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JOINT REDUCTIONS OF MONOMIAL IDEALS AND MULTIPLICITY OF COMPLEX ANALYTIC MAPS

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Abstract. We characterize the joint reductions of a set of monomial ideals in the ring $\mathcal{O}_n$ of complex analytic functions defined in a neighbourhood of the origin in $\mathbb{C}^n$. We also study an integer $\sigma(I_1,\dots,I_n)$ attached to a family of ideals $I_1,\dots,I_n$ in a Noetherian local ring that extends the usual notion of mixed multiplicity. If $I_1,\dots,I_n$ are monomial ideals of $\mathcal{O}_n$, then we obtain a characterization of the families $g_1,\dots,g_n$ such that $g_i \in I_i$, for all $i=1,\dots,n$, and that $e(g_1,\dots,g_n)=\sigma(I_1,\dots,I_n)$.

1. Introduction

The computation of the integral closure of ideals is one of the central problems in commutative algebra (see [4], [7] or [22]). A key role in the context of this problem is played by the reductions of an ideal, which were defined by Northcott and Rees in [11] (see Section 2). These ideals are very useful in the computation of multiplicities of ideals. For instance, if I is an ideal of $\mathbb{C}[[x_1,\dots,x_n]]$ of finite colength generated by monomials, then the author obtained in [2] a canonical reduction of I that allowed to compute the multiplicity of I in an effective way (we refer [5] for a different approach to the computation of the multiplicity of a monomial ideal).

The notion of reduction of an ideal was generalized by Rees in [14] thus giving the notion of joint reduction of ideals. This notion simplifies the task of computing the mixed multiplicities of ideals, defined by Teissier and Risler in [18]. By a result of Swanson [17], joint reductions of ideals of finite colength are characterized via an equality of mixed multiplicities. This result extends the celebrated Rees' multiplicity theorem (see [7, p. 222]).

In Section 2 we consider an integer attached to an ample class of n-tuples of ideals $I_1,\dots,I_n$ in a Noetherian local ring of dimension n (see Definition 2.4). This integer, that we denote by $\sigma(I_1,\dots,I_n)$, extends the notion of mixed multiplicity of ideals of finite colength defined by Teissier and Risler in [18]. However $\sigma(I_1,\dots,I_n)$ is not defined for arbitrary n-tuples of ideals. We point out that, as a consequence of Proposition 2.9, the integer $\sigma(I_1,\dots,I_n)$ is equal to the multiplicity defined by Rees in [15, p. 181] for certain sets of ideals not necessarily of finite colength.

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In the study of $\sigma(I_1,\dots,I_n)$ we apply results developed by Rees [14] and Swanson [17] concerning joint reductions, mixed multiplicities and integral closures of ideals.

Let us denote by $\mathcal{O}_n$ the ring of analytic function germs $f:(\mathbb{C}^n,0)\to\mathbb{C}$. Let $I_1,\dots,I_n$ be monomial ideals of $\mathcal{O}_n$. Then we give in Section 3 a combinatorial characterization of the joint reductions of $I_1,\dots,I_n$ (see Proposition 3.7). If we assume that $\sigma(I_1,\dots,I_n)<\infty$, then we will apply this result to characterize those analytic maps $g=(g_1,\dots,g_n):(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$ such that $g_i\in I_i$, for all $i=1,\dots,n$, and such that $e(g_1,\dots,g_n)=\sigma(I_1,\dots,I_n)$ (see Theorem 3.10), where $e(g_1,\dots,g_n)$ is the Samuel multiplicity of the ideal of $\mathcal{O}_n$ generated by $g_1,\dots,g_n$. This characterization is expressed via the respective Newton polyhedra of $I_1,\dots,I_n$. The set of such maps is denoted by $\mathcal{R}(I_1,\dots,I_n)$.

If $I_1,\dots,I_n$ are monomial and integrally closed ideals of $\mathcal{O}_n$, then, at the end of the paper, we give a result where an important part of the integral closure of the ideals generated by the components of a map of $\mathcal{R}(I_1,\dots,I_n)$ is computed.

The results that we show in this article will be applied, in a subsequent work, to problems in singularity theory concerning invariants of analytic functions $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$. This is the main reason that we fix the setup of this work in $\mathcal{O}_n$ instead of the ring of formal power series $\mathbb{C}[[x_1,\dots,x_n]]$.

2. Joint reductions of ideals and mixed multiplicities

Let R be a commutative ring. We denote by $\overline{I}$ the integral closure of an ideal I of R. If J and I are ideals of R such that $J\subseteq I$, then J is said to be a *reduction* of I if there exists an integer $r\geqslant 0$ such that $I^{r+1}=JI^r$. This definition is due to Northcott and Rees [11]. It is known that J is a reduction of I if and only if $\overline{I}=\overline{J}$ (see [7, p. 6]). The notion of reduction was generalized by Rees in [14] by defining the notion of joint reduction of a set of ideals.

Definition 2.1. [14] Let $I_1,\dots,I_p$ be ideals of R. Let $g_1,\dots,g_p$ be elements of R such that $g_i\in I_i$, for all $i=1,\dots,p$. The p-tuple $(g_1,\dots,g_p)$ is termed a *joint reduction of* $(I_1,\dots,I_p)$ if and only if the ideal

$$g_1I_2\cdots I_p+g_2I_1I_3\cdots I_p+\cdots+g_pI_1\cdots I_{p-1}$$

is a reduction of $I_1\cdots I_p$.

Let (R,m) be a Noetherian local ring of dimension n. If an ideal I of R is m-primary then we will also say that I has *finite colength*. If I is an ideal of R of finite colength, then we denote by $e(I)$, or by $e(I;R)$, the multiplicity of I in the sense of Samuel (see [7, p. 214]). We will denote the colength of I in R by $\ell(R/I)$.

If $I_1,\dots,I_n$ are ideals of R of finite colength, we denote indistinctly by $e(I_1,\dots,I_n)$ or by $e(I_1,\dots,I_n;R)$ the mixed multiplicity of $I_1,\dots,I_n$ defined by Teissier and Risler in [18]. We also refer to [7, §17] or [17] for the definitions and fundamental results concerning mixed multiplicities of ideals. Here we recall briefly the definition of $e(I_1,\dots,I_n)$. Let us denote by $\mathbb{Z}_+$ the set of non-negative integers. Under the conditions exposed above, let us consider the function $H:\mathbb{Z}_+^n \to \mathbb{Z}_+$ given by

$$H(r_1,\dots,r_n)=\ell\left(\frac{R}{I_1^{r_1}\cdots I_n^{r_n}}\right), \tag{1}$$

for all $(r_1,\dots,r_n)\in\mathbb{Z}_+^n$. Then, it is proven in [18] that there exists a polynomial P in n variables, say $x_1,\dots,x_n$, with rational coefficients and of degree n such that

$$H(r_1,\dots,r_n)=P(r_1,\dots,r_n),$$

for all sufficiently large $r_1,\dots,r_n\in\mathbb{Z}_+$. Moreover, the coefficient of the monomial $x_1\cdots x_n$ in P is an integer. This integer is called the *mixed multiplicity* of $I_1,\dots,I_n$ and is denoted by $e(I_1,\dots,I_n)$.

We remark that if $I_1,\dots,I_n$ are all equal to a given ideal I of R, then $e(I_1,\dots,I_n)=e(I)$. We will need the following known result (see [7, p. 345] or [17, Lemma 2.4]).

Lemma 2.2. *Let R be a Noetherian local ring of dimension $n\geqslant 1$. Let $I_1,\dots,I_n$ be ideals of R of finite colength. Let $g_1,\dots,g_n$ be elements of R such that $g_i\in I_i$, for all $i=1,\dots,n$, and that the ideal $\langle g_1,\dots,g_n\rangle$ has also finite colength. Then*

$$e(g_1,\dots,g_n)\geqslant e(I_1,\dots,I_n).$$

Rees proved in [13] that if $J\subseteq I$ are ideals of a quasi-unmixed Noetherian local ring R, then J is a reduction of I if and only if $e(I)=e(J)$ (see also [7, p. 222]). Moreover, Rees proved in [14, Theorem 2.4] that if $(g_1,\dots,g_n)$ is a joint reduction of $(I_1,\dots,I_n)$, where $I_1,\dots,I_n$ is a set of ideals of finite colength of a local Noetherian ring R, then $e(g_1,\dots,g_n)=e(I_1,\dots,I_n)$ (see also [7, p. 343]). The converse of this result is a nice result of Swanson that we now state.

Theorem 2.3. [17] *Let R be a quasi-unmixed Noetherian local ring. Let $I_1,\dots,I_s$ be ideals and let g_i be an element of I_i, for $i=1,\dots,s$. Suppose that the ideals $I_1,\dots,I_s$ and $\langle g_1,\dots,g_s\rangle$ have the same height s and the same radical. If*

$$e(\langle g_1,\dots,g_s\rangle R_{\mathfrak p};R_{\mathfrak p})=e(I_1R_{\mathfrak p},\dots,I_sR_{\mathfrak p};R_{\mathfrak p}),$$

for each prime ideal $\mathfrak p$ minimal over $\langle g_1,\dots,g_s\rangle$, then $(g_1,\dots,g_s)$ is a joint reduction of $(I_1,\dots,I_s)$.

We now define an invariant, defined in terms of mixed multiplicities of ideals, that is attached to a set of ideals in a Noetherian local ring. The ideals that we consider are not assumed to have finite colength.

Definition 2.4. Let (R,m) be a Noetherian local ring of dimension n. Let $I_1,\dots,I_n$ be ideals of R. Then we define

$$\sigma(I_1,\dots,I_n)=\max_{r\in\mathbb{Z}_+} e(I_1+m^r,\dots,I_n+m^r). \tag{2}$$

The set of integers $\{e(I_1+m^r,\dots,I_n+m^r) : r\in\mathbb{Z}_+\}$ is not bounded in general; therefore $\sigma(I_1,\dots,I_n)$ is not always finite for any family of ideals $I_1,\dots,I_n$. The finiteness of $\sigma(I_1,\dots,I_n)$ is characterized in Proposition 2.9. We remark that if I_i has finite colength, for all $i=1,\dots,n$, then $\sigma(I_1,\dots,I_n)$ equals the mixed multiplicity $e(I_1,\dots,I_n)$.

Proposition 2.5. *Let (R,m) be a quasi-unmixed Noetherian local ring of dimension n. Let $I_1,\dots,I_n$ be ideals of R such that $\sigma(I_1,\dots,I_n)<\infty$ and let $g_1,\dots,g_n$ be elements of R such that $g_i\in I_i$, for all $i=1,\dots,n$, and $\langle g_1,\dots,g_n\rangle$ is an ideal of finite colength. Then $\sigma(I_1,\dots,I_n)=e(g_1,\dots,g_n)$ if and only if there exists an integer $r_0\geqslant 1$ such that $(g_1,\dots,g_n)$ is a joint reduction of $(I_1+m^r,\dots,I_n+m^r)$, for all $r\geqslant r_0$.*

Proof. The *if* part follows as a direct consequence of the expression of mixed multiplicities as the multiplicity of a joint reduction (see the paragraph before Theorem 2.3).

Conversely, if $\sigma(I_1,\dots,I_n)=e(g_1,\dots,g_n)$ then $(g_1,\dots,g_n)$ is a joint reduction of $(I_1+m^r,\dots,I_n+m^r)$, for all $r\gg 0$, as a consequence of Theorem 2.3. □

By virtue of the previous result we give the following definition.

Definition 2.6. Let (R,m) be a local ring of dimension n and let $I_1,\dots,I_n$ be ideals of R. Let $g_i\in I_i$, for $i=1,\dots,n$. We say that $g_1,\dots,g_n$ is a *σ-joint reduction* of $(I_1,\dots,I_n)$ when there exists an integer $r_0\geqslant 1$ such that $(g_1,\dots,g_n)$ is a joint reduction of $(I_1+m^r,\dots,I_n+m^r)$, for all $r\geqslant r_0$.

We will use the following auxiliary result, whose proof appears in [7, p. 134].

Lemma 2.7. *Let (R,m) be a Noetherian local ring and let I be an ideal of R. Then*

$$\overline{I}=\bigcap_{r\geqslant 1}\overline{I+m^r}.$$

Proposition 2.8. *Let (R,m) be a Noetherian local ring of dimension n and let $I_1,\dots,I_n$ be ideals of R. Let $g_i\in I_i$, for $i=1,\dots,n$. If $g_1,\dots,g_n$ is a σ-joint reduction of $(I_1,\dots,I_n)$ then $(g_1,\dots,g_n)$ is a joint reduction of $(I_1,\dots,I_n)$.*

Proof. When $n=1$ the result follows easily from Lemma 2.7. Let us suppose that $n\geqslant 2$. Let us define the ideals

$$P_r=g_1(I_2+m^r)\cdots(I_n+m^r)+\cdots+g_n(I_1+m^r)\cdots(I_{n-1}+m^r)$$
$$Q_r=(I_1+m^r)\cdots(I_n+m^r).$$

Then there exists an integer $r_0\geqslant 1$ such that

$$\overline{Q_r}=\overline{P_r},\qquad \text{for all } r\geqslant r_0. \tag{3}$$

If $j,s\in\{1,\dots,n\}$, we define

$$L_j=\sum_{1\leqslant i_1<\cdots<i_j\leqslant n} I_{i_1}\cdots I_{i_j},\qquad\qquad L_j^s=\sum_{\substack{1\leqslant i_1<\cdots<i_j\leqslant n\\ i_j\neq s}} I_{i_1}\cdots I_{i_j},$$

where in the definition of L_j^s we suppose that $j \leqslant n-1$. Then, a simple computation shows that

$$(4)\quad L_n + m^{r(n-1)}L_1 \subseteq Q_r = L_n + m^r L_{n-1} + \cdots + m^{r(n-1)}L_1 + m^{rn} \subseteq L_n + m^{r+n-1}$$

and that

$$(5)\quad P_r = g_1 L^1_{n-1} + \cdots + g_n L^n_{n-1} + \sum_{i=1}^{n} g_i \left(m^r L^i_{n-2} + \cdots + m^{(n-2)r} L^i_1 + m^{(n-1)r} \right).$$

Let J denote the ideal of R generated by $g_1, \ldots, g_n$. Then

$$(6)\quad g_1 L^1_{n-1} + \cdots + g_n L^n_{n-1} + m^{(n-1)r} J \subseteq P_r \subseteq g_1 L^1_{n-1} + \cdots + g_n L^n_{n-1} + m^{r+n-1}.$$

Then, from Lemma 2.7 and the inclusions given in (4) and (6) we obtain the equalities

$$(7)\qquad \overline{L_n} = \bigcap_{r \geqslant 1} \overline{Q_r}, \qquad \overline{g_1 L^1_{n-1} + \cdots + g_n L^n_{n-1}} = \bigcap_{r \geqslant 1} \overline{P_r}.$$

Therefore, from (3) we have

$$\overline{g_1 L^1_{n-1} + \cdots + g_n L^n_{n-1}} = \bigcap_{r \geqslant r_0} \overline{P_r} = \bigcap_{r \geqslant r_0} \overline{Q_r} = \overline{L_n} = \overline{I_1 \cdots I_n}.$$

This implies that $g_1 L^1_{n-1} + \cdots + g_n L^n_{n-1}$ is a reduction of $I_1 \cdots I_n$, or equivalently, that $(g_1, \ldots, g_n)$ is a joint reduction of $(I_1, \ldots, I_n)$. $\square$

In Example 2.10 we show that the converse of Proposition 2.8 does not hold in general.

Let (R, m) be a local ring of dimension n with $k = R/m$ an infinite field. Let $I_1, \ldots, I_n$ be ideals of R. Let us consider a generating system $a_{i1}, \ldots, a_{is_i}$ of I_i, for $i = 1, \ldots, n$. Let $s = s_1 + \cdots + s_n$. We say that a property holds for *sufficiently general* elements of $I_1 \oplus \cdots \oplus I_n$ if there exists a non-empty Zariski-open set U in k^s such that all elements $(g_1, \ldots, g_n) \in I_1 \oplus \cdots \oplus I_n$ satisfy the said property provided that $g_i = \sum_j u_{ij} a_{ij}$, $i = 1, \ldots, n$, where $(u_{11}, \ldots, u_{1s_1}, \ldots, u_{n1}, \ldots, u_{ns_n}) \in U$.

Proposition 2.9. *Let $I_1, \ldots, I_n$ be ideals of a Noetherian local ring (R, m) such that the residue field $k = R/m$ is infinite. Then $\sigma(I_1, \ldots, I_n) < \infty$ if and only if there exist elements $g_i \in I_i$, for $i = 1, \ldots, n$, such that $\langle g_1, \ldots, g_n \rangle$ has finite colength. In this case, we have that $\sigma(I_1, \ldots, I_n) = e(g_1, \ldots, g_n)$ for sufficiently general elements $(g_1, \ldots, g_n) \in I_1 \oplus \cdots \oplus I_n$.*

Proof. The *if* part is immediate. Let us suppose that $\sigma(I_1, \ldots, I_n) < \infty$. Then there exists a positive integer r_0 such that

$$\sigma(I_1, \ldots, I_n) = e(I_1 + m^r, \ldots, I_n + m^r),$$

for all $r \geqslant r_0$. By the definition of joint reduction we have that if $(a_1, \ldots, a_n)$ is a joint reduction of $(I_1 + m^r, \ldots, I_n + m^r)$ and P denotes the ideal

$$a_1(I_2 + m^r) \cdots (I_n + m^r) + \cdots + a_n(I_1 + m^r) \cdots (I_{n-1} + m^r),$$

then

$$\overline{(I_1 + m^r) \cdots (I_n + m^r)} = \overline{P} \subseteq \overline{\langle a_1, \dots a_n \rangle}.$$

Therefore, we observe that there exists an integer $s \geqslant 1$ such that $m^s \subseteq \overline{\langle a_1, \dots, a_n \rangle}$, for all joint reduction $(a_1, \dots, a_n)$ of $(I_1 + m^{r_0}, \dots, I_n + m^{r_0})$. We can suppose that $s \geqslant r_0$.

By the theorem of existence of joint reductions (see [17, p. 4] or [7, p. 336]), let us consider elements $g_i \in I_i$, for $i = 1, \dots, n$, and elements $h_i \in m^{s+1}$, for $i = 1, \dots, n$, such that $(g_1, \dots, g_n)$ is a joint reduction of $(I_1, \dots, I_n)$ and that $(g_1+h_1, \dots, g_n+h_n)$ is a joint reduction of $(I_1+m^{s+1}, \dots, I_n+m^{s+1})$. Let J be the ideal of R generated by $g_1+h_1, \dots, g_n+h_n$. Then J has finite colength and $e(J) = e(I_1+m^{s+1}, \dots, I_n+m^{s+1})$.

Since $s \geqslant r_0$, we have

$$e(I_1 + m^{r_0}, \dots, I_n + m^{r_0}) = e(I_1 + m^{s+1}, \dots, I_n + m^{s+1}) = e(J).$$

Then it follows that $(g_1+h_1, \dots, g_n+h_n)$ is a joint reduction of $(I_1+m^{r_0}, \dots, I_n+m^{r_0})$ by Theorem 2.3. But this implies that $m^s \subseteq \overline{J}$, by the definition of s.

Hence we have

$$\overline{J} \subseteq \overline{\langle g_1, \dots, g_n \rangle + m \cdot m^s} \subseteq \overline{\langle g_1, \dots, g_n \rangle + m \cdot J}.$$

By the integral Nakayama's Lemma (see [18, p. 324]), we deduce that

$$\overline{J} \subseteq \overline{\langle g_1, \dots, g_n \rangle}.$$

Then $\langle g_1, \dots, g_n \rangle$ has also finite colength. Moreover we have

$$\sigma(I_1, \dots, I_n) = e(J) \geqslant e(g_1, \dots, g_n) \geqslant e(I_1 + m^{r_0}, \dots, I_1 + m^{r_0}) = \sigma(I_1, \dots, I_n).$$

Hence we have

$$e(g_1, \dots, g_n) = \sigma(I_1, \dots, I_n). \tag{8}$$

By the construction of the elements $g_1, \dots, g_n$ that we have considered, we observe that equality (8) is satisfied for sufficiently general elements of $I_1 \oplus \cdots \oplus I_n$, as a consequence of the theorem of existence of joint reductions. □

If $\sigma(I_1, \dots, I_n) < \infty$ then $I_1 + \cdots + I_n$ is an ideal of finite colength in R, by Proposition 2.9. Obviously the converse does not hold. We also have that $e(I_1 + \cdots + I_n) \leqslant \sigma(I_1, \dots, I_n)$, by Lemma 2.2. As a consequence of Rees' multiplicity theorem (see [7, p. 222]) we have that $e(I_1 + \cdots + I_n) = \sigma(I_1, \dots, I_n)$ if and only if any n-tuple $(g_1, \dots, g_n)$ such that $g_i \in I_i$, for all $i = 1, \dots, n$, and satisfying the equality $e(g_1, \dots, g_n) = \sigma(I_1, \dots, I_n)$ generates a reduction of $I_1 + \cdots + I_n$.

Proposition 2.9 shows that, if $\sigma(I_1, \dots, I_n) < \infty$, then $\sigma(I_1, \dots, I_n)$ is equal to the mixed multiplicity of $I_1, \dots, I_n$ defined by Rees in [15, p. 181] via the notion of general extension of a local ring (see [15, p. 145] and [16]). Therefore, we will refer to $\sigma(I_1, \dots, I_n)$ as the *Rees' mixed multiplicity* of $I_1, \dots, I_n$.

By Propositions 2.5 and 2.8 we have that $\sigma(I_1, \dots, I_n) = e(g_1, \dots, g_n)$, where $(g_1, \dots, g_n)$ is a joint reduction of $(I_1, \dots, I_n)$. However, if $\sigma(I_1, \dots, I_n) < \infty$, not every joint reduction of $I_1, \dots, I_n$ generates an ideal of finite colength. Moreover, if

I is the ideal generated by a joint reduction of $I_1, \ldots, I_n$ and we suppose that I has finite colength then it does not hold in general that $e(I) = \sigma(I_1, \ldots, I_n)$. Both facts are shown in the following example.

Example 2.10. Let us consider in $\mathcal{O}_3$ the ideals $I_1 = I_2 = \langle x, y\rangle$ and $I_3 = \langle z\rangle$ and the elements $g_1 = g_2 = x + y$ and $g_3 = z$, where we have fixed the coordinates x, y, z in $\mathbb{C}^3$. It is obvious that $\sigma(I_1, I_2, I_3) = 1$ and that (g_1, g_2, g_3) is a joint reduction of (I_1, I_2, I_3). However g_1, g_2, g_3 do not generate an ideal of finite colength of $\mathcal{O}_3$.

Let us consider the elements $g_1' = x + y + x^3$, $g_2' = x + y + y^3$, $g_3' = z$. Then we observe that (g_1', g_2', g_3') is also a joint reduction of (I_1, I_2, I_3). These elements generate an ideal of finite colength of $\mathcal{O}_3$ but $\sigma(I_1, I_2, I_3) = 1$ and $e(g_1', g_2', g_3') = 3$.

Let (R, m) be a Noetherian local ring of dimension n such that the residue field R/m is infinite. The mixed multiplicity of ideals, as introduced by Risler and Teissier [18] and studied by Rees [14] and Swanson [17], is defined for n ideals $I_1, \ldots, I_n$ of finite colength in R. By the theorem of existence of joint reductions (see [7, p. 336]), we have

$$e(I_1, \ldots, I_n) = e(g_1, \ldots, g_n), \tag{9}$$

where $(g_1, \ldots, g_n)$ is a sufficiently general element of $I_1 \oplus \cdots \oplus I_n$.

We observe that the function defined in (1) and that leads to the definition of $e(I_1, \ldots, I_n)$ is well defined if and only if I_i has finite colength, for all $i = 1, \ldots, n$. However, the multiplicity on the right hand side of (9) could be computed in cases where some of the ideals I_i has not finite colength. By Proposition 2.9, this multiplicity is equal to $\sigma(I_1, \ldots, I_n)$.

If I, J are two ideals of finite colength of R, then we can define for all $i \in \{0, 1, \ldots, n\}$ the multiplicity

$$e_i(I, J) = e(I, \ldots, I, J, \ldots, J), \tag{10}$$

where I is repeated $n - i$ times and J is repeated i times, for all $i = 0, 1, \ldots, n$. If I and J are arbitrary ideals, we define analogously the number $\sigma_i(I, J)$ by replacing in (10) the mixed multiplicity $e(I, \ldots, I, J, \ldots, J)$ by $\sigma(I, \ldots, I, J, \ldots, J)$ (of course, for arbitrary ideals I and J the resulting numbers are not always finite for all $i = 0, 1, \ldots, n$).

If J is an ideal of R, let $J^\infty = \{x \in R : x^s J = 0, \text{ for some } s \geqslant 1\}$. As can be seen in the paper [20] of Trung, one can also define a family of mixed multiplicities $\{e_i(I|J) : i = 0, 1, \ldots, r\}$ of a pair of ideals I, J, where I is assumed to have finite colength, J is an arbitrary ideal of R and $r = \dim(R/(0 : J^\infty)) - 1$. These numbers arise from the coefficients of the homogeneous part of highest degree of the polynomial that coincides with the length function of the bigraded ring

$$R(I|J) = \bigoplus_{(u,v)\in\mathbb{Z}_+^2} I^u J^v / I^{u+1} J^v.$$

We refer to [9], [20], [21] and [23] for the details about this definition.

Let $\ell(J)$ denote the analytic spread of J. The multiplicities $e_i(I|J)$ are not all positive for all $i = 0, 1, \dots, r$. In fact, Trung proved that $e_i(I|J) = 0$, for all $i \geqslant \ell(J)$ (see [20, Corollary 3.6]). Moreover, if $i \in 0, 1, \dots, \mathrm{ht}(J) - 1$, then it is proved in [20, Proposition 4.1] that

$$e_i(I|J) = e\big(a_1, \dots, a_{n-i}, b_1, \dots, b_i\big), \tag{11}$$

where $(a_1, \dots, a_{n-i}, b_1, \dots, b_i)$ is a sufficiently general element of $I \oplus \cdots \oplus I \oplus J \oplus \cdots \oplus J$. We remark that relation (11) shows that $e_i(I|J) = \sigma_i(I, J)$, for all $i \in 0, 1, \dots, \mathrm{ht}(J) - 1$, by Proposition 2.9. However, we show a simple example where the multiplicity on the right hand part of (11) can be positive for $i = \ell(J)$ and therefore it can be expressed as a Rees' mixed multiplicity.

Example 2.11. Let I, J be the ideals in $\mathcal{O}_3$ given by $I = \langle x, y, z \rangle$, $J = \langle x^2, y^2 \rangle$. Then $\ell(J) = 2$ (see [2, Theorem 2.3]) and $\sigma_2(I, J) = \sigma(I, J, J) = 4$.

3. Mixed multiplicities and non-degeneracy

Throughout the remaining text, if no confusion arises, we will denote the maximal ideal of $\mathcal{O}_n$ by m instead of m_n. We say that an ideal I of $\mathcal{O}_n$ is a *monomial ideal* when I is generated by a family of monomials x^k such that $k \in \mathbb{Z}^n_+$, $k \neq 0$. Let $I_1, \dots, I_n$ be a sequence of monomial ideals in $\mathcal{O}_n$ such that $\sigma(I_1, \dots, I_n) < \infty$. In this section we characterize the sets of functions $g_1, \dots, g_n \in \mathcal{O}_n$ such that $g_i \in I_i$, for all $i = 1, \dots, n$, and that $e(g_1, \dots, g_n) = \sigma(I_1, \dots, I_n)$. In order to show our results we will introduce first some definitions and notation.

Let $h \in \mathcal{O}_n$, let us suppose that the Taylor expansion of h around the origin is given by $h = \sum_k a_k x^k$. We define the *support* of h, denoted by $\mathrm{supp}(h)$, as the set of those $k \in \mathbb{Z}^n_+$ such that $a_k \neq 0$. If A is a compact subset of $\mathbb{R}^n_+$, then we denote by h_A the polynomial given by the sum of all terms $a_k x^k$ such that $k \in \mathrm{supp}(h) \cap A$. If $\mathrm{supp}(h) \cap A = \emptyset$, then we set $h_A = 0$. If I is a monomial ideal of $\mathcal{O}_n$, we define the *support* of I, denoted by $\mathrm{supp}(I)$, as the set of $k \in \mathbb{Z}^n_+$ such that $x^k \in I$.

We say that a subset Γ of $\mathbb{R}^n_+$ is a *Newton polyhedron* when there exists some $B \subseteq \mathbb{Q}^n_+$ such that Γ is equal to the convex hull in $\mathbb{R}^n_+$ of the set $\{k + v : k \in B,\, v \in \mathbb{R}^n_+\}$. In this case we say that Γ is the *Newton polyhedron determined by* B and we also denote Γ by $\Gamma(B)$. A Newton polyhedron Γ is termed *convenient* when Γ intersects each coordinate axis in a point different from the origin. In this case, we denote by $\mathrm{V}_n(\Gamma)$ the n-dimensional volume of the set $\mathbb{R}^n_+ \smallsetminus \Gamma$.

If $h \in \mathcal{O}_n$, the *Newton polyhedron of* h is defined as $\Gamma(h) = \Gamma\big(\mathrm{supp}(h)\big)$. Let J be an ideal of $\mathcal{O}_n$, let us suppose that J is generated by the elements $h_1, \dots, h_p$. Then the *Newton polyhedron of* J, denoted by $\Gamma(J)$, is defined as the convex hull of the union $\Gamma(h_1) \cup \cdots \cup \Gamma(h_p)$. It is easy to check that the definition of $\Gamma(J)$ does not depend on the chosen generating system of J.

If $\Gamma^1, \dots, \Gamma^p$ are Newton polyhedra in $\mathbb{R}^n_+$, then we define the *Minkowski sum* of $\Gamma^1, \dots, \Gamma^p$ as

$$\Gamma^1 + \cdots + \Gamma^p = \{k_1 + \cdots + k_p : k_i \in \Gamma^i, \text{ for all } i = 1, \dots, p\}.$$

This set is again a Newton polyhedron, since it is known that $\Gamma^1 + \cdots + \Gamma^p = \Gamma(I_1 \cdots I_p)$, whenever $\Gamma^i = \Gamma(I_i)$, for some monomial ideal $I_i \in \mathcal{O}_n$, $i = 1, \dots, p$ (see for instance [6]).

Let us fix a Newton polyhedron $\Gamma \subseteq \mathbb{R}^n_+$. Given a vector $v \in \mathbb{R}^n_+ \smallsetminus \{0\}$ we define

$$\ell(v, \Gamma) = \min\big\{\langle v, k\rangle : k \in \Gamma\big\}.$$

We say that a subset Δ of Γ is a *face* of Γ if there exists a vector $v \in \mathbb{R}^n_+ \smallsetminus \{0\}$ such that Δ is expressed as

$$\Delta = \big\{k \in \Gamma : \langle v, k\rangle = \ell(v, \Gamma)\big\}. \tag{12}$$

We will denote the set on the right hand side of (12) by $\Delta(v, \Gamma)$ and we will also say that Δ is the face of Γ *supported* by v. We have that $\Delta(v, \Gamma)$ is a compact face of Γ if and only if all components of v are non-zero. If I is an ideal of $\mathcal{O}_n$, then we denote by $\overline{\Gamma}(I)$ the union of the compact faces of $\Gamma(I)$. Moreover, we will denote the face $\Delta(v, \Gamma(I))$, for a given $v \in \mathbb{R}^n_+ \smallsetminus \{0\}$, by $\Delta(v, I)$.

Definition 3.1. Let $I_1, \dots, I_p$ be monomial ideals in $\mathcal{O}_n$. Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be an analytic map germ such that $g_i \in I_i$, for all $i = 1, \dots, p$. Let $v \in \mathbb{R}^n_+ \smallsetminus \{0\}$ and let $\Delta_i = \Delta(v, I_i)$, for all $i = 1, \dots, p$. We say that g satisfies the (K_v) *condition with respect to* $I_1, \dots, I_p$ when

$$\big\{x \in \mathbb{C}^n : (g_1)_{\Delta_1}(x) = \cdots = (g_p)_{\Delta_p}(x) = 0\big\} \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}.$$

Then the map g is termed *non-degenerate with respect to* $I_1, \dots, I_p$ when g satisfies the (K_v) condition with respect to $I_1, \dots, I_p$ for all $v \in (\mathbb{R}_+ \smallsetminus \{0\})^n$.

Under the conditions of the above definition, we observe that if there exists some $i_0 \in \{1, \dots, p\}$ such that g_{i_0} is equal to a monomial x^k, for some $k \in \mathbb{Z}^n_+$, $k \neq 0$, and $I_{i_0} = \langle x^k\rangle$, then the map g is automatically non-degenerate with respect to $I_1, \dots, I_p$.

If $L \subseteq \{1, \dots, n\}$, $L \neq \emptyset$, we define $\mathbb{C}^n_L = \{x \in \mathbb{C}^n : x_i = 0, \text{ for all } i \notin L\}$. The set $\mathbb{R}^n_L$ is defined analogously. If $h \in \mathcal{O}_n$ and the Taylor expansion of h around the origin is given by $h = \sum_k a_k x^k$, we denote by h^L the function obtained as the sum of those terms $a_k x^k$ such that $k \in \operatorname{supp}(h) \cap \mathbb{R}^n_L$. If $\operatorname{supp}(h) \cap \mathbb{R}^n_L = \emptyset$, then we set $h^L = 0$. If $g = (g_1, \dots, g_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is an analytic map germ, then we denote by g^L the map $(g^L_1, \dots, g^L_p) : (\mathbb{C}^n_L, 0) \to (\mathbb{C}^p, 0)$. In some occasions we will identify $\mathbb{C}^n_L$ with $\mathbb{C}^r$, where $r = |L|$.

Let $L = \{i_1, \dots, i_r\} \subseteq \{1, \dots, n\}$, then we denote by $\mathcal{O}_{n,L}$ the subring of $\mathcal{O}_n$ generated by the functions of $\mathcal{O}_n$ depending, at most, on the variables $x_{i_1}, \dots, x_{i_r}$. We denote by m_L the maximal ideal of $\mathcal{O}_{n,L}$. We observe that the map $\mathcal{O}_n \to \mathcal{O}_{n,L}$ given by $h \mapsto h^L$, $h \in \mathcal{O}_n$, is a ring epimorphism. If I is a monomial ideal of $\mathcal{O}_n$ then we denote by I^L the ideal of $\mathcal{O}_{n,L}$ generated by all monomials x^k such that $k \in \operatorname{supp}(I) \cap \mathbb{R}^n_L$. If $\operatorname{supp}(I) \cap \mathbb{R}^n_L = \emptyset$, then we set $I^L = 0$.

Definition 3.2. Let $I_1, \dots, I_p$ be monomial ideals of $\mathcal{O}_n$ such that $I_1 + \cdots + I_p$ is an ideal of finite colength in $\mathcal{O}_n$. Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be an analytic map germ such that $g_i \in I_i$, for all $i = 1, \dots, p$. We say that g is *strongly non-degenerate with respect* *to* $I_1, \dots, I_p$ when for all $L \subseteq \{1, \dots, n\}$, $L \neq \emptyset$, the map $g^L : (\mathbb{C}^n_L, 0) \to (\mathbb{C}^p, 0)$ is non-degenerate with respect to the non-zero ideals of the sequence of ideals $I^L_1, \dots, I^L_p$.

We remark that, since we are assuming in the above definition that $I_1 + \cdots + I_p$ is an ideal of finite colength, then the set of non-zero ideals in the sequence $I^L_1, \dots, I^L_p$ is non-empty, for all $L \subseteq \{1 \dots, n\}$, $L \neq \emptyset$.

Definitions 3.1 and 3.2 are motivated by the notion of Newton non-degenerate ideal (see the paragraph after Remark 3.4), which in turn has its origin in the notion of Newton non-degenerate function. This kind of functions were studied by Kouchnirenko [10] and Yoshinaga [24], among other authors, with the aim of obtaining information about the topology of a given function $h \in \mathcal{O}_n$ (like the Milnor number of h, in the case that h has an isolated singularity at the origin, or the topological determinacy of h) in terms of the Newton polyhedron of h.

Under the conditions of Definition 3.2, we denote the set of analytic maps $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ such that $g_i \in I_i$, for all $i = 1, \dots, p$, and such that g is strongly non-degenerate with respect to $I_1, \dots, I_p$ by $\mathcal{R}(I_1, \dots, I_p)$. Let us remark that if $g \in \mathcal{R}(I_1, \dots, I_p)$ then g_i does not need to have the same Newton polyhedron as I_i, for all $i = 1, \dots, p$.

Example 3.3. Let us consider the ideals I_1, I_2, I_3 of $\mathcal{O}_3$ and the polynomials g'_1, g'_2, g'_3 given in Example 2.10. Then we have that the map $g' : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ defined by $g' = (g'_1, g'_2, g'_3)$ is non-degenerate with respect to I_1, I_2, I_3. If $L = \{1, 2\}$, then $\{i : I^L_i \neq 0\} = \{1, 2\}$. We observe that the map $h = ((g'_1)^L, (g'_2)^L)$ is not non-degenerate with respect to I^L_1, I^L_2, since h does not satisfy the (K_v) condition for $v = (1, 1)$. Therefore g' is not strongly non-degenerate with respect to I_1, I_2, I_3.

Remark 3.4. Let $\Gamma^1, \dots, \Gamma^p$ be a family of Newton polyhedra in $\mathbb{R}^n_+$. It is well known that if Δ is a compact face of $\Gamma^1 + \cdots + \Gamma^p$, then Δ is uniquely expressed as $\Delta_1 + \cdots + \Delta_p$, where Δ_i is face of Γ^i, for all $i = 1, \dots, p$. This expression is a consequence of the following relations:

$$\begin{aligned} \ell(v, \Gamma^1 + \cdots + \Gamma^p) &= \ell(v, \Gamma^1) + \cdots + \ell(v, \Gamma^p) \\ \Delta(v, \Gamma^1 + \cdots + \Gamma^p) &= \Delta(v, \Gamma^1) + \cdots + \Delta(v, \Gamma^p), \end{aligned}$$

for all $v \in \mathbb{R}^n_+ \smallsetminus \{0\}$. Therefore, under the hypothesis of Definition 3.1, the set of non-redundant (K_v) conditions that a non-degenerate map with respect to $I_1, \dots, I_p$ must satisfy is parameterized by the set of compact faces of $\Gamma(I_1) + \cdots + \Gamma(I_p)$. Hence the definition of strongly non-degenerate map with respect to $I_1, \dots, I_p$ consists of a finite set of conditions.

Here we recall the definition of Newton non-degenerate ideal (see [2] or [3]). Let I be an ideal of $\mathcal{O}_n$ and let $g_1, \dots, g_r$ be a generating system of I. Then the ideal I is said to be *Newton non-degenerate* when for each compact face Δ of $\Gamma(I)$ we have

$$\big\{x \in \mathbb{C}^n : (g_1)_\Delta(x) = \cdots = (g_r)_\Delta(x)\big\} \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}.$$

It is straightforward to see that this definition does not depend on the generating system of I. We observe that any monomial ideal is Newton non-degenerate. We recall that, given a function $h \in \mathcal{O}_n$, then h is said to be *Newton non-degenerate* when the ideal of $\mathcal{O}_n$ generated by $x_1\frac{\partial h}{\partial x_1}, \dots, x_n\frac{\partial h}{\partial x_n}$ is Newton non-degenerate.

Moreover, we also have that an ideal I of $\mathcal{O}_n$ is Newton non-degenerate if and only if I admits a generating system $g_1, \dots, g_r$ such that the map $(g_1, \dots, g_r) : (\mathbb{C}^n, 0) \to (\mathbb{C}^r, 0)$ is non-degenerate with respect to $I, \dots, I$, with I repeated r times (see Definition 3.1). If I is an ideal of finite colength, then this condition is equivalent to saying that $(g_1, \dots, g_r) \in \mathcal{R}(I, \dots, I)$, where I is repeated r times (see also Corollary 3.8).

The next result shows a numerical characterization of the Newton non-degeneracy condition (we refer to [1] for the definition and characterization of the Newton non-degeneracy condition in the context of submodules of the free module $\mathcal{O}_n^p$, $p \geqslant 1$).

Theorem 3.5. [2, 3] *Let I be an ideal of $\mathcal{O}_n$ of finite colength. Then $e(I) \geqslant n!\mathrm{V}_n\big(\Gamma(I)\big)$ and equality holds if and only if I is a Newton non-degenerate ideal.*

Given an ideal J of $\mathcal{O}_n$ and a fixed coordinate system in $\mathbb{C}^n$, we denote by J_0 the ideal of $\mathcal{O}_n$ generated by all monomials x^k such that $k \in \Gamma(J)$. The ideal J_0 is integrally closed (see [7, p. 11] or [19]). Therefore, from the inclusions $J \subseteq \overline{J} \subseteq \overline{J_0} = J_0$, we deduce that $\Gamma(J) = \Gamma(\overline{J})$.

Proposition 3.6. *Let I be a Newton non-degenerate ideal of $\mathcal{O}_n$ and let $J \subseteq I$. Then the following conditions are equivalent:*

(1) *J is a reduction of I;*
(2) *J is Newton non-degenerate and $\Gamma(J) = \Gamma(I)$;*
(3) *there exists a generating system $g_1, \dots, g_r$ of J such that, for all compact face Δ of $\Gamma(I)$, we have*

$$\big\{x \in \mathbb{C}^n : (g_1)_\Delta(x) = \cdots = (g_r)_\Delta(x)\big\} \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}. \tag{13}$$

Proof. We point out that the ideal I is not assumed to have finite colength. Let us see (1) $\Rightarrow$ (2). Suppose that J is a reduction of I. Then $\overline{I} = \overline{J}$ and, in particular, we have that $\Gamma(I) = \Gamma(J)$. Moreover we also deduce that

$$\overline{I + m^r} = \overline{\overline{I} + m^r} = \overline{\overline{J} + m^r} = \overline{J + m^r}, \tag{14}$$

for all $r \geqslant 1$. Using relation (14) and the fact that $\overline{I + m^r}$ is a monomial ideal of finite colength, it follows that

$$n!\mathrm{V}_n\big(\Gamma(J + m^r)\big) = n!\mathrm{V}_n\big(\Gamma(I + m^r)\big) = e(I + m^r) = e(J + m^r),$$

by Theorem 3.5. Therefore the ideal $J + m^r$ is Newton non-degenerate, for all $r \geqslant 1$, by virtue of Theorem 3.5. Let r_0 a positive integer such that each compact face Δ of $\Gamma(J)$ is a compact face of $\Gamma(J + m^r)$, for all $r \geqslant r_0$. Therefore, by writing down the condition that $J + m^r$ is Newton non-degenerate, for all $r \geqslant r_0$, we conclude that J is Newton non-degenerate.

Let us see (2) $\Rightarrow$ (1). We will see that item (2) implies that $\overline{I} = \overline{J}$. In particular, we will have that J is a reduction of I, since $J \subseteq I$ (see [7, p. 6]). As before, let us consider a big enough positive integer r_0 such that each compact face of $\Gamma(J)$ is a compact face of $\Gamma(J+m^r)$, for all $r \geqslant r_0$. Then we have that $J+m^r$ is Newton non-degenerate, for all $r \geqslant r_0$. Hence $e(J+m^r) = n!\mathrm{V}_n(\Gamma(J+m^r))$, for all $r \geqslant r_0$. This implies, by Rees' multiplicity theorem, that

$$\overline{J+m^r} = (J+m^r)_0 = (J_0+m^r)_0 = \overline{J_0+m^r}, \text{ for all } r \geqslant r_0.$$

By Lemma 2.7, we have

$$\overline{J} = \bigcap_{r\geqslant r_0} \overline{J+m^r} = \bigcap_{r\geqslant r_0} \overline{J_0+m^r} = \overline{J_0} = J_0. \tag{15}$$

Since $J \subseteq I$ and $\Gamma(I) = \Gamma(J)$, then $\overline{J} \subseteq \overline{I} \subseteq I_0 = J_0$. Then relation (15) implies that $\overline{I} = \overline{J}$.

The implication (2) $\Rightarrow$ (3) is obvious. In order to see the implication (3) $\Rightarrow$ (2) it suffices to prove that $\Gamma(I) = \Gamma(J)$. Let $g_1, \dots, g_r$ be a generating system of J verifying the inclusion (13), for all compact face Δ of $\Gamma(I)$. In particular, if Δ is a vertex of $\Gamma(I)$, then this condition must be satisfied for Δ. This implies that if Δ is any vertex of $\Gamma(I)$, then some function $(g_i)_\Delta$ is not identically zero. Thus $\Gamma(I) \subseteq \Gamma(J)$. But since we assume that $J \subseteq I$, we have that $\Gamma(I) = \Gamma(J)$. □

The previous proposition gives the family of all reductions of a given monomial ideal. Rees and Sally [16] defined the *core* of an ideal I in a commutative ring as the intersection of all reductions of I; it is denoted by $\mathrm{core}(I)$. In particular, by Proposition 3.6, the computation of the core of a monomial and integrally closed ideal I in $\mathcal{O}_n$, or in $\mathbb{C}[[x_1, \dots, x_n]]$, reduces to compute the intersection of all ideals J of $\mathcal{O}_n$ such that $\Gamma(I) = \Gamma(J)$ and J is Newton non-degenerate. We remark that the study of the core of an ideal is quite an active research topic in commutative algebra (see for instance [8] or [12]).

In the next result we show a characterization of the joint reductions of a family of monomial ideals.

Proposition 3.7. *Let $I_1, \dots, I_p$ be monomial ideals of $\mathcal{O}_n$. Let $g_1, \dots, g_p \in \mathcal{O}_n$ such that $g_i \in I_i$, for all $i = 1, \dots, p$. Then the following conditions are equivalent:*

1. *$(g_1, \dots, g_p)$ is a joint reduction of $(I_1, \dots, I_p)$;*
2. *the map $g = (g_1, \dots, g_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is non-degenerate with respect to $I_1, \dots, I_p$.*

Proof. Let us consider the ideal J of $\mathcal{O}_n$ given by

$$J = g_1 I_2 \cdots I_p + g_2 I_1 I_3 \cdots I_p + \cdots + g_p I_1 \cdots I_{p-1}. \tag{16}$$

By Definition 2.1, we have that $(g_1, \dots, g_p)$ is a joint reduction of $(I_1, \dots, I_p)$ if and only if J is a reduction of the monomial ideal $I_1 \cdots I_p$. Let I denote the ideal $I_1 \cdots I_p$, then $J \subseteq I$. Therefore, item (1) holds if and only if J satisfies item (3) of Proposition 3.6 with respect to $\Gamma(I)$.

Let $\Gamma = \Gamma(I)$, we remark that Γ is equal to the Minkowski sum $\Gamma(I_1) + \cdots + \Gamma(I_p)$. Let B denote the set $\{1, \dots, p\}$. From the definition of J we have that there exist finite subsets $S_1, \dots, S_p \subseteq \mathbb{Z}_+^n$ such that the set $\mathcal{J}$ of functions given by

$$\begin{aligned}\mathcal{J} =& \left\{g_1 x^{k_2+\cdots+k_p} : k_i \in S_i, i \in B, i \neq 1\right\} \cup \left\{g_2 x^{k_1+k_3+\cdots+k_p} : k_i \in S_i, i \in B, i \neq 2\right\} \\ &\cup \cdots \cup \left\{g_p x^{k_1+\cdots+k_{p-1}} : k_i \in S_i, i \in B, i \neq p\right\}\end{aligned}$$

is a generating system of J. Let us fix a compact face Δ of $\Gamma(I)$. Then Δ is expressed univocally as $\Delta = \Delta_1 + \cdots + \Delta_p$, where Δ_i is a compact face of $\Gamma(I_i)$, for all $i = 1, \dots, p$.

If h is an element of $\mathcal{J}$, then there exists an $i_0 \in B$ such that

$$h = g_{i_0} x^{k_1+\cdots+k_{i_0-1}+k_{i_0+1}+\cdots+k_p},$$

for some $k_i \in S_i$, $i \neq i_0$. Therefore h_Δ is expressed as

$$h_\Delta = \left(g_{i_0}\right)_{\Delta_{i_0}} \left(x^{k_1}\right)_{\Delta_1} \cdots \left(x^{k_{i_0-1}}\right)_{\Delta_{i_0-1}} \left(x^{k_{i_0+1}}\right)_{\Delta_{i_0+1}} \cdots \left(x^{k_p}\right)_{\Delta_p}.$$

Then the set of common zeros of $\{h_\Delta : h \in \mathcal{J}\}$ in $(\mathbb{C} \smallsetminus \{0\})^n$ is equal to the set of common zeros of $\{(g_i)_{\Delta_i} : i = 1, \dots, p\}$ in $(\mathbb{C} \smallsetminus \{0\})^n$. This fact shows that item (3) of Proposition 3.6 applied to the ideals J and I holds if and only if the map g is non-degenerate with respect to $I_1, \dots, I_p$. Thus the equivalence between (1) and (2) follows. □

Corollary 3.8. *Let $I_1, \dots, I_p$ be monomial ideals of finite colength of $\mathcal{O}_n$. Let $g_1, \dots, g_p \in \mathcal{O}_n$ such that $g_i \in I_i$, for all $i = 1, \dots, p$. Let $g = (g_1, \dots, g_p)$, then $g \in \mathcal{R}(I_1, \dots, I_p)$ if and only if g is non-degenerate with respect to $I_1, \dots, I_p$.*

Proof. The *only if* part is obvious. Let us suppose that g is non-degenerate with respect to $I_1, \dots, I_p$. Therefore $(g_1, \dots, g_p)$ is a joint reduction of $(I_1, \dots, I_p)$, by Proposition 3.7. This means that J is a reduction of $I_1 \cdots I_p$, where J is the ideal defined in (16). In particular, for a given $L \subseteq \{1, \dots, n\}$, $L \neq \emptyset$, we have that J^L is a reduction of $(I_1 \cdots I_p)^L = I_1^L \cdots I_p^L$, since reductions are stable under ring morphisms. Therefore $(g_1^L, \dots, g_p^L)$ is a joint reduction of $(I_1^L, \dots, I_p^L)$. We have that $I_i^L \neq 0$, for all $i = 1, \dots, p$, since each ideal I_i has finite colength. Then the result follows as a consequence of Proposition 3.7. □

Given an integer $r \geqslant 1$ and a subset $L \subseteq \{1, \dots, n\}$, we denote by $\delta_{L,r}$ the convex hull in $\mathbb{R}^n$ of $\{re_i : i \in L\}$, where $e_1, \dots, e_n$ denotes the canonical basis in $\mathbb{R}^n$.

If I is an ideal of $\mathcal{O}_n$, $I \neq 0$, then we denote by $\mathrm{ord}(I)$ the maximum of those integers $s \geqslant 1$ such that $I \subseteq m^s$.

Lemma 3.9. *Let $I_1, \dots, I_n$ be monomial ideals in $\mathcal{O}_n$ such that $I_1 + \cdots + I_n$ has finite colength. Let us consider, for a given integer $r \geqslant 1$, the ideal $Q_r = (I_1 + m^r) \cdots (I_n + m^r)$. Then, there exists an integer $r_0 \geqslant 1$ such that for all $r \geqslant r_0$ the following hold:*

(1) *every compact face of $\Gamma(I_1 \cdots I_n)$ is a compact face of Q_r;*

(2) *let* Δ *be a face of* $\Gamma(Q_r)$ *not intersecting* $\overline{\Gamma}(I_1\cdots I_n)$*, let us write* Δ *as* $\Delta = \Delta_1+\cdots+\Delta_n$*, where* Δ_i *is a face of* I_i+m^r*, for all* $i=1,\dots,n$*, and let* $S=\{i:\Delta_i\cap\overline{\Gamma}(I_i)\neq\emptyset\}$*; then* $S\neq\emptyset$ *and there exists some* $L\subsetneq\{1,\dots,n\}$ *such that*

$$\Delta=\sum_{i\in S}\Delta_i+\big(n-|S|\big)\delta_{L,r}$$

and Δ_i *is a face of* $\overline{\Gamma}(I_i^L)$ *if* $I_i^L\neq 0$*.*

Proof. Let us define, for a given integer $j\in\{1,\dots,n\}$, the ideal

$$L_j=\sum_{1\leqslant i_1<\cdots<i_j\leqslant n} I_{i_1}\cdots I_{i_j}.$$

Since the ideal L_1 has finite colength, then there exists an integer $r_0\geqslant 1$ such that $m^{r_0}\subseteq L_1$. Then, for any integer $r\geqslant r_0$, we observe that Q_r is expressed as

$$Q_r=L_n+m^rL_{n-1}+\cdots+m^{r(n-1)}L_1. \tag{17}$$

Relation (17) shows that we can increase the integer r in order to have that any compact face of L_n is a compact face of Q_r. Then item (1) holds.

If $v=(v_1,\dots,v_n)\in\mathbb{R}^n$ then we define $v_0=\min_i v_i$. We also define $L(v)=\{i: v_i=v_0\}$. For any vector $v\in(\mathbb{R}_+\smallsetminus\{0\})^n$ and any $r\geqslant 1$ we have

$$\ell(v,m^r)=rv_0 \quad\text{and}\quad \Delta(v,m^r)=\delta_{L(v),r}. \tag{18}$$

Let us suppose that $r>\operatorname{ord}(I_i^L)$, for all $i=1,\dots,n$ and all $L\subseteq\{1,\dots,n\}$, $L\neq\emptyset$. Let $v\in(\mathbb{R}_+\smallsetminus\{0\})^n$ and let $i\in\{1,\dots,n\}$ such that $I_i^{L(v)}\neq 0$. Then

$$\ell(v,I_i)\leqslant \ell(v,I_i^{L(v)})=\operatorname{ord}(I_i^{L(v)})v_0<rv_0=\ell(v,m^r).$$

In particular, there exists an integer $r_1\geqslant r_0$ such that for all $r\geqslant r_1$ we have

$$\Delta(v,I_i+m^r)\cap\Delta(v,I_i)\neq\emptyset, \tag{19}$$

for all vector $v\in(\mathbb{R}_+\smallsetminus\{0\})^n$ and all i such that $I_i^{L(v)}\neq 0$.

Let us consider an integer $r_2\geqslant r_1$ such that each compact face of $\Gamma(I_i)$ is a compact face of $\Gamma(I_i+m^r)$, for all $i=1,\dots,n$ and all $r\geqslant r_2$. Then the number of compact faces of $\Gamma(I_i+m^r)$ does not depend on r, if $r\geqslant r_2$, for all $i=1,\dots,n$. In particular, there exists an integer $r_3\geqslant r_2$ such that the number of compact faces of $\Gamma(Q_r)$ does not depend on r if $r\geqslant r_3$.

For each face Δ of $\Gamma(Q_{r_3})$, let us choose a vector v_Δ such that $\Delta=\Delta(v_\Delta,Q_{r_3})$. Let us consider the decomposition $\Delta=\Delta_1+\cdots+\Delta_n$, where $\Delta_i=\Delta(v_\Delta,I_i+m^{r_3})$, for all $i=1,\dots,n$.

Let us suppose that Δ is face of $\Gamma(Q_{r_3})$ such that $\Delta\cap\overline{\Gamma}(I_1\cdots I_n)=\emptyset$. Then the set $S=\{i:\Delta_i\cap\overline{\Gamma}(I_i)\neq\emptyset\}$ is non-empty, by (17). Moreover, if L denotes the set $L(v_\Delta)$, then $\{i: I_i^L\neq 0\}\subseteq S$, by (19). In particular, if $i\notin S$ then $I_i^L=0$ and $\Delta_i=\delta_{L,r_3}$, by (18).

We remark that, for a given $i\in\{1,\dots,n\}$, any face of $\Gamma(I_i+m^r)$, for $r\geqslant r_2$, is determined by its intersection with $\Gamma(I_i)$ and its intersection with the family of the coordinate axis. Then the vector v_Δ is integrated in a natural way in a family of vectors v_Δ^r, for $r \geqslant r_3$, satisfying

$$\begin{aligned}\Delta(v_\Delta^r, I_i + m^r) \cap \overline{\Gamma}(I_i) &= \Delta_i \cap \overline{\Gamma}(I_i), \text{ for all } i \in S\\ \Delta(v_\Delta^r, I_i + m^r) \cap \overline{\Gamma}(m^r) &= \delta_{L,r}, \text{ for all } i \notin S\\ L(v_\Delta^r) &= L.\end{aligned}$$

Then we can consider an integer $r_\Delta \geqslant r_3$ such that if $i \in S$ verifies that $I_i^L \neq 0$, then

$$\Delta(v_\Delta^r, I_i + m^r) \subseteq \Delta(v_\Delta^r, I_i) \cap \mathbb{R}^n_L,$$

for all $r \geqslant r_\Delta$. Hence, if $r \geqslant r_\Delta$, the face Δ is written as

$$\Delta = \sum_{i \in S} \Delta_i + (n - |S|)\delta_{L,r},$$

where Δ_i is a face of $\overline{\Gamma}(I_i^L)$ for all $i \in S$ such that $I_i^L \neq 0$. □

Theorem 3.10. *Let $I_1, \dots, I_n$ be monomial ideals of $\mathcal{O}_n$. Suppose that $\sigma(I_1, \dots, I_n)$ is finite. Let $g_1, \dots, g_n \in \mathcal{O}_n$ such that $g_i \in I_i$, for all $i = 1, \dots, n$. Then the following conditions are equivalent:*

(1) *the ideal $\langle g_1, \dots, g_n \rangle$ has finite colength and $\sigma(I_1, \dots, I_n) = e(g_1, \dots, g_n)$;*
(2) *$g \in \mathcal{R}(I_1, \dots, I_n)$.*

Proof. Let g denote the map $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ given by $g = (g_1, \dots, g_n)$. For a given $r \geqslant 1$ we define the ideals

$$\begin{aligned}P_r &= g_1(I_2 + m^r) \cdots (I_n + m^r) + \cdots + g_n(I_1 + m^r) \cdots (I_{n-1} + m^r)\\ Q_r &= (I_1 + m^r) \cdots (I_n + m^r).\end{aligned}$$

Let us see that (1) implies (2). By Nakayama's Lemma we can suppose that g_i is a polynomial, for all $i = 1, \dots, n$. By Proposition 2.5, $(g_1, \dots, g_n)$ is a σ-joint reduction of $(I_1, \dots, I_n)$. In particular, it is a joint reduction of $(I_1, \dots, I_n)$, by Proposition 2.8. Therefore g is non-degenerate with respect to $(I_1, \dots, I_n)$, by Proposition 3.7.

Let r_0 be an integer such that P_r is a joint reduction of Q_r, for all $r \geqslant r_0$. Let us fix a subset $L \subsetneq \{1, \dots, n\}$, $L \neq \emptyset$, and an integer $r \geqslant r_0$. Since reductions are stable under ring morphisms, we have that P_r^L is a reduction of Q_r^L. Therefore the map g^L is non-degenerate with respect to $(I_1 + m^r)^L, \dots, (I_n + m^r)^L$, by Proposition 3.7. Let us remark that $(I_i + m^r)^L \neq 0$, for all $i = 1, \dots, n$.

Let $C = \{i : I_i^L \neq 0\}$. The condition $\sigma(I_1, \dots, I_n) < \infty$ implies that $I_1 + \cdots + I_n$ has finite colength. Therefore $C \neq \emptyset$. Without loss of generality we can suppose that $C = \{1, \dots, s\}$, for some $1 \leqslant s \leqslant n$. We have to see that $(g_1^L, \dots, g_s^L) : (\mathbb{C}^n_L, 0) \to (\mathbb{C}^s, 0)$ is non-degenerate with respect to $I_1^L, \dots, I_s^L$.

Since g_i is a polynomial, for all $i = 1, \dots, n$, let us assume that

$$\operatorname{supp}(g_i) \cap \Gamma(m^r) = \emptyset, \text{ for all } i = 1, \dots, n. \tag{20}$$

Let $H = (I_1^L + m_L^r) \cdots (I_s^L + m_L^r)$. Then $Q_r^L = H m_L^{r(n-s)}$. In particular, we have

$$\Gamma(Q_r^L) = \Gamma(H) + \Gamma(m_L^{r(n-s)}). \tag{21}$$

By Lemma 3.9 (1) we can suppose that r_0 is big enough in order to have that each compact face of $I_1^L \cdots I_s^L$ is a compact face of H. This fact together with (21) implies that if v is a vector of $(\mathbb{R}_+ \smallsetminus \{0\})^q$, where $q = |L|$, then the set

$$\Delta(v, I_1^L \cdots I_s^L) + \Delta(v, m_L^{r(n-s)}) \tag{22}$$

is a compact face of $\Gamma(Q_r^L)$.

By hypothesis the map g^L is non-degenerate with respect to $(I_1 + m^r)^L, \dots, (I_n + m^r)^L$. Then g^L verifies the (K_v) condition with respect to these ideals (see Definition 3.1). Therefore, writing down this condition and considering (20) and (22), we have

$$\big\{x \in \mathbb{C}_L^n : (g_1^L)_{\Delta_1}(x) = \cdots = (g_s^L)_{\Delta_s}(x) = 0\big\} \subseteq \big\{x \in \mathbb{C}_L^n : \prod_{i \in L} x_i = 0\big\},$$

where $\Delta_i = \Delta(v, I_i^L)$, for all $i = 1, \dots, s$. This shows that the map $(g_1^L, \dots, g_s^L) : (\mathbb{C}_L^n, 0) \to (\mathbb{C}^s, 0)$ is non-degenerate with respect to $I_1^L, \dots, I_s^L$. Since we started from an arbitrary $L \subsetneq \{1, \dots, n\}$, it follows that $g \in \mathcal{R}(I_1, \dots, I_n)$.

Let us see that (2) implies (1). Let us suppose that $g \in \mathcal{R}(I_1, \dots, I_n)$. By Proposition 2.5 and Proposition 3.7, item (1) holds if and only if there exists an integer r_0 such that g is non-degenerate with respect to $I_1 + m^r, \dots, I_n + m^r$, for all $r \geqslant r_0$.

Let r_0 be an integer such that items (1) and (2) of Lemma 3.9 hold for all $r \geqslant r_0$. Let us fix an integer $r \geqslant r_0$ and let us fix a compact face Δ of $\Gamma(Q_r)$. Let us write Δ as $\Delta = \Delta_1 + \cdots + \Delta_n$, where Δ_i is a face of $\Gamma(I_i + m^r)$, for all $i = 1, \dots, n$. We have to see that

$$\big\{x \in \mathbb{C}^n : (g_1)_{\Delta_1}(x) = \cdots = (g_n)_{\Delta_n}(x) = 0\big\} \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}. \tag{23}$$

Let $\Delta' = \Delta \cap \Gamma(I_1 \cdots I_n)$ and let $\Delta'_i = \Delta_i \cap \Gamma(I_i)$, for all $i = 1, \dots, n$. If $\Delta' \neq \emptyset$, then $\Delta' = \Delta'_1 + \cdots + \Delta'_n$ and $(g_i)_{\Delta_i} = (g_i)_{\Delta'_i}$, for all $i = 1, \dots, n$. Thus inclusion (23) holds, since g is non-degenerate with respect to $I_1, \dots, I_n$ by hypothesis.

Let us suppose that $\Delta' = \emptyset$. By Lemma 3.9, there exists a subset $L \subsetneq \{1, \dots, n\}$ such that, if $S = \{i : \Delta'_i \neq \emptyset\}$ and $C_L = \{i : I_i^L \neq 0\}$, then $C_L \subseteq S$ and Δ is written as

$$\Delta = \sum_{i \in S} \Delta_i + (n - |S|)\delta_{L,r}.$$

Let us suppose that $C_L = \{i_1, \dots, i_s\}$, for some $1 \leqslant i_1 < \cdots < i_s \leqslant n$, $s \leqslant t$, where $t = |S|$. Therefore we have

$$\Delta = \Delta^1 + \Delta^2,$$

where Δ^1 is a face of $m^{r(n-t)} I_{i_1}^L \cdots I_{i_s}^L$ and $\Delta^2 = \sum_{i \in S \smallsetminus C_L} \Delta_i$.

Then we observe that the set of common zeros of $(g_1)_{\Delta_1}, \dots, (g_n)_{\Delta_n}$ is contained in the set of common zeros of $(g_{i_1})_{\Delta'_{i_1}}, \dots, (g_{i_s})_{\Delta'_{i_s}}$.

Since Δ_i' is a face of I_i^L, for all $i \in C_L$, then $(g_i)_{\Delta_i'} = (g_i^L)_{\Delta_i'}$, for all $i \in C_L$. Then the inclusion (23) follows, since the map $(g_{i_1}^L, \dots, g_{i_s}^L) : (\mathbb{C}_L^n, 0) \to (\mathbb{C}^s, 0)$ is non-degenerate with respect to $I_{i_1}^L, \dots, I_{i_s}^L$, by hypothesis. $\square$

Let us suppose that $I_1, \dots, I_n$ are ideals of finite colength of $\mathcal{O}_n$. Then Rees showed in [14] that the mixed multiplicity $e(I_1, \dots, I_n)$ can be computed in terms of Samuel multiplicities via the following formula:

$$e(I_1, \dots, I_n) = \frac{1}{n!} \sum_{\substack{J \subseteq \{1,\dots,n\} \\ J \neq \emptyset}} (-1)^{n-|J|} e\bigg(\prod_{j \in J} I_j \bigg).$$

If we assume that I_i is a monomial ideal for all $i = 1, \dots, n$, then $e(\prod_{j \in J} I_j)$ can be computed effectively using [2], for all $J \subseteq \{1, \dots, n\}$, $J \neq \emptyset$. That is, we can apply [2, Theorem 5.1] to deduce that if f_J denotes the polynomial given by the sum of all x^k such that k is a vertex of $\Gamma(\prod_{j \in J} I_j)$, for all non-empty $J \subseteq \{1, \dots, n\}$, then

$$e(I_1, \dots, I_n) = \frac{1}{n!} \sum_{\substack{J \subseteq \{1,\dots,n\} \\ J \neq \emptyset}} (-1)^{n-|J|} \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle x_1 \frac{\partial f_J}{\partial x_1}, \dots, x_n \frac{\partial f_J}{\partial x_n} \rangle}.$$

Thus we have an effective method to compute the mixed multiplicity $e(I_1, \dots, I_n)$ when I_i are monomial ideals of finite colength of $\mathcal{O}_n$. Let us suppose now that some of these ideals do not have finite colength but still $\sigma(I_1, \dots, I_n) < \infty$. Then, by the above discussion, the effective computation of $\sigma(I_1, \dots, I_n)$ reduces to compute some $r \geqslant 1$ such that $\sigma(I_1, \dots, I_n) = e(I_1 + m^r, \dots, I_n + m^r)$. If $g = (g_1, \dots, g_n) \in \mathcal{R}(I_1, \dots, I_n)$, then we found in the proof of Theorem 3.10 that g is non-degenerate with respect to $I_1 + m^r, \dots, I_n + m^r$, when r is an integer such that $\Gamma(Q_r)$ satisfy conditions (1) and (2) of Lemma 3.9. Hence $e(g_1, \dots, g_n) = e(I_1 + m^r, \dots, I_n + m^r)$ and therefore $\sigma(I_1, \dots, I_n) = e(I_1 + m^r, \dots, I_n + m^r)$. Obviously, the problem of finding an integer r satisfying these conditions is easy when $n = 2$, and needs a more careful analysis in higher dimensions.

To end the paper we show a result about the computation of the monomials which are integral over the ideal generated by the components of a given map of $\mathcal{R}(I_1, \dots, I_n)$.

Proposition 3.11. *Let $I_1, \dots, I_n$ monomial ideals of $\mathcal{O}_n$ such that $\sigma(I_1, \dots, I_n) < \infty$. Let $g = (g_1, \dots, g_n) \in \mathcal{R}(I_1, \dots, I_n)$. Then*

$$I_1 \cap \cdots \cap I_n \subseteq \overline{\langle g_1, \dots, g_n \rangle}.$$

Proof. Let J be the ideal of $\mathcal{O}_n$ generated by $g_1, \dots, g_n$. Let x^k be a monomial in $\mathcal{O}_n$. By Rees' multiplicity theorem we know that $x^k \in \overline{J}$ if and only if $e(J) = e(J, x^k)$ (see [7, p. 222]).

By a result of Northcott-Rees (see [7, p. 166] or [11]), we can consider general $\mathbb{C}$-linear combinations $h_1, \dots, h_n$ of $g_1, \dots, g_n, x^k$ such that the ideal H generated by $h_1, \dots, h_n$ is a reduction of $J + \langle x^k \rangle$. Then $e(H) = e(J, x^k)$. Therefore, let A be a squared matrix of size n with entries in $\mathbb{C}$ and let B be a row matrix with n columns with entries in $\mathbb{C}$ such that

$$\begin{bmatrix} g_1 & \cdots & g_n & x^k \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} h_1 & \cdots & h_n \end{bmatrix}. \tag{24}$$

Since the coefficients of A are generic, we can suppose that A is invertible. In particular, multiplying both sides of (24) by A^{-1}, we obtain

$$\begin{bmatrix} g_1 & \cdots & g_n & x^k \end{bmatrix} \begin{bmatrix} \mathbf{I}_n \\ BA^{-1} \end{bmatrix} = \begin{bmatrix} h_1 & \cdots & h_n \end{bmatrix} A^{-1}, \tag{25}$$

where $\mathbf{I}_n$ denotes the identity matrix of size n. We observe that the entries of the left hand side of (25) are of the form $g_1 + \alpha_1 x^k, \dots, g_n + \alpha_n x^k$, for some $\alpha_1, \dots, \alpha_n \in \mathbb{C}$. Relation (25) implies that $H = \langle g_1 + \alpha_1 x^k, \dots, g_n + \alpha_n x^k \rangle$. Then, we have

$$e(J) \geqslant e(J, x^k) = e(H) = e(g_1 + \alpha_1 x^k, \dots, g_n + \alpha_n x^k), \tag{26}$$

for some $\alpha_1, \dots, \alpha_n \in \mathbb{C}$. If $x^k \in I_1 \cap \cdots \cap I_n$, then $e(g_1 + \alpha_1 x^k, \dots, g_n + \alpha_n x^k) \geqslant \sigma(I_1, \dots, I_n)$, by Lemma 2.2. But by Theorem 3.10, the equality $e(J) = \sigma(I_1, \dots, I_n)$ holds, since we assume that $g \in \mathcal{R}(I_1, \dots, I_n)$. Then (26) implies that $e(J) = e(J, x^k)$ and hence $x^k \in \overline{J}$. □

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