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A VANISHING THEOREM FOR ORIENTED INTERSECTION MULTIPLICITIES

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Abstract. Let A be a regular local ring containing $1/2$, which is either equicharacteristic, or is smooth over a d.v.r. of mixed characteristic. We prove that the product maps on derived Grothendieck-Witt groups of A satisfy the following property: given two elements with supports which do not intersect properly, their product vanishes. This gives an analogue for "oriented intersection multiplicities" of Serre's vanishing result for intersection multiplicities. It also suggests a Vanishing Conjecture for arbitrary regular local rings containing $1/2$, which is analogous to Serre's (which was proved independently by Roberts, and Gillet and Soulé).

1. Introduction

Let A be a regular local ring of dimension d and let M and N be finitely generated A-modules such that $M \otimes N$ is of finite length. Serre defined the intersection multiplicity $\chi_A(M,N)$ by

$$\chi_A(M,N) = \sum_{i=0}^{d} (-1)^i \ell_A[\mathrm{Tor}_i^A(M,N)].$$

Here ℓ_A denotes the length of an A-module of finite length. It has been shown that the intersection multiplicity satisfies Serre's Vanishing Conjecture: if $\dim M + \dim N < d$, then $\chi(M,N) = 0$. Serre proved this in [18] for A which is equicharacteristic, or is smooth [1] over a d.v.r. of mixed characteristic; the general case was proved independently by Roberts [16], and by Gillet and Soulé [11], [12].

On the other hand, let X be a smooth variety of dimension n over a field k, V, W be irreducible subvarieties of X and P be an irreducible component of $V \cap W$ that has the right codimension, i.e. such that $\mathrm{codim}(P,X) = \mathrm{codim}(V,X) + \mathrm{codim}(W,X)$. If $CH^*(X)$ denotes the Chow ring of X and $\{V\}, \{W\}, \{P\}$ are the classes of the varieties V, W, P in that ring, then it is well known that

$$\{V\} \cdot \{W\} = \chi_{\mathcal{O}_{X,P}}(\mathcal{O}_{V,P}, \mathcal{O}_{W,P})\{P\} + \beta \tag{1}$$

for a cycle β whose support does not contain P.

Some years ago, Barge and Morel introduced a generalization of the Chow groups called the *oriented Chow groups* or *Chow-Witt groups* ([3]; see also [5] for more details), which admit homomorphisms to the "usual" Chow groups. In a recent paper ([4]), the first author showed that for a smooth variety X over a field of characteristic

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[1] Here, the local ring A is said to be smooth over a d.v.r. if its completion is a power series ring over the corresponding complete d.v.r.

$\neq 2$, the total Chow-Witt group of X, denoted by $\widetilde{CH}^*(X)$, admits a graded ring structure, such that the natural map $\widetilde{CH}^*(X) \to CH^*(X)$ is a ring homomorphism.

If V is a closed algebraic subset of X of pure codimension n, let $GW^n_V(X)$ denote the n-th Grothendieck-Witt group of perfect complexes on X supported in V (see [20] for more information). In [4], it is shown that there is a natural homomorphism

$$\alpha_V : GW^n_V(X) \to \widetilde{CH}^n(X);$$

moreover, if W is another closed algebraic subset of pure codimension m such that $V \cap W$ is of pure codimension $m+n$, we have a commutative diagram ([4], Theorem 7.6)

$$\begin{tikzcd}
GW^n_V(X) \times GW^m_W(X) \arrow[r, "\star"] \arrow[d, "\alpha_V \times \alpha_W"'] & GW^{m+n}_{V\cap W}(X) \arrow[d, "\alpha_{V\cap W}"] \\
\widetilde{CH}^n(X) \times \widetilde{CH}^m(X) \arrow[r, "\cdot"'] & \widetilde{CH}^{m+n}(X)
\end{tikzcd}$$

Here the bottom row is the multiplication in the Chow-Witt ring, and $\star$ denotes the product on the Grothendieck-Witt groups, induced by the usual tensor structure on perfect complexes. The existence of this commutative diagram is an analogue for Chow-Witt groups of the formula (1) for Serre's intersection multiplicity.

Given this result, it is natural to ask whether a version of Serre's Vanishing Conjecture is true for the Grothendieck-Witt groups. We may thus formulate:

Conjecture. *Let* $(A, \mathfrak{m})$ *be a regular local ring of dimension* n *containing* $1/2$. *Let* Z *and* T *be closed subsets of* $\operatorname{Spec}(A)$ *such that* $\dim Z + \dim T < n$ *and* $Z \cap T = \mathfrak{m}$. *Then the multiplication*

$$GW^i_Z(A) \times GW^j_T(A) \to GW^{i+j}_{\mathfrak{m}}(A)$$

is zero for any $i, j \in \mathbb{N}$.

As evidence, we have the following theorem (see Theorems 4.2, 5.1):

Theorem. *Let* $(A, \mathfrak{m})$ *be a regular local ring of dimension* n, *such that* $1/2 \in A$. *Assume further that either* A *contains a field, or* A *is smooth over a discrete valuation ring of mixed characteristic.*

Let Z *and* T *be closed subsets of* $\operatorname{Spec}(A)$ *with* $\dim Z + \dim T < n$ *and* $Z \cap T = \mathfrak{m}$. *Then the multiplication*

$$GW^i_Z(A) \times GW^j_T(A) \to GW^{i+j}_{\mathfrak{m}}(A)$$

is zero for any $i, j \in \mathbb{N}$.

The principal ingredients of the proof are the "calculus" of derived Grothendieck-Witt and Witt groups, some geometric normalization lemmas, and a result of Gille and Hornbostel giving the vanishing of maps between Witt groups defined by extension of support, in some special situations ([9], Theorem 0.1).

We do not recall the definitions of the Grothendieck-Witt groups and Witt groups of a triangulated category with duality. We instead refer the reader to [1] for information on Witt groups, with detailed references, and to [20] for the definition, and basic properties of Grothendieck-Witt groups, of triangulated categories with duality. We will also use the products on Grothendieck-Witt and Witt groups defined in [10], which are naturally induced from the tensor structure on perfect complexes. We remark that the hypothesis that all schemes are over $\mathbb{Z}[1/2]$ is built into the foundations of the theory of Grothendieck-Witt and Witt groups of triangulated categories with a duality, in their present form.

2. Geometric lemmas

The following theorem is well known (see [13], theorem 14.4):

Theorem 2.1. *Let $(A, \mathfrak{m})$ be a d-dimensional Noetherian local ring and suppose that $k = A/\mathfrak{m}$ is an infinite field. Let $\mathfrak{q} = (u_1, \ldots, u_s)$ be an $\mathfrak{m}$-primary ideal. Then if $y_i = \sum a_{ij} u_j$ for $1 \leq i \leq d$ are d sufficiently general linear combinations of $u_1, \ldots, u_s$, the ideal $\mathfrak{b} = (y_1, \ldots, y_d)$ is a reduction of $\mathfrak{q}$ and $\{y_1, \ldots, y_d\}$ is a system of parameters of A.*

Proof. We give a sketch of the proof in order to make precise what "sufficiently general" means. As a first step, the proof shows that there is an homogeneous ideal $Q \subset k[x_1, \ldots, x_s]$ such that

$$k[x_1, \ldots, x_s]/Q \simeq \bigoplus_{n \geq 0} \mathfrak{q}^n/\mathfrak{q}^n\mathfrak{m} = \mathrm{gr}_{\mathfrak{q}}(A) \otimes_{A/\mathfrak{q}} k$$

and $\dim k[x_1, \ldots, x_s]/Q = d = \dim A$. Now let $X = \mathrm{Spec}(k[z_{ij}])$ with $1 \leq i \leq d$, $1 \leq j \leq s$. Any k-rational point $(b_{ij}) \in X(k)$ gives an homomorphism

$$\varphi_{(b_{ij})} : k[x_1, \ldots, x_d] \to k[x_1, \ldots, x_s]/Q$$

defined by $\varphi_{(b_{ij})}(x_l) = \sum b_{lj} x_j$. We say that (b_{ij}) is good if $k[x_1, \ldots, x_s]/Q$ is a finite $k[x_1, \ldots, x_d]$-module (under $\varphi_{(b_{ij})}$). The proof shows that there exists a polynomial $D \in k[z_{ij}]$ such that if $U = \mathrm{Spec}(k[z_{ij}]_D)$ then any $(b_{ij}) \in U(k)$ is good. Then it is easy to see that $(a_{ij}) \in A$ has the desired property if the residue $(\overline{a}_{ij})$ is in $U(k)$. $\square$

From now on, we assume that every field appearing until the end of Section 3 is infinite.

If $k[[z_1, \ldots, z_m]] \to k[[x_1, \ldots, x_n]]$ is a homomorphism between power series algebras over k induced by $z_i \mapsto \sum_j a_{ij} x_j$ for some $a_{ij} \in k$, we call the induced morphism $\mathrm{Spec}(k[[x_1, \ldots, x_n]]) \to \mathrm{Spec}(k[[z_1, \ldots, z_m]])$ a linear projection. Such linear projections are determined by points in $\mathbb{A}^{mn}(k)$.

The following two corollaries to theorem 2.1 are obvious:

Corollary 2.2. *Let $X = \mathrm{Spec}(k[[x_1, \ldots, x_n]])$, $Y = \mathrm{Spec}(k[[z_1, \ldots, z_{n-1}]])$ and let $Z = V(f) \subset X$. Then any sufficiently general linear projection $p : X \to Y$ has the property that $p_{|_Z} : Z \to Y$ is finite.*

Corollary 2.3. *Let $\overline{x} = (x_1, \ldots, x_n)$ and $\overline{y} = (y_1, \ldots, y_n)$. Consider an ideal $\langle I(\overline{x}), J(\overline{y})\rangle$ in the ring $k[[\overline{x}, \overline{y}]]$. Suppose that*

(i) $\dim k[[\overline{x}, \overline{y}]]/\langle I(\overline{x}), J(\overline{y})\rangle \leq n - 1$

(ii) $\langle I(\overline{x}), J(\overline{y}), x_1 - y_1, \ldots, x_n - y_n\rangle$ *is* $\langle \overline{x}, \overline{y}\rangle$*-primary.*

Then for any sufficiently general elements $a_{ij} \in k$ $(1 \leq i \leq n-1,\ 1 \leq j \leq n)$ *the ideal* $\langle I(\overline{x}), J(\overline{y}), l_1(\overline{x}) - l_1(\overline{y}), \ldots, l_{n-1}(\overline{x}) - l_{n-1}(\overline{y})\rangle$ *where* $l_i(\overline{x}) = \sum_{j=1}^n a_{ij}x_j$ *and* $l_i(\overline{y}) = \sum_{j=1}^n a_{ij}y_j$ *is* $\langle \overline{x}, \overline{y}\rangle$*-primary.*

Corollary 2.4. *Let* $X = \mathrm{Spec}(k[[x_1, \ldots, x_n]])$ *and* $Y = \mathrm{Spec}(k[[z_1, \ldots, z_{n-1}]])$. *Let* $Z, T \subset X$ *be closed subsets such that* $\dim Z + \dim T < \dim X$ *and* $Z \cap T$ *is supported on the closed point. Then for any sufficiently general linear projection* $p : X \to Y$ *we have that* $Z \neq p^{-1}(p(Z))$ *and* $p^{-1}(p(Z)) \cap T$ *is also supported on the closed point.*

Proof. We may interpret $Z \cap p^{-1}p(T)$, $T \cap p^{-1}p(Z)$ in terms of $Z \times_Y T$. Thus, let T and Z be respectively defined by ideals I and J. Using Corollary 2.3, we can find a nonempty open subset $U \subset \mathrm{Spec}(k[z_{ij}])$ such that for any rational point $(b_{ij}) \in U(k)$ the associated projection p has the property that the ideal $\langle I(\overline{x}), J(\overline{y}), l_1(\overline{x}) - l_1(\overline{y}), \ldots, l_{n-1}(\overline{x}) - l_{n-1}(\overline{y})\rangle$ is $\langle \overline{x}, \overline{y}\rangle$-primary. Therefore $p^{-1}p(Z) \cap T$ and $Z \cap p^{-1}p(T)$ are supported on the closed point. Further, $Z \to p(Z)$, $T \to p(T)$ are finite, since the ideal $\langle I(\overline{x}), l_1(\overline{x}), \ldots, l_{n-1}(\overline{x})\rangle$ is $\langle \overline{x}\rangle$-primary and the ideal $\langle J(\overline{y}), l_1(\overline{y}), \ldots, l_{n-1}(\overline{y})\rangle$ is $\langle \overline{y}\rangle$-primary. Hence $p^{-1}p(Z) \neq Z$ and $p^{-1}p(T) \neq T$, since the fibre of p over the closed point of Y is 1-dimensional. □

3. The zero theorem for transfers

We recall (see [6], Defn. 2.16) that if X is a Noetherian, Gorenstein $\mathbb{Z}[1/2]$-scheme of finite Krull dimension (e.g. quasi-projective over a field or a complete discrete valuation ring), and $Z \subset X$ is a closed subscheme, then $\widetilde{W}^i_Z(X)$ (resp. $\widetilde{GW}^i_Z(X)$) denotes the i-th derived Witt group (resp. i-th derived Grothendieck-Witt group) of the full subcategory $D^b_Z(X) \subset D^b(X)$ of the bounded derived category of X consisting of complexes with (coherent) homology supported (set-theoretically) in Z, with the natural duality structure (essentially Grothendieck-Serre duality). If X is regular, this coincides with $W^i_Z(X)$ (resp. $GW^i_Z(X)$), similarly defined using perfect complexes. The trace in duality theory leads to transfer maps in certain situations, with the expected properties. For Witt groups, a good reference is [7]. A closer look at the arguments used there shows that a transfer morphism with the expected properties can also be defined for Grothendieck-Witt groups.

The next theorem is due to Gille and Hornbostel ([9], Theorem 0.1).

Theorem 3.1. *Let* R *be a Gorenstein ring of finite Krull dimension,* $t \in R$ *a non zero-divisor and* $\pi : R \to R/tR$ *the quotient map. Suppose that* π *has a flat splitting* $q : R/tR \to R$. *Then the transfer morphism*

$$Tr_{(R/tR)/R} : \widetilde{W}^i(R/Rt) \to \widetilde{W}^{i+1}(R)$$

is zero for all $i \in \mathbb{Z}$.

In fact, the proof of their result ultimately boils down to Lemma 2.5 in their paper, proved using a specific computation at the level of forms, given by an isometry. This formula in fact yields a stronger conclusion, for coherent Witt groups with supports (analogous to the stronger assertion Theorem 2.1 in their paper):

Theorem 3.2. *Let* R *be a Gorenstein ring of finite Krull dimension,* $t \in R$ *a non zero-divisor and* $\pi : R \to R/tR$ *the quotient map. Suppose that* π *has a flat splitting*

$q : R/tR \to R$. Let $J \subset R$ be an ideal containing t, and let $\widetilde{J} = q(J/tR)R$. Then the transfer morphism

$$Tr_{(R/tR)/R} : \widetilde{W}^i_{J/tR}(R/Rt) \to \widetilde{W}^{i+1}_{\widetilde{J}}(R)$$

is zero for all $i \in \mathbb{Z}$.

For Grothendieck-Witt groups, the situation is similar. First recall that for any triangulated category $\mathcal{C}$ with duality and any $i \in \mathbb{Z}$ there is an exact sequence ([20], Theorem 2.6):

$$\begin{tikzcd} GW^i(\mathcal{C}) \arrow[r, "f"] & K_0(\mathcal{C}) \arrow[r, "H"] & GW^{i+1}(\mathcal{C}) \arrow[r] & W^{i+1}(\mathcal{C}) \arrow[r] & 0 \end{tikzcd}$$

where f is induced by the forgetful functor and H is the hyperbolic functor. Using this sequence, we can prove

Theorem 3.3. *Let R be a Gorenstein ring of finite Krull dimension, $t \in R$ a non zero-divisor and $\pi : R \to R/tR$ the quotient map. Suppose that π has a flat splitting $q : R/tR \to R$. Let $J \subset R$ be an ideal containing t, and let $\widetilde{J} = q(J/tR)R$. Then the transfer morphism*

$$Tr_{(R/tR)/R} : \widetilde{GW}^i_{J/tR}(R/Rt) \to \widetilde{GW}^{i+1}_{\widetilde{J}}(R)$$

is zero for all $i \in \mathbb{Z}$.

Proof. The arguments of [9], Lemma 2.8 show that for any $x \in \widetilde{GW}^i_{J/tR}(R/Rt)$ the transfer $Tr_{(R/tR)/R}(x)$ is isometric to $[K(t), l_1] \star q^*(x)$ where $[K(t), l_1]$ is the class in $GW^1(R)$ of the Koszul complex

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "t"] & R \arrow[r] & 0 \end{tikzcd}$$

endowed with the form

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "t"] \arrow[d, "-1"'] & R \arrow[r] \arrow[d, "1"] & 0 \\ 0 \arrow[r] & R \arrow[r, "-t"'] & R \arrow[r] & 0 \end{tikzcd}$$

and $\star$ denotes the action of the Grothendieck-Witt groups of the derived category of bounded perfect complexes on the Grothendieck-Witt groups of the derived category of bounded complexes with coherent homology (see [10],§3). It turns out that $[K(t), l_1] = H(R)$ in $GW^1(R)$ (use for example [10], §2.4) where R denotes the complex with R concentrated in degree 0 and zeroes elswhere. This complex carries a symmetric form (the identity for example) and can be seen as an element of $GW^0(R)$. This shows that R is in the image of f. The exact sequence above gives $H(R) = 0$ in $GW^1(R)$. Hence $[K(t), l_1] = 0$ and the result is proved.

□

Theorem 3.4. *Let R be a Gorenstein ring of finite Krull dimension, $t \in R$ a non zero-divisor and $J \subset R$ an ideal containing t. Then for any $i \in \mathbb{N}$ the transfer morphism*

$$Tr_{(R/tR)/R} : \widetilde{GW}^{i}_{J/tR}(R/Rt) \to \widetilde{GW}^{i+1}_{J}(R)$$

is an isomorphism.

Proof. We follow the arguments of [8], §3. First remark that the filtration by the codimension of the support yields a Gersten complex in Grothendieck-Witt groups by [17], Remark 8.3. As in Gille's work, we are then reduced to show the result for a local Gorenstein ring. The result then follows from the computation of the Grothendieck-Witt groups of such a ring. This computation goes as in our Lemma 4.1 below (using also [6], Lemma 4.4). □

Corollary 3.5. *Let R, t be as in Theorem 3.2. Let $X = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(R/tR)$ and Z a closed subscheme of Y. Let $p : X \to Y$ be the flat splitting of the inclusion $i : Y \to X$. Then the extension of support*

$$e : \widetilde{GW}^{i}_{Z}(X) \to \widetilde{GW}^{i}_{p^{-1}(Z)}(X)$$

is zero.

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
\widetilde{GW}^{i}_{Z}(X) \arrow[r, "e"] & \widetilde{GW}^{i}_{p^{-1}(Z)}(X) \\
\widetilde{GW}^{i-1}_{Z}(Y). \arrow[u, "Tr_{Y/X}"] \arrow[ur, "Tr_{Y/X}"'] &
\end{tikzcd}$$

The diagonal transfer $Tr_{Y/X} : \widetilde{GW}^{i-1}_{Z}(Y) \to \widetilde{GW}^{i}_{p^{-1}(Z)}(X)$ is zero by theorem 3.3 and the vertical transfer $Tr_{Y/X} : \widetilde{GW}^{i-1}_{Z}(Y) \to \widetilde{GW}^{i}_{Z}(X)$ is an isomorphism by the above theorem. □

Finally, we have the following proposition, analogous to the key step in Quillen's proof of the Gersten conjecture (see also [9], [19] or [15]):

Proposition 3.6. *Let $X = \mathrm{Spec}(k[[x_1, \ldots, x_n]])$, $Z \subset X$ a proper closed subset and $Y = \mathrm{Spec}(k[[z_1, \ldots, z_{n-1}]])$. Then for any $i \in \mathbb{N}$ and any sufficiently general linear projection $p : X \to Y$ the extension of support*

$$\widetilde{GW}^{i}_{Z}(X) \to \widetilde{GW}^{i}_{p^{-1}(p(Z))}(X)$$

is zero.

Proof. As Z is a proper closed subset of X there exists a non-zero non-unit $t \in k[[x_1, \ldots, x_n]]$ such that $Z \subset V(t)$. Let $j : V(t) \to X$ be the inclusion. Any sufficiently general linear projection $p : X \to Y$ is flat and has the property that $p_{|_{V(t)}} : V(t) \to Y$ is finite. Consider the following fibre product:

$$\begin{tikzcd}
X' \arrow[r, "f"] \arrow[d, "p'"'] & X \arrow[d, "p"] \\
V(t) \arrow[ur, "j"] \arrow[r, "p_{|V(t)}"'] & Y.
\end{tikzcd}$$

The inclusion $j : V(t) \to X$ induces a closed immersion $i' : V(t) \to X'$ such that $fi' = j$. Observe that $V(t)$ is also a principal divisor in X' ([19], Theorem 5.23). As closed subsets, we have $p^{-1}(p(Z)) = f(p')^{-1}(Z)$ and then it is enough to show that $\widetilde{GW}^i_Z(X) \to \widetilde{GW}^i_{f(p')^{-1}(Z)}(X)$ is zero to get the result. We have the following commutative diagram:

$$\begin{tikzcd}
\widetilde{GW}^i_Z(X) \arrow[r, equal] & \widetilde{GW}^i_Z(X) \arrow[r, "e"] & \widetilde{GW}^i_{f(p')^{-1}(Z)}(X) \\
\widetilde{GW}^{i-1}_Z(V(t)) \arrow[u, "j_*"] \arrow[r, "(i')_*"'] & \widetilde{GW}^i_{i'(Z)}(X',\mathcal{L}) \arrow[u, "f_*"] \arrow[r, "e"'] & \widetilde{GW}^i_{(p')^{-1}(Z)}(X',\mathcal{L}) \arrow[u, "f_*"]
\end{tikzcd}$$

where e is the extension of support, and $\mathcal{L}$ is the relative dualizing sheaf for f (see [7], §4 for the functoriality of the transfer). By Theorem 3.4, we know that j_* is an isomorphism. Since X' is local and $\mathcal{L}$ is a locally free sheaf, we can trivialize it. Therefore, Corollary 3.5 shows that $e : \widetilde{GW}^i_{i'(Z)}(X',\mathcal{L}) \to \widetilde{GW}^i_{(p')^{-1}(Z)}(X',\mathcal{L})$ is zero. □

4. The theorem in the equicharacteristic case

Let $(A,\mathfrak{m})$ be a regular local ring. The proof of our main result will require the computation of the Grothendieck-Witt groups with support on the closed point. For any $i \in \mathbb{Z}$, we have already seen that there is an exact sequence ([20], Theorem 2.6)

$$GW^i_{\mathfrak{m}}(A) \xrightarrow{f} K_0(D^b_{\mathfrak{m}}(A)) \xrightarrow{H} GW^{i+1}_{\mathfrak{m}}(A) \longrightarrow W^{i+1}_{\mathfrak{m}}(A) \longrightarrow 0$$

Lemma 4.1. *(i) Let $(A,\mathfrak{m})$ be a regular local ring. Then*

$$GW^n_{\mathfrak{m}}(A) \simeq \begin{cases} GW(A/\mathfrak{m}) & \text{if} \quad n \equiv \dim A \pmod 4 \\ 0 & \text{if} \quad n \equiv \dim A + 1 \pmod 4. \end{cases}$$

If $n \equiv \dim A + 2 \pmod 4$, then $H : K_0(D^b_{\mathfrak{m}}(A)) \to GW^n_{\mathfrak{m}}(A)$ is an isomorphism. Finally, if $n \equiv \dim A + 3 \pmod 4$ there is an exact sequence

$$0 \longrightarrow K_0(D^b_{\mathfrak{m}}(A)) \xrightarrow{\cdot 2} K_0(D^b_{\mathfrak{m}}(A)) \xrightarrow{H} GW^n_{\mathfrak{m}}(A) \longrightarrow 0\,.$$

(ii) If $(A,\mathfrak{m}) \to (B,\mathfrak{n})$ is a flat homomorphism of regular local rings such that $\mathfrak{m}B = \mathfrak{n}$, so that $\dim A = \dim B$, then for any $n \equiv \dim A \pmod 4$, there is

a commutative diagram

$$\begin{tikzcd} GW^n_{\mathfrak{m}}(A) \arrow[r, "\cong"] \arrow[d] & GW(A/\mathfrak{m}) \arrow[d] \\ GW^n_{\mathfrak{n}}(B) \arrow[r, "\cong"] & GW(B/\mathfrak{n}) \end{tikzcd}$$

If $n \not\equiv \dim A \pmod 4$, *then the map* $GW^n_{\mathfrak{m}}(A) \to GW^n_{\mathfrak{n}}(B)$ *is induced by the map in* K*-theory* $K_0(D^b_{\mathfrak{m}}(A)) \to K_0(D^b_{\mathfrak{n}}(B))$.

Proof. The arguments of [2], §6 use equivalences of triangulated categories. Therefore, the same method shows that $GW^{\dim A}_{\mathfrak{m}}(A) \simeq GW^{fl}(A)$. The latter is in turn isomorphic to $GW(A/\mathfrak{m})$ by [14], Theorem 6.10. This also proves that the forgetful functor

$$f : GW^{\dim A}_{\mathfrak{m}}(A) \to K_0(D^b_{\mathfrak{m}}(A))$$

is surjective. The above exact sequence and Lemma 4.1 show then that $GW^n_{\mathfrak{m}}(A) = 0$ if $n \equiv \dim A + 1 \pmod 4$, which in turn implies that

$$H : K_0(D^b_{\mathfrak{m}}(A)) \to GW^n_{\mathfrak{m}}(A)$$

is an isomorphism if $n \equiv \dim A + 2 \pmod 4$. For the remaining case, we have to compute the composition $f \circ H$ where f is the forgetful functor and H is the homomorphism of the above line. An easy computation shows that it is the multiplication by 2. The proof of (ii) follows from the remark that a flat morphism induces a commutative diagram of exact sequences linking K_0, GW and W.

□

Theorem 4.2. *Let* $(A, \mathfrak{m})$ *be a regular local ring of dimension* n *containing a field of characteristic* $\neq 2$. *Let* Z *and* T *be closed subsets of* $\operatorname{Spec}(A)$ *with* $Z \cap T = \mathfrak{m}$ *and* $\dim Z + \dim T < n$. *Then the multiplication*

$$GW^i_Z(A) \times GW^j_T(A) \to GW^{i+j}_{\mathfrak{m}}(A)$$

is zero for any $i, j \in \mathbb{N}$.

Proof. Let $\widehat{A}$ be the completion of A (for the $\mathfrak{m}$-adic valuation).

Using lemma 4.1, we see that $GW^n_{\mathfrak{m}}(A) \simeq GW^n_{\widehat{\mathfrak{m}}}(\widehat{A})$ for all n, and the following diagram commutes,

$$\begin{tikzcd} GW^i_Z(A) \times GW^j_T(A) \arrow[r] \arrow[d] & GW^{i+j}_{\mathfrak{m}}(A) \arrow[d] \\ GW^i_{\widehat{Z}}(\widehat{A}) \times GW^j_{\widehat{T}}(\widehat{A}) \arrow[r] & GW^{i+j}_{\widehat{\mathfrak{m}}}(\widehat{A}) \end{tikzcd}$$

where the vertical arrows are induced by the completion. Hence it is enough to prove the result for a complete regular local ring. Therefore we can suppose that $A = k[[x_1, \dots, x_n]]$ for a field k.

Next, we observe that if k' is an infinite algebraic extension of k which is an increasing union of finite algebraic subextensions of odd degree, then the natural map $GW(k) \to GW(k')$ is injective, since the map on Grothendieck-Witt groups for a finite extension of fields of odd degree is injective (standard transfer argument). By applying lemma 4.1(ii) to $k[[x_1, \dots, x_n]] \to k'[[x_1, \dots, x_n]]$, we thus further reduce to the case when the field k is infinite.

Let $B = k[[z_1, \dots, z_{n-1}]]$. Then using Corollary 2.4 and Proposition 3.6, we see that there exists a linear projection $p : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ such that:

(1) The extension of support $e : GW^i_Z(A) \to GW^i_{p^{-1}(p(Z))}(A)$ is zero.

(2) $p^{-1}p(Z) \cap T = \mathfrak{m}$.

The conclusion follows from the following commutative diagram:

$$\begin{tikzcd}
GW^i_Z(A) \times GW^j_T(A) \arrow[r] \arrow[d, "e\times Id"'] & GW^{i+j}_{\mathfrak{m}}(A) \arrow[d, equal] \\
GW^i_{p^{-1}(p(Z))}(A) \times GW^j_T(A) \arrow[r] & GW^{i+j}_{\mathfrak{m}}(A).
\end{tikzcd}$$

□

5. The case of a regular local ring smooth over a d.v.r. of mixed characteristic

The proof in the case of a regular local ring smooth over a d.v.r. of mixed characteristic is similar in many respects to that in the equicharacteristic case. Hence, we will be sketchy, except at points where there are some new features in the proof in this case.

First, as in the proof of Theorem 4.2, it suffices to reduce to the case of a complete local ring, i.e., we may assume that $A \cong \Lambda[[x_1, \dots, x_n]]$ where Λ is a complete d.v.r. of mixed characteristic, and $1/2 \in \Lambda$.

Next, we claim that it suffices to treat the case when Λ has an infinite residue field. This is similar to the argument in the equicharacteristic case. If Λ has a finite residue field k, and k' is a finite extension of k of odd degree, then (from Cohen structure theory, for example) we can find an over-ring Λ' which is also a complete discrete valuation ring, finite and unramified over Λ, with residue field k'. Since the map of Grothendieck-Witt groups $GW(k) \to GW(k')$ is injective (since we have a transfer here as well), it suffices to obtain the result for $A \otimes_\Lambda \Lambda'$. We may pass to a direct limit over a tower of such odd extensions, and obtain a new local ring, smooth over a d.v.r. with infinite residue field, and it suffices to prove the result for the completion of this new local ring.

Thus, we have reduced the proof of the Main Theorem in the mixed characteristic case to the following result.

Theorem 5.1. *Let Λ be a complete d.v.r. of mixed characteristic containing $1/2$, with infinite residue field, and let $A = \Lambda[[x_1, \dots, x_n]]$, with maximal ideal $\mathfrak{m}$. Let Z and T be closed subsets of $\mathrm{Spec}(A)$ such that $\dim Z + \dim T < n + 1 = \dim A$ and $Z \cap T = \mathfrak{m}$. Then the multiplication*

$$GW^i_Z(A) \times GW^j_T(A) \to GW^{i+j}_{\mathfrak{m}}(A)$$

is zero for any $i, j \in \mathbb{N}$.

Proof. We first consider the case when Z, T are both contained in the closed fiber $\overline{X}$ of $X \to \mathrm{Spec}(\Lambda)$. Let $f : \overline{X} \to X$ be the inclusion. Then we have isomorphisms (since $\overline{X}$ is a principal divisor)

$$f_* : GW^{i-1}_Z(\overline{X}) \cong GW^i_Z(X), \ \ f_* : GW^{j-1}_T(\overline{X}) \to GW^j_T(X).$$

If $\alpha \in GW^{i-1}_Z(\overline{X})$, $\beta \in GW^{i-1}_T(\overline{X})$, then by the projection formula ([7], Theorem 5.2), we have

$$f_*(\alpha) \cdot f_*(\beta) = f_*(\alpha \cdot f^* f_* \beta) \in GW^{i+j}_{\{\mathfrak{m}\}}(X).$$

So it suffices to prove that the composition

$$GW^{j-1}_T(\overline{X}) \xrightarrow{f_*} GW^j_T(X) \xrightarrow{f^*} GW^j_T(\overline{X}).$$

is 0. For this, it suffices to prove the vanishing of the further composition with the isomorphism

$$f_* : GW^j_T(\overline{X}) \to GW^{j+1}_T(X).$$

Now if $1 \in W(\overline{X})$ is the unit form, then

$$f_* f^* f_*(\beta) = f_*(f^* f_* \beta \cdot 1) = f_* \beta \cdot f_* 1 = f_* \beta \cdot f_*(f^* f_* 1).$$

So it suffices to show that $f^* f_*(1) \in GW^1(\overline{X})$ vanishes. But, regarding X as a scheme over $\mathrm{Spec}(\Lambda)$, clearly $1 \in GW(\overline{X})$ is the pullback of $1 \in GW(\mathrm{Spec}(k))$, where k is the residue field of Λ, and the element $f^* f_*(1)$ is similarly the pullback of the corresponding element in $GW^1(\mathrm{Spec}(k))$. But $GW^1(\mathrm{Spec}(k)) = 0$.

Thus, in the case when Z, T are both contained in the closed fiber $\overline{X}$, the theorem holds. So we may assume that (say) Z is not contained in the closed fiber; in particular, $n > 0$.

Let $Z = Z' \cup Z''$ where all irreducible components of Z' dominate $\mathrm{Spec}(\Lambda)$, and $Z'' = Z \cap \overline{X}$. Let $\overline{Z'} = Z' \cap \overline{X}$ be the closed fiber of $Z' \to \mathrm{Spec}(\Lambda)$, so that $\dim \overline{Z'} = \dim Z' - 1$. Let $\overline{T}$ be the closed fiber of $T \to \mathrm{Spec}(\Lambda)$.

Let $Y = \mathrm{Spec}(\Lambda[[z_1, \ldots, z_{n-1}]])$. Let $\overline{Y}$ denote the closed fiber of Y over $\mathrm{Spec}(\Lambda)$. We consider morphisms $p : X \to Y$ of Λ-schemes induced by continuous homomorphisms with $z_i \mapsto \sum_j a_{ij} x_j$, with $a_{ij} \in \Lambda$. For $a \in \Lambda$, let $\overline{a}$ denote its image in the residue field k.

By Corollary 2.3 applied to the ideals of $\overline{Z'}$ and $\overline{T}$ in $\overline{X} = \mathrm{Spec}(k[[x_1, \ldots, x_n]])$, we see that for general $a_{ij} \in \Lambda$, if $p : X \to Y$ is the corresponding morphism, then $p^{-1}p(Z') \cap T = \{\mathfrak{m}\}$, and $p \mid_{Z'} : Z' \to Y$ is finite (since the fiber of $Z' \to Y$ over the closed point is the corresponding fiber of $\overline{Z'} \to \overline{Y}$, it is quasi-finite).

Let $\overline{p^{-1}p(Z')}$ be the closed fiber of $p^{-1}p(Z')$, and let $\widetilde{Z} = Z'' \cup \overline{p^{-1}p(Z')}$. Then $\widetilde{Z}$ is the closed fiber of $p^{-1}p(Z') \cup Z''$, and it has dimension at most that of Z.

We now claim that the image of the extension of support map

$$GW^i_Z(X) \to GW^i_{p^{-1}p(Z') \cup Z''}(X)$$

has image contained in that of the similar map

$$GW^i_{\widetilde{Z}}(X) \to GW^i_{p^{-1}p(Z') \cup Z''}(X).$$

From the exact sequence

$$GW^i_{\widetilde{Z}}(X) \to GW^i_{p^{-1}p(Z') \cup Z''}(X) \to GW^i_{p^{-1}p(Z'') \setminus \widetilde{Z}}(X \setminus \widetilde{Z}),$$

and the excision isomorphism

$$GW^i_{p^{-1}p(Z')\cup Z''\setminus\widetilde{Z}}(X\setminus\widetilde{Z}) \to GW^i_{p^{-1}p(Z'')\setminus\overline{X}}(X\setminus\overline{X}),$$

it suffices to show that the map

$$GW^i_Z(X) \to GW^i_{p^{-1}p(Z')\setminus\overline{X}}(X\setminus\overline{X})$$

is 0. This in turn follows from the fact that

$$GW^i_{Z\setminus\overline{X}}(X\setminus\overline{X}) \to GW^i_{p^{-1}p(Z'')\setminus\overline{X}}(X\setminus\overline{X})$$

is 0, by Proposition 3.6 applied to the affine scheme $X\setminus\overline{X}$.

From the commutative diagram

$$\begin{tikzcd}
GW^i_Z(A)\times GW^j_T(A) \arrow[r] \arrow[d, "e\times Id"'] & GW^{i+j}_{\mathfrak{m}}(A) \arrow[d, equal] \\
GW^i_{p^{-1}(p(Z)')\cup Z''}(A)\times GW^j_T(A) \arrow[r] & GW^{i+j}_{\mathfrak{m}}(A),
\end{tikzcd}$$

we thus see that it suffices to prove the result with Z replaced by $\widetilde{Z}$, i.e., in the special case when Z is contained in the closed fiber. By a similar argument, we further reduce to the case when T is also contained in the closed fiber; now we are in the first case, already dealt with.

□

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References

[1] P. Balmer, *Witt groups*, in Handbook of K-theory. Vol. 1, 2, 539–576, Springer, Berlin (2005).

[2] P. Balmer and C. Walter, *A Gersten-Witt spectral sequence for regular schemes*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 1, 127–152.

[3] J. Barge and F. Morel, *Groupe de Chow des cycles orientés et classe d'Euler des fibrés vectoriels*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), no. 4, 287–290.

[4] J. Fasel, *The Chow-Witt ring*, Doc. Math. **12** (2007) 275–312 (electronic).

[5] ———, *Groupes de Chow-Witt (in french)* (2007). Preprint.

[6] S. Gille, *On Witt groups with support*, Math. Ann. **322** (2002), no. 1, 103–137.

[7] ———, *A transfer morphism for Witt groups*, J. Reine Angew. Math. **564** (2003) 215–233.

[8] ———, *The general dévissage theorem for Witt groups of schemes*, Arch. Math. (Basel) **88** (2007), no. 4, 333–343.

[9] S. Gille and J. Hornbostel, *A zero theorem for the transfer of coherent Witt groups*, Math. Nachr. **278** (2005), no. 7-8, 815–823.

[10] S. Gille and A. Nenashev, *Pairings in triangular Witt theory*, J. Algebra **261** (2003), no. 2, 292–309.

[11] H. Gillet and C. Soulé, *K-théorie et nullité des multiplicités d'intersection*, C. R. Acad. Sci. Paris Sér. I Math. **300** (1985), no. 3, 71–74.

[12] ———, *Intersection theory using Adams operations*, Invent. Math. **90** (1987), no. 2, 243–277.

[13] H. Matsumura, Commutative ring theory, Vol. 8 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, second edition (1989), ISBN 0-521-36764-6. Translated from the Japanese by M. Reid.

[14] H.-G. Quebbemann, W. Scharlau, and M. Schulte, *Quadratic and Hermitian forms in additive and abelian categories*, J. Algebra **59** (1979), no. 2, 264–289.

[15] D. Quillen, *Higher algebraic K-theory. I*, in Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), 85–147. Lecture Notes in Math., Vol. 341, Springer, Berlin (1973).

[16] P. Roberts, *The vanishing of intersection multiplicities of perfect complexes*, Bull. Amer. Math. Soc. (N.S.) **13** (1985), no. 2, 127–130.

[17] M. Schlichting, *Hermitian K-theory, derived equivalences and Karoubi's fundamental theorem* (2008). Preprint available at `http://www.math.lsu.edu/ mschlich/research/prelim.html`.

[18] J.-P. Serre, Algèbre locale. Multiplicités, Vol. 11 of *Cours au Collège de France, 1957–1958, rédigé par Pierre Gabriel. Seconde édition, 1965. Lecture Notes in Mathematics*, Springer-Verlag, Berlin (1965).

[19] V. Srinivas, Algebraic K-theory, Vol. 90 of *Progress in Mathematics*, Birkhäuser Boston Inc., Boston, MA, second edition (1996), ISBN 0-8176-3702-8.

[20] C. Walter, *Grothendieck-Witt groups of triangulated categories* (2003). Preprint available at `www.math.uiuc.edu/K-theory/0643/`.

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