Math. Res. Lett.
Volume 21, Number 05, 1069–1109, 2014

On the existence of curves with A_k-singularities on $K3$ surfaces

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Let (S, H) be a general primitively polarized $K3$ surface. We prove the existence of irreducible curves in $|\mathcal{O}_S(nH)|$ with A_k-singularities and corresponding to regular points of the equisingular deformation locus. Our result is optimal for $n = 1$. As a corollary, we get the existence of irreducible curves in $|\mathcal{O}_S(nH)|$ of geometric genus $g \geq 1$ with a cusp and nodes or a simple tacnode and nodes. We obtain our result by studying the versal deformation family of the m-tacnode. Moreover, using results on Brill–Noether theory of curves on $K3$ surfaces, we provide a regularity condition for families of curves with only A_k-singularities in $|\mathcal{O}_S(nH)|$.

1. Introduction

Let S be a complex smooth projective $K3$ surface and let H be a globally generated line bundle of sectional genus $\mathrm{p} = \mathrm{p}_a(H) \geq 2$ and such that H is not divisible in $\operatorname{Pic} S$. The pair (S, H) is called a *primitively polarized* $K3$ *surface of genus* p. It is well-known that the moduli space $\mathcal{K}_{\mathrm{p}}$ of primitively polarized $K3$ surfaces of genus p is non-empty, smooth and irreducible of dimension 19. Moreover, if $(S, H) \in \mathcal{K}_{\mathrm{p}}$ is a very general element (meaning that it belongs to the complement of a countable union of Zariski closed proper subsets), then $\operatorname{Pic} S \cong \mathbb{Z}[H]$. If $(S, H) \in \mathcal{K}_{\mathrm{p}}$, we denote by $\mathcal{V}^S_{nH,1^\delta} \subset |\mathcal{O}_S(nH)| = |nH|$ the so called Severi variety of δ-nodal curves, defined as the Zariski closure of the locus of *irreducible* and reduced curves with exactly δ nodes as singularities. More generally, we will denote by $\mathcal{V}^S_{nH,1^{d_2},2^{d_3},\ldots,(m-1)^{d_m}}$ the Zariski closure of the locus in $|nH|$ of reduced and irreducible curves with exactly d_k singularities of type A_{k-1}, for every $2 \leq k \leq m$, and no further singularities. We recall that an A_k-singularity is

1991 *Mathematics Subject Classification.* 14B07, 14H10, 14J28.
Key words and phrases. versal deformations, tacnodes, Severi varieties, $K3$ surfaces, A_k-singularities.

a plane singularity of analytic equation $y^2 - x^{k+1}$. Every plane singularity of multiplicity 2 is an A_k-singularity, for some k.

The Severi variety $\mathcal{V}^S_{nH,1^\delta} \subset |\mathcal{O}_S(nH)|$ is a well-behaved variety. By [25], we know that $\mathcal{V}^S_{nH,1^\delta}$ is smooth of the expected dimension at every point $[C] \in \mathcal{V}^S_{nH,1^\delta}$ corresponding to a δ-nodal curve, i.e., the tangent space $T_{[C]}\mathcal{V}^S_{nH,1^\delta}$ has dimension $\dim(|nH|) - \delta$ for every $\delta \leq \dim(|nH|) = \mathrm{p}_a(nH)$. The existence of nodal curves of every allowed genus in the primitive linear system $|H|$ on a general primitively polarized K3 surface has been proved first by Mumford, cf. [22]. Later Chen proved the non-emptiness of $\mathcal{V}^S_{nH,1^\delta}$ in the case (S, H) is a general primitively polarized $K3$ surface, $n \geq 1$ and $\delta \leq \dim(|nH|) = \mathrm{p}_a(nH)$ [7]. Chen's existence theorem is obtained by degeneration techniques. A very general primitively polarized $K3$ surface $S_t \subset \mathbb{P}^{\mathrm{p}}$ of genus p is degenerated in $\mathbb{P}^{\mathrm{p}}$ to the union of two rational normal scrolls $S_0 = R_1 \cup R_2$, intersecting transversally along a smooth elliptic normal curve E. Rational nodal curves on S_t are obtained by deformation from suitable reduced curves $C_0 = C^1 \cup C^2 \subset S_0$ having tacnodes at points of E and nodes elsewhere. A key ingredient in the proof of Chen's theorem is the Caporaso–Harris description of the locus of $(m-1)$-nodal curves in the versal deformation space Δ_m of the m-tacnode (or A_{2m-1}-singularity). The question we ask in this paper is the following.

Main Problem. *With the notation above, assume that $C = C_1 \cup C_2 \subset R_1 \cup R_2$ is any curve having an m-tacnode at a point p of E. Then, which kinds of curve singularities on S_t may be obtained by deforming the m-tacnode of C at p?*

Theorem 3.3, which is to be considered the main result of this paper, completely answers this question. It proves that, under suitable hypotheses, *the m-tacnode of C at p deforms to d_k singularities of type A_{k-1}, for every $2 \leq k \leq m$ and $d_k \geq 0$ such that $\sum_k d_k(k-1) = m-1$.* By trivial dimensional reasons, no further singularities on S_t may be obtained by deforming the m-tacnode of $C \subset R_1 \cup R_2$. The result is a local result, obtained by studying the versal deformation family of the m-tacnode, with the same approach as in [2, Section 2.4]. In particular, the result holds for any flat family $\mathcal{X} \to \Delta$ of regular surfaces, with smooth total space $\mathcal{X}$ and special fiber $\mathcal{X}_0 = A \cup B$ having two irreducible components A and B intersecting transversally, and it can be applied to curves $C' \subset \mathcal{X}_0$ with several tacnodes on E and any kind of singularities on $\mathcal{X}_0 \setminus E$, cf. Corollary 3.12 and Remark 3.13. Section 3 is completely devoted to the proof of Theorem 3.3. In Section 4, inspired by [7], we apply Theorem 3.3, more precisely Corollary 3.12, to a family of $K3$ surfaces with suitable central fiber $\mathcal{X}_0 = R_1 \cup \tilde{R}_2$, by deforming curves $C_0 = C^1 \cup C^2 \subset R_1 \cup \tilde{R}_2$ ad hoc constructed, and we obtain the following result.

Theorem 1.1. *Let (S, H) be a general primitively polarized $K3$ surface of genus $\mathrm{p} = \mathrm{p}_a(H) = 2l + \epsilon \geq 3$, with $l \geq 1$ and $\epsilon = 0, 1$. Then, for every $n \geq 1$ and for every $(m-1)$-tuple of non-negative integers $d_2, \ldots, d_m$ satisfying*

$$(1) \qquad \sum_{k=2}^{m}(k-1)d_k = \begin{cases} 2n(l-1+\epsilon)+2-\epsilon, \text{ if } (n,\mathrm{p}) \neq (2,3),\ (2,4), \\ 2n(l-1+\epsilon)+1-\epsilon, \text{ if } (n,\mathrm{p}) = (2,3),\ (2,4), \end{cases}$$

there exist reduced irreducible curves C in the linear system $|nH|$ on S such that:

- *C has d_k singularities of type A_{k-1}, for every $k = 3, \ldots, m$, and $\delta + d_2$ nodes, where $\delta = \dim(|nH|) - \sum_{k=2}^{m}(k-1)d_k$, and no further singularities;*
- *C corresponds to a regular point of the equisingular deformation locus $ES(C)$. Equivalently, $\dim(T_{[C]}ES(C)) = 0$.*

Finally, the singularities of C may be smoothed independently. In particular, under the hypothesis (1), for any $d'_k \leq d_k$ and for any $\delta' \leq \delta$, there exist curves C in the linear system $|nH|$ on S with d'_k singularities of type A_{k-1}, for every $k = 3, \ldots, m$, and $\delta' + d'_2$ nodes as further singularities and corresponding to regular points of their equisingular deformation locus.

The notion of equisingular deformation locus and regularity is recalled in Definition 2.3 and Remark 2.4. In Corollaries 4.1 and 4.2 we observe that Theorem 1.1 is optimal if $n = 1$ and that, for $n \geq 1$, it implies the existence of curves of every geometric genus $g \geq 1$ with a cusp and nodes or a 2-tacnode and nodes as further singularities. By [6], this is not possible for $(g, n) = (0, 1)$. Finally, in the next section, we recall some standard results and terminology of deformation theory that will be useful later, focusing our attention on properties of equisingular deformations of curves with only A_k-singularities on $K3$ surfaces. In Section 2, we also provide the following regularity condition.

Proposition 1.2. *Let S be a $K3$ surface with $\operatorname{Pic} S \cong \mathbb{Z}[H]$, let $\mathrm{p} = \mathrm{p}_a(H)$ and $n \geq 1$ an integer. Assume that $C \in |nH|$ is a reduced and irreducible curve on S having precisely $d_k \geq 0$ singularities of type A_{k-1}, for each $k \geq 2$,* *and no further singularities, such that*

$$\sum_k (k-1)d_k = \deg T^1_C < \frac{p+2}{2} = \frac{H^2}{4} + 2, \quad \text{if } n = 1 \text{ or} \tag{2}$$

$$\sum_k (k-1)d_k = \deg T^1_C < 2(n-1)(p-1) = (n-1)H^2, \text{ if } n \geq 2, \tag{3}$$

where T^1_C is the first cotangent bundle of C. Then $[C]$ is a regular point of $ES(C)$ and the singularities of C may be smoothed independently.

The previous proposition is obtained by results on Brill–Noether theory of curves on $K3$ surfaces [16, 19, 20]. In particular, its proof does not require any degeneration argument of surfaces or curve singularities and is thus independent of the other results in this paper. Proposition 1.2 together with Theorem 1.1 provide sufficient conditions for the variety $\mathcal{V}^S_{nH,1^{d_2},2^{d_3},\ldots,(m-1)^{d_m}}$ to be non-empty and regular; see Remark 4.4.

2. Tangent spaces and a new regularity condition

In this section, we recall some properties of the equisingular and equigeneric deformation loci of a reduced curve on an arbitrary smooth projective $K3$ surface S and, in particular, of a curve with only A_k-singularities. Finally, at the end of the section, we prove Proposition 1.2.

Let S be a smooth projective $K3$ surface and let D be a Cartier divisor on S of arithmetic genus $\mathrm{p}_a(D)$. Assume that $|D| = |\mathcal{O}_S(D)|$ is a Bertini linear system, i.e., a linear system without base points and whose general element corresponds to a smooth curve. (In fact, by [23], every irreducible curve D on S such that $D^2 \geq 0$ defines a Bertini linear system on S.) If $C \in |D|$ is a reduced curve, we consider the following standard exact sequence of sheaves on C:

$$0 \longrightarrow \Theta_C \longrightarrow \Theta_S|_C \xrightarrow{\alpha} \mathcal{N}_{C|S} \xrightarrow{\beta} T^1_C \longrightarrow 0, \tag{4}$$

where $\Theta_C = \mathfrak{hom}(\Omega^1_C, \mathcal{O}_C)$ is the tangent sheaf of C, $\Theta_S|_C$ is the tangent sheaf of S restricted to C, $\mathcal{N}_{C|S} \simeq \mathcal{O}_C(C)$ is the normal bundle of C in S, and T^1_C is the first cotangent sheaf of C. The latter is supported on the singular locus $\mathrm{Sing}(C)$ of C, and its stalk $T^1_{C,p}$ at every singular point p of C is the versal deformation space of the singularity (see [8, (3.1)], [17, 24] or [15]). Identifying $H^0(C, \mathcal{N}_{C|S})$ with the tangent space $T_{[C]}|D|$, the induced map

$$(5)\qquad H^0(\beta):\ H^0(C,\mathcal{N}_{C|S}) \longrightarrow H^0(C,T^1_C) = \oplus_{p\in \mathrm{Sing}(C)} T^1_{C,p}$$

is classically identified with the differential at $[C]$ of the versal map from an analytic neighborhood of $[C]$ in $|D|$ to an analytic neighborhood of the origin in $H^0(C,T^1_C)$. By this identification and by the fact that the origin in $T^1_{C,p}$ is the only point parametrizing singularities analytically equivalent to the singularity of C at p [8, Lemma (3.21)], we have that the global sections of the kernel $\mathcal{N}'_{C|S}$ of the sheaf map β in (4) are infinitesimal deformations of C that are analytically equisingular, i.e., infinitesimal deformations of C preserving the analytic class of every singularity of C [8, Definition (3.9)]. For this reason, $\mathcal{N}'_{C|S}$ is usually called the *equisingular normal sheaf of C in S* [24, Proposition 1.1.9 (ii)]. Let J be the Jacobian ideal of C. By a straightforward computation, $J\otimes\mathcal{N}_{C|S} = \mathcal{N}'_{C|S}$ and, consequently, $\dim(H^0(C,T^1_C)) = \deg(J) = \sum_{p\in C}\deg(J_p)$, where J_p is the localization of J at p. Keeping in mind the versal property of T^1_C, the following definition makes sense.

Definition 2.1. We say that the singularities of C may be smoothed independently if the map $H^0(\beta)$ in (5) is surjective or, equivalently, if $h^0(C,\mathcal{N}'_{C|S}) = h^0(C,\mathcal{N}_{C|S}) - \deg(J)$. If this happens, we also say that the Jacobian ideal imposes linearly independent conditions to the linear system $|D|$.

Remark 2.2. If C is an irreducible reduced curve in a Bertini linear system $|D|$ on a smooth projective $K3$ surface S, then $h^1(C,\mathcal{N}_{C|S}) = h^1(C,\mathcal{O}_C(C)) = h^1(C,\omega_C) = 1$, where ω_C denotes the dualizing sheaf of C. In particular, by the short exact sequence of sheaves on C

$$0 \longrightarrow \mathcal{N}'_{C|S} \longrightarrow \mathcal{N}_{C|S} \longrightarrow T^1_C \longrightarrow 0,$$

we have that $h^1(C,\mathcal{N}'_{C|S}) \geq 1$, and the singularities of C may be smoothed independently if and only if $h^1(C,\mathcal{N}'_{C|S}) = 1$.

The locus in $|D|$ of deformations of C preserving the analytic class of singularities coincides with the locus of formally locally trivial deformations in the Zariski topology or locally trivial deformations in the étale topology [8, Proposition (3.23)]. In general, this locus is a proper subset of the Zariski locally closed subset $ES(C)\subset|D|$ parametrizing topologically equisingular deformations of C in $|D|$, i.e., deformations of C preserving the equisingular class of every singularity of C. For the notion of equisingular deformation of a plane singularity, we refer to [8, Definition (3.13)]. The *equisingular deformation locus* $ES(C)$ of C in $|D|$ has a natural structure of scheme, representing a suitable deformation functor [14, Section 2]. The tangent space $T_{[C]}ES(C)$ to $ES(C)$ at the point $[C]$ corresponding to C, is well understood. In particular, there exists an ideal sheaf I, named the *equisingular ideal* of C, such that $J \subset I$ and

$$T_{[C]}ES(C) \simeq H^0(C, I \otimes \mathcal{O}_C(C)).$$

Definition 2.3. We say that $[C]$ is a regular point of $ES(C)$ if $ES(C)$ is smooth of the expected dimension at $[C]$, equivalently if

$$\dim(T_{[C]}ES(C)) = \dim(H^0(C, I \otimes \mathcal{O}_C(C))) = \dim(H^0(C, \mathcal{O}_C(C))) - \deg I.$$

In this case, we also say that the equisingular ideal imposes linearly independent conditions to curves in $|D|$.

We also recall the inclusion $J \subset I \subset A$, where A is the conductor ideal.

Throughout this paper we will be interested in curves with A_k-singularities. An A_k-singularity has analytic equation $y^2 = x^{k+1}$. Every plane curve singularity of multiplicity 2 is an A_k-singularity for a certain $k \geq 1$. In particular, two singularities of multiplicity 2 are analytically equivalent if and only if they are topologically equivalent.

Remark 2.4. The equisingular ideal I of an A_k-singularity of equation $y^2 = x^{k+1}$ coincides with the Jacobian ideal $J = I = (y, x^k)$ [26, Proposition 6.6]. It follows that, if $C \in |D|$ is a reduced curve on S with only A_k-singularities, then $W \subset |D|$ is the linear system of curves passing through every A_k-singularity $p \in C$ and tangent there to the reduced tangent cone to C at p with multiplicity k and $\mathcal{W} \subset H^0(S, \mathcal{O}_S(D))$ is the vector space such that $\mathbb{P}(\mathcal{W}) = W$, then the tangent space

$$T_{[C]}ES(C) \simeq H^0(C, \mathcal{N}_{C|S} \otimes I) = H^0(C, \mathcal{N}_{C|S} \otimes J) = H^0(C, \mathcal{N}'_{C|S})$$

to $ES(C)$ at the point $[C]$ is isomorphic to $r_C(\mathcal{W})$, where $r_C : H^0(S, \mathcal{O}_S(D)) \to H^0(C, \mathcal{O}_C(D))$ is the natural restriction map. In particular, every A_k-singularity imposes at most $k = \dim(\mathbb{C}[x,y]/(y, x^k))$ linearly independent conditions to $|D|$, and the equisingular deformation locus $ES(C)$ of C in $|D|$ is regular at $[C]$ if and only if the singularities of C may be smoothed independently. If $C \in |D|$ is reduced and irreducible with d_k singularities of type A_{k-1}, $k = 2, \dots, m$, and no further singularities, then the reduced support of $ES(C)$ is an open set in one irreducible component V of the variety $\mathcal{V}^S_{D,1^{d_2},2^{d_3},\dots,(m-1)^{d_m}}$ introduced in Section 1. In particular, we have that

$$T_{[C]}V \subset T_{[C]}ES(C) \simeq H^0(C, \mathcal{N}'_{C|S}).$$

We say that V is regular at $[C]$ if $ES(C)$ is regular at $[C]$, in which case we have that $T_{[C]}V = T_{[C]}ES(C)$ and $\dim(T_{[C]}V) = \dim(T_{[C]}ES(C)) = h^0(C, \mathcal{N}'_{C|S}) = \dim(|D|) - \sum_k d_k(k-1)$. Moreover, V is said to be regular if it is regular at every point corresponding to an irreducible and reduced curve with d_k singularities of type A_{k-1}, $k = 2, \dots, m$, and no further singularities. Finally, we say that $\mathcal{V}^S_{D,1^{d_2},2^{d_3},\dots,(m-1)^{d_m}}$ is regular if all its irreducible components are regular. In particular, if $\mathcal{V}^S_{D,1^{d_2},2^{d_3},\dots,(m-1)^{d_m}}$ is regular, all its irreducible components are generically smooth of the expected dimension.

If k is odd, an A_k-singularity is also called a $\frac{k+1}{2}$-tacnode whereas, if k is even, an A_k-singularity is said to be a cusp. Moreover, by classical terminology, A_1-singularities are nodes, A_2-singularities are ordinary cusps and A_3-singularities are called simple tacnodes. As we already observed, for every $\delta \leq \mathrm{p}_a(D)$, the Severi variety $\mathcal{V}^S_{D,1^\delta}$ of δ-nodal curves is a regular variety, i.e., is smooth of the expected dimension at every point $[C]$ corresponding to a curve with exactly δ nodes as singularities [25].

Now we may prove our regularity condition for curves with only A_k-singularities on a $K3$ surface S with $\operatorname{Pic} S \cong \mathbb{Z}[H]$.

Proof of Proposition 1.2. Assume that $[C]$ is *not* a regular point of $ES(C)$. Then, by Remarks 2.2 and 2.4, we must have $h^1(\mathcal{N}'_{C|S}) \geq 2$. Now consider $\mathcal{N}'_{C|S}$ as a torsion sheaf on S and define $\mathcal{A} := \mathfrak{ext}^1(\mathcal{N}'_{C|S}, \mathcal{O}_S)$. Then $\mathcal{A}$ is a rank one torsion-free sheaf on C and a torsion sheaf on S. Moreover, by [16, Lemma 2.3], being S a $K3$ surface, we have that $h^0(\mathcal{A}) = h^1(\mathcal{N}'_{C|S}) \geq 2$ and

$$\deg \mathcal{A} = C^2 - \deg \mathcal{N}'_{C|S} = \deg T^1_C = \sum_k (k-1)d_k.$$

By [16, Proposition 2.5 and proof of Theorem I at p. 749], the pair $(C, \mathcal{A})$ may be deformed to a pair $(C', \mathcal{A}')$ where $C' \sim C$ is smooth, and $\mathcal{A}'$ is a line bundle on C' with $h^0(\mathcal{A}') \geq h^0(\mathcal{A})$ and $\deg \mathcal{A}' = \deg \mathcal{A}$. In other words, there is a smooth curve in $|nH|$ carrying a $g^1_{\deg T^1_C}$. If $n = 1$ then, by Lazarsfeld's famous result [20, Corollary 1.4], no curve in $|H|$ carries any g^1_d with $2d < \mathrm{p}_a(H) + 2$.

Now assume that $n \geq 2$. By [19, Theorem 1.3], the minimal gonality of a smooth curve in a complete linear system $|L|$ on any $K3$ surface is either $\lfloor \frac{p_a(L)+3}{2} \rfloor = \lfloor \frac{L^2}{4} \rfloor + 2$ (the gonality of a generic curve of genus $p_a(L)$) or the minimal integer d such that $2 \leq d < \lfloor \frac{p_a(L)+3}{2} \rfloor$ and there is an effective divisor D satisfying $D^2 \geq 0$, $(L^2, D^2) \neq (4d-2, d-1)$ and

$$2D^2 \overset{\text{(i)}}{\leq} L.D \leq D^2 + d \overset{\text{(ii)}}{\leq} 2d,$$

with equality in (i) if and only if $L \sim 2D$ and $L^2 \leq 4d$ and equality in (ii) if and only if $L \sim 2D$ and $L^2 = 4d$. If $L \sim nH$ with $n \geq 2$ and $\operatorname{Pic} S \cong \mathbb{Z}[H]$, one easily verifies that the minimal integer satisfying these conditions is $d = (n-1)H^2 = 2(n-1)(p_a(H)-1)$ (with $D = H$). The result follows. $\square$

Remark 2.5. As far as we know, the previously known regularity condition for curves C as in the statement of the proposition above is given by

$$\sum_k k^2 d_k \leq n^2 H^2, \tag{6}$$

which has been deduced from [18, Corollary 2.4]. This result is very different from Proposition 1.2 and we will not compare the two results here.

We conclude this section with a naive upper-bound on the dimension of the equisingular deformation locus of an irreducible curve with only A_k-singularities on a smooth $K3$ surface. This result is a simple application of Clifford's theorem, and for nodal curves it reduces to Tannenbaum's proof that Severi varieties of irreducible nodal curves on $K3$ surfaces have the expected dimension [25].

Lemma 2.6. *Let $|D|$ be a Bertini linear system on a smooth projective $K3$ surface S. Let $C \in |D|$ be a reduced and irreducible genus g curve with only A_k-singularities, τ of which are (not necessarily ordinary) cusps. Then*

$$\dim T_{[C]} ES(C) \leq g - \tau/2.$$

Proof. Let C and S be as in the statement. By Remark 2.4, since C has only A_k-singularities, we have that $T_{[C]} ES(C) = H^0(C, \mathcal{N}'_{C|S})$. Moreover, by standard deformation theory (see, e.g., [24, (3.51)]), if $\phi : \widetilde{C} \to C \subset S$ is the normalization map, we have the following exact sequence of line bundles on $\widetilde{C}$

$$0 \longrightarrow \Theta_{\widetilde{C}}(Z) \xrightarrow{\phi_*} \phi^*\Theta_S \longrightarrow \mathcal{N}'_\phi \longrightarrow 0\,. \tag{7}$$

Here $\phi_* : \Theta_{\widetilde{C}} \to \phi^*\Theta_S$ is the differential map of ϕ, having zero divisor Z, and $\mathcal{N}'_\phi \simeq \mathcal{N}_\phi/\mathcal{K}_\phi$ is the quotient of the normal sheaf $\mathcal{N}_\phi$ of ϕ by its torsion subsheaf $\mathcal{K}_\phi$ (with support on Z). By (7), using that S is a $K3$ surface, we have that $h^1(\widetilde{C}, \mathcal{N}'_\phi) = h^1(\widetilde{C}, \Theta^{-1}_{\widetilde{C}}(-Z)) \geq 1$. Moreover, again by [24, p. 174], one has

$$\mathcal{N}'_\phi \simeq \phi^*\mathcal{N}'_{C|S} \text{ and hence } h^0(C, \mathcal{N}'_{C|S}) \leq h^0(\widetilde{C}, \mathcal{N}'_\phi).$$

Finally, by applying Clifford's theorem, we deduce the desired inequality

$$h^0(C, \mathcal{N}'_{C|S}) \leq h^0(\widetilde{C}, \mathcal{N}'_\phi) \leq \frac{1}{2}\deg \mathcal{N}'_\phi + 1 = \frac{1}{2}(2g-2-\tau) + 1 = g - \frac{\tau}{2}.$$

□

3. Smoothing tacnodes

In this section, by using classical deformation theory of plane curve singularities, we will find sufficient conditions for the existence of curves with A_k-singularities on smooth projective complex surfaces that we may obtain as deformations of a "suitable" reducible surface.

Let $\mathcal{X} \to \mathbb{A}^1$ be a flat family of projective surfaces with smooth total space $\mathcal{X}$. Assume moreover that $\mathcal{X} \to \mathbb{A}^1$ has smooth and regular general fiber $\mathcal{X}_{\mathrm{t}}$ and reducible central fiber $\mathcal{X}_0 = A \cup B$, consisting of two irreducible components A and B with $h^1(\mathcal{O}_A) = h^1(\mathcal{O}_B) = h^1(\mathcal{O}_{\mathcal{X}_{\mathrm{t}}}) = 0$ and intersecting transversally along a smooth curve $E = A \cap B$. Let D be a Cartier divisor on $\mathcal{X}$. We denote by $D_t = D \cap \mathcal{X}_{\mathrm{t}}$ the restriction of D to the fiber $\mathcal{X}_{\mathrm{t}}$. Notice that, since $\mathcal{X}_0 = A \cup B$ is a reducible surface, the Picard group $\mathrm{Pic}(\mathcal{X}_0)$ of $\mathcal{X}_0$ is the fiber product of the Picard groups $\mathrm{Pic}(A)$ and $\mathrm{Pic}(B)$ over $\mathrm{Pic}(E)$. In particular, we have that

$$|\mathcal{O}_{\mathcal{X}_0}(D)| = \mathbb{P}(H^0(\mathcal{O}_A(D)) \times_{H^0(\mathcal{O}_E(D))} H^0(\mathcal{O}_B(D))).$$

From now on, for every curve $C \subset \mathcal{X}_0$, we will denote by C_A and C_B the restrictions of C to A and B, respectively. Let p be a point of E. Choose local analytic coordinates (x, z) of A at p and (y, z) of B at p in such a way that the equation of $\mathcal{X}$ at p, by using coordinates (x, y, z, t), is given by $xy = t$.

Now assume there exists a divisor $C = C_A \cup C_B \subset \mathcal{X}_0$, with $[C] \in |D_0|$, such that C_A and C_B are both smooth curves, tangent to E at a point $p \in E$ with multiplicity m and intersecting E transversally outside p. Local analytic equations of C at p are given by

$$\begin{cases} y + x - z^m = 0, \\ xy = t, \\ t = 0, \end{cases} \tag{8}$$

with $m \geq 2$. Since the singularity of C at p is analytically equivalent to the tacnode of local equation

$$f(y, z) = (y - z^m)y = 0, \tag{9}$$

we say that *C has an m-tacnode at p.*

Definition 3.1. We say that the m-tacnode of C at p imposes linearly independent conditions to $|D_0|$ if the linear system $W_{p,m} \subset |D_0|$ parametrizing curves $F_A \cup F_B \subset \mathcal{X}_0$, such that F_A and F_B are tangent to E at p with multiplicity m, has codimension m (which is the expected codimension).

Remark 3.2. We remark that, if the m-tacnode of C at p imposes linearly independent conditions to $|D_0|$, then, for every $r \leq m$, the locus $W_{p,r} \subset |D_0|$ parametrizing curves with an r-tacnode at p is non-empty of codimension exactly r. In particular, the general element of an analytic neighborhood of $[C]$ in $|D_0|$ intersects E transversally at m points close to p.

We now introduce the main result of this paper.

Theorem 3.3. *Let $\{d_2, \ldots, d_m\}$ be an $(m-1)$-tuple of non-negative integers such that*

$$\sum_{j=2}^{m} (j-1)d_j = m - 1.$$

Using the notation above, assume that:

(1) $\dim(|D_0|) = \dim(|D_t|)$*;*

(2) *the linear system $W_{p,m-1} \subset |D_0|$ of curves with an $(m-1)$-tacnode at $p \in E$ has dimension* $\dim(|D_0|) - m + 1$.

Denote by $\mathcal{V}^{\mathcal{X}_t}_{D_t,1^{d_2},2^{d_3},\ldots,(m-1)^{d_m}} \subset |D_t|$ *the Zariski closure of the locus in* $|D_t|$ *of irreducible curves with exactly* d_j *singularities of type* A_{j-1}*, for every* $2 \leq j \leq m$*, and no further singularities. Then, for a general* $t \neq 0$*, there exists a non-empty irreducible component* V_t *of* $\mathcal{V}^{\mathcal{X}_t}_{D_t,1^{d_2},2^{d_3},\ldots,(m-1)^{d_m}} \subset |D_t|$ *whose general element* $[C_t] \in V_t$ *is a regular point of* V_t*, i.e.,* $\dim(T_{[C_t]}V_t) = \dim(T_{[C_t]}ES(C_t)) = h^0(C_t, \mathcal{N}'_{C_t|\mathcal{X}_t}) = \dim(|D_t|) - \sum_{j=2}^{m}(j-1)d_j$.

The proof of this theorem will occupy us until Corollary 3.12. In the remainder of the section, we will discuss several consequences and applications of Theorem 3.3.

We want to obtain curves with A_k-singularities on the general fiber $\mathcal{X}_{\mathrm{t}}$ of $\mathcal{X}$ as deformations of $C \subset \mathcal{X}_0$. The moduli space of deformations of C in $\mathcal{X}$ is contained in an irreducible component $\mathcal{H}$ of the relative Hilbert scheme $\mathcal{H}^{\mathcal{X}|\mathbb{A}^1}$ of the family $\mathcal{X} \to \mathbb{A}^1$. Let $\pi_{\mathcal{H}} : \mathcal{H} \to \mathbb{A}^1$ be the natural map from $\mathcal{H}$ to $\mathbb{A}^1$. By the hypothesis of regularity on the fibers of the family $\mathcal{X}$, we have that the general fiber $\mathcal{H}_t$ of $\pi_{\mathcal{H}}$ coincides with the linear system $|\mathcal{O}_{\mathcal{X}_{\mathrm{t}}}(D_t)|$, whereas, in general, the central fiber $\mathcal{H}_0$ of $\pi_{\mathcal{H}}$ consists of several irreducible components of the Hilbert scheme of $\mathcal{X}_0$, only one of which, call it $\mathcal{H}_0^0$, can be generically identified with $|\mathcal{O}_{\mathcal{X}_0}(D_0)|$. This happens because the limit line bundle on $\mathcal{X}_0$ of a line bundle on $\mathcal{X}_{\mathrm{t}}$ is unique only up to twisting with a multiple of $\mathcal{O}_{\mathcal{X}}(A)$ (see, e.g. [3, Section 2.2]). Moreover, by standard deformation theory (cf. e.g. [24, Proposition 4.4.7]), the hypothesis $\dim(|D_0|) = \dim(|D_t|)$ ensures smoothness of $\mathcal{H}$ at the point $[C]$ corresponding to C. Again, since C is a local complete intersection in the smooth variety $\mathcal{X}$ (see [24, Proposition 1.1.9]), we have the same exact sequence introduced in the previous section

$$0 \longrightarrow \Theta_C \longrightarrow \Theta_{\mathcal{X}}|_C \xrightarrow{\alpha} \mathcal{N}_{C|\mathcal{X}} \xrightarrow{\beta} T^1_C \longrightarrow 0, \tag{10}$$

where T^1_C is the first cotangent sheaf of C and where the kernel $\mathcal{N}'_{C|\mathcal{X}}$ of β is called the equisingular normal sheaf of C in $\mathcal{X}$, cf. [24, Proposition 1.1.9]. By hypothesis, C is smooth outside $C \cap E = C_A \cap C_B$, it has an m-tacnode at p and nodes at the other intersection points of C_A and C_B. So, at every node r of C we have that

$$T^1_{C,r} \simeq \mathbb{C}[x,y]/(x,y) \simeq \mathbb{C},$$

while the stalk of T^1_C at p is given by

$$T^1_{C,p} \simeq \mathbb{C}^{2m-1} \simeq \mathbb{C}[y,z]/J_f,$$

where $J_f = (2y - z^m, mz^{m-1}y)$ is the Jacobian ideal of $f(y,z) = (y - z^m)y$ [8]. In particular, choosing

$$\{1,\, z,\, z^2,\, \dots,\, z^{m-1},\, y,\, yz,\, yz^2,\, \dots,\, yz^{m-2}\}$$

as a base for $T^1_{C,p}$ and using the same notation as in [2, 8], the versal deformation family $\mathcal{C}_p \to T^1_{C,p}$ of the singularity of C at p has equation

$$\mathcal{C}_p : F(y, z; \underline{\alpha}, \underline{\beta}) = y^2 + \Big(\sum_{i=0}^{m-2} \alpha_i z^i + z^m\Big) y + \sum_{i=0}^{m-1} \beta_i z^i = 0, \tag{11}$$

while the versal family $\mathcal{C}_r \to T^1_{C,r}$ of the node has equation

$$xy + t = 0.$$

Denote by $\mathcal{D} \to \mathcal{H}$ the universal family parametrized by $\mathcal{H}$ and by $\mathcal{C}_q \to T^1_{C,q}$ the versal family parametrized by $T^1_{C,q}$. By versality, for every singular point q of C there exist analytic neighborhoods U_q of $[C]$ in $\mathcal{H}$, U'_q of q in $\mathcal{D}$ and V_q of $\underline{0}$ in $T^1_{C,q}$ and a map $\phi_q : U_q \to V_q$ such that the family $\mathcal{D}|_{U_q} \cap U'_q$ is isomorphic to the pull-back of $\mathcal{C}_q|_{V_q}$, with respect to ϕ_q,

$$\begin{array}{ccccccccc}
\mathcal{C}_q & \hookleftarrow & \mathcal{C}_q|_{V_q} & \longleftarrow & U_q \times_{V_q} \mathcal{C}_q|_{V_q} & \xrightarrow{\simeq} & \mathcal{D}|_{U_q} \cap U'_q & \hookrightarrow & \mathcal{D} \\
\downarrow & & \downarrow & & & \searrow & \downarrow & & \downarrow \\
T^1_{C,q} & \hookleftarrow & V_q & & \xleftarrow{\phi_q} & & U_q & \hookrightarrow & \mathcal{H}.
\end{array} \tag{12}$$

Furthermore, by the standard identifications of the tangent space $T_{[C]}\mathcal{H}$ at $[C]$ to the relative Hilbert scheme with $H^0(C, \mathcal{N}_{C|\mathcal{X}})$ and of the versal deformation space of a plane singularity with its tangent space at the origin, the natural map

$$H^0(\beta) : H^0(C, \mathcal{N}_{C|\mathcal{X}}) \to H^0(C, T^1_C) = \oplus_{q \in \mathrm{Sing}(C)} T^1_{C,q}$$

induced by (10) is identified with the differential $d\phi_{[C]}$ at $[C]$ of the versal map

$$\phi = \oplus_{q \in \mathrm{Sing}(C)} \phi_q : \cap_{q \in \mathrm{Sing}(C)} U_q \subset \mathcal{H} \to H^0(C, T^1_C).$$

We want to obtain the existence of curves with the desired singularities on $\mathcal{X}_t$ in $|D_t|$ by versality. In particular, we will prove that the locus, in the image of ϕ, of curves with d_j singularities of type A_{j-1}, for every j as in the statement of Theorem 3.3, is non-empty. In order to do this, we observe that, no matter how we deform $C \subset \mathcal{X}_0$ to a curve on $\mathcal{X}_t$, the nodes of C (lying on E) are necessarily smoothed. Thus singularities of type A_k may arise by deformation of the tacnode of C at p only. For this reason, we may restrict our attention to the versal map ϕ_p of (12) and its differential

$$d\phi_p : H^0(C, \mathcal{N}_{C|\mathcal{X}}) \to H^0(C, T^1_C) \to T^1_{C,p}$$

at the point $[C] \in \mathcal{H}^0_0$, where, as above, $\mathcal{H}^0_0$ is the irreducible component of the central fiber $\mathcal{H}_0$ of the relative Hilbert scheme $\mathcal{H}^{\mathcal{X}|\mathbb{A}^1}$ containing $[C]$.

We first study the kernel of $d\phi_p$.[1] Let $r_C : H^0(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0}(D_0)) \to H^0(C, \mathcal{O}_C(D_0)) = H^0(C, \mathcal{N}_{C|\mathcal{X}_0})$ be the natural restriction map, $W_{p,m-1}$ the linear system of curves in $|\mathcal{O}_{\mathcal{X}_0}(D_0)|$ with an $(m-1)$-tacnode at p, as in Definition 3.1, and $\mathcal{W}_{p,m-1} \subset H^0(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0}(D_0))$ the vector space such that $\mathbb{P}(\mathcal{W}_{p,m-1}) = W_{p,m-1}$.

Lemma 3.4. *We have*

$$\ker(d\phi_p) = \ker(d\phi_{[C]}) = H^0(C, \mathcal{N}'_{C|\mathcal{X}}) = r_C(\mathcal{W}_{p,m-1}) = H^0(C, \mathcal{N}'_{C|\mathcal{X}_0}). \tag{13}$$

More generally, let $C' = C'_A \cup C'_B \in |D_0|$ be any reduced curve and $x \in C' \cap E$ a singular point of C' on E. We have that, if K_x is the kernel of the natural map $H^0(C', \mathcal{N}_{C'|\mathcal{X}}) \to T^1_{C',x}$, then

$$H^0(C', \mathcal{N}'_{C'|\mathcal{X}}) \subseteq K_x \subseteq H^0(C', \mathcal{N}_{C'|\mathcal{X}_0}) \text{ and } H^0(C', \mathcal{N}'_{C'|\mathcal{X}}) = H^0(C', \mathcal{N}'_{C'|\mathcal{X}_0}). \tag{14}$$

Finally, using the notation above, if C' has an m-tacnode at x, then

$$K_x \subseteq r_{C'}(\mathcal{W}_{x,m-1}), \text{ with equality if } \dim(W_{x,m-1}) = \dim(|D_0|) - m + 1. \tag{15}$$

Proof. From what we observed above, we have that $\ker(d\phi_{[C]}) = \ker(H^0(\beta)) = H^0(C, \mathcal{N}'_{C|\mathcal{X}})$, where $\mathcal{N}'_{C|\mathcal{X}}$ is the equisingular normal sheaf of C in $\mathcal{X}$. Moreover we have the inclusion $\ker(d\phi_{[C]}) \subseteq \ker(d\phi_p)$. We want to prove that equality holds and that $H^0(C, \mathcal{N}'_{C|\mathcal{X}}) = r_C(\mathcal{W}_{p,m-1}) = H^0(C, \mathcal{N}'_{C|\mathcal{X}_0})$.

[1]The kernel and the image of $d\phi_p$ are also computed in [7, Theorem 2.3]. We give a different and more detailed proof of these two results. This will make the proof of Theorem 1.1 shorter.

Consider the localized exact sequence

$$0 \longrightarrow \mathcal{N}'_{C|\mathcal{X},\,p} \longrightarrow \mathcal{N}_{C|\mathcal{X},\,p} \longrightarrow T^1_{C,p} \longrightarrow 0. \tag{16}$$

Using local analytic coordinates x, y, z, t at p as in (8), we may identify:

- the local ring $\mathcal{O}_{C,\,p} = \mathcal{O}_{\mathcal{X},p}/\mathcal{I}_{C|\mathcal{X},p}$ of C at p with $\mathbb{C}[x,y,z]/(f_1,f_2)$, where $f_1(x,y,z) = x+y+z^m$ and $f_2(x,y,z) = xy$;

- the $\mathcal{O}_{C,p}$-module $\mathcal{N}_{C|\mathcal{X},\,p}$ with the free $\mathcal{O}_{\mathcal{X},\,p}$-module $\mathfrak{hom}_{\mathcal{O}_{\mathcal{X},\,p}}$ $(\mathcal{I}_{C|\mathcal{X},\,p}, \mathcal{O}_{C,p})$, generated by the morphisms f_1^* and f_2^*, defined by

$$f_i^*(s_1(x,y,z)f_1(x,y,z) + s_2(x,y,z)f_2(x,y,z)) = s_i(x,y,z), \text{ for } i = 1,\, 2$$

and, finally;

- the $\mathcal{O}_{C,p}$-module

$$\begin{aligned}(\Theta_{\mathcal{X}}|_C)_p &\simeq \Theta_{\mathcal{X},p}/(I_{C,p}\otimes\Theta_{\mathcal{X},p})\\ &\simeq \langle \partial/\partial x, \partial/\partial y, \partial/\partial z, \partial/\partial t\rangle_{\mathcal{O}_{C,\,p}}/\langle \partial/\partial t - x\partial/\partial y - y\partial/\partial x\rangle\end{aligned}$$

  with the free $\mathcal{O}_{\mathcal{X},\,p}$-module generated by the derivatives $\partial/\partial x, \partial/\partial y, \partial/\partial z$.

With these identifications, the localization $\alpha_p : (\Theta_{\mathcal{X}}|_C)_p \to \mathcal{N}_{C|\mathcal{X},\,p}$ of the sheaf map α from (10) is defined by

$$\begin{aligned}\alpha_p(\partial/\partial x) = &\quad \Big(s = s_1f_1 + s_2f_2 \rightarrowtail \partial s/\partial x =_{\mathcal{O}_{C,p}} s_1\partial f_1/\partial x + s_2\partial f_2/\partial x\Big)\\ = &\quad f_1^* + yf_2^*,\\ \alpha_p(\partial/\partial y) = &\quad f_1^* + xf_2^* \text{ and}\\ \alpha_p(\partial/\partial z) = &\quad mz^{m-1}f_1^*.\end{aligned}$$

By definition of $\mathcal{N}'_{C|\mathcal{X}}$, a section $s \in \mathcal{N}_{C|\mathcal{X},\,p}$ is equisingular at p, i.e., $s \in \mathcal{N}'_{C|\mathcal{X},\,p}$, if and only if there exists a section

$$u = u_x(x,y,z)\partial/\partial x + u_y(x,y,z)\partial/\partial y + u_z(x,y,z)\partial/\partial z \in \Theta_{\mathcal{X}}|_{C\;p}$$

such that $s = \alpha_p(u)$. Hence, locally at p, first-order equisingular deformations of C in $\mathcal{X}$ have equations

$$\begin{cases} x + y + z^m + \epsilon(u_x + u_y + mz^{m-1}u_z) = 0, \\ xy + \epsilon(yu_x + xu_y) = 0. \end{cases} \tag{17}$$

The first equation above gives an infinitesimal deformation of the Cartier divisor cutting C on $\mathcal{X}_0$, while the second equation gives an infinitesimal deformation of $\mathcal{X}_0$ in $\mathcal{X}$. More precisely, by [4, Section 2], the equation $xy + \epsilon(yu_x + xu_y) = 0$ is the local equation at p of an equisingular deformation of $\mathcal{X}_0$ in $\mathcal{X}$ preserving the singular locus E. But $\mathcal{X}_0$ may be deformed in $\mathcal{X}$ only to a fiber and $\mathcal{X}_0$ is the only singular fiber of $\mathcal{X}$. It follows that the polynomial $yu_x(x, y, z) + xu_y(x, y, z)$ in the second equation of (17) must be identically zero. In particular, we obtain that

$$\ker(d\phi_p) \subset H^0(C, \mathcal{N}_{C|\mathcal{X}_0}).$$

Since every infinitesimal deformation of C in $\mathcal{X}_0$ preserves the nodes of C lying on E, i.e., all nodes of C, we deduce that $\ker(d\phi_p) \subseteq \ker(d\phi_{[C]})$ and thus $\ker(d\phi_p) = \ker(d\phi_{[C]}) = H^0(C, \mathcal{N}'_{C|\mathcal{X}}) \subset H^0(C, \mathcal{N}_{C|\mathcal{X}_0})$. Moreover, since the natural linear map $H^0(C, \mathcal{N}_{C|\mathcal{X}_0}) \to H^0(T^1_C)$ is the restriction of $d\phi_{[C]}$ to $H^0(C, \mathcal{N}_{C|\mathcal{X}_0})$, with kernel $H^0(C, \mathcal{N}'_{C|\mathcal{X}_0})$, we also obtain the equality $H^0(C, \mathcal{N}'_{C|\mathcal{X}_0}) = H^0(C, \mathcal{N}'_{C|\mathcal{X}})$. This in particular is consistent with the very well-known fact that there do not exist deformations

$$\begin{array}{ccccc} C & \subset & \mathcal{C} & \subset & \mathcal{X} \\ \downarrow & & \downarrow & \swarrow & \\ \underline{0} & \in & \mathbb{A}^1 & & \end{array} \tag{18}$$

of C in $\mathcal{X}$ preserving the nodes of C, except for deformations of C in $\mathcal{X}_0$ (see [10, Section 2] for a proof).

Notice finally that, by the argument above, the inclusions

$$H^0(C, \mathcal{N}'_{C|\mathcal{X}}) \subset \ker(d\phi_p) \subset H^0(C, \mathcal{N}_{C|\mathcal{X}_0})$$

hold independently of the kind of singularities of C on E. This proves (14).

Now it remains to show that $H^0(C, \mathcal{N}'_{C|\mathcal{X}}) = r_C(\mathcal{W}_{p,m-1})$. Consider the first equation of (17). By the fact that the polynomial $yu_x(x, y, z)$

$+xu_y$ (x,y,z) is identically zero, we deduce that

- $u_x(0,0,0) = u_y(0,0,0) = 0$ and
- for every $n \geq 1$, no z^n-terms appear in $u_x(x,y,z)$ and $u_y(x,y,z)$, no y^n-terms and $y^n z^m$-terms appear in $u_x(x,y,z)$ and, finally, no x^n-terms and $x^n z^m$-terms appear in $u_y(x,y,z)$.

In particular, local equations at p on B of equisingular infinitesimal deformations of C are given by

$$\begin{cases} yq(y,z) + z^m + \epsilon m z^{m-1} u_z(x,y,z) = 0, \\ x = 0, \end{cases} \tag{19}$$

where $q(y,z)$ is a polynomial with variables y and z, and similarly on A. This proves that $H^0(C, \mathcal{N}'_{C|\mathcal{X}}) \subset r_C(\mathcal{W}_{p,m-1})$ and more generally the inclusion in (15). The opposite inclusion $r_C(\mathcal{W}_{p,m-1}) \subset H^0(C, \mathcal{N}'_{C|\mathcal{X}})$ follows from a naive-dimensional computation. Indeed, by the hypothesis (2), $\dim(r_C(\mathcal{W}_{p,m-1})) = \dim(W_{p,m-1}) = \dim(|D_0|) - m + 1$. On the other hand, if $W \subset W_{E,m}$ is the irreducible component containing the point $[C]$, then $\dim(W) \geq \dim(|D_0|) - m + 1$ and W is contained in the Zariski closure of the family of locally trivial deformations of C. Thus, W is the Zariski closure of the locus of locally trivial deformations of C. Its tangent space at $[C]$ is isomorphic to $H^0(C, \mathcal{N}'_{C|\mathcal{X}})$ and (13) is proved. The same argument shows that $K_x = r_{C'}(\mathcal{W}_{x,m-1})$ in (15) if $\dim(W_{x,m-1}) = \dim(|D_0|) - m + 1$. The lemma is proved. □

We now describe the image of $d\phi_p$.

Lemma 3.5. *Let $\alpha_0, \ldots, \alpha_{m-2}, \beta_0, \ldots, \beta_{m-1}$ be coordinates on $T^1_{C,p}$ as above. Then the image $H_p \subset T^1_{C,p}$ of $d\phi_p$ is given by the equations*

$$H_p = d\phi_p(H^0(C, \mathcal{N}_{C|\mathcal{X}})) : \beta_1 = \cdots = \beta_{m-1} = 0. \tag{20}$$

Moreover, the image of $H^0(C, \mathcal{N}_{C|\mathcal{X}_0})$ under $d\phi_p$ is the linear subspace of H_p given by

$$d\phi_p(H^0(C, \mathcal{N}_{C|\mathcal{X}_0})) = T_0 V_{1^m} : \beta_0 = \beta_1 = \cdots = \beta_{m-1} = 0 \tag{21}$$

and coincides with the tangent space at 0 of the locus V_{1^m} of m-nodal curves.

From the following remark to the end of the section, $V_p \subset T^1_C$ and $U_p \subset \mathcal{H}_{\mathcal{X}|\mathbb{A}^1}$ are analytic neighborhood as in diagram (12).

Remark 3.6. Notice that, because of the choice we have made for the local analytic equation of C at p, we have that the locus $V_{1^m} \subset T^1_{C,p}$ of m-nodal curves is the linear space defined by the equations $\beta_0 = \beta_1 = \cdots = \beta_{m-1} = 0$, coinciding with its tangent space at 0. In general, the locus V_{1^m} of m-nodal curves in the versal deformation space of the m-tacnode $T^1_{C,p}$ is a smooth variety, parametrizing equigeneric deformations. Lemma 3.5 shows in general that, if C is as in Theorem 3.3, then $d\phi_p(H^0(C, \mathcal{N}_{C|\mathcal{X}_0})) = T_0 V_{1^m}$ and consequently $\phi_p(\mathcal{H}^0_0 \cap U_p) = V_{1^m} \cap V_p$.

Proof of Lemma 3.5. We first observe that $\dim(H_p) = m$. Indeed, by Lemma 3.4 and hypothesis (2) of Theorem 3.3, we have that

$$\dim(H_p) = h^0(C, \mathcal{N}_{C|\mathcal{X}}) - \dim(r_C(\mathcal{W}_{p,m-1})) = m.$$

Moreover, since $\ker(d\phi_p) \subset H^0(C, \mathcal{N}_{C|\mathcal{X}_0})$ and $H^0(C, \mathcal{N}_{C|\mathcal{X}_0})$ has codimension 1 in $H^0(C, \mathcal{N}_{C|\mathcal{X}})$, we have that H_p contains $d\phi_p(H^0(C, \mathcal{N}_{C|\mathcal{X}_0}))$ as a codimension-1 linear subspace. Again by the hypothesis (2) of Theorem 3.3 and Remark 3.2, the $(m-1)$-linear space $d\phi_p(H^0(C, \mathcal{N}_{C|\mathcal{X}_0}))$ is contained in the tangent space $T_{\underline{0}} V_{1^m} \subset T^1_{C,p}$ at $\underline{0}$ to the Zariski closure $V_{1^m} \subset T^1_{C,p}$ of the locus of m-nodal curves. Now it is easy to verify that, using the coordinates (11) on $T^1_{C,p}$, the equations of V_{1^m} are $\beta_0 = \cdots = \beta_{m-1} = 0$. Hence

$$d\phi_p(H^0(C, \mathcal{N}_{C|\mathcal{X}_0})) = T_{\underline{0}} V_{1^m} = V_{1^m} = \phi_p(\mathcal{H}^0_0 \cap U_p) \tag{22}$$

(cf. Remark 3.6). By (22), in order to find a base of H_p, it is enough to find the image by $d\phi_p$ of the infinitesimal deformation $\sigma \in H^0(C, \mathcal{N}_{C|\mathcal{X}}) \setminus H^0(C, \mathcal{N}_{C|\mathcal{X}_0})$ having equations

$$\begin{cases} x + y + z^m = 0, \\ xy = \epsilon. \end{cases} \tag{23}$$

The image of σ is trivially the point corresponding to the curve $y(y + z^m) = \epsilon$. Thus, the equations of $H_p \subset T^1_{C,p}$ are given by (20). ☐

Now let $d_2, \ldots, d_m$ be non-negative integers such that $m = \sum_{j=2}^{m} (j-1)d_j + 1$, as in the statement of Theorem 3.3. Let $V_p \subset T^1_{C,p}$ be the analytic open set as in (12) and denote by

$$V_{1^{d_2}, 2^{d_3}, \ldots, (m-1)^{d_m}} \subset V_p \subset T^1_{C,p}$$

the Zariski closure of the locus in V_p of curves with exactly d_j singularities of type A_j, for every j, and no further singularities. The following proposition implies Theorem 3.3, as indicated below.

Proposition 3.7. *We have*

$$V_{1^{d_2},2^{d_3},\dots,(m-1)^{d_m}} \cap H_p = B_1 \cup B_2 \cup \cdots \cup B_k, \tag{24}$$

where every B_i is an irreducible and reduced affine curve containing 0*, whose general element corresponds to a curve with exactly d_j singularities of type A_{j-1}, for every j, and no further singularities.*

Remark 3.8. The number k is explicitly determined by a combinatorial argument at the end of the proof of Claim 3.9.

Proof of Theorem 3.3. By Lemmas 3.4 and 3.5, we know that the image $\phi_p(U_p) \subset V_p \subset T^1_{C,p}$ of $U_p \subset \mathcal{H}$ by ϕ_p is an m-dimensional subvariety of V_p that is smooth at $\underline{0}$. By Remark 2.4 and the openness of versality (more precisely, by the properties [8, (3.5), (3.6)] of versal deformation families), up to restricting V_p, we have that $V_{1^{d_2},2^{d_3},\dots,(m-1)^{d_m}} \subset V_p$ is a variety of pure codimension $\sum_j (j-1)d_j = m-1$, which is non-empty by (24). Moreover, up to restricting V_p again, we may assume that $\underline{0}$ is contained in every irreducible component of $V_{1^{d_2},2^{d_3},\dots,(m-1)^{d_m}}$. Hence $V_{1^{d_2},2^{d_3},\dots,(m-1)^{d_m}} \cap \phi_p(U_p)$ is non-empty and each of its irreducible components has dimension $\geq 2m - 1 - 2m + 2 = 1$. By recalling that $H_p = d\phi_p(H^0(C, \mathcal{N}_{C|\mathcal{X}})) = T_{\underline{0}}\phi_p(U_p)$ and $\phi_p(U_p)$ is smooth at $\underline{0}$, we see that (24) implies that the intersection

$$V_{1^{d_2},2^{d_3},\dots,(m-1)^{d_m}} \cap \phi_p(U_p) = B'_1 \cup B'_2 \cup \cdots \cup B'_k,$$

has pure dimension 1. Now notice that, by semicontinuity, since $\phi_p^{-1}(\underline{0})$ is smooth of (maximal) codimension m in the relative Hilbert scheme $\mathcal{H}_{\mathcal{X}|\mathbb{A}^1}$, the same is true for the fiber $\phi_p^{-1}(x)$, for $x \in V_p$, up to restricting V_p. More precisely, again by the properties [8, (3.5), (3.6)] of versal deformation families, if $x \in B'_i$, with $x \neq \underline{0}$, is a point sufficiently close to $\underline{0}$ of any irreducible component of $V_{1^{d_2},2^{d_3},\dots,(m-1)^{d_m}} \cap \phi_p(U_p)$, and $[C_t] \in \phi_p^{-1}(B'_i) \cap \mathcal{H}_t$, then we have that $T_{[C_t]}(\phi_p^{-1}(B'_i)) \simeq H^0(C_t, \mathcal{N}'_{C_t|\mathcal{X}})$ and thus $\phi_p^{-1}(B'_i \setminus \underline{0})$ is the equisingular deformation locus of C_t in $\mathcal{X}$ *scheme theoretically*, cf. Definition 2.3 and Remark 2.4. In particular, $\phi_p^{-1}(B'_i \setminus \underline{0}) \cap \mathcal{H}_t$ is the equisingular deformation locus of C_t in $\mathcal{X}_{\mathrm{t}}$,

$$T_{[C_t]}(\phi_p^{-1}(B'_i) \cap \mathcal{H}_t) \simeq H^0(C_t, \mathcal{N}'_{C_t|\mathcal{X}_t})$$

and

$$h^0(C_t, \mathcal{N}'_{C_t|\mathcal{X}_t}) = h^0(C_t, \mathcal{N}'_{C_t|\mathcal{X}}) - h^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = \dim(|D_t|) - m + 1.$$

This proves that $[C_t]$ is a regular point of an irreducible component V of the variety $\mathcal{V}^{\mathcal{X}_t}_{D_t, 1^{d_2}, 2^{d_3}, \ldots, (m-1)^{d_m}}$. □

Proof of Proposition 3.7. We use the same approach as Caporaso and Harris in [2, Section 2.4]. Equation (11), defining the versal family $\mathcal{C}_p$ of the m-tacnode, has degree 2 in y. In particular, for every point $x = (\underline{\alpha}, \underline{\beta}) \in T^1_{C,p}$, the corresponding curve $\mathcal{C}_{p,x} : F(y, z; \underline{\alpha}, \underline{\beta}) = 0$ is a double cover of the z-axis. Moreover, the discriminant map

$$\Delta : T^1_{C,p} \to \mathcal{P}^{2m-1} = \left\{ z^{2m} + \sum_{i=0}^{2m-2} a_i z^i, a_i \in \mathbb{C} \right\}$$

from $T^1_{C,p}$ to the affine space of monic polynomials of degree $2m$ with no $2m-1$ term, defined by

$$\Delta(\underline{\alpha}, \underline{\beta})(z) = \Big(\sum_{i=0}^{m-2} \alpha_i z^i + z^m \Big)^2 - 4\Big(\sum_{i=0}^{m-1} \beta_i z^i \Big),$$

is an isomorphism (see [2, p. 179]). Thus, we may study curves in the versal deformation family of the m-tacnode in terms of the associated discriminant polynomial. In particular, for every point $(\underline{\alpha}, \beta_0, 0, \ldots, 0) \in H_p$, which we will shortly denote by $(\underline{\alpha}, \beta_0) := (\underline{\alpha}, \beta_0, 0, \ldots, 0)$, the corresponding discriminant polynomial is given by

$$\Delta(\underline{\alpha}, \beta_0)(z) = \Big(\sum_{i=0}^{m-2} \alpha_i z^i + z^m \Big)^2 - 4\beta_0 = \Big(\nu(z) - 2\sqrt{\beta_0}\Big)\Big(\nu(z) + 2\sqrt{\beta_0}\Big), \tag{25}$$

where we set $\nu(z) := \sum_{i=0}^{m-2} \alpha_i z^i + z^m$. From now on, since we are interested in deformations of the m-tacnodal curve $C \subset \mathcal{X}_0$ to curves on $\mathcal{X}_t$, with $t \neq 0$, we will always assume $\beta_0 \neq 0$, being $V_{1^m} : \beta_0 = \beta_1 = \cdots = \beta_{m-1} = 0$ the image in $T^1_{C,p}$ of infinitesimal (and also effective) deformations of C in $\mathcal{X}_0$. Writing down explicitly the derivatives of the polynomial $F(y, z; \underline{\alpha}, \underline{\beta})$, one may verify that a point $x = (\underline{\alpha}, \beta_0)$ parametrizes a curve $\mathcal{C}_{p,x}$ with an A_k-singularity at the point (z_0, y_0) if and only if z_0 is a root of multiplicity $k+1$ of the discriminant polynomial $\Delta(\underline{\alpha}, \beta_0)(z)$. Our existence problem is thus equivalent to the following.

Claim 3.9. *Let $d_2, \dots, d_m$ be an $(m-1)$-tuple of non-negative integers satisfying (1). The locus of points $(\underline{\alpha}, \beta_0) \in H_p$ such that the discriminant polynomial $\Delta(\underline{\alpha}, \beta_0)(z)$ has exactly d_j roots of multiplicity j, for every $2 \leq j \leq m$, and no further multiple roots, is non-empty of pure dimension* 1.

The claim will be proved right below. □

To prove the last claim we need the following auxiliary result, whose proof is postponed to Appendix A.

Lemma 3.10. *Let $m \geq n \geq 2$ be integers and $(d_2^+, d_2^-, \dots, d_n^+, d_n^-)$ be a $2(n-1)$-tuple of non-negative integers satisfying*

$$m = \sum_{j=2}^{n} (j-1)(d_j^+ + d_j^-) + 1 \geq 2 \text{ and} \tag{26}$$

$$m \geq \sum_{j=2}^{n} j d_j^{\pm} > 0. \tag{27}$$

Then there exists a triple of permutations (τ^+, τ^-, σ) of m indices, such that $\tau^{\pm}$ has cyclic structure $\Pi_{j=2}^{n} j^{d_j^{\pm}}$, σ is a cycle of order m and $\sigma\tau^+\tau^- = 1$.

Definition 3.11. An admissible $2(n-1)$-tuple is a $2(n-1)$-tuple of non-negative integers $(d_2^+, d_2^-, \dots, d_n^+, d_n^-)$ satisfying (26) and (27).

Proof of Claim 3.9. Let $d_2, \dots, d_m$ be an $(m-1)$-tuple of non-negative integers satisfying (1). Assume that there exists a point $(\underline{\alpha}, \beta_0) \in H_p$ such that the discriminant polynomial $\Delta(\underline{\alpha}, \beta_0)(z)$ has the desired properties. Then, since $\nu(z) - 2\sqrt{\beta_0}$ and $\nu(z) + 2\sqrt{\beta_0}$ cannot have common factors for $\beta_0 \neq 0$, there exist non-negative integers d_j^+, d_j^- such that $d_j = d_j^+ + d_j^-$ and the d_j roots of multiplicity j of the discriminant $\Delta(\underline{\alpha}, \beta_0)(z)$ are distributed as d_j^+ roots of $\nu(z) + 2\sqrt{\beta_0}$ and d_j^- roots of $\nu(z) - 2\sqrt{\beta_0}$. The obtained $2(m-1)$-tuple of non-negative integers $(d_2^+, d_2^-, \dots, d_m^+, d_m^-)$ is admissible (see Definition 3.11). The polynomial $\nu(z) = z^m + \sum_{i=0}^{m-2} \alpha_i z^i$ defines a degree m covering $\nu : \mathbb{P}^1 \to \mathbb{P}^1$, totally ramified at ∞ and with further $d_j^{\pm}$ ramification points of order $j-1$ over $\pm 2\sqrt{\beta_0}$, for every $2 \leq j \leq m$. We get in this way

$$(m-1) + \sum_{j=2}^{m} (j-1) d_j^+ + \sum_{j=2}^{m} (j-1) d_j^- = 2m - 2$$

ramification points of ν. Hence ν has no further ramification by the Riemann–Hurwitz formula. The branch points of ν are three, consisting of ∞, $-2\sqrt{\beta_0}$

and $2\sqrt{\beta_0}$, if both sums $\sum_j d_j^+$ and $\sum_j d_j^-$ are positive, while the branch points of ν are only two if $\sum_j d_j^+ = 0$ or $\sum_j d_j^- = 0$.

Consider first the case that ν has only two ramification points, say ∞ and $2\sqrt{\beta_0}$. Then we have that $\sum_j d_j^+ = 0$ and $d_j = d_j^-$, for every j. In particular, using the conditions (26) and (27), we find that $d_j = 0$ for $j \neq m$ and $d_m = 1$. It follows that $\nu(z) - 2\sqrt{\beta_0} = (z-\lambda)^m$, for a certain λ. But the only λ such that $(z-\lambda)^m$ has no degree $m-1$ term, is $\lambda = 0$. Thus, we get $\nu(z) = z^m + 2\sqrt{\beta_0}$. On the other hand, for every fixed $\beta_0 \neq 0$, if $\nu(z) = z^m + 2\sqrt{\beta_0}$, then the associated discriminant $\Delta(\underline{\alpha}, \beta_0) = z^m(z^m + 4\sqrt{\beta_0})$ has a root of multiplicity m and no further multiple roots. *This proves the claim under the hypothesis $d_m = 1$ and $d_j = 0$ for $j \neq m$. More precisely, this shows that the Zariski closure of the locus in $H_p \subset T^1_{C,p}$ of curves with an A_{m-1}-singularity is the smooth curve given by the equations*

$$(28) \qquad V_{(m-1)^1} \cap H_p : \begin{cases} \alpha_i &= 0, \quad \text{for every} \quad 1 \leq i \leq m-2, \\ \alpha_0^2 &= 4\beta_0. \end{cases}$$

Now consider the general case, i.e., assume that $\sum_j d_j^\pm > 0$. Then the polynomial $\nu(z)$ defines an m-covering $\nu : \mathbb{P}^1 \to \mathbb{P}^1$ having branch points at ∞, $-2\sqrt{\beta_0}$ and $+2\sqrt{\beta_0}$ with monodromy permutations σ, τ^+ and τ^- respectively, where σ is an m-cycle while $\tau^\pm$ has cyclic structure $\Pi_{j=2}^m j^{d_j^\pm}$. Moreover, the group $\langle \sigma, \tau^+, \tau^- \rangle$ is trivially transitive and, by the theory of coverings of $\mathbb{P}^1$, we have that $\sigma\tau^+\tau^- = 1$. In fact, the class $\{\nu\Phi | \Phi : \mathbb{P}^1 \to \mathbb{P}^1 \text{ automorphism}\}$ of degree m coverings isomorphic to ν and with branch locus at ∞, $-2\sqrt{\beta_0}$ and $+2\sqrt{\beta_0}$, is uniquely determined by the conjugacy class of the triple of permutations (τ^+, τ^-, σ) (cf. e.g., [21, Corollary III.4.10]). Hence, for every $\beta_0 \neq 0$ and for every fixed $(2m-2)$-tuple $(d_2^+, d_2^-, \ldots, d_m^+, d_m^-)$, there exist at most finitely many polynomials $\nu(z)$ with the properties above, up to a change of variable. Moreover, a change of variable that transforms $\nu(z)$ in a polynomial with the same properties corresponds to an automorphism $\Phi : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\Phi(z) = az + b$ with $a^m = 1$ and $b = 0$, as $\nu(z)$ is monic and has no z^{m-1}-term.

We may now prove the non-emptiness in the general case. Let $d_2, \ldots, d_m$ be non-negative integers satisfying (1). Then, no matter how we choose non-negative integers $d_i^\pm$, with $d_i = d_i^+ + d_i^-$, we have that the $(2m-2)$-tuple $\underline{d} = (d_2^+, d_2^-, \ldots, d_m^+, d_m^-)$ is admissible. Furthermore, by Lemma 3.10, there exist finitely many triples of permutations (τ^+, τ^-, σ) of m indices such that $\tau^\pm$ has cyclic structure $\Pi_{j=2}^m j^{d_j^\pm}$, σ is a cycle of order m and $\sigma\tau^+\tau^- = 1$. Since the group $\langle \sigma, \tau^+, \tau^- \rangle$ is trivially transitive, again by the general theory of coverings of $\mathbb{P}^1$, for every $\gamma \in \mathbb{C}$ there exists an m-covering $\nu_{(\underline{d},(\tau^+,\tau^-,\sigma),\gamma)} : \mathbb{P}^1 \to \mathbb{P}^1$ with branch points $\infty, -\gamma, +\gamma$ and monodromy permutations σ, τ^+, τ^-, respectively. Up to a change of variables in the domain, we may always assume that ν is defined by a monic polynomial $\nu_{(\underline{d},(\tau^+,\tau^-,\sigma),\gamma)}$ (z) with no z^{m-1}-term. *This proves that the locus of points* $(\underline{\alpha}, \beta_0) \in H_p$ *such that the discriminant polynomial* $\Delta(\underline{\alpha}, \beta_0)(z)$ *has the desired properties is non-empty. We finally want to see that it has pure dimension* 1*, by writing its parametric equations explicitly.* We notice that, from what we observed above, the polynomials,

$$\nu_{(\underline{d},(\tau^+,\tau^-,\sigma),\gamma)}\left(\frac{z}{\zeta}\right), \text{ where } \zeta \text{ is an } m\text{th root of unity},$$

are all polynomials with no z^{m-1}-term, and define an m-covering of $\mathbb{P}^1$ isomorphic to $\nu_{(\underline{d},(\tau^+,\tau^-,\sigma),\gamma)}$ (whose monodromy is conjugated with (τ^+, τ^-, σ)). More generally, if

$$\nu_{(\underline{d},(\tau^+,\tau^-,\sigma),1)}(z) = z^m + \sum_{i=0}^{m-2} c_{i,(\underline{d},(\tau^+,\tau^-,\sigma),1)} z^i, \tag{29}$$

then, for every $\gamma \in \mathbb{C} \setminus 0$, the polynomials

$$\nu(z) := u^m \nu_{(\underline{d},(\tau^+,\tau^-,\sigma),1)}\left(\frac{z}{u}\right) = z^m + \sum_{i=0}^{m-2} \alpha_i z^i, \ u^m = \gamma \tag{30}$$

are all the monic polynomials without z^{m-1}-term such that $\nu^2(z) - 4\gamma^2$ has the same kind of multiple roots with the desired distribution, and defining an m-covering of $\mathbb{P}^1$ with monodromy in the conjugacy class of (τ^+, τ^-, σ). We have thus proved that the reduced and irreducible curve of parametric equations

$$\alpha_i = u^{m-i} c_{i,(\underline{d},(\tau^+,\tau^-,\sigma),1)}, \text{ for } i = 0, 1, \ldots, m-2, \text{ and } \beta_0 = \frac{u^{2m}}{4}, \ u \in \mathbb{C}, \tag{31}$$

is an irreducible component of $V_{1^{d_2}, 2^{d_3}, \ldots, (m-1)^{d_m}} \cap H_p$. In particular,

$$V_{1^{d_2}, 2^{d_3}, \ldots, (m-1)^{d_m}} \cap H_p = B_1 \cup \cdots \cup B_k$$

is a reduced curve with k irreducible components, where k is the number of pairs $(\underline{d}, [\tau^+, \tau^-, \sigma])$, such that $\underline{d} = (d_2^+, d_2^-, \ldots, d_m^+, d_m^-)$ is an admissible

$(2m-2)$-tuple with $d_j^+ + d_j^- = d_j$, for every j, and $[\tau^+, \tau^-, \sigma] = \{(\rho\tau^+\rho^{-1}, \rho\tau^-\rho^{-1}, \rho\sigma\rho^{-1}) \mid \rho \in \mathfrak{S}_m\}$, where $\mathfrak{S}_m$ denotes the symmetric group of order m and σ, τ^+ and $\tau^- \in \mathfrak{S}_m$ are permutations such that σ is an m cycle, $\tau^\pm$ has cyclic structure $\Pi_{j=2}^m j^{d_j^\pm}$ and $\sigma\tau^+\tau^- = 1$. □

Theorem 3.3 is a local result. It describes all possible deformations of an m-tacnode of a reduced curve $C' \in |D_0|$ at a double point x of $\mathcal{X}_0$, as precisely stated in the following.

Corollary 3.12. *Let $C' \subset |D_0|$ be any reduced curve with an m-tacnode at a point $x \in E$ and possibly further singularities. Then, fixing coordinates $(\underline{\alpha}, \underline{\beta})$ on $T^1_{C',x}$ as in (11), we have that the image H'_x of the morphism*

$$d\phi_{[C'],x} : H^0(C', \mathcal{N}_{C'|\mathcal{X}}) \to T^1_{C',x}$$

is contained in the linear space $H_x : \beta_1 = \cdots = \beta_{m-1} = 0$. Moreover, it contains in codimension 1 the image Γ'_x of $H^0(C', \mathcal{N}_{C'|\mathcal{X}_0})$, which is in turn contained in Γ_x, with $\Gamma_x = T_{\underline{0}}V_{1^m} : \beta_0 = \beta_1 = \cdots = \beta_{m-1} = 0$. If $\Gamma'_x = \Gamma_x$ then $H'_x = H_x$ and, for every $(m-1)$-tuple of integers $d_2, \ldots, d_m$ such that $\sum_j (j-1)d_j = m-1$, the intersection $V_{1^{d_2}, 2^{d_3}, \ldots, (m-1)^{d_m}} \cap H'_x$ is the curve given by (24). In particular, there exist curves $C'_t \in |D_t|$ with d_j singularities of type A_{j-1} at a neighborhood of x, for every $j = 2, \ldots, m$. By (15), a sufficient condition in order that $\Gamma'_x = \Gamma_x$ (equivalently $H'_x = H_x$) is that $\dim(W_{x,m-1}) = \dim(|D_0|) - m + 1$.

Proof. By hypothesis we have that $C' = C'_A \cup C'_B$, with C'_A and C'_B smooth at x and tangent to E with multiplicity m. Trivially, every deformation of C' in $\mathcal{X}_0$ has on E tacnodes of order m_i at points p_i close to x with $\sum_i m_i = m$. In particular, every deformation of C' in $\mathcal{X}_0$ is equigeneric at p [8, Definition 3.13]. This implies that the image $\Gamma'_x = d\phi_{[C'],x}(H^0(C', \mathcal{N}_{C'|\mathcal{X}_0})$ is contained in the tangent space $\Gamma_x = T_{\underline{0}}V_{1^m} : \beta_0 = \beta_1 = \cdots = \beta_{m-1}$ at $\underline{0}$ of the locus V_{1^m} of m-nodal curves. By (14) and the fact that $H^0(C', \mathcal{N}_{C'|\mathcal{X}_0})$ is a codimension 1 subspace $H^0(C', \mathcal{N}_{C'|\mathcal{X}})$, we have that Γ'_x has codimension 1 in H'_x. Moreover, if $\sigma \in H^0(C', \mathcal{N}_{C'|\mathcal{X}}) \setminus H^0(C', \mathcal{N}_{C'|\mathcal{X}_0})$ is the infinitesimal deformation given by (23), where now $x + y + z^m = xy = 0$ is the local equation of C' at x, then we have that the image point of σ is contained in H'_x. Thus $H'_x \subseteq H_x$, with $H_x : \beta_1 = \cdots = \beta_{m-1} = 0$. The corollary follows now by Proposition 3.7 and by versality, arguing as in the proof of Theorem 3.3. □

Remark 3.13. Corollary 3.12 is helpful if one wants to deform a reduced curve $C' \in |D_0|$ with tacnodes at points of E and further singularities, by smoothing together all singularities. In the next section, it will be applied to the case where $\mathcal{X}_0$ is a reducible stable $K3$ surface and $C' \subset \mathcal{X}_0$ is a reduced reducible curve with nodes and tacnodes (at singular points of $\mathcal{X}_0$) as singularities. In [11] the curve $C' \subset \mathcal{X}_0$ is also allowed to have a space triple point at a point of E. In these applications the fact that the family $\mathcal{X} \to \mathbb{A}^1$ is a family of $K3$ surfaces is useful in the construction of the limit curve, but not in the deformation argument of the curve, where only versality is used.

In order to help the reader understand the deformation argument in Theorem 1.1 (especially in the Case 1.4.2), we want to point out the following difference of behavior between curve singularities at smooth points of $\mathcal{X}_0$ and at points of E. For curves on smooth surfaces we have Definition 2.1. Assume now that the curve $C' \subset \mathcal{X}_0$ has several tacnodes at points $p_1, \ldots, p_r$ of E of order $m_1, .., m_r$. Then the natural map

$$H^0(C', \mathcal{N}_{C'|\mathcal{X}}) \to \oplus_i H_{p_i} \subset \oplus_i T^1_{C',i}$$

is not surjective. Indeed, $\dim(\oplus_i H_i) = \sum_i m_i$, but the kernel of the map has dimension $\geq h^0(C', \mathcal{N}_{C'|\mathcal{X}}) - \sum_i m_i - r + 1$. This is not an obstruction to simultaneously deform the tacnodes to desired singularities of type A_k. A sufficient condition in order to be able to smooth the tacnodes independently is that the linear system W of curves in $|D_0|$ with an $(m_i - 1)$-tacnode at p_i for every i has codimension $\sum_i (m_i - 1)$ in $|D_0|$, as expected. Indeed, if this happens, then, by (15), we have that $h^0(C', \mathcal{N}'_{C'|\mathcal{X}_0}) = h^0(C', \mathcal{N}_{C'|\mathcal{X}_0}) - \sum_i (m_i - 1)$. Thus, by Corollary 3.12, the map

$$H^0(C', \mathcal{N}_{C'|\mathcal{X}_0}) \to \oplus_i \Gamma_{p_i}$$

is surjective and hence the map $H^0(C', \mathcal{N}_{C'|\mathcal{X}}) \to H_{p_i}$ is a surjection for any i. This implies that tacnodes may be deformed independently by arguing as in [1, Lemma 4.4] (see Case 1.4.2 of the proof of Theorem 1.1 for details).

We finally observe that the deformation argument of Theorem 1.1 also works if the limit curve has singularities different than nodes on the smooth locus of $\mathcal{X}_0$, as long as $H^0(C', N'_{C'|\mathcal{X}})$ has the “expected dimension”.

Remark 3.14. The curve (24) in Proposition 3.7 has nicer geometric properties in several cases. Consider the case $d_j = 0$ for every $j \geq 3$, i.e., deformations of the m-tacnode to curves with $m-1$ nodes. Then, by [2, Section 2.4], there is only one possible conjugacy class $[\tau^+, \tau^-, \sigma]$ and the curve (24) is smooth and irreducible. Similarly, in the case $d_m = 1$ and $d_j = 0$ for every $j \leq m-1$, the curve in $H_p \subset T^1_{C,p}$ parametrizing deformations of the m-tacnode to an A_{m-1}-singularity is defined by (28). In particular it is smooth and irreducible. This is no longer true in general.

In the special case that $m = 2l+1$ is odd and $d_j^+ = d_j^-$ for every j, the parametric equations (31) of (24) have a simpler feature, since the polynomial (29) is odd. Indeed, under the hypotheses, the polynomial $-\nu_{(\underline{d},(\tau^+,\tau^-,\sigma),1)}(-z)$ is monic and with no $(m-1)$-term and defines an m-covering of $\mathbb{P}^1$ with monodromy in the conjugacy class (τ^+,τ^-,σ). Moreover, the discriminant polynomial $(-\nu_{(\underline{d},(\tau^+,\tau^-,\sigma),1)}(-z))^2 - 4$ has d_j roots of multiplicity j, distributed as d_j^+ roots of $\nu_{(\underline{d},(\tau^+,\tau^-,\sigma),1)}(-z)+1$ and d_j^- roots of $\nu_{(\underline{d},(\tau^+,\tau^-,\sigma),1)}(-z)-1$, for every j, and no further multiple roots. By uniqueness, there exists $\zeta \in \mathbb{C}$ with $\zeta^m = 1$ such that

$$-\nu_{(\underline{d},(\tau^+,\tau^-,\sigma),1)}(-z) = \nu_{(\underline{d},(\tau^+,\tau^-,\sigma),1)}(\zeta z),$$

from which we deduce that $c_{i,(\underline{d},(\tau^+,\tau^-,\sigma),1)} = 0$, for every even i and $\zeta = 1$. Then equations (31) become

$$\begin{array}{rclcl} \alpha_0 & = & 0, & & \alpha_1 = t^l c_{1,(\underline{d},(\tau^+,\tau^-,\sigma),1)}, \\ \alpha_2 & = & 0, & & \alpha_3 = t^{l-1} c_{3,(\underline{d},(\tau^+,\tau^-,\sigma),1)}, \\ \vdots & & \vdots & & \vdots \\ \alpha_{m-2} & = & 0 & \text{and} & \beta_0 = \frac{t^m}{4},\ t = u^2 \in \mathbb{C}. \end{array} \tag{32}$$

Remark 3.15 (Multiplicities and base changes). In the same way as in [10, Section 1] for families of curves with only nodes and ordinary cusps, it is possible to define a relative Severi–Enriques variety $\mathcal{V}^{\mathcal{X}|\mathbb{A}^1}_{D,1^{d_2},2^{d_3},\dots,(m-1)^{d_m}} \subset \mathcal{H}$ in the relative Hilbert scheme $\mathcal{H}$, whose general fiber is the variety $\mathcal{V}^{\mathcal{X}_t}_{D_t,1^{d_2},2^{d_3},\dots,(m-1)^{d_m}} \subset |D_t| = \mathcal{H}_t$. Theorem 3.3 proves that, whenever its hypotheses are verified, the locus of locally trivial deformations $W_{E,m} \subset |D_0|$ of C in $|D_0|$ is one of the irreducible components of the special fiber $\mathcal{V}_0$ of $\mathcal{V}^{\mathcal{X}|\mathbb{A}^1}_{D,1^{d_2},2^{d_3},\dots,(m-1)^{d_m}} \to \mathbb{A}^1$.

The multiplicity m_C of $W_{E,m}$, as irreducible component of $\mathcal{V}_0$, coincides with the intersection multiplicity at $\underline{0}$ of the curve

$$V_{1^{d_2},2^{d_3},\dots,(m-1)^{d_m}} \cap H_p = B_1 \cup \cdots \cup B_k$$

with the linear space $d\phi_p(H^0(C,\mathcal{N}_{C|\mathcal{X}_0})) : \beta_0 = \beta_1 = \cdots = \beta_{m-1} = 0$.

Furthermore, the minimum of all intersection multiplicities at $\underline{0}$ of the irreducible components B_i with $d\phi_p(H^0(C, \mathcal{N}_{C|\mathcal{X}_0}))$ is the geometric multiplicity m_g of C defined in [10, Problem 1 and Definition 1]. In particular, $m_g \leq m_C$. When $d_2 = m-1$ and $d_j = 0$ for $j \neq 2$, we know by [2] that $k = 1$ and B_1 is smooth at $\underline{0}$ and tangent to $d\phi_p(H^0(C, \mathcal{N}_{C|\mathcal{X}_0})$ with multiplicity $m_C = m$.

By Corollary 3.12 and Remark 3.13, a similar result holds for reduced curves $C' \in |D_0|$ with tacnodes on E and singularities of type A_k on the smooth locus of $\mathcal{X}_0$, as long as $H^0(C', N'_{C'|\mathcal{X}})$ has the "expected dimension". In this case the multiplicity $m_{C'}$ may be computed as in [1, Lemma 4.4].

We finally observe that it is an abuse of terminology to say that the m-tacnodal curve $C \subset \mathcal{X}_0$ deforms to a curve $C_t \in |\mathcal{O}_{X_t}(D_t)|$ with d_j singularities of type A_{j-1} for every j. This is true only up to a base change. More precisely, let

$$\begin{array}{ccc} \tilde{\mathcal{X}} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \xrightarrow{\nu_{m_g}} & \mathbb{A}^1 \end{array}$$

be the family of surfaces obtained from $\mathcal{X} \to \mathbb{A}^1$ by a base change of order m_g. Observe that $\tilde{\mathcal{X}}$ is an m_g-cover of $\mathcal{X}$ totally ramified along the central fiber. In particular, by substituting $\nu_{m_g}(u) = u^{m_g} = t$ in the local equation $xy = t$ of $\mathcal{X}$ at a point $p \in E = A \cap B$, one finds that $\tilde{\mathcal{X}}$ is singular exactly along the singular locus $\tilde{E} \simeq E$ of the central fiber $\tilde{\mathcal{X}}_0 \simeq \mathcal{X}_0$. By blowing-up $m_g - 1$ times $\tilde{X}$ along E one obtains a family of surfaces

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{h} & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \xrightarrow{\nu_{m_g}} & \mathbb{A}^1 \end{array}$$

with smooth total space, having general fiber $\mathcal{X}'_u \simeq \tilde{\mathcal{X}}_u \simeq \mathcal{X}_{u^m}$, and whose central fiber has a decomposition into irreducible components $\mathcal{X}'_0 = \mathcal{E}_0 \cup \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_{m_g-1} \cup \mathcal{E}_m$ where $\mathcal{E}_0 = A$, $\mathcal{E}_m = B$ and $\mathcal{E}_i$ is a $\mathbb{P}^1$-bundle on the curve $\mathcal{E}_i \cap \mathcal{E}_{i-1} \simeq E$, for every $1 \leq i \leq m-1$. In this new family of surfaces $\mathcal{X}' \to \mathbb{A}^1$, we have that the pull-back curve $h^*(C) \in |\mathcal{O}_{\mathcal{X}'_0}(h^*(D))|$ of C deforms to a curve $C_u \in |\mathcal{O}_{\mathcal{X}_u}(h^*(D))|$ with the wished singularities. Equivalently, the divisor $m_g C \in |\mathcal{O}_{\mathcal{X}_0}(m_g D_0)|$ deforms to a reduced curve in $|\mathcal{O}_{\mathcal{X}_t}(m_g D_t)|$ having m_g irreducible components, each of which is a curve with the desired singularities.

Example 3.16. In the previous remark, take $m = 5$, $d_3 = 2$ and $d_j = 0$, for $j \neq 3$. Then $m_C = m_g = 5$. Roughly speaking, if the curve C in Theorem 3.3 has a 5-tacnode at the point $p \in E \subset \mathcal{X}_0$, then it appears as limit of curves with two ordinary cusps on $\mathcal{X}_{\mathrm{t}}$ with multiplicity 5.

Proof. If $m = 5$, $d_3 = 2$ and $d_j = 0$ for $j \neq 3$, then $d_3^+ = d_3^- = 1$ and $(\tau^+, \tau^-, \sigma) = ((123)(345)(654321))$, up to conjugation. In particular, $V_{2^2} \cap H_p$ is an irreducible curve whose parametric equations are given by equation (32) in Remark 3.14. In particular, it intersects $\beta_0 = \beta_1 = \cdots = \beta_5 = 0$ with multiplicity 5 at $\underline{0}$. We finally want to prove that $V_{2^2} \cap H_p$ is smooth at $\underline{0}$, by explicitly computing (32). The following argument has been suggested to us by the referee. Let $\nu(z)$ be a degree 5 polynomial with no z^4-term. If we require

$$\nu(z) = 1 + (z-a)^3(z-a')(z-a'') = 1 + (z-b)^3(z-b')(z-b''),$$

by solving the corresponding polynomial equations, we obtain that

- a is any root of $8x^5 - 3$,
- a' and a'' are the two roots of $3x^2 + 9ax + 8a^2$,
- $b = -a$, $b' = -a'$ and $b'' = -a''$.

Thus we find that $\nu(z) = 1 + \frac{1}{3}(z-a)^3(3z^2 + 9az + 8a^2) = z^5 - \frac{10a^2}{3}z^3 + 5a^4 z$. In particular, the polynomial $\nu(z)$ is odd, as expected, and equations (32) become

$$\alpha_0 = \alpha_2 = 0,\ \alpha_1 = 5a^4t^2,\ a_3 = -\frac{10a^2}{3}t \text{ and } \beta_0 = \frac{t^5}{4},\ t \in \mathbb{C}, \tag{33}$$

with a any fixed 5th root of $\frac{3}{8}$, proving smoothness. □

The corollary below follows directly from equations (28) and it is an easy generalization of [10, Lemmas 2 and 6].

Corollary 3.17. *Independently of m, if in Remark 3.15 we have that $d_m = 1$ and $d_j = 0$ for $j \neq m$, then $m_C = m_g = 2$. Roughly speaking, the m-tacnodal curve C in Theorem 3.3 appears as limit of an A_{m-1}-singularity on $\mathcal{X}_t$ with multiplicity* 2.

As already observed in Remark 3.15, Examples 3.16 and Corollary 3.17 may be generalized to curves $C' \subset \mathcal{X}_0$ with several tacnodes on E and singularities A_k on the smooth locus of $\mathcal{X}_0$, under the hypothesis that

$H^0(C', \mathcal{N}'_{C'|\mathcal{X}_0})$ has the expected dimension. But computing the multiplicity $m_{C'}$ in the general case is a non-easy exercise.

4. An application to general $K3$ surfaces

This section is devoted to the proof of Theorem 1.1. We also point out several corollaries of it. The degeneration argument we will use has been introduced in [5] and also used in [7]. In the following (S, H) will denote a general primitively polarized $K3$ surface of genus $\mathrm{p} = \mathrm{p}_a(H)$. We will show the existence of curves on S with A_k-singularities as an application of Theorem 3.3, more precisely of Corollary 3.12. In particular we will study deformations of suitable curves with tacnodes and nodes on the union of two ad hoc constructed rational normal scrolls, cf. Remark 3.13. We point out that our argument is strongly inspired by the one in [7] but is not the same.

Let $\mathrm{p} = 2l + \varepsilon \geq 3$ be an integer with $\varepsilon = 0, 1$ and $l \geq 1$ and let $E \subset \mathbb{P}^{\mathrm{p}}$ be an elliptic normal curve of degree $\mathrm{p}+1$. Consider two line bundles $L_1, L_2 \in \mathrm{Pic}^2(E)$ with $L_1 \neq L_2$. We denote by R_1 and R_2 the rational normal scrolls of degree $\mathrm{p}-1$ in $\mathbb{P}^{\mathrm{p}}$ generated by the secants of the divisors in $|L_1|$ and $|L_2|$, respectively. We have that

$$R_i \cong \begin{cases} \mathbb{P}^1 \times \mathbb{P}^1 & \text{if } \mathrm{p} = 2l+1 \text{ is odd and } \mathcal{O}_E(1) \not\sim (l+1)L_i, \\ \mathbb{F}_1 & \text{if } \mathrm{p} = 2l \text{ is even,} \\ \mathbb{F}_2 & \text{if } \mathrm{p} = 2l+1 \text{ is odd and } \mathcal{O}_E(1) \sim (l+1)L_i. \end{cases}$$

We will only need to consider the first two cases. In the first case, where $R_1 \cong R_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$, we let σ_i and F_i be the classes of the two rulings of R_i, for $i = 1, 2$. In the second case, where $R_1 \cong R_2 \cong \mathbb{F}_1$, we let σ_i be the section of negative self-intersection and F_i be the class of a fiber. Then the embedding of R_i into $\mathbb{P}^{\mathrm{p}}$ is given by the line bundle $\sigma_i + lF_i$ for $i = 1, 2$ and R_1 and R_2 intersect transversally along the curve $E \sim -K_{R_i} \sim 2\sigma_i + (3-\varepsilon)F_i$, which is anticanonical in each R_i (cf. [5, Lemma 1]). In particular, $R := R_1 \cup R_2$ is a variety with normal crossings and, by [9, Section 2], we have that the first cotangent bundle $T^1_R \simeq \mathcal{N}_{E|R_1} \otimes \mathcal{N}_{E|R_2}$ of R is a line bundle on E of degree 16. Let now $\mathcal{U}_{\mathrm{p}}$ be the component of the Hilbert scheme of $\mathbb{P}^{\mathrm{p}}$ containing R. Then we have that $\dim(\mathcal{U}_{\mathrm{p}}) = \mathrm{p}^2 + 2\mathrm{p} + 19$ and, by [5, Theorems 1 and 2], the general point $[S] \in \mathcal{U}_{\mathrm{p}}$ represents a smooth, projective $K3$ surface S of degree $2\mathrm{p}-2$ in $\mathbb{P}^{\mathrm{p}}$ such that $\mathrm{Pic}\, S \cong \mathbb{Z}[\mathcal{O}_S(1)] = \mathbb{Z}[H]$.

In the proof of Theorem 1.1 we will consider general deformations $\mathcal{S} \to T$ of $R = \mathcal{S}_0$ over a one-dimensional disc T contained in $\mathcal{U}_{\mathrm{p}}$. Now $\mathcal{S}$ is smooth except for 16 rational double points $\xi_1, \dots, \xi_{16}$ lying on E; these are the zeroes of the section of the first cotangent bundle T^1_R of R that is the image by the natural map $H^0(R, \mathcal{N}_{R|\mathbb{P}^{\mathrm{p}}}) \to H^0(R, T^1_R)$ of the first-order embedded deformation determined by $\mathcal{S} \to T$, cf. [5, pp. 644–647]. Blowing-up $\mathcal{S}$ at these points and contracting the obtained exceptional components (all isomorphic to $\mathbb{F}_0$) on R_2, we get a smooth family of surfaces $\mathcal{X} \to T$, such that $\mathcal{X}_{\mathrm{t}} \simeq \mathcal{S}_t$ and $\mathcal{X}_0 = R_1 \cup \tilde{R}_2$, where $\tilde{R}_2$ is the blowing-up of R_2 at the points $\xi_1, \dots, \xi_{16}$, with new exceptional curves $E_1, \dots, E_{16}$.

Proof of Theorem 1.1. Let $\mathrm{p} = 2l + \epsilon \geq 3$, with $\epsilon = 0, 1$, let E be a smooth elliptic curve and $n \geq 1$ an integer.

Case 1. We first prove the theorem under the assumption $(n, \mathrm{p}) \neq (2, 3)$, $(2, 4)$. The proof will be divided into 4 steps.

Step 1.1. We construct two suitable rational normal scrolls R_1 and R_2. Let $L_1 \neq L_2$ be two degree 2 line bundles on E such that

$$(n-1)(l-1+\varepsilon)L_1 \sim (n-1)(l-1+\varepsilon)L_2. \tag{34}$$

Note that there is no requirement if $n = 1$, and that the hypothesis $\mathrm{p} \geq 5$ if $n = 2$ ensures that we can choose $L_1 \neq L_2$. Now fix any general point $r \in E$ and embed E as an elliptic normal curve of degree $\mathrm{p} + 1$ by the very ample line bundle

$$\mathcal{O}_E(1) := \Big(2n(l-1+\varepsilon) + 3 - \varepsilon\Big)r - (n-1)(l-1+\varepsilon)L_i. \tag{35}$$

When $\mathrm{p} = 2l + 1$ is odd, the condition $\mathcal{O}_E(1) \not\sim (l+1)L_i$ is equivalent to $(nl+1)L_i \not\sim 2(nl+1)r$, which is certainly verified for a general point r. Hence, letting R_1 and R_2 be the two rational normal scrolls in $\mathbb{P}^{\mathrm{p}}$ spanned by L_1 and L_2 as above, we have that $R_1 \cong R_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ when $\mathrm{p} = 2l + 1$ is odd and $R_1 \cong R_2 \cong \mathbb{F}_1$ when $\mathrm{p} = 2l$ is even.

Step 1.2. We next construct a special curve $C \subset R = R_1 \cup R_2$, inspired by [7]. Using the notation above, let M_i be the divisor on R_i defined by $M_i = \sigma_i + [n(l-1+\varepsilon) + 1 - \varepsilon]F_i$. By (35) and the fact that $L_i|_E \sim F_i|_E$, we have that $[2n(l-1+\varepsilon) + 3 - \varepsilon]r \in |\mathcal{O}_E(M_i)|$. Since $H^0(R_i, \mathcal{O}_{R_i}(M_i)) \cong H^0(E, \mathcal{O}_E(M_i))$ we deduce that there exists a unique (necessarily smooth and irreducible) curve $C^i_n \subset R_i$ such that

$$C^i_n \in \Big|M_i\Big| \quad \text{and} \quad C^i_n \cap E = \Big(2n(l-1+\varepsilon) + 3 - \varepsilon\Big)r, \tag{36}$$

for both $i = 1, 2$. Now we fix a general point $q_0 \in E$ and we denote by H the hyperplane class of $\mathbb{P}^{\mathrm{p}}$. If $n = 1$ then $M_i \sim C^i_1 \sim H$ on R_i and $q_0 \notin C^i_1$

for $i = 1, 2$. More generally, by (34)–(36), we have that

$$(nH - C_n^1)|_E \sim (nH - C_n^2)|_E,$$

for every $n \geq 1$. In particular, if $n \geq 2$, there exists a curve $D^1 \cup D^2 \subset R$ with $D^i = \cup_{j=1}^{n-1} C_j^i \subset R_i$, where every $C_j^i \in |\sigma_i + (1-\varepsilon)F_i|$ is a (necessarily smooth and irreducible) curve on R_i, such that

$$C_j^1 \cap E = q_{2j-2} + (2-\varepsilon)q_{2j-1} \text{ and } C_j^2 \cap E = (2-\varepsilon)q_{2j-1} + q_{2j},$$

with $i = 1, 2$ and $1 \leq j \leq n-1$, and where $q_1, q_2, \ldots, q_{2n-2} = q_0$ are distinct points on E. Notice that the curve $D^1 \cup D^2 \subset R$ is uniquely determined by q_0 if p is odd, while for p even there are finitely many curves like $D^1 \cup D^2$. Now, for $i = 1, 2$, let $C^i \in |\mathcal{O}_{R_i}(nH)|$ be the curve defined by

$$C^i = \begin{cases} C_1^i \text{ if } n = 1, \\ D^i \cup C_n^i = C_1^i \cup C_2^i \cup \cdots \cup C_{n-1}^i \cup C_n^i \text{ if } n \geq 2. \end{cases}$$

Observe that, if $n \geq 2$, because of the generality of q_0, we may assume that all irreducible components C_j^i of D^i intersect C_n^i transversally for $i = 1, 2$. In particular, we have that the singularities of $C := C^1 \cup C^2 \in |\mathcal{O}_R(nH)|$ consist of a $(\mathrm{p}+1)$-tacnode at $r \in E$ if $n = 1$, and are given by nodes on $R \setminus E$ and nodes and tacnodes on E if $n \geq 2$.

Step 1.3. We now construct a general deformation $\mathcal{S} \to T$ of R, whose general fiber $\mathcal{S}_t$ is a smooth projective K3 surface, and a smooth birational modification of it

$$(37) \qquad \begin{array}{ccc} \mathcal{X} & \xrightarrow{\pi} & \mathcal{S} \\ & \searrow & \downarrow \\ & & T \end{array}$$

as above. Let $\xi_1 + \cdots + \xi_{16} \in |T_R^1|$ be a general divisor if $n = 1$ and a general divisor such that $\xi_1 = q_0$ and $\xi_l \neq q_m$, for every $l \geq 2$ and $m \geq 1$, if $n \geq 2$. By the generality of q_0, we have that $\xi_1 + \cdots + \xi_{16}$ is a general member of $|T_R^1|$ also for $n \geq 2$. By the surjectivity of the natural map $H^0(R, \mathcal{N}_{R|\mathbb{P}^{\mathrm{p}}}) \to H^0(R, T_R^1)$ (see [5, Corollary 1]), by [5, Theorems 1 and 2] and related references (precisely, [9, Remark 2.6] and [13, Section 2]), we deduce that there exists a deformation $\mathcal{S} \to T$ of $\mathcal{S}_0 = R$ whose general fiber is a smooth projective $K3$ surface $\mathcal{S}_t$ in $\mathbb{P}^{\mathrm{p}}$ with $\mathrm{Pic}(\mathcal{S}_t) \cong \mathbb{Z}[\mathcal{O}_{\mathcal{S}_t}(1)] \cong \mathbb{Z}[H]$ and such that $\mathcal{S}$ is singular exactly at the points $\xi_1 \ldots, \xi_{16} \in E$. Let $\mathcal{X} \to T$ be the smooth family obtained from $\mathcal{S} \to T$ as above and $\pi : \mathcal{X} \to \mathcal{S}$ the induced birational morphism. We recall that it has special fiber $\mathcal{X}_0 = R^1 \cup \tilde{R}^2$, where $\tilde{R}^2$ is the blowing up of R_2 at $\xi_1 \dots, \xi_{16}$ and $R_1 \cap \tilde{R}^2 = E$.

Step 1.4. Let $\tilde{C}$ and $\pi^*(C)$ be the strict transform and the pull-back of C with respect to the natural morphism $\pi : \mathcal{X} \to \mathcal{S}$. Using the ideas developed in Section 3, we will prove that $\pi^*(C)$ deforms into a family of curves in the $\mathcal{X}_t$'s enjoying the required properties.

In the case $n = 1$, the result is a straightforward application of Theorem 3.3. Indeed, for $n = 1$, by the generality of $r \in E$, the curve $\tilde{C} \simeq \pi^*(C)$ is a $(\mathrm{p}+1)$-tacnodal curve satisfying all hypotheses of Theorem 3.3.

Assume now that $n \geq 2$. In this case $\pi^*(C) = \tilde{C} \cup E_{q_0}$, where $E_{q_0} = E_1 \subset \tilde{R}_2$ is the (-1)-curve corresponding to $q_0 = \xi_1$. By abusing notation, we denote every irreducible component of $\tilde{C}$ as the corresponding irreducible component of C. In particular, we set $\tilde{C} = C^1_n \cup D^1 \cup C^2_n \cup D^2$ and $D^i = \cup_{j=1}^{n-1} C^i_j$, for every i. The singularities of $\pi^*(C)$ on $\mathcal{X}_0 \setminus E$ are given by the singularities of the strict transform $\tilde{C}$ of C and a further node at $x_0 = E_{q_0} \cap \tilde{C}$. We want to obtain curves in $|\mathcal{O}_{\mathcal{X}_t}(nH_t)|$ with the desired singularities as deformations of $\pi^*(C)$. We first observe that every deformation of $\pi^*(C)$ in $|\mathcal{O}_{\mathcal{X}_t}(nH_t)|$ is an irreducible curve. This may easily be verified using that the divisor H_t generates $\mathrm{Pic}(\mathcal{X}_t)$. In particular, no matter how we deform $\pi^*(C)$ to a curve on $\mathcal{X}_t$, at least one node of $\pi^*(C)$ on $(C^1_n \cup C^2_n) \cap (D^1 \cup D^2) \cap (R \setminus E)$ must be smoothed. Moreover, the smoothed node may be chosen arbitrarily, as will be clear by the following argument. Let $q \in (C^1_n \cup C^2_n) \cap (D^1 \cup D^2) \cap (\mathcal{X}_0 \setminus E)$ be any fixed point and consider the natural morphism

$$\Phi : H^0(\pi^*(C), \mathcal{N}_{\pi^*(C)|\mathcal{X}}) \to T = \oplus_{x \text{ tacnode}} T^1_{\pi^*(C),x} \oplus_{y \neq q \text{ node on } \mathcal{X}_0 \setminus E} T^1_{\pi^*(C),y}$$

obtained by composing the morphism $H^0(\pi^*(C), \mathcal{N}_{\pi^*(C)|\mathcal{X}}) \to H^0(\pi^*(C), T^1_{\pi^*(C)})$ with the projection $H^0(\pi^*(C), T^1_{\pi^*(C)}) \to T$. By Remark 2.4 and Lemma 3.4, the kernel of Φ is contained in the subspace $r_{\pi^*(C)}(\mathcal{W}) \subset H^0(\pi^*(C), \mathcal{N}_{\pi^*(C)|\mathcal{X}_0})$, where $r_{\pi^*(C)} : H^0(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0}(\pi^*(C))) \to H^0(\pi^*(C), \mathcal{N}_{\pi^*(C)|\mathcal{X}_0})$ is the restriction map and $\mathcal{W} \subset H^0(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0}(\pi^*(C)))$ is the subspace associated with the linear system $W \subset |\mathcal{O}_{\mathcal{X}_0}(\pi^*(C))|$ of curves passing through every node $y \neq q$ of $\pi^*(C)$ on $\mathcal{X}_0 \setminus E$ and having an $(m-1)$-tacnode at every m-tacnode of $\pi^*(C)$.

We now want to show that

$$\ker(\Phi) = r_{\pi^*(C)}(\mathcal{W}) = H^0(\pi^*(C), \mathcal{N}'_{\pi^*(C)|\mathcal{X}_0}) = H^0(\pi^*(C), \mathcal{N}'_{\pi^*(C)|\mathcal{X}}) = \{0\}. \tag{38}$$

The equality $H^0(\pi^*(C), \mathcal{N}'_{\pi^*(C)|\mathcal{X}_0}) = H^0(\pi^*(C), \mathcal{N}'_{\pi^*(C)|\mathcal{X}})$ follows from Lemma 3.4. Moreover, we observe that, in order to prove (38), it is enough to show that the linear system W consists of the unique curve $W = \{\pi^*(C)\}$. Indeed, if this last equality is true, then $\ker(\Phi) = r_{\pi^*(C)}(\mathcal{W}) = \{0\}$. Moreover, the equality $W = \{\pi^*(C)\}$ also implies that every curve D in W contains the point q, too. In other words, every infinitesimal deformation of $\pi^*(C)$ in $\mathcal{X}_0$ preserving every tacnode and the nodes y different from q on $\mathcal{X}_0 \setminus E$, also preserves q. Using that the nodes of $\pi^*(C)$ on E are trivially preserved by every section of $H^0(\pi^*(C), \mathcal{N}_{\pi^*(C)|\mathcal{X}_0})$, we have that $r_{\pi^*(C)}(\mathcal{W}) \subset H^0(\pi^*(C), \mathcal{N}'_{\pi^*(C)|\mathcal{X}_0})$. Since the other inclusion holds trivially, we obtain (38).

Now the equality $W = \{\pi^*(C)\}$ is a straightforward application of Bezout's theorem. Let $B = B_1 \cup B_2$, where $B_1 = B|_{R_1}$ and $B_2 = B|_{\tilde{R}_2}$, be a curve in the linear system W. Assume that $q \in \tilde{R}_2$. The other case is similar. We first observe that the intersection number $B_i.C_n^i$ is given by $B_i.C_n^i = n^2(l-1+\varepsilon) + nl$. Moreover, by the hypothesis $B \in W$, we have that the intersection multiplicity of B_1 and C_n^1 at r is given by $\mathrm{mult}_r(B_1, C_n^1) = 2n(l-1+\varepsilon) + 2 - \varepsilon$. Furthermore, the intersection $B_1 \cap C_n^1$ contains the intersection points $C_n^1 \cap C_j^1$, for $1 \leq j \leq n-1$. We deduce that the cardinality of the intersection $B_1 \cap C_n^1$ is at least equal to

$$2n(l-1+\varepsilon) + 2 - \varepsilon + (n-1)M_i\big(\sigma_i + (\varepsilon-1)F_i\big) = n^2(l-1+\varepsilon) + nl + 1.$$

Thus, by Bezout's Theorem, $C_n^1 \subset B_1$. Since B is a Cartier divisor, it follows that the intersection multiplicity of B_2 with E at r is given by $\mathrm{mult}_r(B_2, E) = \mathrm{mult}_r(B_1, E) \geq \mathrm{mult}_r(C_n^1, E) = 2n(l-1+\epsilon) + 3 - \epsilon$. Moreover, B_2 contains the points, different from q, arising from the intersection of $C_n^2 \cap C_j^2$, with $j \leq n-1$. Using again Bezout's theorem, we find that $C_n^2 \subset B_2$ and, in particular $q \in B$. It remains to prove that $C_j^i \subset B^i$, for $i = 1, 2$ and $j \leq n-1$. We observe that, if $j \leq n-1$, then $B_i.C_j^i = nl$. Now consider the intersection $B_2 \cap C_{n-1}^2$. It contains the point q_{2n-3} with multiplicity $1-\epsilon$; the point x_0; the $(n-2)(1-\epsilon)$ points arising from the intersection $C_j^2 \cap C_{n-1}^2$, for $j \leq n-2$; the intersection points $C_{n-2}^2 \cap C_n^2$. This amounts to a total of $nl+1$ points. Thus, $C_{n-1}^2 \subset B_2$. Similarly we have that $E_{q_0} \subset B_2$. Then B_1 passes through the further points $q_0 = q_{2n-2} = E_{q_0} \cap E$ and q_{2n-3} with multiplicity $2-\epsilon$. This implies by Bezout that $C_{n-1}^1 \subset B_1$. Applying this argument $2(n-2)$ more times, one obtains that $B = \pi^*(C)$ and thus Φ is injective.

The rest of the proof will be divided according to the parity of p.

Case 1.4.1: $\mathrm{p} = 2l+1$ *is odd.* Then all singularities of $\pi^*(C)$ are nodes except for the point $r \in E$ that is a $(2nl+2)$-tacnode. Moreover, by Corollary 3.12 (and using the notation therein), the image of Φ is contained in the linear space

$$T' = H_r \oplus_{y \neq q \text{ node on } R\setminus E} T^1_{\pi^*(C),y} \subset T.$$

As Φ is injective, the image of Φ has dimension $h^0(\pi^*(C), \mathcal{N}_{\pi^*(C)|\mathcal{X}}) = 2n^2l+2$. Since $\dim(H_r) = 2nl+2$ by Corollary 3.12 and the curve $\pi^*(C)$ has exactly $2nl(n-1)$ nodes on $R \setminus E$ different from q, the image of Φ must coincide with T'. Again by Corollary 3.12 and by versality, we deduce that the curve $\pi^*(C)$ may be deformed to a curve $C_t \in |\mathcal{O}_{\mathcal{X}_t}(nH_t)|$ preserving all nodes of $\pi^*(C)$ on $R \setminus E$ except q and deforming the $(2nl+2)$-tacnode to d_k singularities of type A_{k-1}, for every sequence (d_k) of non-negative integers such that $\sum_k (k-1)d_k = 2nl+1$. Moreover, by the fact that $\ker(\Phi) = H^0(\pi^*(C), \mathcal{N}'_{\pi^*(C)|\mathcal{X}}) = \{0\}$, we obtain that the family of curves $C_t \in |\mathcal{O}_{\mathcal{X}_t}(nH_t)|$ constructed in this way is, scheme theoretically, a generically smooth curve $\mathcal{B} \subset \mathcal{H}_{\mathcal{X}|\mathbb{A}^1}$ in the relative Hilbert scheme. By the openness of versality (more precisely, by the properties [8, (3.5) and (3.6)] of versal deformation families), if $[C_t] \in \mathcal{B}$ is a general point, then $T_{[C_t]}\mathcal{B} \simeq H^0(C_t, \mathcal{N}'_{C_t|\mathcal{X}})$. In particular, we obtain that

$$\dim(T_{[C_t]}ES(C_t)) = h^0(C_t, \mathcal{N}'_{C_t|\mathcal{X}_0}) = h^0(C_t, \mathcal{N}'_{C_t|\mathcal{X}}) - 1 = 0.$$

This proves the theorem in the case p is odd.

Case 1.4.2: $\mathrm{p} = 2l$ *is even.* In this case $\pi^*(C)$ has a $(2n(l-1)+3)$-tacnode at r, a 2-tacnode at q_{2j-1}, for every $j = 1, \ldots, n-1$, and nodes elsewhere. In particular, $\pi^*(C)$ has $2(n-1)(nl-n+1)+(n-1)(n-2)$ nodes on $R \setminus E$ different from q. Again by Corollary 3.12 (and using the notation therein), $\Phi(H^0(\pi^*(C), \mathcal{N}_{\pi^*(C)|\mathcal{X}}))$ is contained in the linear subspace

$$T' = H_r \oplus_{j=1}^{n-1} H_{q_{2j-1}} \oplus_{y \neq q \text{ node on } R\setminus E} T^1_{\pi^*(C),y}$$

of T. In this case the image of Φ does not coincide with T' unless $n=1$. For $n=1$ the theorem follows as before. If $n>1$ we observe that $\Phi(H^0(\pi^*(C), \mathcal{N}_{\pi^*(C)|\mathcal{X}}))$ contains $\Phi(H^0(\pi^*(C), \mathcal{N}_{\pi^*(C)|\mathcal{X}_0}))$ as a codimension 1 subspace. Moreover, by Corollary 3.12 and a straightforward dimension count, we have that

$$\Phi(H^0(\pi^*(C), \mathcal{N}_{\pi^*(C)|\mathcal{X}_0})) = \Gamma_r \oplus_{j=1}^{n-1} \Gamma_{q_{2j-1}} \oplus_{y \neq q \text{ node on } R\setminus E} T^1_{\pi^*(C),y},$$

where $\Gamma_r \subset H_r$ is the locus of $(2n(l-1)+3)$-nodal curves and $\Gamma_{q_{2j-1}} \subset H_{q_{2j-1}}$ is the locus of 2-nodal curves, for every $j = 1, \dots, n-1$, cf. Remark 3.6. It follows that

$$\Phi(H^0(\pi^*(C), \mathcal{N}_{\pi^*(C)|\mathcal{X}})) = \Omega \oplus_{y \neq q \text{ node on } R \setminus E} T^1_{\pi^*(C),y},$$

where $\Omega \subset H_r \oplus_{j=1}^{n-1} H_{q_{2j-1}}$ is a linear subspace containing $\Gamma_r \oplus_{j=1}^{n-1} \Gamma_{q_{2j-1}}$ as a codimension 1 subspace. Moreover, again by Corollary 3.12 and the surjectivity of the map $H^0(\pi^*(C), \mathcal{N}_{\pi^*(C)|\mathcal{X}_0})) \to \Gamma_x$, with $x = r$ or $x = q_{2j-1}$, we have that the projection maps

$$\rho_r : \Omega \to H_r \text{ and } \rho_{q_{2j-1}} : \Omega \to H_{q_{2j-1}}$$

are surjective, for every j. By [2, Section 2.4], the locus of 1-nodal curves in $H_{q_{2j-1}}$ is a smooth curve simply tangent to $\Gamma_{q_{2j-1}}$ at $\underline{0}$. Let $(d_2, \dots, d_m)$ be any $(m-1)$-tuple of non-negative integers such that $\sum_{k=2}^{m}(k-1)d_k = 2n(l-1)+2$. By Proposition 3.7 again, the locus $V_{1^{d_2}, 2^{d_3}, \dots, (m-1)^{d_m}} \subset H_r$ of points corresponding to curves with d_k singularities of type A_{k-1}, for every k, is a reduced (possibly reducible) curve intersecting Γ_r only at $\underline{0}$. It follows that the locus of curves in Ω with $d_2 + n - 1$ nodes and d_k singularities of type A_{k-1} for every $k \geq 3$ is a reduced (possibly reducible) curve. Parametric equations of this curve may be explicitly computed for selected values of $d_2, \dots, d_m$ (see, e.g., Remark 3.14, Example 3.16 and Corollary 3.17) by arguing exactly as in [1, proof of Lemma 4.4, pp. 381–382]. By versality, the curve $\pi^*(C)$ may be deformed to a curve $C_t \in |\mathcal{O}_{\mathcal{X}_t}(nH_t)|$, preserving all nodes of $\pi^*(C)$ on $R \setminus E$ except q and deforming every simple tacnode of $\pi^*(C)$ to a node and the $(2n(l-1)+3)$-tacnode at r to d_k singularities of type A_{k-1}. As before, by the fact that $h^0(\pi^*(C), \mathcal{N}'_{\pi^*(C)|\mathcal{X}}) = 0$ and properties [8, (3.5) and (3.6)] of versal deformation families, we obtain that $\dim(T_{[C_t]}ES(C_t)) = 0$, for a general t.

Case 2. We finally consider the cases $(n, \mathrm{p}) = (2,3)$ and $(2,4)$. Let E be a general elliptic normal curve of degree $\mathrm{p}+1$ in $\mathbb{P}^{\mathrm{p}}$ and $R_1 = Q_1$ and $R_2 = Q_2$ be two general rational normal scrolls intersecting transversally along E. Let $X \to T$ be a one-parameter family of very general primitively polarized $K3$ surfaces with special fiber X_0 and double points $p_1, \dots, p_{16} \in E$. Consider on $X_0 := Q_1 \cup Q_2$ the curve $\cup_{i=1}^{2} \cup_{j=1}^{n} C_j^i$ constructed in [7, Section 3.2]. The theorem follows in this case by studying deformations of this curve by the same techniques as before. Details are left to the reader. □

In the case $n = 1$, Theorem 1.1 proves that the variety $\mathcal{V}^S_{H,1^{d_2},2^{d_3},\dots,(m-1)^{d_m}}$ is non-empty whenever it has non-negative expected dimension. The precise statement is the following.

Corollary 4.1. *Let (S, H) be a general primitively polarized $K3$ surface of genus $\mathrm{p} = \mathrm{p}_a(H)$. Then, for every $(m-1)$-tuple of non-negative integers $d_2, \dots, d_m$ such that*

$$\sum_{j=2}^{m} (j-1)d_j \leq \dim(|H|) = \mathrm{p}, \tag{39}$$

there exist reduced irreducible curves C in the linear system $|H|$ on S having d_j singularities of type A_{j-1} for every $j = 2, \dots, m$, and no further singularities and corresponding to regular points of their equisingular deformation locus $ES(C_t)$. Equivalently, $\dim(T_{[C_t]}ES(C_t)) = \dim(|H|) - \sum_{j=2}^{m}(j-1)d_j$. *In particular, the variety $\mathcal{V}^S_{H,1^{d_2},2^{d_3},\dots,(m-1)^{d_m}}$ is non-empty whenever* (39) *is satisfied.*

The previous result is optimal for $g = 0$, because, by [6], we know that all the rational curves in $|H|$ are nodal (and nodes are the worst expected singularities of a rational curve in $|H|$). Theorem 1.1 also proves the existence of divisors in $\mathcal{V}^S_{nH,1^\delta}$, parametrizing curves with a tacnode or a cusp and nodes, whenever they have non-negative expected dimension.

Corollary 4.2. *Let (S, H) be a general primitively polarized $K3$ surface of genus $\mathrm{p} = \mathrm{p}_a(H) \geq 3$ and let $\delta \leq \mathrm{p} - 1$. Then the Severi variety $\mathcal{V}^S_{nH,1^\delta}$ of reduced and irreducible δ-nodal curves contains two non-empty generically smooth divisors V_{tac} and V_c, whose general point of every irreducible component corresponds to a curve with a simple tacnode and $\delta - 2$ nodes and an ordinary cusp and $\delta - 1$ nodes, respectively. In particular, the varieties $\mathcal{V}^S_{nH,1^{\delta-2},3^1}$ and $\mathcal{V}^S_{nH,1^{\delta-1},2^1}$, are non-empty.*

Remark 4.3. The existence of a further non-empty generically smooth divisor $V_{trip} \subset \mathcal{V}^S_{nH,1^\delta}$, whose general element in every irreducible component corresponds to a curve with a triple point and $\delta - 3$ nodes, has been proved in [11, Corollary 4.2] under the assumption $(n, \mathrm{p}) \neq (1, 4)$. The case $(n, \mathrm{p}) = (1, 4)$ has been studied in [12, Proposition 2.2]. It is unknown if $\mathcal{V}^S_{nH,1^\delta}$ may contain divisors W different from V_{trip}, V_{tac} and V_c and parametrizing curves with singularities different than nodes.

We finally observe that Theorem 1.1 together with Proposition 1.2 provide sufficient conditions for the variety $\mathcal{V}^S_{nH,1^{d_2},2^{d_3},\dots,(m-1)^{d_m}}$ to be non-empty and regular. In the following remark we write explicitly the corresponding existence and regularity condition for $n = 1$. The case $n \geq 2$ is left to the reader.

Remark 4.4. By Theorem 1.1 and Proposition 1.2 in the case $n = 1$, we have that, if (S, H) is a general primitively polarized $K3$ surface of genus p and

$$\sum_{j=2}^{m}(j-1)d_j < \frac{\mathrm{p}+2}{2}, \tag{40}$$

then the variety $\mathcal{V}^S_{H,1^{d_2},2^{d_3},\dots,(m-1)^{d_m}} \subset |H|$ is non-empty and regular (cf. Definition 2.3 and Remark 2.4). This condition of existence and regularity is certainly an improvement of (6) but is not optimal. Indeed, by Mumford and Tannebaum [22] and [25], we know that the Severi variety of δ-nodal curves in $|H|$ is non-empty and regular for every $\delta \leq \mathrm{p}$. When (40) is not satisfied, the existence of irreducible components $V \subset \mathcal{V}^S_{H,1^{d_2},2^{d_3},\dots,(m-1)^{d_m}}$ of dimension bigger that the expected would imply the reducibility of the variety.

Appendix A: Proof of Lemma 3.10

The proof is by induction on m.

Base case of the induction. We first prove the lemma in the special case of an admissible $2(n-1)$-tuple satisfying $\sum_{j=2}^n d_j^- = 1$ or $\sum_{j=2}^n d_j^+ = 1$. So assume, by symmetry, that $\sum_{j=2}^n d_j^- = 1$. Then there is an index i_0 such that $d_{i_0}^- = 1$ and $d_j^- = 0$ for all $j \neq i_0$. The question is whether there is a permutation τ^+ of cyclic structure $\Pi_{j=2}^n j^{d_j^+}$ and a cycle $\tau^- = \sigma_{i_0}$ of order i_0 such that $\tau^+\sigma_{i_0}$ is cyclic of order m. Let $\sigma_{i_0} = (1\ 2\ \cdots\ i_0)$. By (26), we have $\sum_{j=2}^n (j-1)d_j^+ = m - i_0$. This implies that we can construct a permutation τ^+ of the desired cyclic structure such that each cycle contains precisely one integer in the set $\{1, 2, \dots, i_0\}$. It is then easily seen that $\tau^+(1\ 2\ \cdots\ i_0)$ is an m-cycle.

Induction step. The base cases of the induction are all cases where $\sum_{j=2}^n d_j^- = 1$ or $\sum_{j=2}^n d_j^+ = 1$, which have been treated above. Now let $(d_2^+, d_2^-, \dots, d_n^+, d_n^-)$ be an admissible $2(n-1)$-tuple such that both $\sum_{j=2}^n d_j^{\pm} \geq 2$. By symmetry we may assume that $\sum_{j=2}^n jd_j^- \geq \sum_{j=2}^n jd_j^+$.

Set $i_0 := \min\{j|\ d_j^- > 0\}$. We claim that the $2(n-1)$-tuple

$$(d'^+_2, d'^-_2, \dots, d'^+_{i_0}, d'^-_{i_0}, \dots, d'^+_n, d'^-_n) = (d_2^+, d_2^-, \dots, d_{i_0}^+, d_{i_0}^- - 1, \dots, d_n^+, d_n^-)$$

is admissible. Indeed, set $m' := \sum_{j=2}^n (j-1)(d'^+_j + d'^-_j) + 1 = m - i_0 + 1$. Then $m' \geq 2$ since $\sum_{j=2}^n d'^+_j = \sum_{j=2}^n d_j^+ \geq 2$. Clearly,

$$\sum_{j=2}^n j d'^-_j = \sum_{j=2}^n j d_j^- - i_0 \leq m - i_0 < m - i_0 + 1 = m' \tag{A.1}$$

by (27). Assume that $\sum_{j=2}^n j d'^+_j = \sum_{j=2}^n j d_j^+ > m' = m - i_0 + 1$. Then we have that

$$m - i_0 + 2 \leq \sum_{j=2}^n j d_j^+ \leq \sum_{j=2}^n j d_j^-, \quad \text{whence}$$

$$\begin{aligned} 2m - 2i_0 + 4 &\leq \sum_{j=2}^n j(d_j^+ + d_j^-) \\ &= 2\sum_{j=2}^n (j-1)(d_j^+ + d_j^-) \\ &\quad - \sum_{j=2}^n (j-2)(d_j^+ + d_j^-) \\ &= 2(m-1) - \sum_{j=2}^n (j-2)(d_j^+ + d_j^-), \end{aligned}$$

by (26). It follows that $0 \leq \sum_{j=2}^n (j-2) d_j^- \leq \sum_{j=2}^n (j-2)(d_j^+ + d_j^-) \leq 2i_0 - 6$. In particular, we obtain that $i_0 \geq 3$. Moreover, by definition of i_0, we must have $2(i_0 - 2) \leq (i_0 - 2)\sum_{j=2}^n d_j^- \leq \sum_{j=2}^n (j-2) d_j^- \leq 2i_0 - 6$, getting a contradiction. Therefore, we have proved our claim that the $2(n-1)$-tuple

$$(d'^+_2, d'^-_2, \dots, d'^+_{i_0}, d'^-_{i_0}, \dots, d'^+_n, d'^-_n) = (d_2^+, d_2^-, \dots, d_{i_0}^+, d_{i_0}^- - 1, \dots, d_n^+, d_n^-)$$

is admissible.

By induction, there exist permutations $\tau^\pm$ in the symmetric group $\mathfrak{S}_{m-i_0+1}$ of order $m - i_0 + 1$ of cyclic structures $\Pi_{j=2}^n j^{d'^\pm_j}$, respectively, such that

$$\tau^+ \tau^- = \Big(1\ 2\ \cdots (m - i_0 + 1)\Big).$$

The number of distinct integers from $\{1, 2, \ldots, m - i_0 + 1\}$ appearing in the permutation τ^- is $\sum_{j=2}^{n} j d'^{-}_{j}$, which is less than $m - i_0 + 1$ by (A.1). Hence there exists an $x \in \{1, 2, \ldots, m - i_0 + 1\}$ not appearing in τ^-. Then the permutation

$$\alpha^- = \tau^- \Big((m - i_0 + 2)\ (m - i_0 + 3)\ \cdots\ m\ x \Big)$$

has cyclic structure $\Pi_{j=2}^{n} j^{d_j^-}$ and

$$\tau^+\alpha^- = \Big(1\ 2\ \cdots\ (m - i_0 + 1)\Big)\Big((m - i_0 + 2)\ (m - i_0 + 3)\ \cdots\ m\ x\Big)$$

is cyclic of order m, as desired. □

Remark A.1. In general, given an admissible $(2n-2)$-tuple, we have several conjugacy classes of triples of permutations satisfying Lemma 3.10. For example, if $m = 7$, $d_2^+ = d_3^+ = 1$, $d_4^- = 1$ and $d_j^{\pm} = 0$ otherwise, then the two triples

$$((267)(15), (1234), (1, 672, 345)^{-1}) \text{ and } ((365)(17), (1234), (1, 256, 347)^{-1})$$

satisfy Lemma 3.10 and are not conjugated.

Acknowledgments

The first author is indebted with J. Harris for invaluable conversations on deformation theory of curve singularities. She also benefited from conversations with F. van der Wyck. Both authors want to express deep gratitude to C. Ciliberto and T. Dedieu for many stimulating questions and suggestions. Finally, the authors are very grateful to the referee for his careful reading and many comments. He pointed out a mistake in the original version of the paper and provided a series of suggestions that ultimately improved the exposition and readability.

Both authors want to thank the Department of Mathematics of the University of Calabria and the Department of Mathematics of the University of Bergen for hospitality and for financial support. The first author was also supported by GNSAGA of INdAM and by the PRIN 2008 'Geometria delle varietà algebriche e dei loro spazi di moduli', co-financied by MIUR.

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Received November 4, 2011