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A REMARK ON CONICAL KÄHLER–EINSTEIN METRICS

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Abstract. We give some non-existence results for Kähler–Einstein metrics with conical singularities along a divisor on Fano manifolds. In particular, we show that the maximal possible cone angle is in general smaller than the invariant $R(M)$. We study this discrepancy from the point of view of log K-stability.

1. Introduction

Given a Fano manifold M and a smooth anticanonical divisor $D \subset M$, the existence of a Kähler–Einstein (KE) metric on M with conical singularities along D has received considerable attention recently. Interest in such metrics goes back to at least McOwen [10] on Riemann surfaces, and Tian [18] for higher dimensions. The renewed interest has been sparked by a proposal by Donaldson [3, 5] to use such singular metrics in a continuity method for finding smooth KE metrics, which has recently led to a solution of the problem of when KE metrics exist on Fano manifolds [2]. There is by now a large body of work on such conical KE metrics, see for instance Mazzeo–Rubinstein [9], Song–Wang [13], Li–Sun [8], and many others.

In this paper, we give some simple calculations implying non-existence results. A KE metric ω on M with conical singularities along a divisor $D \in |-K_M|$ satisfies the equation

$$\mathrm{Ric}(\omega) = \beta\omega + (1-\beta)[D], \tag{1.1}$$

where the cone angle is $2\pi\beta$ for some $\beta \in (0, 1]$, and $[D]$ denotes the current of integration along D. Let us write

$$R(M, D) = \sup\{\beta > 0 \,|\, \text{there is a cone-singularity solution of (1.1)}\}. \tag{1.2}$$

Let M_1 and M_2 be the blowup of $\mathbf{P}^2$ in one or two points, respectively.

Theorem 1. *On M_1, for any smooth $D \in |-K_{M_1}|$ we have $R(M_1, D) \leq 12/15$. On M_2, if $D \in |-K_{M_2}|$ passes through the intersection of two (-1)-curves, then $R(M_2, D) \leq 7/9$.*

Recall that for any Fano manifold M one can define an invariant $R(M) \in (0, 1]$

$$R(M) = \sup\{t \,|\, \exists\omega \in c_1(M) \text{ such that } \mathrm{Ric}(\omega) > t\omega\}. \tag{1.3}$$

We computed in [15] that $R(M_1) = 6/7$, and the invariant for all toric Fano manifolds has been computed by Li [7] (see also Tian [17] for earlier results bounding $R(M)$). In particular, $R(M_2) = 21/25$. In [15], we proved that if $\alpha \in c_1(M)$ is a Kähler form, then the equation

$$\mathrm{Ric}(\omega) = \beta\omega + (1-\beta)\alpha \tag{1.4}$$

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can be solved if and only if $\beta < R(M)$. In relation to conical KE metrics, i.e., when replacing α by a current of integration along a smooth divisor, Donaldson [3] conjectured the following.

Conjecture 2. *Suppose $D \in |-K_M|$ is smooth. For all $0 < \beta < R(M)$ there exists a cone-singularity solution to* (1.1)*, and there is no solution for $R(M) < \beta < 1$. In other words, $R(M, D) = R(M)$ for any smooth $D \in |-K_M|$.*

Since $12/15 < 6/7 = R(M_1)$, and $7/9 < 21/25 = R(M_2)$, our result gives counterexamples to this conjecture.

An important generalization of Equation (1.1) was studied by Song–Wang [13], where D is allowed to be an element of the linear system $|-mK_M|$ for some $m > 0$. In Section 3 we will give a non-existence result for conical KE metrics along such D, complementing the results of Song–Wang to some extent.

The proof of Theorem 1 will be given in Section 2. It is based on a log K-stability calculation of Li [6], together with the result of Berman [1] which says that log K-stability is a necessary condition for the existence of a conical KE metric. In Section 4 we will give a discussion of the difference between $R(M)$ and $R(M, D)$ from the point of view of algebro-geometric stability conditions.

2. Proof of Theorem 1

We will use the notion of log K-stability, which was introduced in [3] (see also [14] for a related notion for asymptotically cuspidal metrics instead of conical ones). In particular, we will use the calculation in Li [6], where this stability notion is analyzed for toric manifolds. We quickly recall his result. A toric Fano manifold M can be viewed as a reflexive lattice polytope P in $\mathbf{R}^n$. For instance M_1, the blowup of $\mathbf{P}^2$ in one point, corresponds to the convex hull of the points $(0, -1), (-1, 0), (-1, 2), (2, -1)$ in $\mathbf{R}^2$, shown in Figure 1.

The lattice points in P correspond to sections of K_M^{-1}, giving a decomposition of $H^0(M, K_M^{-1})$ into one-dimensional weight spaces of the torus action. Let us write $\{s_1, \ldots, s_N\}$ for these sections, corresponding to lattice points $\{\alpha_1, \ldots, \alpha_N\}$. Given an anticanonical divisor D, we can write

$$D = \left\{ \sum_{i=1}^{N} a_i s_i = 0 \right\}, \tag{2.1}$$

for some coefficients a_i. Define $P_D \subset P$ to be the convex hull of those weights α_i, for which $a_i \neq 0$.

Let us choose $\lambda \in \mathbf{Z}^n$ giving the weights of a one-parameter subgroup in $(\mathbf{C}^*)^n$. Note that P naturally lives in the dual of the Lie algebra of the torus, so here we are identifying this $\mathbf{R}^n$ with its dual, using the Euclidean inner product. This λ defines a test-configuration for the pair (M, D), which is simply a product configuration on M, but degenerates D. Let us write

$$W(\lambda) = \max_{p \in P_D} \langle p, \lambda \rangle, \tag{2.2}$$

and let $P_{\mathrm{c}} \in P$ denote the barycenter of P. For any $\beta \in [0, 1]$, the Futaki invariant, denoted by $F(M, \beta D, \lambda)$ is computed in Li [6] (see also Section 4 for more details). The

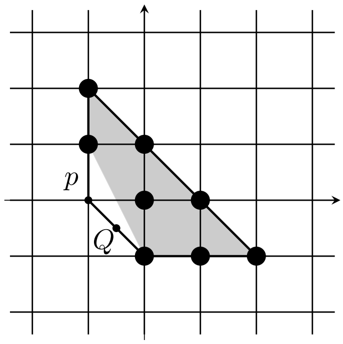

FIGURE 1. The polytope corresponding to M_1, with the sections vanishing at p highlighted, and P_D shaded.

calculation there assumes that D is generic, so that $a_i \neq 0$ for all i and so $P_D = P$, but the same argument works if $P_D \neq P$. The result is

Theorem 3 (Li [6]).

$$F(M, \beta D, \lambda) = -\Big[\beta\langle P_{\mathrm{c}}, \lambda\rangle + (1-\beta)W(\lambda)\Big]\mathrm{Vol}(P). \tag{2.3}$$

The sign convention is such that logarithmic K-stability requires

$$F(M, \beta D, \lambda) < 0. \tag{2.4}$$

In particular, Berman [1] has shown that (2.4) is necessary for a conical KE metric to exist with angle $2\pi\beta$ along D.

Proof of Theorem 1. Let $D \subset M_1$ be a smooth anticanonical divisor. Suppose that D intersects the exceptional divisor at the point p. We can choose a torus action on M_1 for which p is a fixed point. The toric polytope P can be chosen to be the convex hull of the points $(0,-1), (-1,0), (-1,2), (2,-1)$, so the center of mass is given by

$$P_{\mathrm{c}} = \left(\frac{1}{12}, \frac{1}{12}\right). \tag{2.5}$$

Let us write $\{s_1, \ldots, s_N\}$ for the sections of $K_{M_1}^{-1}$ giving eigenvectors of the torus action, and let us assume that s_N is the section corresponding to the weight $(-1,0)$. We can assume that p corresponds to the vertex $(-1,0)$, meaning that the space of sections of $K_{M_1}^{-1}$ which vanish at p are spanned by the sections $s_1, \ldots, s_{N-1}$. In Figure 1, we have indicated the lattice points corresponding to the sections $s_1, \ldots, s_{N-1}$.

This implies that

$$D = \left\{\sum_{i=1}^{N-1} a_i s_i = 0\right\}, \tag{2.6}$$

for some coefficients a_i, and in particular

$$P_D \subset \mathrm{conv}\{(0,-1), (-1,1), (-1,2), (2,-1)\}, \tag{2.7}$$

where "conv" denotes convex hull. Let us choose $\lambda = (-2,-1)$, and consider the test-configuration corresponding to the one-parameter subgroup of $(\mathbf{C}^*)^2$ generated by λ.

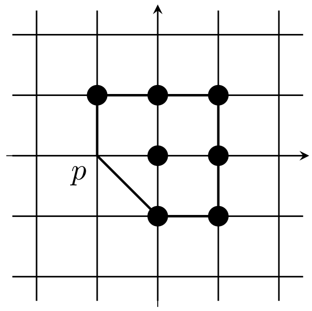

FIGURE 2. The polytope corresponding to M_2, with the sections vanishing at p highlighted.

Using Theorem 3 we can compute

$$F(M_1, \beta D, \lambda) = -\Big[\frac{-3}{12}\beta + 1 - \beta\Big], \tag{2.8}$$

and $F(M_1, \beta D, \lambda) < 0$ implies $\beta < 12/15$. Theorem 4.2 of Berman [1] implies that there is no conical metric solution of (1.1) for $\beta \geq 12/15$.

For the manifold M_2 we can argue similarly. We have drawn the corresponding polytope P in Figure 2. We can assume that we chose our torus action in such a way, that the anticanonical divisor D meets two exceptional divisors at the point corresponding to the vertex p. It follows that D is given as the zero set of a linear combination of the sections corresponding to the lattice points

$$(-1, 1), (0, -1), (0, 0), (0, 1), (1, -1), (1, 0), (1, 1). \tag{2.9}$$

The barycenter of P is

$$P_{\mathrm{c}} = \left(\frac{2}{21}, \frac{2}{21}\right). \tag{2.10}$$

Let us once again choose $\lambda = (-2, -1)$, and compute

$$F(M_2, \beta D, \lambda) = -\Big[\frac{-6}{21}\beta + (1 - \beta)\Big], \tag{2.11}$$

We find that $F(M_2, \beta D, \lambda) < 0$ implies $\beta < 7/9$. Once again, Berman's theorem [1] implies that there is no conical metric solution of (1.1) for $\beta \geq 7/9$. □

Remark 4. Many other similar examples can be given. In general, if P_{c} is the barycenter of the moment polytope P, let Q be the intersection of the ray from P_{c} through the origin O, with the boundary of P. It is shown by Li [7] that

$$R(M) = \frac{|OQ|}{|P_{\mathrm{c}}Q|}. \tag{2.12}$$

Using the formula in Theorem 3 is it easy to see that we will get

$$F(M, R(M)D, \lambda) < 0 \tag{2.13}$$

for a suitable λ whenever P_D does not contain the point Q, as shown in Figure 1.

3. Pluri-anticanonical divisors

Instead of letting D be an anticanonical divisor, we can allow D to be a smooth divisor in the linear system $|-mK_M|$ for some $m > 1$. In this case, Song–Wang [13] have shown that for any $\beta \in (0, R(M))$ there exists an $m > 0$ such that for any smooth divisor $D \in |-mK_M|$ there is a conical KE metric ω satisfying the equation

$$\mathrm{Ric}(\omega) = \beta\omega + \frac{1-\beta}{m}[D]. \tag{3.1}$$

We give a related result in the converse direction. In particular, we obtain bounds on how large such m have to be in terms of β.

Theorem 5. *On the manifold M_1, for any $m > 0$ there is a smooth divisor $D \in |-mK_{M_1}|$ such that a cone-singularity solution of* (3.1) *must satisfy*

$$\beta < \frac{12m}{14m+1} < R(M_1). \tag{3.2}$$

Similarly on M_2 there is a smooth divisor $D \in |-mK_{M_2}|$ such that a solution of (3.1) *must satisfy*

$$\beta < \frac{21m}{25m+2} < R(M_2). \tag{3.3}$$

Proof. We can use the same toric calculation as in the proof of Theorem 1, using the polytope in Figure 1. The only difference is that sections of $K_{M_1}^{-m}$ correspond to lattice points in $P \cap \frac{1}{m}\mathbf{Z}^2$.

Let us write $s_0, \ldots, s_N$ for the corresponding sections, ordered in such a way that $s_0, \ldots, s_{m-1}$ correspond to the lattice points along the edge joining $(-1, 0)$ and $(0, -1)$, except for the point $(0, -1)$. In other words these are the m sections corresponding to the lattice points

$$(-1,0), \left(-\frac{m-1}{m}, -\frac{1}{m}\right), \left(-\frac{m-2}{m}, \frac{-2}{m}\right), \ldots, \left(-\frac{1}{m}, -\frac{m-1}{m}\right) \tag{3.4}$$

We will take D to be of the form

$$D = \left\{\sum_{i=m}^{N} a_i s_i = 0\right\}, \tag{3.5}$$

for generic choice of a_i. This will be a smooth section by Bertini's theorem, since the base locus of the corresponding linear system consists of only the point p, and we can check directly that the general element is smooth at p. In fact to be smooth at p we only need the coefficient corresponding to the lattice point $\left(-1, \frac{1}{m}\right)$ to be non-zero. The divisor D will meet the exceptional divisor with multiplicity m at the point p.

We now take $\lambda = (-m-1, -m)$. Again using Theorem 3 (or rather a slight generalization which works for pluri-anticanonical divisors), we obtain

$$F(M_1, \beta D, \lambda) = -\Big[\frac{-2m-1}{12}\beta + m(1-\beta)\Big]. \tag{3.6}$$

The inequality $F(M_1, \beta D, \lambda) < 0$ implies

$$\beta < \frac{12m}{14m+1}. \tag{3.7}$$

Note that for any m, we have $\frac{12m}{14m+1} < R(M_1)$, since $R(M_1) = 6/7$. It is also worth pointing out that for $m > 1$ the divisor D we use here is quite special, since a generic element in $|-mK_{M_1}|$ will meet the exceptional divisor in m distinct points.

The calculation for M_2 is completely analogous, the only difference is that in that case $P_c = \left(\frac{2}{21}, \frac{2}{21}\right)$ as in the proof of Theorem 1. The divisor D in this case will meet the (-1)-curve, which intersects the two exceptional divisors, with multiplicity m at the point p. $\square$

4. Stability conditions

By definition $t < R(M)$ if and only if there is a metric $\omega \in c_1(M)$, and a smooth positive form $\alpha \in c_1(M)$ such that

$$\mathrm{Ric}(\omega) = t\omega + (1-t)\alpha. \tag{4.1}$$

We showed in [15] that the solvability of (4.1) for a given t is independent of the choice of $\alpha \in c_1(M)$. The reasoning behing Conjecture 2 is the natural expectation that the same holds if we allow α to be a current supported on a divisor. We have seen that this is not the case for the manifolds M_1 and M_2.

To understand the counterexamples from the point of view of algebraic geometry, we will compare log K-stability with an analogous notion of stability where the current $[D]$ is replaced by a smooth form in $c_1(M)$. We plan to flesh out these ideas in more detail in future work, so for now we just give a brief sketch.

A test-configuration for M is obtained by embedding $M \hookrightarrow \mathbf{P}^{N_r}$ using the linear system $|-rK_M|$ for some $r > 0$, and then acting on $\mathbf{P}^{N_r}$ by a $\mathbf{C}^*$-action λ. The flat limit

$$M_0 = \lim_{t\to 0} \lambda(t)\cdot M \tag{4.2}$$

is invariant under the action λ, and this can be used to define (see Donaldson [4] for details) the Futaki invariant $\mathrm{Fut}(M, \lambda)$. Our sign convention, in order to match with Li [6], is such that K-semistability means $\mathrm{Fut}(M, \lambda) \leq 0$ for all such test-configurations.

In [5], Donaldson outlined a modification of this, which is conjecturally equivalent to the existence of KE metrics on M with conical singularities along a divisor $D \in |-mK_M|$ for some $m > 0$. Given a test-configuration as above, we have $D \subset M \subset \mathbf{P}^{N_r}$, and we can take the flat limit

$$D_0 = \lim_{t\to 0} \lambda(t)\cdot D. \tag{4.3}$$

Suppose that $\lambda(t) = t^A$ for some $A \in \sqrt{-1}\mathfrak{su}(N_r+1)$ with integer eigenvalues. For real t, the one parameter group of automorphisms $\lambda(t)$ is induced by the gradient flow of the function

$$H_A = \frac{A_{ij} Z^i \overline{Z}^j}{|Z|^2}, \tag{4.4}$$

where the Z^i are homogeneous coordinates on $\mathbf{P}^{N_r+1}$. It is well known that the function

$$f(t) = \int_{\lambda(t)\cdot D} H_A\, \omega_{\mathrm{FS}}^{n-1} \tag{4.5}$$

is increasing in t, where n is the dimension of M, and ω_{FS} is the Fubini–Study metric. One defines the Chow weight to be

$$\mathrm{Ch}(D,\lambda)=\lim_{t\to 0} f(t). \tag{4.6}$$

The Chow weight $\mathrm{Ch}(M,\lambda)$ is defined similarly by integrating over $\lambda(t)\cdot M$.

The relevant modified Futaki invariant when looking for KE metrics on M with conical singularities along D, is

$$\mathrm{Fut}(M,\beta D,\lambda)=\beta\mathrm{Fut}(M,\lambda)+\frac{1-\beta}{m}\left[\mathrm{Ch}(D,\lambda)-\frac{\mathrm{Vol}(D)}{\mathrm{Vol}(M)}\mathrm{Ch}(M,\lambda)\right]. \tag{4.7}$$

Here, as before, the parameter $\beta\in(0,1]$ determines the cone angle.

If we want to replace D with a smooth positive form $\alpha\in c_1(M)$, then it is natural to define an analogous Chow weight as follows, as was also remarked on in Donaldson [5]. Let us write $\iota: M\hookrightarrow \mathbf{P}^{N_r+1}$ for our initial embedding, and $\varphi_t=\lambda(t)\circ\iota$. One can then check that the function

$$f(t)=\int_M \alpha\wedge\varphi_t^*(H_A\omega_{\mathrm{FS}}^{n-1}) \tag{4.8}$$

is monotonic in t, and we define

$$\mathrm{Ch}(\alpha,\lambda)=\lim_{t\to 0} f(t). \tag{4.9}$$

Then in analogy with (4.7) we define

$$\mathrm{Fut}(M,\beta\alpha,\lambda)=\beta\mathrm{Fut}(M,\lambda)+(1-\beta)\left[\mathrm{Ch}(\alpha,\lambda)-\frac{\mathrm{Vol}(D)}{\mathrm{Vol}(M)}\mathrm{Ch}(M,\lambda)\right]. \tag{4.10}$$

The main point that we want to make is the following.

Theorem 6. *Suppose that $\alpha\in c_1(M)$ is a smooth positive form as above, and $D\in|-K_M|$. Then we have*

$$\lim_{t\to 0}\int_M \alpha\wedge\varphi_t^*(H_A\omega_{\mathrm{FS}}^{n-1})\le \lim_{t\to 0}\int_{\lambda(t)\cdot D} H_A\omega_{\mathrm{FS}}^{n-1}. \tag{4.11}$$

In other words, we have

$$\mathrm{Fut}(M,\beta\alpha,\lambda)\le \mathrm{Fut}(M,\beta D,\lambda) \tag{4.12}$$

for all $\beta\in[0,1]$, and all $\mathbf{C}^*$-actions λ.

Proof. First let us suppose that α is the pullback of a Fubini–Study metric, i.e., $\alpha=\frac{1}{k}\Phi^*\omega_{\mathrm{FS}}$ for some embedding $\Phi: M\hookrightarrow\mathbf{P}^{N_k}$ using the linear system $|-kK_M|$. In this case we can write α as an average of the currents of integration $\frac{1}{k}[C]$ as the divisor C varies over $|-kK_M|$. This follows from Lemma 3.1 in Shiffman–Zelditch [12]. In fact the relevant measure $d\mu$ on the linear system $|-kK_M|$ is induced by the inner product on $H^0(K_M^{-k})$, for which the embedding Φ is given by orthonormal sections.

This implies that

$$\int_M \alpha\wedge\varphi_t^*(H_A\omega_{\mathrm{FS}}^{n-1})=\frac{1}{k}\int_{C\in|-kK_M|}\left(\int_{\lambda(t)\cdot C} H_A\omega_{\mathrm{FS}}^{n-1}\right)d\mu. \tag{4.13}$$

For a fixed $C \in |-kK_M|$, the limit

$$\lim_{t\to 0} \int_{\lambda(t)\cdot C} H_A \omega^{n-1} \tag{4.14}$$

is the Chow weight $\mathrm{Ch}(C, \lambda)$. For any integer w, let us write $E_w \subset |-kK_M|$ for the set

$$E_w = \{C \in |-kK_M| \; ; \; Ch(C,\lambda) \geq w\}. \tag{4.15}$$

This is a Zariski closed subset, since the weight can only jump up under specialization. In fact under an embedding of $|-kK_M|$ into a projective space using the Chow line bundle, E_w is the intersection with a linear subspace. It follows that if we let $w_{\min}$ be the largest w for which $E_w = |-kK_M|$, then $E_{w_{\min}+1}$ has measure zero in $|-kK_M|$. From the monotone convergence theorem we obtain

$$\begin{aligned} \lim_{t\to 0} \int_M \alpha \wedge \varphi_t^*(H_A \omega_{\mathrm{FS}}^{n-1}) &= \frac{1}{k} \int_{C\in|-kK_M|} \left(\lim_{t\to 0} \int_{\lambda(t)\cdot C} H_A \omega_{\mathrm{FS}}^{n-1} \right) d\mu \\ &= \frac{1}{k} \int_{C\in|-kK_M|\setminus E_{w_{\min}+1}} w_{\min}\, d\mu \\ &= \frac{1}{k} w_{\min}. \end{aligned} \tag{4.16}$$

On the other hand, for a divisor $D \in |-K_M|$ we have $kD \in |-kK_M|$, and so $kD \in E_w$ for some $w \geq w_{\min}$. It follows that

$$\lim_{t\to 0} \int_{\lambda(t)\cdot D} H_A \omega_{\mathrm{FS}}^{n-1} = \frac{1}{k} \lim_{t\to 0} \int_{\lambda(t)\cdot kD} H_A \omega_{\mathrm{FS}}^{n-1} = \frac{1}{k} w \geq \frac{1}{k} w_{\min}. \tag{4.17}$$

Comparing this with (4.16) we obtain the result for such α.

Now suppose that $\alpha \in c_1(M)$ is an arbitrary smooth positive form. From the asymptotic expansion of the Bergman kernel (see Ruan [11], Tian [16], Zelditch [19]), we know that we can approximate α with forms of the type $\frac{1}{k}\Phi^*\omega_{\mathrm{FS}}$. In particular, we can choose $\alpha_k \in c_1(M)$ for which our arguments above apply, and

$$\alpha = \alpha_k + \sqrt{-1}\partial\bar{\partial} f_k, \tag{4.18}$$

where $|f_k| < \frac{1}{k}$. For any t we have

$$\int_M (\alpha - \alpha_k) \wedge \varphi_t^*(H_A \omega_{\mathrm{FS}}^{n-1}) = \int_M f_k \, \varphi_t^*(\sqrt{-1}\partial\bar{\partial} H_A \wedge \omega_{\mathrm{FS}}^{n-1}). \tag{4.19}$$

For some constant A we have

$$-A\omega_{\mathrm{FS}}^n < \sqrt{-1}\partial\bar{\partial} H_A \wedge \omega_{\mathrm{FS}}^{n-1} < A\omega_{\mathrm{FS}}^n, \tag{4.20}$$

so

$$\left| \int_M f_k \, \varphi_t^*(\sqrt{-1}\partial\bar{\partial} H_A \wedge \omega_{\mathrm{FS}}^{n-1}) \right| < \frac{A}{k} \mathrm{Vol}(M). \tag{4.21}$$

It follows that

$$\lim_{t\to 0} \int_M \alpha \wedge \varphi_t^*(H_A \omega_{\mathrm{FS}}^{n-1}) = \lim_{t\to 0} \int_M \alpha_k \wedge \varphi_t^*(H_A \omega_{\mathrm{FS}}^{n-1}) + O(1/k). \tag{4.22}$$

Since we already showed that

$$\lim_{t\to 0}\int_M \alpha_k \wedge \varphi_t^*(H_A\omega_{\mathrm{FS}}^{n-1}) \leq \lim_{t\to 0}\int_{\lambda(t)\cdot D} H_A\omega_{\mathrm{FS}}^{n-1}, \tag{4.23}$$

and k was arbitrary, we get

$$\lim_{t\to 0}\int_M \alpha \wedge \varphi_t^*(H_A\omega_{\mathrm{FS}}^{n-1}) \leq \lim_{t\to 0}\int_{\lambda(t)\cdot D} H_A\omega_{\mathrm{FS}}^{n-1}, \tag{4.24}$$

and so the result follows for arbitrary smooth positive $\alpha \in c_1(M)$. □

Remark 7. It is clear from the proof that if D is chosen to be in a special position, in particular if it passes through more non-minimal critical points of H_A than a generic D would, then one would expect strict inequality to hold in (4.12). This means that if we can find a cone-singularity solution of (4.1), with $\alpha = [D]$ for some divisor $D \in |-K_M|$, then we expect to be able to solve the same equation with any smooth positive form $\alpha \in c_1(M)$, at least if there are no holomorphic vector fields on M. The converse, however, need not be true if D is in special position. This is exactly what happens in the examples that we have for M_1 and M_2. In particular for M_1, if D is any smooth anticanonical divisor, then we can choose a $\mathbf{C}^*$-action on M_1 for which D is in special position and gives a discrepancy between $R(M_1)$ and $R(M_1, D)$.

It also follows from the proof that if we fix the $\mathbf{C}^*$-action λ, then for a generic divisor D, we will have equality in (4.12). A special case of this can be observed in Theorem 3, where for generic D we have $P_D = P$. Indeed in this case the formula matches up with the result we obtained in [15] for the case of a smooth positive $\alpha \in c_1(M)$, which was formulated in terms of the derivative of the twisted Mabuchi functional.

It is interesting to speculate on what happens with the conical KE metrics on M_1, as $\beta \to 12/15$. Along the test-configuration that we used in the proof of Theorem 1, the divisor D degenerates into a divisor D_0 given by the union of a conic passing through the exceptional divisor, and a line which is tangent to the conic. We expect that M_1 admits a cone-singularity solution of

$$\mathrm{Ric}(\omega) = \beta\omega + (1-\beta)[D_0] \tag{4.25}$$

in a suitable sense with $\beta = 12/15$, to which the conical KE metrics solving (1.1) on $M_1 \setminus D$ degenerate as $\beta \to 12/15$. Moreover, we expect that one can find Kähler–Ricci solitons (or extremal metrics) with conical singularities along D_0 in a suitable sense even for $\beta > 12/15$. This would be a natural extension of Donaldson's deformation result in [3] to the case when there exist vector fields preserving the divisor. Finally these conical Kähler–Ricci solitons (or extremal metrics) should converge to the smooth Kähler–Ricci soliton (or extremal metric), which is known to exist on M_1, as $\beta \to 1$.

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