $T_{1,2,\ldots,n}$ as a quadratic subquotient of another quadratic space $\tilde{T}_{1,2,\ldots,n}$ whose dimension depends only on the dimensions $\dim l_i \cap l_{i+1}$. Property (3) then follows from Lemma 1 above.

Remark. As already mentioned in 1.2, there is a quasi-isomorphism $\Phi\colon C \to C^*$. To define $\tilde{T}_{1,2,\dots,n}$ we factor Φ as

$$C \xrightarrow{\ \alpha\ } D \xrightarrow{\ \tilde{\Phi}\ } D^* \xrightarrow{\ \alpha^*\ } C^*$$

where α is a quasi-isomorphism and $\tilde{\Phi}$ an isomorphism. The construction of such a factorization (see 6.3) is quite general and is useful in other situations (see e.g. [Wa], [So], [Ke]).

1.4. Here is a sheaf-theoretic interpretation due to A. Beilinson (private communication). He interprets $T_{1,2,\dots,n}$ as $H^1(D,P)$, where D is the filled n-gon and P is the following subsheaf of the constant sheaf V_D with fibre V. Cyclically label the vertices of D by $\mathbb{Z}/n\mathbb{Z}$ and give D the corresponding orientation. Let U be the interior of D. Then P has fibre V on U, l_i on the edge $(i, i+1)$, and $l_{i-1} \cap l_i$ at the vertex i.

Write $j_U\colon U \to D$ for the inclusion. Let F_U be the constant sheaf on U with fibre F. The symplectic form on V induces a map $P \otimes P \to j_{U,!}F_U$, and thereby

$$\cup\colon \mathrm{Sym}^2 H^1(D,P) \to H^2(D, j_{U,!}F_U) = F. \tag{4}$$

In section 8, which can mostly be read independently from the rest of the work, we show that (4) is non-degenerate and in fact is *minus* our original bilinear form.

1.4.1. From this perspective, dihedral symmetry (1) is already clear, since the cyclic symmetry is manifest, while reversing the order of the Lagrangians reverses the orientation of D and so changes the identification $H^2(D, j_! F_U) = F$ by a sign.

1.4.2. In 8.3 we explain how the chain condition (2) may be proved via a "bordism" between three polygons, in analogy to the proof of the additivity of the index of manifolds.

1.5. In the electronic version www.arxiv.org/math.SG/0505561/ of this paper, we explain how our quadratic form occurs naturally in the context of the Weil representation, which originally motivated this work. The relationship between the Maslov index and the Weil representation is well known (see [LV],[Li],[Pe],[Ra],[Sou],...), with $n = 3$ Lagrangians being the key case. Our definitions allow a direct approach for any n.

1.6. I am grateful to V. Drinfeld for suggesting this subject, and to him and D. Arinkin for much useful advice. I also thank A. Beilinson for explaining his sheaf-theoretic reformulation, and M. Kamgarpour and B. Wieland for many interesting discussions.

2. The Quadratic Space.

2.1. Preliminaries. Our Lagragians $l_1, \dots, l_n$ are indexed by $\mathbb{Z}/n\mathbb{Z}$. Think of $\mathbb{Z}/n\mathbb{Z}$ as the vertices of a graph whose set $\mathbb{E}$ of edges consists of pairs of consecutive numbers $\{i, i+1\}$, $i \in \mathbb{Z}/n\mathbb{Z}$. So the graph looks like an n-sided polygon.

An element

$$v = (v_{\{i,i+1\}}) \in \bigoplus_{\{i,i+1\}\in\mathbb{E}} V$$

may be thought of as a function $a\colon \mathbb{E} \to V$. We can form the 'derivative'

$$\partial v = (\partial v_i) \in \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} V \qquad \partial v_i = v_{\{i,i+1\}} - v_{\{i-1,i\}}. \tag{5}$$

Conversely, given $w = (w_i) \in \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} V$, there is the obvious notion of an 'anti-derivative' $\hat{w}$

$$\hat{w} = (\hat{w}_{\{i,i+1\}}) \in \bigoplus_{\{i,i+1\} \in \mathbb{E}} V \qquad \text{such that} \quad \partial(\hat{w}) = w. \tag{6}$$

An anti-derivative exists so long as $\sum_{i \in \mathbb{Z}/n\mathbb{Z}} w_i = 0$ in which case $\hat{w}$ is unique up to adding a constant function.

2.2. Definitions.

2.2.1. Definition. Let $K_{1,2,\ldots,n}$ be the kernel of the natural summation

$$K_{1,2,\ldots,n} = \ker \left[\bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} l_i \xrightarrow{\Sigma} V \right]. \tag{7}$$

Any $w = (w_i) \in K_{1,2,\ldots,n}$ has an anti-derivative $\hat{w} \in \bigoplus_{\{i,i+1\} \in \mathbb{E}} V$, as in 2.1.

2.2.2. Definition. Define a bilinear form $q_{1,2,\ldots,n}$ on $K_{1,2,\ldots,n}$ by the formula

$$q_{1,2,\ldots,n}(v, w) = \sum_{i \in \mathbb{Z}/n\mathbb{Z}} B(v_i, \hat{w}_{\{i,i+1\}}) \tag{8}$$

for any choice of anti-derivative $\hat{w}$ (it is simple to check that the right-hand side of (8) is independent of this choice.)

Remark 1. The bilinear form may equivalently be defined by

$$q_{1,2,\ldots,n}(v, w) = \sum_{i \in \mathbb{Z}/n\mathbb{Z}} B(v_i, \hat{w}_{\{i-1,i\}}). \tag{9}$$

Indeed, the difference is

$$\sum_{i \in \mathbb{Z}/n\mathbb{Z}} B(v_i, \hat{w}_{\{i,i+1\}} - \hat{w}_{\{i-1,i\}}) = \sum_{i \in \mathbb{Z}/n\mathbb{Z}} B(v_i, w_i),$$

and $B(v_i, w_i) = 0$ since v_i, w_i lie in the same Lagrangian l_i.

Remark 2. In the definition (8) of $q_{1,2,\ldots,n}$, one may concretely choose $\hat{w}_{\{i,i+1\}} = \sum_{j=1}^{i} w_j$, in which case the definition takes the simple form

$$q_{1,2,\ldots,n}(v, w) = \sum_{i \geq j \geq 1}^{n} B(v_i, w_j) = \sum_{i > j > 1}^{n} B(v_i, w_j). \tag{10}$$

We often use this version for calculations, but the natural symmetries of $q_{1,2,\ldots,n}$ are obscured.

Proposition 2. *The bilinear form $q_{1,2,\ldots,n}$ is symmetric.*

Proof. "Summation by parts." Explicitly,

$$\begin{aligned} q_{1,2,\ldots,n}(v,w) &= \sum_{i\in\mathbb{Z}/n\mathbb{Z}} B(v_i, \hat{w}_{\{i,i+1\}}) \\ &= \sum_{i\in\mathbb{Z}/n\mathbb{Z}} B(\hat{v}_{\{i,i+1\}} - \hat{v}_{\{i-1,i\}}, \hat{w}_{\{i,i+1\}}) \\ &= \sum_{i\in\mathbb{Z}/n\mathbb{Z}} B(\hat{w}_{\{i,i+1\}} - \hat{w}_{\{i-1,i\}}, \hat{v}_{\{i-1,i\}}) \\ &= \sum_{i\in\mathbb{Z}/n\mathbb{Z}} B(w_i, \hat{v}_{\{i-1,i\}}) = q_{1,2,\ldots,n}(w,v). \end{aligned}$$

The last equality is (9). □

2.2.3. Definition of the Quadratic Space $(T_{1,2,\ldots,n}, q_{1,2,\ldots,n})$**.**

$$T_{1,2,\ldots,n} = K_{1,2,\ldots,n}\big/ \ker q_{1,2,\ldots,n}. \tag{11}$$

The induced non-degenerate bilinear form on $T_{1,2,\ldots,n}$ will still be called $q_{1,2,\ldots,n}$.

Let us give an explicit description of $T_{1,2,\ldots,n}$. The derivative (5) restricts to give a map

$$\partial\colon \bigoplus_{\{i,i+1\}\in\mathbb{E}} l_i \cap l_{i+1} \to K_{1,2,\ldots,n}. \tag{12}$$

It is clear from the definition (8) that $\ker q_{1,2,\ldots,n} \supset \operatorname{im}\partial$.

Proposition 3. *We have* $\ker q_{1,2,\ldots,n} = \operatorname{im}\partial$*. In other words,* $T_{1,2,\ldots,n}$ *is the cohomology* $H^0(C)$ *at the center term of the complex*

$$C = [\bigoplus_{\{i,i+1\}\in\mathbb{E}} l_i \cap l_{i+1} \xrightarrow{\ \partial\ } \bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}} l_i \xrightarrow{\ \Sigma\ } V] \tag{13}$$

which we consider to lie in degrees $-1, 0, 1$.

The proof will be given in section 3, where we also show that $q_{1,2,\ldots,n}$ is induced by a quasi-isomorphism between C and C^*, and give a formula for the dual form on $T^*_{1,2,\ldots,n}$.

Corollary.

$$\dim T_{1,2,\ldots,n} = (n-2)\frac{\dim V}{2} - \sum_{i\in\mathbb{Z}/n\mathbb{Z}} \dim l_i \cap l_{i+1} + 2\dim \bigcap_{i\in\mathbb{Z}/n\mathbb{Z}} l_i. \tag{14}$$

2.2.4. Definition. The symbol $\tau(l_1, l_2, \ldots, l_n)$ denotes the class of the quadratic space $(T_{1,2,\ldots,n}, q_{1,2,\ldots,n})$ in $W(F)$, called *the Maslov index* of $l_1, \ldots, l_n$.

In section 7 we will verify that this Maslov index equals Kashiwara's.

3. Proof of Proposition 3; The Dual Form; Homotopy Equivalence of C and C^*.

In this section we give an algebraic proof of Proposition 3 (a sheaf-theoretic argument is given in section 8). The proof allows us to write down a formula for the dual form on $T^*_{1,2,\ldots,n}$ in 3.2. It also implies that the complexes C and C^* are isomorphic in the derived category of complexes of vector spaces, this isomorphism inducing $q_{1,2,\ldots,n}$ on $H^0(C)$. In 3.3 we give an explicit, though non-canonical, symmetric quasi-isomorphism $C \to C^*$.

3.1. *Proof of Proposition 3.* Let Φ be the linear map $\Phi\colon H^0(C) \to H^0(C^*) = H^0(C)^*$ such that

$$q_{1,2,\ldots,n}(v,w) = \langle v, \Phi w\rangle\,.$$

We want to show that Φ is an isomorphism.

The situation is expressed by the following commutative diagram, in which the top row is C and the bottom row C^*; every row is a complex.

(15)
$$\begin{array}{ccccccc}
 & & \bigoplus_{\{i,i+1\}\in\mathbb{E}} l_i\cap l_{i+1} & \xrightarrow{\partial} & \bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}} l_i & \xrightarrow{\Sigma} & V \\
 & & \downarrow{\scriptstyle\mathrm{incl}} & & \downarrow{\scriptstyle\mathrm{incl}} & & \| \\
 & & \bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}} l_i & \xrightarrow{\partial} & \bigoplus_{\{i,i+1\}\in\mathbb{E}} l_i + l_{i+1} & \xrightarrow{\Sigma} & V \\
 & & \downarrow{\scriptstyle\mathrm{incl}} & & \downarrow{\scriptstyle\mathrm{incl}} & & \| \\
V & \xrightarrow{\mathrm{diag}} & \bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}} V & \xrightarrow{\partial} & \bigoplus_{\{i,i+1\}\in\mathbb{E}} V & \xrightarrow{\Sigma} & V \\
\downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle-\beta} & & \\
V^* & \xrightarrow{\mathrm{diag}} & \bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}} l_i^* & \xrightarrow{\partial^*} & \bigoplus_{\{i,i+1\}\in\mathbb{E}} (l_i\cap l_{i+1})^*. & &
\end{array}$$

Here "incl" means summand-by-summand inclusion and $\beta\colon x \mapsto B(-,x)$.

By definition (8) of $q_{1,2,\ldots,n}$, $\Phi\colon H^0(C) \to H^0(C^*)$ factors as

(16)
$$\Phi = \beta \circ \partial^{-1} \circ \mathrm{incl} \circ \mathrm{incl}.$$

Denote by W the cohomology at the center term of the second row. The first two rows of the diagram constitute a quasi-isomorphism of complexes, because the map "incl" between them is injective with acyclic cokernel. In particular we obtain an isomorphism $\mathrm{incl}\colon H^0(C) \to W$.

On the other hand, the last three rows of the diagram form a short exact sequence of complexes; since the third row is exact, the boundary map $\delta = \mathrm{incl}^{-1} \circ \partial \circ \beta^{-1}\colon H^0(C^*) \to W$ is an isomorphism.

Thus Φ factors through isomorphisms

$$\Phi\colon H^0(C) \xrightarrow{\ \mathrm{incl}\ } W \xrightarrow{\ \delta^{-1}\ } H^0(C^*).$$

□

3.2. The Dual Form. Write $q^*_{1,2,\dots,n}$ for the quadratic form on $T^*_{1,2,\dots,n}$, induced, in the notation of 3.1, by the isomorphism $\Phi^{-1}\colon H^0(C^*) \to H^0(C)$. Let us record a formula for $q^*_{1,2,\dots,n}$ which follows from the factorization (16).

We will not need it for the rest of this article.

3.2.1. Define $S := \{x \in \bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}} V \mid x_{i+1} - x_i \in l_i + l_{i+1}\}$. $T^*_{1,2,\dots,n}$ is a quotient of S: the map $S \to T^*_{1,2,\dots,n}$ is given by the last two rows of (15), in which $S = \ker(-\beta \circ \partial)$ and $T^*_{1,2,\dots,n} = \ker(\partial^*)/\operatorname{im}(\text{diag})$. We will describe $q^*_{1,2,\dots,n}$ pulled back to S.

3.2.2. Suppose first given $x_i, x_{i+1} \in V$ with $x_{i+1} - x_i \in l_i + l_{i+1}$. Define a functional $\varepsilon(x_i, x_{i+1})$ on $l_i + l_{i+1}$ as follows. For $v \in l_i + l_{i+1}$, write $v = a + b$, with $a \in l_i$, $b \in l_{i+1}$. Then

$$\langle \varepsilon(x_i, x_{i+1}), v\rangle := B(a, x_i) + B(b, x_{i+1}).$$

It is easy to verify that this quantity is independent of the choice of a, b.

Proposition 4. *As a bilinear form on S,*

$$q^*_{1,2,\dots,n}(x, y) = \sum_{i\in\mathbb{Z}/n\mathbb{Z}} \langle \varepsilon(x_i, x_{i+1}), y_{i+1} - y_i\rangle .$$

3.3. An Explicit Quasi-Isomorphism. Here is one particular quasi-isomorphism $\Phi\colon C \to C^*$ inducing $q_{1,2,\dots,n}$ on $H^0(C)$:

(17)
$$\begin{array}{ccccc}
\bigoplus_{\{i,i+1\}\in\mathbb{E}} l_i \cap l_{i+1} & \xrightarrow{\partial} & \bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}} l_i & \xrightarrow{\Sigma} & V \\
\downarrow{\scriptstyle \Phi_{-1}} & & \downarrow{\scriptstyle \Phi_0} & & \downarrow{\scriptstyle \Phi_1} \\
V^* & \xrightarrow{\Sigma^*} & \bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}} l_i^* & \xrightarrow{\partial^*} & \bigoplus_{\{i,i+1\}\in\mathbb{E}} (l_i \cap l_{i+1})^*
\end{array}$$

where

(18)
$$\begin{aligned}
\langle \Phi_{-1}(a), v\rangle &= \langle a, \Phi_1(v)\rangle = B(a_{\{n,1\}}, v) \\
\langle \Phi_0(a), b\rangle &= \langle a, \Phi_0(b)\rangle = \tfrac{1}{2} \sum_{i\ge j\ge 1}^{n} \left(B\left(a_i, b_j\right) + B\left(b_i, a_j\right)\right).
\end{aligned}$$

The fact that Φ induces $q_{1,2,\dots,n}$ follows from formula (10).

Remark. $\Phi = \Phi^*$. The isomorphism

$$H^{-1}(C) \oplus H^0(C) \oplus H^1(C) \longrightarrow H^{-1}(C)^* \oplus H^0(C)^* \oplus H^1(C)^*$$

implicit in diagram (15) is symmetric, so any $\Phi\colon C \to C^*$ inducing it may be symmetrized by $\Phi \mapsto \frac{1}{2}(\Phi + \Phi^*)$.

4. Dihedral Symmetry (1).

The space $K_{1,2,\ldots,n}$ can be canonically identified with $K_{2,3,\ldots,n,1}$ and $K_{n,n-1,\ldots,1}$, as is obvious from definition (7).

Proposition 5. *Under these identifications,* $q_{2,3,\ldots,n,1} = q_{1,2,\ldots,n} = -q_{n,n-1,\ldots,1}$.

Proof. The first equality (cyclic symmetry) is obvious from the definition (8) of $q_{1,2,\ldots,n}$. Reversing the order of the Lagrangians is equivalent to replacing ∂ by $-\partial$, and therefore $\hat{w}$ by $-\hat{w}$ in equation (8). □

5. The Chain Condition (2).

Proposition 6. *Fix* $k \in \{2, \ldots, n\}$.

1. *If* $l_1 \cap l_k = 0$ *then* $T_{1,2,\ldots,k} \oplus T_{1,k,\ldots,n} \cong T_{1,2,\ldots,n}$ *isometrically.*
2. *Without conditions,* $T_{1,2,\ldots,k} \oplus T_{1,k,\ldots,n}$ *is a quadratic subquotient of* $T_{1,2,\ldots,n}$.

Therefore, by Lemma 1, $\tau(l_1, l_2, \ldots, l_k) + \tau(l_1, l_k, \ldots, l_n) = \tau(l_1, l_2, \ldots, l_n)$.

Proof. The equivalence is induced by the natural map

$$\bigoplus_{i\in\{1,2,\ldots,k\}} l_i \oplus \bigoplus_{i\in\{1,k,\ldots,n\}} l_i \xrightarrow{\ s\ } \bigoplus_{i\in\{1,2,\ldots,n\}} l_i$$

that is the identity on each summand. More precisely, consider the map of short exact sequences

$$\begin{array}{ccccc} l_1 \oplus l_k & \longrightarrow & \bigoplus_{i\in\{1,2,\ldots,k\}} l_i \oplus \bigoplus_{i\in\{1,k,\ldots,n\}} l_i & \xrightarrow{\ s\ } & \bigoplus_{i\in\{1,2,\ldots,n\}} l_i \\ \downarrow{\scriptstyle\Sigma} & & \downarrow{\scriptstyle\Sigma\oplus\Sigma} & & \downarrow{\scriptstyle\Sigma} \\ V & \xrightarrow{(\mathrm{id},-\mathrm{id})} & V \oplus V & \xrightarrow{\ s\ } & V \end{array}$$

The snake lemma gives an exact sequence

$$l_1 \cap l_k \xrightarrow{(r,-r)} K_{1,2,\ldots,k} \oplus K_{1,k,\ldots,n} \xrightarrow{\ s\ } K_{1,2,\ldots,n} \xrightarrow{\ \delta\ } V/(l_1 + l_k). \tag{19}$$

Here $r = (-\mathrm{id}, \mathrm{id})\colon l_1 \cap l_k \to l_1 \oplus l_k$ and the boundary δ is given by

$$\delta(v) = \sum_{i=1}^{k} v_i \mod (l_1 + l_k). \tag{20}$$

Lemma 7. *The map* $s\colon K_{1,2,\ldots,k} \oplus K_{1,k,\ldots,n} \to K_{1,2,\ldots,n}$ *is an isometry.*

Proof. One sees immediately from the explicit formula (10) that s restricted to each summand is an isometry, and that for any $(v, w) \in K_{1,2,\ldots,k} \oplus K_{1,k,\ldots,n}$ we have $q_{1,2,\ldots,n}(s(v,0), s(0,w)) = 0$. No more is required. □

Transverse Case. If $l_1 \cap l_k = 0$ then, by the exactness of (19), s is an isometric isomorphism. Passing to the non-degenerate quotients establishes part 1 of the proposition.

General Case. Consider the map $r\colon l_1 \cap l_k \to K_{1,2,\ldots,k}$ as in (19). By Proposition 3, the image lies in $\ker q_{1,2,\ldots,k}$, so, since s is an isometry, $\operatorname{im} s \circ r$ is isotropic in $K_{1,2,\ldots,n}$ and $\operatorname{im} s \subset (\operatorname{im} s \circ r)^{\perp}$.

Lemma 8. *The image of* s *is exactly* $(\operatorname{im} s \circ r)^{\perp}$.

Proof. According to formula (10), if $v \in l_1 \cap l_k$ and $w \in K_{1,2,\ldots,n}$, then

$$q_{1,2,\ldots,n}(s \circ r(v), w) = \sum_{j=1}^{k} B(v, w_j).$$

This quantity vanishes for all $v \in l_1 \cap l_k$ if and only if $\sum_{i=1}^{k} w_k$ lies in $l_1 + l_k$; according to (20) this just means $\delta(w) = 0$ or equivalently $w \in \operatorname{im} s$. □

Let I denote the image of $\operatorname{im} s \circ r$ in $T_{1,2,\ldots,n}$. We have constructed a surjective isometry

$$s\colon K_{1,2,\ldots,k} \oplus K_{1,k,\ldots,n} \longrightarrow I^{\perp}/I$$

and therefore

$$s\colon T_{1,2,\ldots,k} \oplus T_{1,k,\ldots,n} \xrightarrow{\cong} I^{\perp}/I.$$

□

6. Local Constancy (3).

In this section, the ground field F is a local field.

6.1. In order to prove (3) we will use the following well known fact:

Lemma 9. *Let $Q(X)$ be the space of non-degenerate symmetric bilinear forms on a fixed vector space X. Then the natural $GL(X)$ action on $Q(X)$ has open orbits. In other words, in a continuous family of quadratic spaces of fixed rank, the isomorphism class is locally constant.*

Proof. It suffices to check that for any $q \in Q(X)$ the map $\lambda\colon GL(X) \to Q(X)$ given by $\lambda(g)(x,x) = q(gx, gx)$ has surjective differential at $1 \in GL(X)$. The differential $\lambda_*\colon \mathfrak{gl}(X) \to \operatorname{Sym}^2(X^*)$ is given by $\lambda_*(A)(x,x) = 2q(Ax, x)$; it is surjective since q is non-degenerate. □

We define in this section another quadratic space $\tilde{T}_{1,2,\ldots,n}$ whose class in $W(F)$ is the same as that of $T_{1,2,\ldots,n}$, but whose dimension is locally constant in $l_1, \ldots, l_n$ when the dimensions $\dim l_i \cap l_{i+1}$ are fixed. Then property (3) follows from Lemma 9. Note that, by (14), the dimension of $T_{1,2,\ldots,n}$ itself varies with $\dim(l_1 \cap \cdots \cap l_n)$.

Remark. One could use $\tilde{T}_{1,2,\ldots,n}$ to define the Maslov index, but then the dihedral symmetry (1) would not be immediately clear.

We will first give a concrete formula (22) for the quadratic form $\tilde{q}_{1,2,\ldots,n}$ on $\tilde{T}_{1,2,\ldots,n}$, but in section 6.3 we describe its origin in a general construction.

6.2. Definition. Define a symmetric bilinear form $\tilde{q}_{1,2,\ldots,n}$ on

$$\tilde{K}_{1,2,\ldots,n} = V^* \oplus \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} l_i \tag{21}$$

by

$$\tilde{q}_{1,2,\ldots,n}(v \oplus a, v \oplus a) = 2 \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \langle v, a_i \rangle + \sum_{i \geq j \geq 1} B(a_i, a_j). \tag{22}$$

Define

$$\tilde{T}_{1,2,\dots,n} = \tilde{K}_{1,2,\dots,n} / \ker \tilde{q}_{1,2,\dots,n} \tag{23}$$

and again write $\tilde{q}_{1,2,\dots,n}$ for the non-degenerate form on $\tilde{T}_{1,2,\dots,n}$.

To describe $\tilde{T}_{1,2,\dots,n}$ explicitly, let

$$\tilde{\partial} = (\Phi_{-1}, \partial)\colon \bigoplus_{\{i,i+1\}\in\mathbb{E}} l_i \cap l_{i+1} \to \tilde{K}_{1,2,\dots,n}.$$

Here $\Phi_{-1}\colon \bigoplus(l_i \cap l_{i+1}) \to V^*$ as in (17,18) and $\partial\colon \bigoplus(l_i \cap l_{i+1}) \to \bigoplus l_i$ as in (12).

Proposition 10. *With notation as above:*

1. *We have* $\ker \tilde{q}_{1,2,\dots,n} = \operatorname{im} \tilde{\partial}$*, and* $\tilde{\partial}$ *is injective. Therefore*

$$\dim \tilde{T}_{1,2,\dots,n} = (n+2)\frac{\dim V}{2} - \sum_{\{i,i+1\}\in\mathbb{E}} \dim l_i \cap l_{i+1}.$$

2. $T_{1,2,\dots,n}$ *is a quadratic subquotient of* $\tilde{T}_{1,2,\dots,n}$ *by the image of* $V^* \subset \tilde{K}_{1,2,\dots,n}$*. Therefore, by Lemma 1,* $T_{1,2,\dots,n}$ *and* $\tilde{T}_{1,2,\dots,n}$ *have the same class in* $W(F)$*.*

Proof of Proposition 10. It is simple to check Proposition 10 directly, making reference to the quasi-isomorphism (17,18); in the notation there,

$$\tilde{q}_{1,2,\dots,n}(v \oplus a, w \oplus b) = \langle v, \Sigma b\rangle + \langle a, \Phi_0 b\rangle + \langle a, \Sigma^* w\rangle .$$

Otherwise, one can feed that quasi-isomorphism into the following general construction.

6.3. *Let C be a complex of finite-dimensional vector spaces. Suppose given a symmetric quasi-isomorphism $\Phi\colon C \to C^*$. Then we construct another such complex D and a commutative diagram*

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & D \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\tilde{\Phi}} \\ C^* & \xleftarrow{\alpha^*} & D^* \end{array} \tag{24}$$

where α is a quasi-isomorphism and $\tilde{\Phi}$ a symmetric isomorphism.

Remark. Instead of vector spaces, one can consider projective modules over an associative ring with anti-involution. The construction still works, if one everywhere replaces "quasi-isomorphism" with "homotopy equivalence" and "acyclic" with "homotopy equivalent to zero."

In fact, [Wa, Theorem 9.4] generalizes this construction to complexes over any "exact category with duality containing $\frac{1}{2}$."

In the context of complexes of locally free modules over a commutative ring, an alternative local construction of a diagram similar to (24) can be found in [So], Corollaire 2.2. See also [Ke].

Construction. The cone of Φ is an acyclic complex including

$$\cdots C^{-1} \oplus (C^2)^* \xrightarrow{f_{-1}} C^0 \oplus (C^1)^* \xrightarrow{f_0} (C^0)^* \oplus C^1 \xrightarrow{f_1} (C^{-1})^* \oplus C^2 \cdots$$

where

$$f_{-1} = f_1^* = \begin{pmatrix} d & 0 \\ -\Phi & d^* \end{pmatrix} \qquad f_0 = \begin{pmatrix} \Phi & d^* \\ d & 0 \end{pmatrix}.$$

Define

$$D^0 = \left(C^0 \oplus (C^1)^*\right) / \operatorname{im} f_{-1}. \tag{25}$$

Then f_0 gives a symmetric isomorphism $f_0\colon D^0 \to (D^0)^*$.

Moreover, we get a chain of complexes

$$\begin{array}{ccccccccccc}
C & = & [\cdots & \xrightarrow{d} & C^{-1} & \xrightarrow{d} & C^0 & \xrightarrow{d} & C^1 & \xrightarrow{d} & \cdots] \\
\alpha\downarrow & & \Phi\downarrow & & \Phi\downarrow & & b\downarrow & & \| & & \| \\
D: & = & [\cdots & \xrightarrow{d^*} & (C^1)^* & \xrightarrow{c} & D^0 & \xrightarrow{e} & C^1 & \xrightarrow{d} & \cdots] \\
\tilde{\Phi}\downarrow & & \| & & \| & & f_0\downarrow & & \| & & \| \\
D^* & = & [\cdots & \xrightarrow{d^*} & (C^1)^* & \xrightarrow{e^*} & (D^0)^* & \xrightarrow{c^*} & C^1 & \xrightarrow{d} & \cdots] \\
\alpha^*\downarrow & & \| & & \| & & b^*\downarrow & & \Phi\downarrow & & \Phi\downarrow \\
C^* & = & [\cdots & \xrightarrow{d^*} & (C^1)^* & \xrightarrow{d^*} & (C^0)^* & \xrightarrow{d^*} & (C^{-1})^* & \xrightarrow{d^*} & \cdots]
\end{array} \tag{26}$$

satisfying our requirements. The maps $b\colon C^0 \to D^0$ and $c\colon (C^1)^* \to D^0$ are the natural ones from the definition of D^0, while $e\colon D^0 \to C^1$ is induced by $(d, 0)\colon C^0 \oplus (C^1)^* \to C^1$. □

6.4. In our particular case, $f_{-1} = \tilde{\partial}$ is injective; we obtain a complex D with $D^0 = \tilde{K}_{1,2,\ldots,n} / \operatorname{im} \tilde{\partial}$ and a symmetric isomorphism $f_0\colon D^0 \to (D^0)^*$ inducing the form $\tilde{q}_{1,2,\ldots,n}$. Moreover, $H^0(D) = H^0(C) = T_{1,2,\ldots,n}$ is the quadratic subquotient of $D^0 = \tilde{T}_{1,2,\ldots,n}$ by the image of c in diagram (26). This completes the proof of Proposition 10. □

7. Relation to Kashiwara's Construction.

Let us recall the construction of Kashiwara's Maslov index, which we denote by $\tau^{\mathrm{Kash}}(l_1, \ldots, l_n)$. First, for $n = 3$, $\tau^{\mathrm{Kash}}(l_1, l_2, l_3) \in W(F)$ is represented by the symmetric bilinear form $q_{1,2,3}^{\mathrm{Kash}}$ on $T_{1,2,3}^{\mathrm{Kash}} = l_1 \oplus l_2 \oplus l_3$ with

$$q_{1,2,3}^{\mathrm{Kash}}(v, w) = \frac{1}{2}\big(B(v_1, w_2 - w_3) + B(v_2, w_3 - w_1) + B(v_3, w_1 - w_2)\big).$$

The definition can then be extended inductively to any n by the chain condition

$$\tau^{\mathrm{Kash}}(l_1, l_2, \ldots, l_n) := \tau^{\mathrm{Kash}}(l_1, l_2, \ldots, l_k) + \tau^{\mathrm{Kash}}(l_1, l_k, \ldots, l_n) \tag{27}$$

for any $k \in \{3, \ldots, n-1\}$.

Proposition 11. $\tau(l_1, l_2, \ldots, l_n) = \tau^{\mathrm{Kash}}(l_1, l_2, \ldots, l_n)$.

Proof. Using the chain properties (2),(27), we need only consider $n = 3$. We show that $T_{1,2,3}$ is a quadratic subquotient of $T_{1,2,3}^{\mathrm{Kash}}$.

Lemma 12. *$I = l_1 \subset T^{\mathrm{Kash}}_{1,2,3}$ is an isotropic subspace;*

$$I^\perp = \{(v_1, v_2, v_3) \text{ with } v_2 - v_3 \in l_1\}.$$

Lemma 13. *The map*

$$(v_1, v_2, v_3) \mapsto (v_2 - v_3, -v_2, v_3)$$

defines an isometric surjection $I^\perp \to K_{1,2,3}$.

The lemmas, which may be checked directly, combine to give an isometric isomorphism from the non-degenerate part of $I^\perp/I$ to $T_{1,2,3}$. Lemma 1 concludes the proof of Proposition 11. □

8. The Maslov Index via Sheaves.

In this section we verify basic properties of Beilinson's construction (see 1.4).

Throughout we use the following notation: if X is a topological space and W a vector space then W_X is the constant sheaf on X with fibre W; if Y is a subset of X let j_Y denote its inclusion. $\mathbb{D}$ is the Verdier dualizing operator.

8.1. Non-Degeneracy. Since $(j_{U,!}F_U)[2]$ is the dualizing complex on D, the map $P \otimes P \to j_{U,!}F_U$ induced by the symplectic form on V defines a map $\phi\colon P \to (\mathbb{D}P)[-2]$. The fact that the bilinear form (4) on $H^1(D, P)$ is non-degenerate follows from the following lemma.

Lemma 14. *The map $\phi\colon P \to (\mathbb{D}P)[-2]$ is an isomorphism.*

Proof. The question is local; it suffices to check that ϕ is an isomorphism over the open sets of the form

$$U_i := U \cup (i-1, i) \cup \{i\} \cup (i, i+1) \qquad i \in \mathbb{Z}/n\mathbb{Z}$$

since these cover D. Write $M := l_{i-1} \cap l_i$ and choose a decomposition

$$V = M \oplus L \oplus L' \oplus M'$$

such that $l_{i-1} = M \oplus L$ and $l_i = M \oplus L'$. Write a, b for the inclusions

$$U \xrightarrow{a} U \cup (i-1, i) \xrightarrow{b} U_i.$$

Then on U_i we have

$$P = b_* a_* M_U \oplus b_! a_* L_U \oplus b_* a_! L'_U \oplus b_! a_! M'_U. \tag{28}$$

To compute $\mathbb{D}P$, note, for example, that $\mathbb{D} b_* a_* M_U = b_! a_! \mathbb{D} M_U = b_! a_! (M^*_U[2])$. Therefore, dualizing (28) and reversing the order of the summands, we obtain

$$(\mathbb{D}P)[-2] = b_* a_* {M'}^* \oplus b_! a_* {L'}^*_U \oplus b_* a_! L^*_U \oplus b_! a_! M^*_U. \tag{29}$$

We observe that ϕ is an upper-triangular matrix with respect to the decompositions (28), (29); moreover, the diagonal entries are isomorphisms, since the symplectic form gives isomorphisms $L' \cong L^*$ and $M' \cong M^*$. □

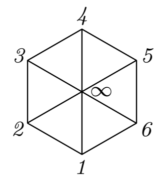

FIGURE 1. Triangulation of D for $n = 6$ Lagrangians.

8.2. Equivalence with Algebraic Definition 2.2.3. The cellular cochain complex arising from the sheaf P is evidently our original complex C in (13). Therefore $H^1(D, P) = T_{1,2,\dots,n}$.

As for the bilinear form, let us calculate explicitly the cup product (4). We triangulate D by introducing a central vertex ∞ and drawing radii, as in Figure 1.

Corresponding to this triangulation, a 1-cochain α with values in P consists of choices of $\alpha_{\{i,i+1\}} \in l_i$ and $\alpha_{\{i,\infty\}} \in V$. The cocycle condition amounts to

$$\alpha_{\{i+1,\infty\}} - \alpha_{\{i,\infty\}} = -\alpha_{\{i,i+1\}}.$$

A cocycle corresponds to an element $v \in T_{1,2,\dots,n}$ by $v_i = \alpha_{\{i,i+1\}}$, so the cocycle condition implies $\alpha_{\{i+1,\infty\}} = -\hat{v}_{\{i,i+1\}}$ in the notation of (6).

A 2-cocycle γ with values in $j_{U,!}F_U$ consists of choices of $\gamma_{\{i,i+1,\infty\}} \in V$. The evaluation $H^2(D, j_! F_U) \to F$ is given by $\gamma \mapsto \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \gamma_{\{i,i+1,\infty\}}$. Given 1-cocycles α, β the 2-cocycle $\gamma = \alpha \cup \beta$ is given by the usual formula

$$\gamma_{\{i,i+1,\infty\}} = B(\alpha_{\{i,i+1\}}, \beta_{\{i+1,\infty\}}).$$

All together, the pairing (4) is given on cocycles by

$$(\alpha, \beta) \mapsto \sum_{i \in \mathbb{Z}/n\mathbb{Z}} B(\alpha_{\{i,i+1\}}, \beta_{\{i+1,\infty\}}) = \sum_{i \in \mathbb{Z}/n\mathbb{Z}} B(v_i, -\hat{w}_{\{i,i+1\}}) = -q_{1,2,\dots,n}(v, w)$$

where $v, w \in T_{1,2,\dots,n}$ are the classes of α, β. The last equality is the definition (8) of $q_{1,2,\dots,n}$.

Remark. The asymmetric formula (10) appears if we choose an asymmetric triangulation of D, namely, if we draw all the diagonals from the vertex 1.

8.3. Chain Condition. Let us write (D_1, P_1), (D_2, P_2), (D_3, P_3) for the polygons and sheaves corresponding to the collections $\{l_1, \dots, l_k\}$, $\{l_1, l_k, \dots, l_n\}$, and $\{l_1, \dots, l_n\}$.

Here is a proof of the chain condition (2) that is very similar to the proof of the additivity of the index of manifolds. Namely, we describe a "bordism" $(Y, \hat{P})$ from $(D_1 \sqcup D_2, P_1 \oplus P_2)$ to (D_3, P_3), and argue that the image of $H^1(Y, \hat{P})$ in $H^1(D_1, P_1) \oplus H^1(D_2, P_2) \oplus H^1(\bar{D}_3, P_3)$ is isotropic of half the total dimension; therefore the total quadratic space is hyperbolic. Here $\bar{D}_3$ is D_3 with opposite orientation.

In Figure 2 we show Y, with $n = 4, k = 3$ for concreteness. The sheaf $\hat{P}$ on Y is constructible with respect to the pictured cell decomposition, with stalks as follows: V on the interior of Y and on the open top and bottom faces; l_1, l_2, l_3, l_4 on the open front, left, back, and right faces respectively. The stalk over any other cell is

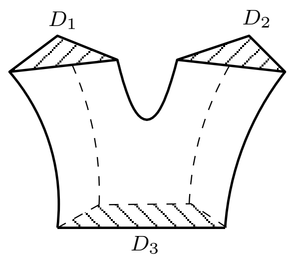

FIGURE 2. A "bordism" Y between $D_1 \sqcup D_2$ and D_3.

the intersection of the stalks over all adjacent cells, and all the restriction maps are inclusions.

Define $\hat{D} := D_1 \sqcup D_2 \sqcup \bar{D}_3 \subset Y$, and write i for $j_{\hat{D}}$ and j for $j_{Y-\hat{D}}$. Let $\hat{U}$ be the interior of Y. We orient Y compatibly with the orientation of $\hat{D}$. The dualizing complex on Y is then $j_{\hat{U},!}F_{\hat{U}}[3]$. The symplectic form induces a pairing $\hat{P} \otimes j_! j^! \hat{P} \to j_{\hat{U},!}F_{\hat{U}}[3]$ and therefore a map

$$\hat{\phi} \colon j_! j^! \hat{P} \to (\mathbb{D}\hat{P})[-3].$$

Lemma 15. *$\hat{\phi}$ is an isomorphism. In particular, it induces isomorphisms*

$$H^m(Y, j_! j^! \hat{P}) \cong H^{3-m}(Y, \hat{P})^*$$

for $m = 0, 1, 2, 3$.

Proof. This can be proved in the same way as Lemma 14. In fact, one can reduce to Lemma 14: the pair $(Y, \hat{P})$ is locally isomorphic to $(D \times I, P \boxtimes F_I)$, where I is the the closed interval, and locally $\hat{\phi}$ is the isomorphism

$$\hat{\phi} = \phi \boxtimes \mathrm{id} \colon P \boxtimes j_{I-\partial I,!}F_{I-\partial I} \to (\mathbb{D}P)[-2] \boxtimes (\mathbb{D}F_I)[-1] = (\mathbb{D}(P \boxtimes F_I))\,[-3].$$

□

Proposition 16. *The image of $\pi \colon H^1(Y, \hat{P}) \to H^1(\hat{D}, i^*\hat{P})$ is isotropic of half the dimension of the total space.*

Proof. It follows from the commutativity of the diagram

$$\begin{array}{ccccc} H^1(Y,\hat{P}) & \xrightarrow{\pi} & H^1(\hat{D}, i^*\hat{P}) & \xrightarrow{\delta} & H^2(Y, j_! j^! \hat{P}) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ H^2(Y, j_! j^! \hat{P})^* & \xrightarrow{\delta^*} & H^1(\hat{D}, i^*\hat{P})^* & \xrightarrow{\pi^*} & H^1(Y, \hat{P})^* \end{array}$$

and the exactness of the rows. □

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