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THE MONGE–AMPÈRE QUASI-METRIC STRUCTURE ADMITS A SOBOLEV INEQUALITY

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Abstract. Sobolev inequalities associated to the Monge–Ampère quasi-metric structure are proved.

1. Introduction and main result

The *Monge–Ampère measure* associated to a twice-differentiable convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is defined as $\mu_\varphi(x) := \det D^2\varphi(x)$. Given $x \in \mathbb{R}^n$ and $t > 0$, a *section* of φ centered at x with height t is the open bounded convex set

$$S_\varphi(x,t) := \{y \in \mathbb{R}^n : \varphi(y) < \varphi(x) + \langle \nabla\varphi(x), y - x\rangle + t\}. \tag{1.1}$$

The relevant compatibility condition between the sections of φ, which from now on we assume to be strictly convex, and its Monge–Ampère measure is the so-called (DC)-*doubling condition*. More precisely, we write $\mu_\varphi \in (\mathrm{DC})_\varphi$ if there exist constants $B \geq 1$ and $0 < \alpha < 1$ such that for all sections $S_\varphi(x,t)$

$$\mu_\varphi(S_\varphi(x,t)) \leq B\mu_\varphi(\alpha S_\varphi(x,t)), \tag{1.2}$$

where $\alpha S_\varphi(x,t)$ denotes α-contraction of $S_\varphi(x,t)$ with respect to its (Euclidean) center of mass x^*. Constants depending only on B and α in (1.2) as well as on dimension n will be called *geometric constants*. The Monge–Ampère quasi-metric structure was introduced by Caffarelli and Gutiérrez [1] in their pioneering work on the linearized Monge–Ampère equation. Remarkably, the strictly convex function φ generates a quasi-metric if and only if μ_φ possesses the (DC)-doubling property, (see, for instance, [4, Section 2]). More precisely, under the (DC)-doubling condition, φ renders a structure of space of homogeneous type, see [4, Section 2] and references there in, in such a way that the function

$$\rho_\varphi(x,y) := \varphi(y) - \varphi(x) - \langle \nabla\varphi(x), y - x\rangle, \quad x, y \in \mathbb{R}^n, \tag{1.3}$$

becomes a quasi-distance in $\mathbb{R}^n$, that is, $\rho_\varphi(x,y) = 0$ if and only if $x = y$; $\rho_\varphi(x,y) \simeq \rho_\varphi(y,x)$; and $\rho_\varphi(x,y) \lesssim \rho_\varphi(x,z) + \rho_\varphi(z,y)$, where the implicit constants are geometric constants. By definition (1.1), the sections of φ can then be realized as the quasi-balls associated to ρ_φ. On the other hand, if ρ_φ generates a quasi-metric then the section of φ satisfy the so-called engulfing property, which, in turn, is equivalent to (DC), see [2, 3, 6, 7].

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We will use the fact that the (DC)-doubling property implies the existence of geometric constants $K_1, K_2 > 0$ such that

$$K_1^n t^n \leq |S_\varphi(x,t)| \mu_\varphi(S_\varphi(x,t)) \leq K_2^n t^n, \quad \forall x \in \mathbb{R}^n, \forall t > 0. \tag{1.4}$$

For the proof this statement, as well as the real analysis associated to φ and further characterizations of (DC), see, for instance, [1], Theorem 8 in [2], Theorem 4 in [3], [7], and [6, Chapter 3].

Suppose that $\mu_\varphi(x) > 0$ for a.e. $x \in \mathbb{R}^n$. The linearized Monge–Ampère operator, denoted by L_φ, is the typically degenerate, elliptic operator defined as

$$L_\varphi(u)(x) := \operatorname{trace}(A_\varphi(x) D^2 u(x)) \quad \text{a.e. } x \in \mathbb{R}^n,$$

with

$$A_\varphi(x) := \mu_\varphi(x)(D^2\varphi(x))^{-1} \quad \text{a.e. } x \in \mathbb{R}^n. \tag{1.5}$$

The study of L_φ is best carried out within the Monge–Ampère structure, see [1–4]. In [8], the author proved Poincaré-type inequalities for the Monge–Ampère quasi-metric structure, which were instrumental in his proof of Harnack's inequality for non-negative solutions to $L_\varphi(u) = 0$ (always under the hypothesis $\mu_\varphi \in (\mathrm{DC})_\varphi$ only).

In [9], Tian and Wang proved that if $\mu \in (\mu_\infty)$ (in their notation, $\mu \in (\mathbb{CG})$, see [9]) and if the sections of φ satisfy certain size conditions (see Lemma 3.3 in [9]), then a power-like decay of the distribution function of Green functions of L_φ holds true. Consequently, a Sobolev inequality associated to A_φ follows. Namely, there exists $p > 1$ such that for every section $S := S_\varphi(x,t)$ there is a $C > 0$ such that whenever $u \in C_0^1(S)$ (that is, u is continuously differentiable and compactly supported in S) we have

$$\left(\frac{1}{\mu_\varphi(S)} \int_S |u|^p \, d\mu_\varphi\right)^{\frac{1}{p}} \leq C \left(\frac{1}{\mu_\varphi(S)} \int_S \langle A_\varphi \nabla u, \nabla u\rangle\right)^{\frac{1}{2}}. \tag{1.6}$$

The importance and variety of the Sobolev inequalities (1.6) have been thoroughly stressed in [9].

Our main result is the Sobolev inequality (1.7) below, which we prove resorting to *neither* the (μ_∞) condition *nor* apriori size condition on the sections of φ. The (μ_∞)-condition is significantly stronger than the (DC)-condition, the gap being comparable to that between A_∞ Muckenhoupt weights and doubling weights, see [4, Section 3] for a thorough discussion and examples. Moreover, under the (DC)-doubling condition only, we can guarantee a (perhaps not optimal but) uniform value of p in (1.6). We prove

Theorem 1. *Assume $\mu_\varphi \in (\mathrm{DC})_\varphi$. Then, there exists a geometric constant $K_6 > 0$ such that for every section $S := S_\varphi(x_0, t)$ and every $u \in C_0^1(S)$, we have*

$$\left(\frac{1}{\mu_\varphi(S)} \int_S |u(x)|^{\frac{2n}{n-1}} \, d\mu_\varphi(x)\right)^{\frac{n-1}{2n}} \leq K_6 \, t^{\frac{1}{2}} \left(\frac{1}{\mu_\varphi(S)} \int_S |\nabla^\varphi u(x)|^2 \, d\mu_\varphi(x)\right)^{\frac{1}{2}}, \tag{1.7}$$

where

$$\nabla^\varphi u(x) := D^2\varphi(x)^{-\frac{1}{2}} \nabla u(x) \quad \forall x \in S. \tag{1.8}$$

By means of a duality argument involving convex conjugates we also obtain the following Sobolev-type inequality, now with respect to Lebesgue measure.

Theorem 2. *Assume $\mu_\varphi \in (\mathrm{DC})_\varphi$. Then, there exists a geometric constant $K_7 > 0$ such that for every section $S := S_\varphi(x_0, t)$ and every $u \in C_0^1(S)$, we have*

$$\left(\frac{1}{|S|}\int_S |u(x)|^{\frac{2n}{n-1}}\,dx\right)^{\frac{n-1}{2n}} \leq K_7\, t^{\frac{1}{2}} \left(\frac{1}{|S|}\int_S |\nabla^\varphi u(x)|^2\,dx\right)^{\frac{1}{2}}. \tag{1.9}$$

2. Estimates for Green functions

The main step in the proof of Theorem 1 is a power-like decay estimate for the distribution function of Green functions of L_φ on the sections of φ. Namely,

Theorem 3. *Suppose $\mu_\varphi \in (\mathrm{DC})_\varphi$. There exists a geometric constant $K_3 \geq 1$ such that for every section $S := S_\varphi(x, t)$ and every $z \in S_\varphi(x, t/2)$ we have*

$$\mu_\varphi(\{y \in S : g_S(z, y) > \lambda\}) \leq K_3 \mu_\varphi(S)^{1-n'} t^{n'} \lambda^{-n'}, \quad \forall \lambda > 0.$$

Here $n' = n/(n-1)$ and g_S denotes the Green function of L_φ in S.

Proof. First, assume that $x = 0$, $\varphi(0) = 0$, and $\nabla\varphi(0) = 0$. With these assumptions we have $\varphi \geq 0$ and $z \in S_\varphi(0, t)$ if and only if $\varphi(z) < t$. Set $S := S_\varphi(0, t)$ and $S_{1/2} := S_\varphi(0, t/2)$.

By the Aleksandrov–Bakelman–Pucci maximum principle (see [4, Lemma 8], [5, Theorem 9.1]), whenever h is a solution to $L_\varphi(h) = H\mu_\varphi$ in S with $h = 0$ on ∂S, then

$$\sup_S |h| \leq C_1 |S|^{1/n} \left(\int_S |H|^n\,d\mu_\varphi\right)^{1/n}, \tag{2.1}$$

where C_1 depends only on the dimension n. On the other hand, note that the problem

$$\begin{cases} L_\varphi(h) &= H\mu_\varphi \quad \text{in } S, \\ h &= 0 \quad \text{on } \partial S, \end{cases} \tag{2.2}$$

is always solvable for $H \in L^n(S, d\mu_\varphi)$ with $h \in W^{2,n}_{loc}(S) \cap C(\overline{S})$ because L_φ has second-order continuous coefficients and $\varphi \in C^2$ with $D^2\varphi > 0$ (see [5, Section 9.6]). That is, we make use of the fact that $\varphi \in C^2$ and consequently the eigenvalues of $(D^2\varphi(x))^{-1}$ will be bounded and bounded away from zero on compact subsets of $\mathbb{R}^n$. Notice that we use this fact only to deduce the existence of solutions.

Again by the maximum principle, if $H \geq 0$ and h solves (2.2), then $h < 0$ (unless $H \equiv 0$, which trivially yields $h \equiv 0$). Fix $z \in S$ and define T_z by

$$T_z : L^n(S, \mu_\varphi) \to \mathbb{R}$$
$$H \mapsto -h(z)$$

Then T_z is a positive linear functional in $L^n(S, \mu_\varphi)^*$ and, from (2.1),

$$\|T_z\|_{L^n(S,\mu_\varphi)^*} \leq C_1 |S|^{1/n}.$$

By the Riesz representation theorem, there is a non-negative function $g_S(z, \cdot) \in L^{n'}(S, \mu_\varphi)$, where $n' = \frac{n}{n-1}$, such that

$$T_z(H) = \int_S g_S(z, y) H(y)\,d\mu_\varphi(y)$$

and

$$\left(\int_S g_S(z,y)^{n'}\,d\mu_\varphi(y)\right)^{\frac{1}{n'}} \le C_1|S|^{1/n}. \tag{2.3}$$

That is, for every $z \in S$ and $h \in W^{2,n}_{loc}(S) \cap C(\overline{S})$, with $h = 0$ on ∂S, we have

$$\begin{aligned} -h(z) &= \int_S g_S(z,y) L_\varphi(h)(y)\,dy \\ &= \int_S g_S(z,y)\mathrm{trace}((D^2\phi(y))^{-1}D^2h(y))\mu_\varphi(y)\,dy. \end{aligned} \tag{2.4}$$

In particular, setting $h := \varphi - t$ in (2.4), for every $z \in S_{1/2}$ we obtain

$$\begin{aligned} \frac{t}{2} \le t - \varphi(z) &= \int_S g_S(z,y)\mathrm{trace}(D^2\varphi(y)^{-1}D^2\varphi(y))\,d\mu_\varphi(y) \\ &= n\int_S g_S(z,y)\,d\mu_\varphi(y). \end{aligned} \tag{2.5}$$

Next, we prove that Green functions $g_S(z,\cdot)$ satisfy a reverse-Hölder inequality uniformly for $z \in S_{1/2}$. Fix $z \in S_{1/2}$, and use (2.3), (1.4), and (2.5) to write

$$\begin{aligned} \left(\frac{1}{\mu_\varphi(S)}\int_S g_S(z,y)^{n'}\,d\mu_\varphi(y)\right)^{\frac{1}{n'}} &\le C_1\frac{\mu_\varphi(S)^{1/n}|S|^{1/n}}{\mu_\varphi(S)} \le C_1K_2\frac{t}{\mu_\varphi(S)} \\ &\le 2C_1K_2n\frac{1}{\mu_\varphi(S)}\int_S g_S(z,y)\,d\mu_\varphi(y). \end{aligned} \tag{2.6}$$

Then, from Chebyshev's inequality and (2.6), for $\lambda > 0$ and always for $z \in S_{1/2}$, we have

$$\begin{aligned} &\mu_\varphi(\{y \in S : g_S(z,y) > \lambda\}) \\ &\quad\le \frac{\mu_\varphi(S)}{\lambda^{n'}}\frac{1}{\mu_\varphi(S)}\int_S g_S(z,y)^{n'}\,d\mu_\varphi(y) \\ &\quad\le \frac{\mu_\varphi(S)}{\lambda^{n'}}\left(\frac{2C_1K_2n}{\mu_\varphi(S)}\int_S g_S(z,y)\,d\mu_\varphi(y)\right)^{n'} \\ &\quad= \frac{(2C_1K_2)^{n'}\mu_\varphi(S)^{1-n'}(t-\varphi(z))^{n'}}{\lambda^{n'}} \\ &\le (2C_1K_2)^{n'}\frac{\mu_\varphi(S)^{1-n'}t^{n'}}{\lambda^{n'}} =: K_3\frac{\mu_\varphi(S)^{1-n'}t^{n'}}{\lambda^{n'}}, \end{aligned}$$

where we also used (2.5). For an arbitrary $x_0 \in \mathbb{R}^n$ and a general section $S_\varphi(x_0,t)$, define

$$\varphi_{x_0}(x) := \varphi(x_0 - x) - \varphi(x_0) + \langle\nabla\varphi(x_0), x\rangle \quad \forall x \in \mathbb{R}^n.$$

Then $\mu_{\varphi_{x_0}}$ verifies the (DC)-doubling property with the same constants as μ_φ does (uniformly in x_0), also $\nabla\varphi_{x_0}(0) = 0$, $\varphi_{x_0}(0) = 0$ and

$$S_{\varphi_{x_0}}(0,t) = x_0 - S_\varphi(x_0,t).$$

Thus, $\mu_{\varphi_{x_0}}(S_{\varphi_{x_0}}(0,t)) = \mu_\varphi(S_\varphi(x_0,t))$ and if we now apply the obtained result to φ_{x_0}, the general case follows by changing variables $x_0 - x \mapsto x$. □

3. Proof of Theorem 1 via Tian-Wang's crucial lemma

What follows is an adaptation, to the context of the Monge–Ampère structure, of the crucial lemma by Tian and Wang (see Lemma 2.1 in [9]). We include details of the proof, as well as some additions, for the sake of completeness and to follow up the geometric constants involved.

Let $S := S_\varphi(x_0, t_0)$ be a section of φ and let $G(z, y)$ denote the Green function of L_φ on $2S := S_\varphi(x_0, 2t_0)$. As before, let us assume that $x_0 = 0$, $\varphi(0) = 0$, and $\nabla\varphi(0) = 0$. By Theorem 3, for $z \in S$, we have

$$\mu_\varphi(\{y \in 2S : G(z, y) > \lambda\}) \leq K_3 \mu_\varphi(2S)^{1-n'} (2t_0)^{n'} \lambda^{-n'}, \quad \forall \lambda > 0. \tag{3.1}$$

Let us set

$$K := K_3 \mu_\varphi(2S)^{1-n'} (2t_0)^{n'} \tag{3.2}$$

and $p := 2n' > 2$ so that (3.1) reads as in Lemma 2.1 of [9], that is,

$$\mu_\varphi(\{y \in 2S : G(z, y) > \lambda\}) \leq K \lambda^{-\frac{p}{2}}, \quad \forall \lambda > 0. \tag{3.3}$$

Next, consider any open set $U \subset S$ and let $\psi_1 = \psi_1(U)$ and $\lambda_1 = \lambda_1(U)$ be the first Dirichlet eigenfunction and eigenvalue of L_φ in U, that is,

$$\begin{cases} L_\varphi(\psi_1) &= -\lambda_1 \psi_1 \mu_\varphi \quad \text{in } U, \\ \psi_1 &= 0 \quad \text{on } \partial U. \end{cases} \tag{3.4}$$

Let G_U denote the Green function of L_φ in U. Note that, since the coefficients of A_φ are continuous and A_φ is positive definite in S, G_U always exists. By the maximum principle, $0 \leq G_U(z, y) \leq G(z, y)$ for every $z, y \in U$ $(z \neq y)$.

Then ψ_1 and G_U can be related by $\psi_1(y) = \lambda_1 \int_U G_U(x, y)\mu_\varphi(x)\,dx$. Normalizing ψ_1 to be non-negative with $\|\psi_1\|_{L^\infty(\bar{U})} = 1$ and taking $y_1 \in U \subset S \subset 2S$ such that $\psi_1(y_1) = 1$, we can write

$$1 \leq \lambda_1(U) \int_U G_U(x, y_1)\psi_1(x)\mu_\varphi(x)\,dx$$

and then, for any $y_0 \in U$, use (3.3) to estimate

$$\begin{aligned} \int_U G_U(x, y_0)\mu_\varphi(x)\,dx &= \int_0^\infty \mu_\varphi(\{x \in U : G_U(x, y_0) > \lambda\})\,d\lambda \\ &\leq \int_0^\infty \min\{\mu_\varphi(U), \mu_\varphi(\{x \in U : G(x, y_0) > \lambda\})\}\,d\lambda \\ &\leq \int_0^\infty \min\{\mu_\varphi(U), K\lambda^{-\frac{p}{2}}\}\,d\lambda = \tau\mu_\varphi(U) + K\int_\tau^\infty \lambda^{-\frac{p}{2}}\,d\lambda \\ &= nK^{\frac{2}{p}}\mu_\varphi(U)^{1-\frac{2}{p}}, \end{aligned} \tag{3.5}$$

where $\tau := (K/\mu_\varphi(U))^{2/p}$. Next, define $c^* := \inf_{U \subset S} \lambda_1(U)\mu_\varphi(U)^{1-\frac{2}{p}}$, and note that $c^* \geq \frac{1}{nK^{2/p}} > 0$. Also define

$$s^* := \inf\left\{\int_S \langle A_\varphi \nabla u, \nabla u\rangle : u \in C_0^1(\bar{S}), \int_S F(u)\mu_\varphi = 1\right\} \tag{3.6}$$

where $F(u) := \int_0^u f(t)\,dt$ and, for a parameter $k > 1$ to be sent to infinity,

$$f(t) := \begin{cases} |t|^{p-1} & \text{if } |t| < k \\ k^{p-1} & \text{if } |t| \geq k. \end{cases} \tag{3.7}$$

In order to make the relevant constants more explicit, we now complement the arguments in [9] by computing an upper bound for s^*. Since $S = S_\varphi(0, t_0)$, set

$$h(x) := \begin{cases} (t_0 - \varphi(x))^2 & \text{if } x \in S, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus S. \end{cases} \tag{3.8}$$

Hence, by using the divergence theorem, the fact that

$$\operatorname{div}(A_\varphi(x)\nabla u(x)) = \operatorname{trace}(A_\varphi(x) D^2 u(x)), \quad \forall u \in C^2,$$

(since the columns of A_φ are divergence free), and choosing $k > t_0^2$ (so that $0 \leq h = |h| \leq t_0^2 < k$ in S) in the definition of f in (3.7), we obtain

$$\begin{aligned}
\int_S \langle A_\varphi(x)\nabla h(x), \nabla h(x)\rangle\,dx &= -\int_S h(x)\ \operatorname{div}(A_\varphi(x)\nabla h(x))\,dx \\
&= \int_S h(x)\ \operatorname{div}(2(t_0 - \varphi(x))A_\varphi(x)\nabla\varphi(x))\,dx \\
&= 2\int_S h(x)[(t_0 - \varphi(x))\ \operatorname{div}(A_\varphi(x)\nabla\varphi(x)) \\
&\quad - \langle A_\varphi(x)\nabla\varphi(x), \nabla\varphi(x)\rangle]\,dx \\
&\leq 2\int_S h(x)(t_0 - \varphi(x))\ \operatorname{div}(A_\varphi(x)\nabla\varphi(x))\,dx \\
&= 2\int_S h(x)(t_0 - \varphi(x))\ \operatorname{trace}(A_\varphi(x)D^2\varphi(x))\,dx \\
&= 2n\int_S h(x)(t_0 - \varphi(x))\mu_\varphi(x)\,dx = 2n\int_S h^{\frac{3}{2}}(x)\mu_\varphi(x)\,dx \\
&\leq 2n\left(\int_S h^p(x)\mu_\varphi(x)\,dx\right)^{\frac{3}{2p}} \mu_\varphi(S)^{\frac{2p-3}{2p}} \\
&= 2np^{\frac{3}{2p}}\mu_\varphi(S)^{\frac{2p-3}{2p}}\left(\int_S F(h(x))\mu_\varphi(x)\,dx\right)^{\frac{3}{2p}},
\end{aligned}$$

which implies

$$s^* \leq 2np^{\frac{3}{2p}}\mu_\varphi(S)^{\frac{2p-3}{2p}}. \tag{3.9}$$

For a fixed k (always large enough), let $v = v_k \in C^1(\bar{S})$ denote the function where the infimum (3.6) is attained. Therefore, v satisfies

$$\begin{cases} L_\varphi(v) = -\hat{\lambda} f(v)\mu_\varphi & \text{in } S, \\ v = 0 & \text{on } \partial S. \end{cases}$$

Here $\hat{\lambda}$ is the Lagrange multiplier associated to the minimization. Take $x' \in S$ such that

$$v(x') = \|v\|_{L^\infty(S)} =: M. \tag{3.10}$$

As in the proof of Lemma 2.1 in [9] we have

$$\hat{\lambda} \leq s^* \leq p\hat{\lambda}. \tag{3.11}$$

For $t \in (0, M)$ (here M is as in (3.10)), set $\Omega_t := \{x \in S : v(x) > M - t\}$. As in the proof of Lemma 2.1 in [9] one gets

$$\mu_\varphi(\Omega_t) \geq \beta \mu_\varphi(\Omega_{t/2})^{\frac{p}{3p-4}}, \tag{3.12}$$

where

$$\beta := \left(\frac{tc^*}{2s^* M^{p-1}} \right)^{\frac{2p}{3p-4}}. \tag{3.13}$$

By iteration of (3.12), for $m \in \mathbb{N}$ we have

$$\mu_\varphi(\Omega_t) \geq \beta^{\sum_{k=0}^{m} \left(\frac{p}{3p-4}\right)^k} \mu_\varphi(\Omega_{t/2^m})^{\left(\frac{p}{3p-4}\right)^m}. \tag{3.14}$$

The next step will be to show that

$$\lim_{m\to\infty} \mu_\varphi(\Omega_{t/2^m})^{\left(\frac{p}{3p-4}\right)^m} = 1. \tag{3.15}$$

In order to show (3.15) we will now deviate from the proof of Lemma 2.1 in [9]. Indeed, instead of using a doubling property for μ_φ we use, yet again, that $\varphi \in C^2$ and $\mu_\varphi > 0$ so that given a compact set Q we have

$$\mu_\varphi(x) \geq \inf_Q \mu_\varphi > 0, \quad \forall x \in Q. \tag{3.16}$$

Continuing as in the proof of Lemma 2.1 in [9], set $a := \|\nabla v\|_{L^\infty(S)}$, then

$$\Omega_{t/2^m} \supset B(x', t2^{-m}/a),$$

here x' is as in (3.10). Now, by (3.16) with $Q := \bar{S}$ setting $\theta := \inf_Q \det D^2\varphi > 0$ we get

$$\mu_\varphi(\Omega_{t/2^m}) \geq \theta |B(x', t2^{-m}/a)| = \theta \omega_n (t/a)^n 2^{-mn}, \quad \forall m \in \mathbb{N}, \tag{3.17}$$

where ω_n denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$. The bound (3.17) and the fact that $p > 2$ then imply (3.15), and, consequently (from (3.14)),

$$\mu_\varphi(\Omega_t) \geq \beta^{\frac{3p-4}{2(p-2)}} = \left(\frac{tc^*}{2s^* M^{p-1}} \right)^{\frac{p}{p-2}}. \tag{3.18}$$

Note that in the proof of (3.15) no a priori rate of convergence as $m \to \infty$ is needed, so we were able to use (3.16) without resorting to any assumptions on a priori structural control of the infimum in (3.16). Also, since we are not using the hypothesis $\mu_\varphi \simeq 1$, we cannot follow the original reasoning in Lemma 2.1 in [9], because μ_φ is doubling on sections of φ, but not necessarily on Euclidean balls.

As in the proof of Lemma 2.1 in [9], setting $\eta := k/M$ (and using (3.18)), we get

$$\begin{aligned} 1 &\geq \left(\frac{c^*}{2s^*} \right)^{\frac{p}{p-2}} \left[\int_0^{1-\eta} t^{\frac{p}{p-2}} \eta^{p-1}\, dt + p \int_{1-\eta}^{1} t^{\frac{p}{p-2}} (1-t)^{p-1}\, dt \right] \\ &=: \left(\frac{c^*}{2s^*} \right)^{\frac{p}{p-2}} R(\eta). \end{aligned} \tag{3.19}$$

Setting $\omega := \{x \in S : v(x) > k\}$, as in the proof of Lemma 2.1 in [9] it follows that

$$\mu_\varphi(\omega) \le pk^{-p}. \tag{3.20}$$

Also, on the set ω we have $L_\varphi(v) = -\hat{\lambda} k^{p-1}\mu_\varphi$, with $v = k$ on $\partial\omega$. Let G_ω denote the Green function of L_φ in ω and let x' be as in (3.10). Hence, by (3.5), (3.20), (3.11), and (3.9)

$$\begin{aligned} M = v(x') &= k + \hat{\lambda} k^{p-1} \int_\omega G_\omega(x, x')\mu_\varphi(x)\,dx \le k + 2\hat{\lambda} k^{p-1} K^{\frac{2}{p}} \mu_\varphi(\omega)^{1-\frac{2}{p}} \\ &\le k + 2\hat{\lambda} k^{p-1} K^{\frac{2}{p}} (pk^{-p})^{1-\frac{2}{p}} = (1 + 2\hat{\lambda} K^{\frac{2}{p}})k \\ &\le (1 + 2s^* K^{\frac{2}{p}})k \le (1 + 4np^{\frac{3}{2p}} \mu_\varphi(S)^{\frac{2p-3}{2p}} K^{\frac{2}{p}})k =: K_4 k. \end{aligned}$$

Consequently, $\eta = \frac{k}{M} \ge \frac{1}{K_4}$ and, from (3.19)

$$s^* \ge \frac{c^*}{2} \inf_{\eta \in [1/K_4, 1]} R(\eta)^{\frac{p-2}{p}} \ge \frac{1}{4K^{2/p}} \inf_{\eta \in [1/K_4, 1]} R(\eta)^{\frac{p-2}{p}} =: K_5 > 0.$$

At this point, given $u \in C_0^1(\bar{S})$, we can take limits as $k \to \infty$ to obtain

$$\left(\int_S u^p \, d\mu_\varphi \right)^{\frac{1}{p}} \le \frac{1}{K_5^{1/2}} \left(\int_S \langle A_\varphi \nabla u, \nabla u \rangle \, dx \right)^{\frac{1}{2}}. \tag{3.21}$$

The seemingly awkward dependence of K and K_5 on $\mu_\varphi(S)$ and t_0 can be circumvented by employing the normalization technique of Caffarelli and Gutiérrez [1]. Indeed, given a section $S = S_\varphi(x_0, t_0)$ let T be an affine transformation normalizing S so that, in the notation of [1, Section 1], we have

$$\psi_\lambda(y) := \frac{1}{\lambda}\varphi(T^{-1}y), \quad B(0,1) \subset S^* := T(S) \subset B(0, n), \tag{3.22}$$

$$\bar{\mu}(y) = \mu_{\Psi_\lambda}(y) = \frac{1}{\lambda^n}|T|^{-2}\mu_\varphi(T^{-1}y), \quad D^2\Psi_\lambda(y) = \frac{1}{\lambda}(T^{-1})^t D^2\varphi(T^{-1}y)T^{-1} \tag{3.23}$$

and $\mu_{\psi_\lambda}(S^*) = 1$ so that

$$\lambda^n |T| = \mu_\varphi(S_\varphi(x_0, t_0)). \tag{3.24}$$

Applying the previous proof to S^* and Ψ_λ and using the fact that

$$c_1 \le \frac{t_0}{\lambda} \le c_2, \tag{3.25}$$

for geometric constants c_1 and c_2 (see Theorem 8 in [2]), the constants K in (3.2) and K_5 are now completely geometric (in particular, they are independent of x_0 and t_0). Consequently, for $\bar{u} \in C_0^1(\bar{S^*})$, it follows that

$$\left(\int_{S^*} \bar{u}(y)^p \bar{\mu}(y)\,dy \right)^{\frac{1}{p}} \le \frac{1}{K_5^{1/2}} \left(\int_{S^*} \langle A_{\Psi_\lambda}(y) \nabla \bar{u}(y), \nabla \bar{u}(y) \rangle \, dy \right)^{\frac{1}{2}}. \tag{3.26}$$

Finally, given $u \in C_0^1(\bar{S})$ we set $y = Tx$ and $u(x) := \bar{u}(Tx)$ for $x \in S_\varphi(x_0, t_0)$, then changing variables in (3.26) by means of (3.22)–(3.25), the Sobolev inequality (1.7) follows. For further considerations on the smoothness assumptions for φ, see Remark 3.2 in [9]. □

4. Proof of Theorem 2 via convex conjugation

Given a strictly convex, twice continuously differentiable $\varphi : \mathbb{R}^n \to \mathbb{R}$, its *Legendre transform* or *convex conjugate*, will be denoted by ψ. Under the hypothesis $\mu_\varphi \in (\mathrm{DC})_\varphi$ we have that $\mu_\psi := \det D^2\psi \in (\mathrm{DC})_\psi$ (with respect to the sections of ψ), where the $(\mathrm{DC})_\psi$-doubling constants for μ_ψ depend only on the ones for μ_φ and dimension n. Also, ψ is a strictly convex twice continuously differentiable function whose domain is $\mathbb{R}^n$ and

$$\nabla\varphi(\nabla\psi(x)) = \nabla\psi(\nabla\varphi(x)) = x \quad \forall x \in \mathbb{R}^n, \tag{4.1}$$

(see [3, Section 5]) which implies that

$$D^2\varphi(\nabla\psi(y))D^2\psi(y) = D^2\psi(\nabla\varphi(x))D^2\varphi(x) = I \quad \forall x, y \in \mathbb{R}^n \tag{4.2}$$

and that, for every Borel set $E \subset \mathbb{R}^n$,

$$|E| = |\nabla\varphi(\nabla\psi(E))| = \mu_\varphi(\nabla\psi(E)) = \mu_\psi(\nabla\varphi(E)). \tag{4.3}$$

Moreover, from Theorem 12 in [3], there exists a geometric constant $K_0 \geq 1$ such that for every $x \in \mathbb{R}^n$ and $t > 0$,

$$\nabla\varphi(S_\varphi(x, t/K_0)) \subset S_\psi(\nabla\varphi(x), t) \subset \nabla\varphi(S_\varphi(x, K_0 t)). \tag{4.4}$$

Next, given a section $S := S_\varphi(x_0, t)$ of φ set $y_0 := \nabla\varphi(x_0)$, $S_\psi := S_\psi(y_0, K_0 t)$ and $S^\varphi := \nabla\psi(S_\psi)$. By (4.4) we have that

$$S = S_\varphi(x_0, t) \subset \nabla\psi(S_\psi(y_0, K_0 t)) = S^\varphi. \tag{4.5}$$

Given $u \in C_0^1(S)$, for $y \in S_\psi$ let us define $v(y) := u(\nabla\psi(y))$. Then, $\mathrm{supp}(v) = \nabla\varphi(\mathrm{supp}(u)) \subset \nabla\varphi(S) \subset S_\psi$, hence $v \in C_0^1(S_\psi)$. By applying (1.7) (with respect to ψ) to v on S_ψ we obtain

$$\begin{aligned}
&\left(\frac{1}{\mu_\psi(S_\psi)}\int_{S_\psi}|v(y)|^{\frac{2n}{n-1}}\,d\mu_\psi(y)\right)^{\frac{n-1}{2n}}\\
&\qquad\leq K_6^\psi\,(K_0 t)^{\frac{1}{2}}\left(\frac{1}{\mu_\psi(S_\psi)}\int_{S_\psi}|\nabla^\psi v(y)|^2\,d\mu_\psi(y)\right)^{\frac{1}{2}}.
\end{aligned}$$

By changing variables $y = \nabla\varphi(x)$, (4.1), (4.2), and (4.3) yield

$$\begin{aligned}
\left(\frac{1}{|S^\varphi|}\int_{S^\varphi}|u(x)|^{\frac{2n}{n-1}}\,dx\right)^{\frac{n-1}{2n}} &\leq K_6^\psi\,(K_0 t)^{\frac{1}{2}}\left(\frac{1}{|S^\varphi|}\int_{S^\varphi}|\nabla^\varphi u(x)|^2\,dx\right)^{\frac{1}{2}}\\
&\leq K_6^\psi\,(K_0 t)^{\frac{1}{2}}\left(\frac{1}{|S|}\int_{S}|\nabla^\varphi u(x)|^2\,dx\right)^{\frac{1}{2}},
\end{aligned}$$

where the last inequality is due to the fact that u (and, therefore, $\nabla^\varphi u$) is supported in S and $S \subset S^\varphi$. By Lemma 5.2(a) in [1], Lebesgue measure is doubling, with uniform constant 2^n, with respect to the sections of any convex function. Then,

$$|S_\varphi(x, t)| \leq 2^n |S_\varphi(x, t/2)| \quad \forall x \in \mathbb{R}^n, \forall t > 0.$$

In particular, recalling (4.5),

$$|S| \leq |S^\varphi| \leq |S_\varphi(x_0, K_0^2 t)| \leq (2K_0^2)^n |S_\varphi(x_0, t)| = (2K_0^2)^n |S|.$$

Therefore,

$$\left(\frac{1}{|S|}\int_S |u(x)|^{\frac{2n}{n-1}}\,dx\right)^{\frac{n-1}{2n}} \le \left(\frac{(2K_0^2)^n}{|S^\varphi|}\int_{S^\varphi} |u(x)|^{\frac{2n}{n-1}}\,dx\right)^{\frac{n-1}{2n}}$$
$$\le K_6^\psi (2K_0^2)^{\frac{(n-1)}{2}} (K_0 t)^{\frac{1}{2}} \left(\frac{1}{|S|}\int_S |\nabla^\varphi u(x)|^2\,dx\right)^{\frac{1}{2}}$$

and (1.9) follows with $K_7 := 2^{\frac{(n-1)}{2}} K_6^\psi K_0^{n-\frac{1}{2}}$. □

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