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SINGULAR YAMABE METRICS AND INITIAL DATA WITH *EXACTLY* KOTTLER–SCHWARZSCHILD–DE SITTER ENDS II. GENERIC METRICS

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Abstract. We present a gluing construction which adds, via a localized deformation, *exactly Delaunay ends* to *generic* metrics with constant positive scalar curvature. This provides time-symmetric initial data sets for the vacuum Einstein equations with positive cosmological constant with *exactly* Kottler–Schwarzschild–de Sitter ends, extending the results in [5].

1. Introduction

There exists very strong evidence suggesting that we live in a world with strictly positive cosmological constant Λ [10, 9]. This leads to a need for a better understanding of the space of solutions of Einstein equations with $\Lambda > 0$. In [5] it has been shown how time-symmetric initial data for such space-times, which contain *asymptotically Delaunay ends*, can be deformed to initial data with *exactly Delaunay ends*. The resulting space-times contain regions of infinite extent on which the metric takes *exactly* the Kottler–Schwarzschild–de Sitter form. Moreover, such ends can be used for creating wormholes, or connecting initial data sets, provided the neck parameters of the ends match. The object of this note is to show that *generic* constant positive scalar curvature metrics can be deformed, by a local deformation, to constant positive scalar curvature metrics containing asymptotically Delaunay ends. The method of [5] is then used to obtain *exactly Delaunay ends*.

We further point out that the neck parameter of each new asymptotically Delaunay end can be arbitrarily prescribed within an interval $(0, \epsilon_0)$, for some $\epsilon_0 > 0$. This flexibility in prescribing the neck sizes guarantees that the exactly Delaunay ends can be matched to perform gluings.

2. Statement of the result

We begin with some terminology. A *static KID* is a function N satisfying

$$D_i D_j N - N R_{ij} - \Delta_g N g_{ij} = 0 \,. \tag{1}$$

We shall say that *there exist no local static KIDs near p* if there exist sequences $0 < \eta_i < \delta_i \to_{i\to\infty} 0$ such that there exist no nontrivial solution of (1) on the annuli

$$A_p(\eta_i, \delta_i) := \bar{B}_p(\delta_i) \setminus B_p(\eta_i) \,,$$

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where $B_p(r)$ is a geodesic ball centered at p of radius r. Observe that (1) forms a set of overdetermined equations, it is natural to expect that for generic metrics on M there will not be any local static KIDs at any point $p \in M$. Precise statements to this effect have been proved in [1], both for unconstrained metrics, and for metrics with constant non-positive scalar curvature.

It is well known that there exists a one parameter family of constant positive scalar curvature conformal metrics on $\mathbb{R} \times S^{n-1}$. In the literature, these metrics are usually referred to as Kottler–Schwarzschild–de Sitter metrics [5] or Delaunay metrics [8] or even Fowler metrics [6]. This family of metrics is parameterized by a parameter which is called the "neck size". Let us describe these briefly since they are at the heart of our result. We assume $n \geq 3$ and we set

$$\epsilon_* := \left(\frac{n-2}{n}\right)^{\frac{4}{n-2}} .$$

For all $\epsilon \in (0, \epsilon_*]$, we define v_ϵ to be the unique solution of

$$\partial_t^2 v_\epsilon - \left(\frac{n-2}{2}\right)^2 v_\epsilon + \frac{n(n-2)}{4}\, v_\epsilon^{\frac{n+2}{n-2}} = 0 \,,$$

with $v_\epsilon = \epsilon$ and $\partial_t v_\epsilon(0) = 0$. The parameter ϵ is called the *neck size* of the associated Delaunay metric

$$g_\epsilon = u_\epsilon^{\frac{4}{n-2}}\, dx^2 = v_\epsilon^{\frac{4}{n-2}}\, (dt^2 + g_{S^{n-1}}) \,, \tag{2}$$

where

$$u_\epsilon(x) := |x|^{\frac{2-n}{2}}\, v_\epsilon(-\log|x|) \,,$$

and where dx^2 denotes the Euclidean metric in $\mathbb{R}^n$ and $g_{S^{n-1}}$ denotes the canonical metric on S^{n-1}. The metric g_ϵ has constant scalar curvature equal to $R \equiv n\,(n-1)$. Observe that, without loss of generality, we can normalize the metrics we are interested in to have (constant) scalar curvature equal to $n\,(n-1)$.

Building on the results in [2, 4, 5], and using a simple perturbation argument, we prove the:

THEOREM 2.1. *Let (M, g) be a smooth n-dimensional Riemannian manifold with constant positive scalar curvature $R \equiv n\,(n-1)$, $n \geq 3$. Let $p \in M$ and suppose that there exist no local static KIDs near p. Then for any $\rho > 0$ there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$ there exists a smooth metric with constant positive scalar curvature $R \equiv n\,(n-1)$ which coincides with g in $M \setminus B_p(\rho)$ and which coincides with a Delaunay metric g_ϵ with neck size parameter ϵ in a punctured ball centered at p.*

As already pointed out, a generic metric will satisfy the "no local static KIDs" property at every point. Since both our construction and that of [5] are purely local, the construction can be repeated at any chosen finite, or countably infinite, collection of points $p_i \in M$ to produce complete constant positive scalar curvature metrics with a countable number of exactly Delaunay ends.

Remark: It would be of interest to generalise this construction to general, not necessarily time-symmetric, general relativistic initial data sets (M, g, K).

2.1. Proof of Theorem 2.1: We choose $p \in M$ and $\rho > 0$. There exists $\bar\rho \le \rho$ such that for all $\delta < \bar\rho$ the linear operator [1]

$$L := \Delta_g + n\,,$$

is injective on $\mathring{\mathcal{C}}^2(\bar B_p(\delta))$, the space of $\mathcal{C}^2$-functions which are defined on $\bar B_p(\delta)$ and which vanish on the boundary of this domain. The injectivity follows easily from the fact that in geodesic normal coordinates the metric g can be expanded as $g_{ij} = \delta_i^j + \mathcal{O}(|x|^2)$ and hence, for all $\delta > 0$ small enough, we can estimate

$$\int_{\bar B_p(\delta)} \Big(\frac{1}{2}\,|\nabla_{\mathring g} v|^2 - 2\,n\,v^2\Big) dv_{\mathring g} \le \int_{\bar B_p(\delta)} (|\nabla_g v|^2 - n\,v^2) dv_g,$$

where $\mathring g$ denotes the Euclidean metric. But, if λ_1 is the first eigenvalue of $-\Delta_{\mathring g}$ on the unit ball of $\mathbb{R}^n$, we have

$$\lambda_1 \int_{\bar B_p(\delta)} v^2\, dv_{\mathring g} \le \delta^2 \int_{\bar B_p(\delta)} |\nabla_{\mathring g} v|^2\, dv_{\mathring g}\,,$$

and we conclude that

$$0 \le \int_{\bar B_p(\delta)} (|\nabla_g v|^2 - n\,v^2) dv_g,$$

for all δ small enough. This clearly implies that L is injective. Since we have assumed that there are no local static KIDs near p, by the arguments in [1] there exists $0 < \eta < \delta < \bar\rho$ such that there are no static KIDs on the annulus

$$A_p(\eta,\delta) = \bar B_p(\delta) \setminus B_p(\eta)\,.$$

Therefore, we conclude that we can choose $0 < \eta < \delta \le \rho$ such that there are no static KIDs on the annulus $A_p(\eta,\delta)$ and the operator L is injective on $\mathring{\mathcal{C}}^2(\bar B_p(\delta))$.

In $\bar B_p(\delta)$, we now deform the metric g to a family of metrics which are conformally flat in a small neighborhood p and which still have constant scalar curvature equal to $R = n\,(n-1)$ in $\bar B_p(\delta)$. The new metrics will match continuously the original ones along the boundary $\partial B_p(\delta)$, but in general their derivatives will not agree there.

Lemma 2.2. *There exists $0 < r_0 \le \eta$ and a family of smooth constant scalar curvature metrics g^r, with $r \in (0, r_0]$, which are defined in $\bar B_p(\delta)$, have constant scalar curvature $R \equiv n\,(n-1)$, are conformally flat in $\bar B_p(r)$, the ball of radius r centered at p (radius computed with respect to the metric g) and for which*

$$L_r := \Delta_{g^r} + n \tag{3}$$

is injective on $\mathring{\mathcal{C}}^2(\bar B_p(\delta))$. Moreover

$$\|g^r - g\|_{L^\infty(B_p(\delta))} + r\,\|\nabla_g (g^r - g)\|_{L^\infty(B_p(\delta))} \le c\,r^\gamma\,, \tag{4}$$

for any $\gamma < 2$, for some constant $c = c(\gamma)$. Furthermore, for any $k \in \mathbb{N}$, the sequence of metrics g^r converges to g in $\mathcal{C}^k$-topology, on compacts of $\bar B_p(\delta) \setminus \{p\}$, as r tends to zero.

[1] Observe that our Laplacian is the sum of second derivatives.

Proof. We agree that the geodesic balls, the gradient and norm are taken with respect to the metric g. We consider geodesic normal coordinates $x := (x_1, \dots, x_n)$ near p. As already mentioned, in these coordinates, the metric g can be expanded as

$$g_{ij} = \delta_i^j + \mathcal{O}(|x|^2)\,.$$

We choose a cutoff function χ which is identically equal to 1 in the unit ball of $\mathbb{R}^n$ and identically equal to 0 outside the ball of radius 2. Given $r \in (0, \delta/2)$, we consider a metric $\bar g^r$ whose coefficients near p are given by

$$\bar g^r_{ij} := \chi(\cdot/r)\,\delta_i^j + (1-\chi(\cdot/r))\,g_{ij}\ .$$

Observe that

$$\|\bar g^r - g\|_{L^\infty} + r\,\|\nabla_g(\bar g^r - g)\|_{L^\infty} + r^2\,\|\nabla_g^2(\bar g^r - g)\|_{L^\infty} \le c\,r^2\ . \tag{5}$$

for some constant $c > 0$ which does not depend on $r \le \delta/2$.

Let us denote by $\bar R_r$ the scalar curvature of the metric $\bar g^r$ and recall that $R = n\,(n-1)$ is the scalar curvature of the metric g. We have $\bar R_r = R$ in $\bar B_p(\delta) \setminus B_p(2r)$ and $\bar R_r = 0$ in $\bar B_p(r)$. Finally, in the annulus $\bar B_p(2r) \setminus B_p(r)$, we only have the estimate

$$\|\bar R_r\|_{L^\infty} \le c\,,$$

for some constant $c > 0$ independent of $r \le \delta/2$. This follows at once from the fact that the expression of the scalar curvature in terms of the coefficients of the metric involves the coefficients of the metric and their derivatives up to order 2 which, thanks to (5), are bounded independently of $r \le \delta/2$.

Now, we explain how to solve the equation

$$\Delta_{\bar g^r}\,u - \frac{n-2}{4(n-1)}\,\bar R_r\,u + \frac{n\,(n-2)}{4}\,u^{\frac{n+2}{n-2}} = 0\,, \tag{6}$$

for r small enough. Once this equation is solved and assuming that the function $u > 0$, the metric

$$g^r := u^{\frac{4}{n-2}}\,\bar g^r\,,$$

will be a constant scalar curvature metric equal to $R \equiv n\,(n-1)$ defined in $\bar B_p(\delta)$.

We set $u = 1 + v$ and rewrite the above equation as

$$L v = (\Delta_g - \Delta_{\bar g^r})\,v + \frac{n-2}{4(n-1)}\,(\bar R_r - R)\,(1+v) - \frac{n(n-2)}{4}\left((1+v)^{\frac{n+2}{n-2}} - 1 - \frac{n+2}{n-2}\,v\right)\ .$$

For short, we denote by $N_r(v)$ the right hand side of this equation and we assume that the function v is small enough, say $\|v\|_{L^\infty} \le 1/2$, so that $1 + v > 0$.

We fix $\ell > n/2$. Using the fact that $\bar R_r - R$ is supported in $B_p(2r)$ and is bounded in this set, it is easy to check that there exists a constant $c > 0$ (which does not depend on $r \in (0, \delta/2)$) such that

$$\|(\bar R_r - R)\,(1+v)\|_{L^\ell} \le c\,(1 + \|v\|_{L^\infty})\,r^{n/\ell}\ ,$$

and

$$\|(\bar R_r - R)\,(v - v')\|_{L^\ell} \le c\,r^{n/\ell}\,\|v - v'\|_{L^\infty}\ .$$

Similarly,

$$\|(1+v)^{\frac{n+2}{n-2}} - 1 - \frac{n+2}{n-2}\,v\|_{L^\ell} \le c\,\|v\|^2_{L^\infty}\ ,$$

and

$$\|(1+v)^{\frac{n+2}{n-2}} - (1+v')^{\frac{n+2}{n-2}} - \frac{n+2}{n-2}\,(v-v')\|_{L^\ell} \leq c\,(\|v\|_{L^\infty} + \|v'\|_{L^\infty})\,\|v-v'\|_{L^\infty}\ ,$$

provided $\|v\|_{L^\infty} \leq 1/2$ and $\|v'\|_{L^\infty} \leq 1/2$.

Thanks to (5) we also check that

$$\|(\Delta_g - \Delta_{\bar{g}^r})\,v\|_{L^\ell} \leq\ c\,r\,\|v\|_{W^{2,\ell}}\ .$$

with perhaps another constant $c > 0$ which again does not depend on $r \in (0, \delta/2)$. Indeed, the difference between these two operators involves a second order differential operator whose coefficients can be estimated by the difference of the coefficients of the two metrics and a first order differential operator whose coefficients can be estimated by the gradient of the difference between the coefficients of the two metrics.

By hypothesis, the operator

$$L := \Delta_g + n\,,$$

is an isomorphism from $\mathring{W}^{2,\ell}(B_p(\delta))$ into $L^\ell(B_p(\delta))$ (Recall that $\mathring{W}^{2,\ell}(B_p(\delta))$ is the completion of the space of smooth functions on $B_p(\delta)$ which vanish on $\partial B_p(\delta)$ with respect to the usual $W^{2,\ell}$-Sobolev norm).

Collecting these we find that

$$\|L^{-1}\,N_r(v)\|_{W^{2,\ell}} \leq c\left(r^{n/\ell} + (r + r^{n/\ell})\,\|v\|_{L^\infty} + \|v\|^2_{L^\infty}\right)\ ,$$

and

$$\text{(7)}\qquad \|L^{-1}\,(N_r(v) - N_r(v'))\|_{W^{2,\ell}} \leq c\left(r + r^{n/\ell} + \|v\|_{L^\infty} + \|v'\|_{L^\infty}\right)\|v-v'\|_{L^\infty}\ .$$

Using the embedding $W^{2,\ell}(B_p(\delta)) \longrightarrow L^\infty(B_p(\delta))$, we conclude, from the fixed point theorem for contraction mappings, that the nonlinear operator

$$v \mapsto L^{-1}\,N_r(v)$$

has a (unique) fixed point v_r in the ball of radius $2\,c\,r^{n/\ell}$ in $W^{2,\ell}(B_p(\delta))$, provided $r > 0$ is small enough, say $r \in (0, r_0]$. Note that the constant $c = c_\ell$ depends on ℓ since it depends on the norm of the Sobolev embedding. This completes the proof of the existence of a positive solution of (6). The fact that v_r (and hence $u_r := 1 + v_r$) is smooth follows from classical elliptic regularity.

The metric which appears in the statement of the result is

$$g^r := (1+v_r)^{\frac{4}{n-2}}\,\bar{g}^r\ .$$

The first estimate in (4) follows from the construction itself and the embedding $W^{2,\ell}(B_p(\delta)) \longrightarrow L^\infty(B_p(\delta))$ when $\ell > n/2$, while the second estimate in (4) follows from the construction and the embedding $W^{2,\ell}(B_p(\delta)) \longrightarrow W^{1,\infty}(B_p(\delta))$ when $\ell > n$.

Perhaps, it is now time to comment about the choice of ℓ. Observe that functions if $W^{2,\ell}$ are in L^∞ provided $\ell > n/2$ while their derivatives are in L^∞ provided $\ell > n$. In order to estimate the nonlinearities in the equation it was easy to work with bounded functions, this is the main reason why we have chosen $\ell > n/2$. Now, in the statement of the result, we claim L^∞ bounds both on the solution itself and its partial derivatives; to obtain those, it would have been tempting to work directly with $\ell > n$. However, with this latter choice we would have obtained the desired L^∞ estimate for the partial derivatives of the solution, but we would not have obtained a good estimate for the L^∞ norm of the solution itself; this is where $\ell > n/2$ was necessary. Observe that, in principle, the solution of the fixed point problem does depend on ℓ. However, if $\ell' > \ell$ and v'_r and v_r are the respective fixed points, it follows easily from (7) that $v'_r = v_r$ for all r small enough.

Finally, reducing r_0 if necessary, we claim that the operator

$$\Delta_{g^r} + n \,,$$

is injective for all $r \in (0, r_0]$. To do so, simply use (5) which implies that the coefficients of $g^r - g$ and their derivatives are bounded in L^∞-norm by a constant times $r^{\gamma-1}$. This implies that

$$\|(\Delta_g - \Delta_{g^r})\, v\|_{L^\ell} \leq \, c\, r^{\gamma-1}\, \|v\|_{W^{2,\ell}} \,.$$

Finally, the claim then follows from a simple perturbation argument since L is an isomorphism between $\mathring{W}^{2,\ell}(B_p(\delta))$ and $L^\ell(B_p(\delta))$ for any $\ell > 1$.

The fact that, for any $k \in \mathbb{N}$, the sequence of metrics g^r converges to g in $\mathcal{C}^k$-topology, on compacts of $\bar{B}_p(\delta) \setminus \{p\}$, as r tends to zero follows from elliptic regularity theory since, for all $r \in (0, \bar{r}]$, the function v_r is a solution of

$$\Delta_g\, v_r + n\, v_r + \frac{n\,(n-2)}{4} \left((1 + v_r)^{\frac{n+2}{n-2}} - 1 - \frac{n+2}{n-2}\, v_r \right) = 0$$

in $\bar{B}_p(\delta) \setminus B_p(\bar{r})$ which vanishes on $\partial B_p(\delta)$ and is bounded by a constant times r^γ in L^∞-norm. □

The following result is essentially borrowed from [2]:

THEOREM 2.3. *Let (M_0, g_0) be a compact Riemannian manifold with smooth boundary ∂M_0. Assume that the scalar curvature is constant equal to $R = n\,(n-1)$, and that the operator*

$$\Delta_{g_0} + n \,,$$

acting on functions $\mathring{W}^{2,\ell}(M_0)$ is injective. Further assume that the metric g_0 is locally conformally flat in a neighborhood of a point $p \in M_0$. Then, there exists $\epsilon_0 > 0$ and for all $\epsilon \in (0, \epsilon_0]$ there exists a complete constant scalar curvature metric defined in $M_0 \setminus \{p\}$ of the form

$$\tilde{g}_\epsilon = e^{\phi_\epsilon}\, g_0 \,,$$

where $\phi_\epsilon = 0$ on ∂M_0. Furthermore, ϕ_ϵ converges to 0 on compacts subsets of $M_0 \setminus \{p\}$ in any $\mathcal{C}^k$-topology, for $k \in \mathbb{N}$.

Finally, in the neighborhood of p where g_0 is conformally flat and where we can use coordinates $x \in \mathbb{R}^n$ with $x = 0$ at p, the metric $\tilde{g}_\epsilon$ can be written as

$$\tilde{g}_\epsilon = \tilde{u}_\epsilon^{\frac{4}{n-2}}\, dx^2 \,,$$

where the function u_ϵ satisfies

$$\tilde{u}_\epsilon(x) = |x|^{\frac{2-n}{2}}\; v_\epsilon(-\log|x| + t_\epsilon)\,(1 + \mathcal{O}(|x|)) \,,$$

for some $t_\epsilon \in \mathbb{R}$, where the error term $\mathcal{O}(|x|)$ depends on ϵ.

REMARK 2.4. *In addition, reducing ϵ_0 if this is necessary, the solutions constructed are "unmarked nondegenerate" (see section 6 of* [2] *or Proposition 10 in* [7] *for a proof). We refer to* [8] *for the definition of unmarked nondegeneracy. In particular this implies that one can use $p \in M_0$ and ϵ as coordinates on the unmarked moduli space.*

Proof. Existence follows from Theorem 1.1 in [2]. The fact that the construction is possible for any small value of the parameter ϵ is not explicit in the statement of Theorem 1.1 in [2] but it is implicit in the proof (see bottom of page 1184). The expansion of the metric close to p follows directly from the construction in [2] but also follows from general results such as [3] and [6] (see (9) page 241 and Theorem 1 page 235 of [6]). □

Finally, we recall the result of [4, Theorem 5.9]: Choose some $k > n/2+4$. If there are no static KIDs on the annulus $A_p(\eta,\delta)$, then there exists $\zeta > 0$ such that if $\tilde{g}$ is a constant scalar curvature metric metric on the annulus $\bar{B}_p(\delta) \setminus B_p(\eta/2)$ satisfying

$$\|g-\tilde{g}\|_{\mathcal{C}^k(A_p(\eta,\delta))} \le \zeta$$

(recall that $A_p(\eta,\delta) = \bar{B}_p(\delta) \setminus B_p(\eta)$), then there exists a constant scalar curvature metric $\hat{g}$ on M which coincides with g in $M \setminus B_p(\delta)$, and which equals $\tilde{g}$ in $\bar{B}_p(\eta) \setminus B_p(\eta/2)$.

We are now in a position to prove Theorem 2.1. We first apply Lemma 2.2 and fix $r \in (0, r_0]$ such that

$$\|g-g^r\|_{\mathcal{C}^k(A)} \le \zeta/2\,.$$

At this stage r is fixed and we next apply Theorem 2.3 with $M_0 = \bar{B}_p(\delta)$ and $g_0 = g^r$ to obtain a family of asymptotically Delaunay metrics $\tilde{g}^{r,\epsilon}$, with neck size $\epsilon \in (0, \tilde{\epsilon}_0]$. Now, $g^{r,\epsilon}$ converges to g^r in $\mathcal{C}^k(A)$ as ϵ tends to zero. Therefore, reducing ϵ_0 if necessary, we can assume that

$$\|g^{r,\epsilon}-g\|_{\mathcal{C}^k(A)} \le \zeta\,,$$

and [4, Theorem 5.9 and Corollary 5.11] apply to produce a smooth constant scalar curvature metric $\mathring{g}^{r,\epsilon}$ which coincides with g away from $B_p(\delta)$, and coincides with $g^{r,\epsilon}$ on $B_p(\eta)$. Observe that this metric is asymptotic to a Delaunay metric as explained in Theorem 2.3. Finally, applying Theorem 3.1 of [5], the metric $\mathring{g}^{r,\epsilon}$ can be deformed to a constant scalar curvature metric which is *exactly Delaunay* in the asymptotically Delaunay region. This completes the proof of our result.

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