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Optimal exponents in weighted estimates without examples

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"This article is dedicated to Prof. Javier Duoandikoetxea in occasion of his 60th birthday"

We present a general approach for proving the optimality of the exponents on weighted estimates. We show that if an operator T satisfies a bound like

$$\|T\|_{L^p(w)} \leq c\,[w]_{A_p}^{\beta} \quad w \in A_p,$$

then the optimal lower bound for β is closely related to the asymptotic behaviour of the unweighted L^p norm $\|T\|_{L^p(\mathbb{R}^n)}$ as p goes to 1 and $+\infty$.

By combining these results with the known weighted inequalities, we derive the sharpness of the exponents, without building any specific example, for a wide class of operators including maximal-type, Calderón–Zygmund and fractional operators. In particular, we obtain a lower bound for the best possible exponent for Bochner–Riesz multipliers. We also present a new result concerning a continuum family of maximal operators on the scale of logarithmic Orlicz functions. Further, our method allows to consider in a unified way maximal operators defined over very general Muckenhoupt bases.

1. Introduction and statement of the main result

1.1. Introduction

A main problem in modern Harmonic Analysis is the study of sharp norm inequalities for some of the classical operators on weighted Lebesgue spaces $L^p(w)$, $1 < p < \infty$. The usual examples include the Hardy–Littlewood (H–L) maximal operator, the Hilbert transform and more general Calderón–Zygmund operators (C–Z operators). Here w denotes a non-negative, locally

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integrable function, that is a weight. The class of weights for which these operators T are bounded on $L^p(w)$ were identified in [17] and in the later works [3, 10]. This class consists of the Muckenhoupt A_p weights defined by the condition

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|} \int_Q w(y)\, dy \right) \left(\frac{1}{|Q|} \int_Q w(y)^{1-p'}\, dy \right)^{p-1} < \infty,$$

where the supremum is taken over all the cubes Q in $\mathbb{R}^n$.

Given any of these operators T, the first part of this problem is to look for quantitative bounds of the norm $\|T\|_{L^p(w)}$ in terms of the A_p constant of the weight. Then, the following step is to find the sharp dependence, typically with respect to the power of $[w]_{A_p}$. In recent years, the answer to this last question has let a fruitful activity and development of new tools in Harmonic Analysis. Firstly, Buckley [1] identified the sharp exponent in the case of the H–L maximal function, i.e.,

$$\|M\|_{L^p(w)} \le c\, [w]_{A_p}^{\frac{1}{p-1}}, \quad w \in A_p \tag{1.1}$$

and $\frac{1}{p-1}$ cannot be replaced with $\frac{1-\varepsilon}{p-1}$, $\varepsilon > 0$. Afterwards, Petermichl [24] showed that

$$\|T\|_{L^p(w)} \le c\, [w]_{A_p}^{\max\{1, \frac{1}{p-1}\}}, \quad w \in A_p \tag{1.2}$$

is sharp when T is any Riesz transform. Similar weighted estimates are known to be true for other classical operators, such as commutators of C–Z operators with BMO functions, the dyadic square function, vector valued maximal operators and fractional integrals. In the case of sharp bounds with respect to the power of the A_p constant of the weight w, the sharpness is most frequently proved by constructing specific examples for each operator.

Throughout the paper, we will use the notation $A \lesssim B$ to indicate that there is a constant $c > 0$ independent of A and B such that $A \le cB$. By $A \sim B$ we mean that both $A \lesssim B$ and $B \lesssim A$ hold.

1.2. Main results

In order to state our main results, we need to introduce the notion of endpoint order for a given operator T. To illustrate the aim of the next definition, consider the following example. Let H be the Hilbert transform.

Then, it is known that the size of its kernel implies (see [25, p. 42]) that the unweighted L^p norm satisfies

$$\|H\|_{L^p(\mathbb{R}^n)} \sim \frac{1}{p-1}. \tag{1.3}$$

The next definition tries to capture this *endpoint order* by looking at the asymptotic behaviour of the L^p norm of a general operator T.

Definition 1.1. Given a bounded operator T on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, we define α_T to be the "endpoint order" of T as follows:

$$\alpha_T =: \sup\left\{\alpha \geq 0 : \forall \varepsilon > 0, \limsup_{p\to 1} (p-1)^{\alpha-\varepsilon}\|T\|_{L^p(\mathbb{R}^n)} = \infty\right\}. \tag{1.4}$$

The analogue of (1.4) for p large is the following. Let γ_T be defined as follows:

$$\gamma_T =: \sup\left\{\gamma \geq 0 : \forall \varepsilon > 0, \limsup_{p\to\infty} \frac{\|T\|_{L^p(\mathbb{R}^n)}}{p^{\gamma-\varepsilon}} = \infty\right\}. \tag{1.5}$$

This definition may have appeared previously in the literature but we are not aware of it.

Now, we can state our main result.

Theorem 1.2. *Let T be an operator (not necessarily linear). Suppose further that for some $1 < p_0 < \infty$ and for any $w \in A_{p_0}$*

$$\|T\|_{L^{p_0}(w)} \leq c\,[w]_{A_{p_0}}^{\beta}. \tag{1.6}$$

Then $\beta \geq \max\left\{\gamma_T; \frac{\alpha_T}{p_0-1}\right\}$.

The novelty here is that we can exhibit a close connection between the weighted estimate and the unweighted behaviour of the operator at the endpoints $p = 1$ and $p = \infty$.

As an application of the method of proof we can derive a lower bound for the optimal exponent that one could expect in a weighted estimate for a maximal operator associated with a generic Muckenhoupt basis $M_{\mathcal{B}}$ (see Section 4). We note that it is not even possible to have an example working for a general basis. The only requirement on the operator $M_{\mathcal{B}}$ is that its L^p norm must blow up when p goes to 1 (no matter the ratio of blow up). Precisely, we have the following theorem.

Theorem 1.3. *Let $\mathcal{B}$ be a Muckenhoupt basis. Suppose in addition that the associated maximal operator $M_{\mathcal{B}}$ satisfies the following weighted estimate:*

$$\|M_{\mathcal{B}}\|_{L^{p_0}(w)} \leq c\,[w]^{\beta}_{A_{p_0,\mathcal{B}}}. \tag{1.7}$$

If $\limsup_{p\to 1^+} \|M_{\mathcal{B}}\|_{L^p(\mathbb{R}^n)} = +\infty$, *then* $\beta \geq \frac{1}{p_0-1}$.

We also obtain new results for a class of maximal functions defined in terms of Orlicz averages (see Section 3.3). For $\Phi_\lambda(t) = t\log(e+t)^\lambda$, $\lambda \in [0,\infty)$, we prove new weighted estimates for the Orlicz maximal operator M_{Φ_λ} which, in addition, are sharp as a consequence of Theorem 1.2. These operators can be seen as continuous versions of the iterated Hardy–Littlewood maximal function. This continuity is reflected in the exponent of the weighted estimates proved in Theorem 3.2. The operators M_{Φ_λ} are relevant in many situations, in particular for the study of the so called "A_p bump conjectures" (see [5, p.187]).

Even in the case where it is not known a sharp weighted estimate, we obtain a lower bound for the exponent of the A_p constant. This is the case of Bochner–Riesz multipliers treated in Section 3.1 and Corollary 3.1.

1.3. Outline

This article is organized as follows. In Section 2, we prove the main result. Then, in Section 3, we show how to derive the sharpness of some weighted estimates for several classical operators. Finally, in Section 4, our method is used to obtain optimal exponents in the case of maximal functions defined over general Muckenhoupt bases.

2. Proof of Theorem 1.2

We present here the proof of the main result. The key tool is the Rubio de Francia's iteration scheme or algorithm to produce A_1 weights with a precise control of the constant of the weight and the main underlying idea comes from extrapolation theory. The same ideas that we use here were already used to prove sharp weighted estimates for the Hilbert transform with A_1 weights in [8]. A more precise and general version was obtained recently in [6]. We remark that the first part of the proof, namely the proof of inequality (2.1) below, is a consequence of the extrapolation result from [6] (see Theorem 3.1, first inequality of (3.2), p. 1889). We choose to include the proof for the sake of completeness. For our inequality (2.4), which is the analogue for large p, we perform a slightly different proof.

Proof of Theorem 1.2. We first consider the bound $\beta \geq \frac{\alpha_T}{p_0-1}$. The first step is to prove the following inequality, which can be seen as an unweighted Coifman–Fefferman type inequality relating the operator T to the Hardy–Littlewood maximal function. We have that

$$\|T\|_{L^p(\mathbb{R}^n)} \leq c\,\|M\|_{L^p(\mathbb{R}^n)}^{\beta(p_0-p)} \quad 1 < p < p_0. \tag{2.1}$$

Lets start by defining, for $1 < p < p_0$, the operator R as follows:

$$R(h) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{M^k(h)}{\|M\|_{L^p(\mathbb{R}^n)}^k}.$$

Then we have

(A) $h \leq R(h)$,

(B) $\|R(h)\|_{L^p(\mathbb{R}^n)} \leq 2\,\|h\|_{L^p(\mathbb{R}^n)}$,

(C) $[R(h)]_{A_1} \leq 2\,\|M\|_{L^p(\mathbb{R}^n)}$.

To verify (2.1), consider $1 < p < p_0$ and apply Holder's inequality to obtain

$$\begin{aligned}
\|T(f)\|_{L^p(\mathbb{R}^n)} &= \Big(\int_{\mathbb{R}^n} |Tf|^p \,(Rf)^{-(p_0-p)\frac{p}{p_0}}\,(Rf)^{(p_0-p)\frac{p}{p_0}}\,dx \Big)^{1/p} \\
&\leq \Big(\int_{\mathbb{R}^n} |Tf|^{p_0}\,(Rf)^{-(p_0-p)}\,dx \Big)^{1/p_0} \Big(\int_{\mathbb{R}^n} (Rf)^p\,dx \Big)^{\frac{p_0-p}{pp_0}}.
\end{aligned}$$

For clarity in the exposition, we denote $w := (Rf)^{-(p_0-p)}$. Then, by the key hypothesis (1.6) together with properties (A) and (B) of the Rubio de Francia's algorithm, we have that

$$\begin{aligned}
\|T(f)\|_{L^p(\mathbb{R}^n)} &\leq c\,[w]_{A_{p_0}}^{\beta} \Big(\int_{\mathbb{R}^n} |f|^{p_0}\,w\,dx \Big)^{1/p_0} \|f\|_{L^p(\mathbb{R}^n)}^{\frac{p_0-p}{p_0}} \\
&\leq c\,[w]_{A_{p_0}}^{\beta} \Big(\int_{\mathbb{R}^n} |f|^{p}\,dx \Big)^{1/p_0} \|f\|_{L^p(\mathbb{R}^n)}^{1-\frac{p}{p_0}} \\
&= c\,[w]_{A_{p_0}}^{\beta}\,\|f\|_{L^p(\mathbb{R}^n)} \\
&= c\,[w^{1-p_0'}]_{A_{p_0'}}^{\beta(p_0-1)}\|f\|_{L^p(\mathbb{R}^n)},
\end{aligned}$$

since $[w]_{A_q} = [w^{1-q'}]_{A_{q'}}^{q-1}$. Now, since $\frac{p_0-p}{p_0-1} < 1$ we can use Jensen's inequality to compute the constant of the weight as follows:

$$[w^{1-p_0'}]_{A_{p_0'}} = \left[(Rf)^{\frac{p_0-p}{p_0-1}}\right]_{A_{p_0'}} \leq [R(f)]_{A_{p_0'}}^{\frac{p_0-p}{p_0-1}} \leq [R(f)]_{A_1}^{\frac{p_0-p}{p_0-1}}.$$

Finally, by making use of property (C), we conclude that

$$\|T(f)\|_{L^p(\mathbb{R}^n)} \leq c\,\|M\|_{L^p(\mathbb{R}^n)}^{\beta(p_0-p)}\,\|f\|_{L^p(\mathbb{R}^n)},$$

which clearly implies (2.1). Once we have proved the key inequality (2.1), we can relate the exponent on the weighted estimate to the endpoint order of T. To that end, we will use the known asymptotic behaviour of the unweighted L^p norm of the maximal function. It is well known that when p is close to 1, there is a dimensional constant c such that

$$\|M\|_{L^p(\mathbb{R}^n)} \leq c\,\frac{1}{p-1}. \tag{2.2}$$

Then, for p close to 1, we obtain

$$\|T\|_{L^p(\mathbb{R}^n)} \leq c\,(p-1)^{-\beta(p_0-p)} \leq c\,(p-1)^{-\beta(p_0-1)}. \tag{2.3}$$

Therefore, multiplying by $(p-1)^{\alpha_T-\varepsilon}$, using the definition of α_T and taking upper limits we have

$$+\infty = \limsup_{p\to 1}\,(p-1)^{\alpha_T-\varepsilon}\|T\|_{L^p(\mathbb{R}^n)} \leq c\,\limsup_{p\to 1}\,(p-1)^{\alpha_T-\varepsilon-\beta(p_0-1)}.$$

This last inequality implies that $\beta \geq \frac{\alpha_T}{p_0-1}$, so we conclude the first part of the proof of the theorem.

For the proof of the other inequality, $\beta \geq \gamma_T$, we follow the same line of ideas, but with a twist involving the dual space $L^{p'}(\mathbb{R}^n)$. Fix p, $p > p_0$. We perform the iteration technique R' as before changing p with p':

$$R'(h) = \sum_{k=0}^{\infty} \frac{1}{2^k}\,\frac{M^k(h)}{\|M\|_{L^{p'}(\mathbb{R}^n)}^k}.$$

Then we have

(A′) $h \leq R'(h)$,

(B′) $\|R'(h)\|_{L^{p'}(\mathbb{R}^n)} \leq 2\,\|h\|_{L^{p'}(\mathbb{R}^n)}$,

(C′) $[R'(h)]_{A_1} \leq 2\,\|M\|_{L^{p'}(\mathbb{R}^n)}$.

Fix $f \in L^p(\mathbb{R}^n)$. By duality there exists a non-negative function $h \in L^{p'}(\mathbb{R}^n)$, $\|h\|_{L^{p'}(\mathbb{R}^n)} = 1$, such that,

$$\begin{aligned}
\|Tf\|_{L^p(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |Tf(x)| h(x)\, dx \\
&\le \int_{\mathbb{R}^n} |Tf| (R'h)^{\frac{p-p_0}{p_0(p-1)}}\, h^{\frac{p(p_0-1)}{p_0(p-1)}}\, dx \\
&\le \left(\int_{\mathbb{R}^n} |Tf|^{p_0} (R'h)^{\frac{p-p_0}{p-1}}\, dx \right)^{1/p_0} \left(\int_{\mathbb{R}^n} h^{p'}\, dx \right)^{1/p_0'} \\
&= \left(\int_{\mathbb{R}^n} |Tf|^{p_0} (R'h)^{\frac{p-p_0}{p-1}}\, dx \right)^{1/p_0}.
\end{aligned}$$

Now, we use the key hypothesis (1.6) and Hölder's inequality to obtain

$$\begin{aligned}
\|Tf\|_{L^p(\mathbb{R}^n)} &\le c\, [(R'h)^{\frac{p-p_0}{p-1}}]^{\beta}_{A_{p_0}} \left(\int_{\mathbb{R}^n} |f|^{p_0} (R'h)^{\frac{p-p_0}{p-1}}\, dx \right)^{1/p_0} \\
&\le c\, [(R'h)^{\frac{p-p_0}{p-1}}]^{\beta}_{A_{p_0}} \left(\int_{\mathbb{R}^n} |f|^{p} dx \right)^{1/p} \left(\int_{\mathbb{R}^n} (R'h)^{p'}\, dx \right)^{\frac{1}{p'}\frac{p-p_0}{p_0(p-1)}} \\
&\le c\, [(R'h)^{\frac{p-p_0}{p-1}}]^{\beta}_{A_{p_0}} \left(\int_{\mathbb{R}^n} |f|^{p} dx \right)^{1/p} \quad \text{by (B}'\text{).} \\
&\le c\, [R'h]^{\beta\frac{p-p_0}{p-1}}_{A_{p_0}} \left(\int_{\mathbb{R}^n} |f|^{p} dx \right)^{1/p} \quad \text{by Jensen's} \\
&\le c\, \|M\|^{\beta\frac{p-p_0}{p-1}}_{L^{p'}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |f|^{p} dx \right)^{1/p} \quad \text{by (C}'\text{).}
\end{aligned}$$

Hence,

$$\|T\|_{L^p(\mathbb{R}^n)} \le c\, \|M\|^{\beta\frac{p-p_0}{p-1}}_{L^{p'}(\mathbb{R}^n)} \quad p > p_0. \tag{2.4}$$

This estimate is similar and, somehow dual, to (2.1). To finish the proof we recall that, for large p, namely $p > p_1 > p_0$, we have the asymptotic estimate, $\|M\|_{L^{p'}(\mathbb{R}^n)} \sim \frac{1}{p'-1} \le p$. Therefore, we have that

$$\|T\|_{L^p(\mathbb{R}^n)} \le c\, p^{\beta\frac{p-p_0}{p-1}} \le c\, p^{\beta}.$$

Since $p > p_1 > p_0 > 1$. As before, dividing by $p^{\gamma_T - \varepsilon}$ and taking upper limits, we obtain

$$+\infty = \limsup_{p\to\infty} \frac{\|T\|_{L^p(\mathbb{R}^n)}}{p^{\gamma_T-\varepsilon}} \le c\, \limsup_{p\to\infty} p^{\beta-\gamma_T+\varepsilon}.$$

This last inequality implies that $\beta \geq \gamma_T$, so we conclude the proof of the theorem. $\square$

Remark 2.1. The techniques used in the proof of Theorem 1.2 actually allow us to deduce sharper results for some particular cases. For the H–L maximal function M, by considering the indicator function of the unit cube, it is easy to conclude that

$$\|M\|_{L^p(\mathbb{R}^n)} \sim (p-1)^{-1}, \tag{2.5}$$

for p close to 1. This precise endpoint behaviour allows us to prove that we cannot replace in the weighted inequality (1.1) the function $t \mapsto t^{(p-1)^{-1}}$ by any other *smaller* growth function φ. To be more precise, the following inequality fails:

$$\|M\|_{L^p(w)} \leq c\,\varphi([w]_{A_p})$$

for any non-decreasing function $\varphi : [0,\infty) \to [0,\infty)$ such that

$$\lim_{t\to\infty} \frac{\varphi(t)}{t^{\frac{1}{p-1}}} = 0.$$

The proof follows the same ideas of Theorem 1.2. We left the details for the interested reader. A similar argument can be used to derive an analogue result for a generic operator T if it is known the precise endpoint behaviour of T.

3. Applications

In this section, we show how to derive from our general result in Theorem 1.2 the sharpness of several known weighted inequalities. This will follow from Theorem 1.2 if we check the appropriate values of α_T and γ_T for each case.

3.1. Operators with large kernel and commutators

Consider any C–Z operator whose kernel K satisfies

$$|K(x,y)| \geq \frac{c}{|x-y|^n} \tag{3.1}$$

for some $c > 0$ and if $x \neq y$ (we can consider the Hilbert transform H as a model example of this phenomenon in $\mathbb{R}$ and the Riesz transforms for $\mathbb{R}^n$,

$n \geq 2$). Then, it is true (see [25, p. 42]) that, for $p \to 1$,

$$\|T\|_{L^p(\mathbb{R}^n)} \sim \frac{1}{p-1}, \tag{3.2}$$

which clearly implies that $\alpha_T = 1$. By duality we can see that $\gamma_T = 1$. Further, for the commutator $[b, T]$ we use the example from [23, Section 5, p. 755]. There, for the choice of $b(x) = \log(|x|)$ and considering the Hilbert transform H, it is shown that

$$\|[b, H]\|_{L^p(\mathbb{R}^n)} \gtrsim \frac{1}{(p-1)^2}, \tag{3.3}$$

which implies that $\alpha_{[b,H]} = 2$. More generally, its k-iteration defined recursively by

$$T_b^k := [b, T_b^{k-1}], \quad k \in \mathbb{N},$$

satisfies that $\alpha_{H_b^k} = \gamma_{H_b^k} = k+1$. The value for $\gamma_{H_b^k}$ follows by duality as in the case of C–Z operators.

We then obtain, as an immediate consequence of Theorem 1.2, that the following known weighted inequalities are sharp (for the proofs, see [11] for the case of C–Z operators and [2] for the case of commutators):

$$\|T\|_{L^p(w)} \leq c\,[w]_{A_p}^{\max\{1, \frac{1}{p-1}\}}, \quad w \in A_p, \tag{3.4}$$

$$\|[b, T]\|_{L^p(w)} \leq c\,\|b\|_{\mathrm{BMO}}\,[w]_{A_p}^{2\max\{1, \frac{1}{p-1}\}}, \quad w \in A_p, \tag{3.5}$$

$$\|T_b^k\|_{L^p(w)} \leq c\,\|b\|_{\mathrm{BMO}}\,[w]_{A_p}^{(k+1)\max\{1, \frac{1}{p-1}\}}, \quad w \in A_p. \tag{3.6}$$

As a final application of this result for large kernels, we present here the following consequence of our Theorem 1.2 for the optimality of weighted estimates of Bochner–Riesz multipliers. For $\lambda > 0$ and $R > 0$, this operator is defined by the formula

$$(B_R^\lambda f)(x) = \int_{\mathbb{R}^n} \Big(1 - (|\xi|/R)^2\Big)_+^\lambda \hat{f}(\xi) e^{2\pi i \xi x}\, d\xi, \tag{3.7}$$

where $\hat{f}$ denotes the Fourier transform of f. For $R = 1$ we write simply B^λ. It is a known fact that this operator has a kernel $K_\lambda(x)$ defined by

$$K_\lambda(x) = \frac{\Gamma(\lambda+1)}{\pi^\lambda} \frac{J_{n/2+\lambda}(2\pi|x|)}{|x|^{n/2+\lambda}}, \tag{3.8}$$

where Γ is the Gamma function and J_η is the Bessel function of integral order η (see [9, p. 352]).

Corollary 3.1. *Let $1 < p < \infty$. Suppose further that the following estimate holds*

$$\|B^{(n-1)/2}\|_{L^p(w)} \leq c\,[w]_{A_p}^{\beta}, \tag{3.9}$$

for any $w \in A_p$ and where the constant c is independent of the weight. Then $\beta \geq \max\left\{1; \frac{1}{p-1}\right\}$.

Proof. We use the known asymptotics for Bessel functions, namely

$$J_\eta(r) = cr^{-1/2}\cos(r-\tau) + O(r^{-3/2})$$

for some constants $c, \tau > 0$, $\tau = \tau_\eta$, and $r > r_0 \gg 1$ (see [25, p. 338, Example 1.4.1, equation (14)]). Combining this with (3.8), we obtain that

$$K_{(n-1)/2}(x) \sim \frac{\cos(|x|-\tau) + \varphi(|x|)}{|x|^n} \tag{3.10}$$

for some $\varphi : \mathbb{R} \to \mathbb{R}$ such that $|\varphi(r)| \lesssim r^{-1}$. We see that this kernel does not satisfy the size condition (3.1). However, (3.10) is sufficient to conclude that $\alpha_{B^{(n-1)/2}} = \gamma_{B^{(n-1)/2}} = 1$. Testing on the indicator function of the unit cube (we use again [25, p. 42]) we obtain, after a change of variables and for some $r_1 \geq r_0$,

$$\|B^{(n-1)/2}\|_{L^p(\mathbb{R}^n)}^p \gtrsim \int_{r>r_1} \frac{|\cos(r-\tau)+\varphi(r)|^p}{r^p}\, dr.$$

We choose $r_2 \geq r_1$ large enough such that $|\varphi(r)| < 1/4$ and consider the set $A = \{r \in R : r > r_2, |\cos(r-\tau)| > 1/2\}$. We obtain that

$$\|B^{(n-1)/2}\|_{L^p(\mathbb{R}^n)}^p \gtrsim \int_A \frac{1}{r^p}\, dr \gtrsim \int_{r>1} \frac{1}{r^p}\, dr \gtrsim \frac{1}{p-1}$$

for p close to 1. The estimate in the middle follows by the monotonicity of the function $t \mapsto t^{-p}$ and taking into account that we can find the exact description of the set A as a union of intervals. The value for $\gamma_{B^{(n-1)/2}} = 1$ follows by duality. □

In particular, this result shows that the claimed weighted norm inequality for the maximal Bochner–Riesz operator from [15] cannot hold (see also [16]).

3.2. Maximal operators and square functions

For $k \in \mathbb{N}$ the kth iteration of the maximal function is defined by $M^k = M(M^{k-1})$. In this case, we have that $\alpha_{M^k} = k$. The case $k = 1$ is (2.5) and an induction argument yields the case $k > 1$. The fact that $\gamma_{M^k} = 0$ is trivial. Then the following weighted inequality is sharp:

$$\|M^k\|_{L^p(w)} \leq c\,[w]_{A_p}^{\frac{k}{p-1}}, \quad w \in A_p. \tag{3.11}$$

We now consider the vector-valued extension of the H–L maximal function. For $1 < q < \infty$ and $1 < p < \infty$, this operator is defined as:

$$\overline{M}_q f(x) = \left(\sum_{j=1}^{\infty} (Mf_j(x))^q\right)^{1/q},$$

where $f = \{f_j\}_{j=1}^{\infty}$ is a vector-valued function. Here, as usual, we adopt the notation $\overline{f}_q := \left(\sum_{j=1}^{\infty} f_j^q\right)^{1/q}$. The fact that $\alpha_{\overline{M}_q} = 1$ can be verified in the same way as in the case $q = 1$. For $\gamma_{\overline{M}_q}$, we can find an example of a vector-valued function satisfying $\|\overline{M}_q f\|_{L^p(\mathbb{R}^n)} \geq cp^{1/q}\|\overline{f}_q\|_{L^p(\mathbb{R}^n)}$ which implies that $\gamma_{\overline{M}_q} = 1/q$. This is already known; see [25, p. 75] for the classic proof. Then the following inequality is sharp:

$$\|\overline{M}_q f\|_{L^p(w)} \leq c\,[w]^{\max\{\frac{1}{q},\frac{1}{p-1}\}}\|\overline{f}_q\|_{L^p(w)}, \quad w \in A_p. \tag{3.12}$$

We include here the case of the dyadic square function S_d defined as

$$S_d f(x) = \left(\sum_{Q\in\Delta} (f_Q - f_{\hat{Q}})^2 \chi_Q(x)\right)^{1/2},$$

where $f_Q = ⨍_Q f(x)\,dx$, Δ is the lattice of dyadic cubes and $\hat{Q}$ stands for the dyadic parent of a given cube Q. We first note that $\alpha_{S_d} = 1$ by looking at the indicator function of the unit cube (as in the case of the maximal function). The value of $\gamma_{S_d} = \frac{1}{2}$ was previously known, see for instance [4, p. 434]. As before, we conclude that the following inequality is sharp:

$$\|S_d f\|_{L^p(w)} \leq c\,[w]_{A_p}^{\max\{\frac{1}{2},\frac{1}{p-1}\}}\|f\|_{L^p(w)}, \quad w \in A_p. \tag{3.13}$$

The proof of inequalities (3.13) and (3.12) can be found in [4].

3.3. Orlicz-type maximal functions

Given a Young function Φ, we define the maximal function as

$$M_\Phi f(x) = \sup_{x\in Q} \|f\|_{\Phi,Q}, \tag{3.14}$$

where $\|f\|_{\Phi,Q}$ is the localized Luxemburg norm on a cube Q. We refer to [5, p. 97] for the precise definitions and properties.

We are interested here in the logarithmic scale given by the functions $\Phi_\lambda(t) := t\log^\lambda(e+t)$, $\lambda \in [0,\infty)$. Note that the case $\lambda = 0$ corresponds to M. The case $\lambda = k \in \mathbb{N}$ corresponds to $M_{L(\log L)^k}$, which is pointwise comparable to M^{k+1} (see, for example, [21]). For non-integer values of λ, we denote by $M_{\Phi_\lambda} = M_{L(\log L)^\lambda}$ the associated maximal operator. We have seen that the sharp exponent in weighted estimates for these operators is $1/(p-1)$ for $\lambda = 0$ and $k/(p-1)$ for $\lambda = k \in \mathbb{N}$. The following theorem provides a sharp bound for these intermediate exponents in $\mathbb{R}_+ \setminus \mathbb{N}$. This theorem is a mixed $A_p - A_\infty$ result involving the Fujii–Wilson A_∞'s constant defined as

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(\chi_Q w)\, dx.$$

Theorem 3.2. *Let $\lambda > 0$, $1 < p < \infty$ and $w \in A_p$. Then*

$$\|M_{\Phi_\lambda}\|_{L^p(w)} \leq c\,[w]_{A_p}^{\frac{1}{p}} [\sigma]_{A_\infty}^{\frac{1}{p}+\lambda}, \tag{3.15}$$

where $\sigma = w^{1-p'}$. As a consequence we have

$$\|M_{\Phi_\lambda}\|_{L^p(w)} \leq c\,[w]_{A_p}^{\frac{1+\lambda}{p-1}}.$$

Furthermore, the exponent is sharp.

Results of this type were proved for first time in [12] and it was used to improve the A_2 theorem from [11].

Proof. We start with the following *weak* (p,p) inequality for the Orlicz maximal function M_Φ. For $t > 0$ and any nonnegative function f, we have that

$$w\left(\{x \in \mathbb{R}^n : M_{\Phi_\lambda} f(x) > t\}\right) \leq c\,[w]_{A_p} \int_{\mathbb{R}^n} \Phi_\lambda\left(\frac{f(x)}{t}\right)^p w(x)\, dx. \tag{3.16}$$

for any $w \in A_p$. The proof follows by standard methods, we first observe that it is enough to assume that $t = 1$. Then, using that for any Φ

$$1 < \|f\|_{\Phi,Q} \qquad \text{if and only if} \qquad 1 < \frac{1}{|Q|}\int_Q \Phi(f),$$

we have that

$$\{x \in \mathbb{R}^n : M_\Phi f(x) > 1\} = \{x \in \mathbb{R}^n : M(\Phi(f))(x) > 1\}.$$

reducing everything to M and then the weak (p,p) estimate follows by well known estimates.

Now, we follow the same ideas from [13, Theorem 1.3]. We write the L^p norm as

$$\|M_{\Phi_\lambda} f\|_{L^p(w)}^p \le c\int_0^\infty t^p w\{x \in \mathbb{R}^n : M_{\Phi_\lambda} f_t(x) > t\}\frac{dt}{t},$$

where $f_t := f\chi_{f>t}$. Since $w \in A_p$, then by the precise open property of A_p classes, we have that $w \in A_{p-\varepsilon}$ where $\varepsilon \sim \frac{1}{[\sigma]_{A_\infty}}$. Moreover, the constants satisfy that $[w]_{A_{p-\varepsilon}} \le c[w]_{A_p}$ (see [13, Theorem 1.2]). We apply (3.16) with $p-\varepsilon$ instead of p to obtain after a change of variable

$$\begin{aligned}
\|M_{\Phi_\lambda} f\|_{L^p(w)}^p &\le c\,[w]_{A_p}\int_{\mathbb{R}^n} f^p \int_1^\infty \frac{\Phi_\lambda(t)^{p-\varepsilon}}{t^p}\frac{dt}{t}\; w\,dx \\
&\le c\,[w]_{A_p}\int_1^\infty \frac{(\log(e+t))^{p\lambda}}{t^\varepsilon}\frac{dt}{t}\;\|f\|_{L^p(w)}^p \\
&\le c\,[w]_{A_p}\left(\frac{1}{\varepsilon}\right)^{\lambda p+1}\|f\|_{L^p(w)}^p \\
&\le c\,[w]_{A_p}[\sigma]_{A_\infty}^{\lambda p+1}\;\|f\|_{L^p(w)}^p.
\end{aligned}$$

Taking p-roots we obtain the desired estimate (3.15).

Regarding the sharpness, we will prove now that the exponent in the term on the right hand side (3.15) cannot be improved. This follows from Theorem 1.2 since it is easy to verify (again by testing on the indicator of the unit cube) that

$$\|M_{\Phi_\lambda}\|_{L^p(\mathbb{R}^n)} \sim \frac{1}{(p-1)^{1+\lambda}}.$$

From this estimate, we conclude that the endpoint order verifies $\alpha_T = 1+\lambda$ for $T = M_{\Phi_\lambda}$. □

3.4. Fractional integral operators

For $0<\alpha<n$, the fractional integral operator or Riesz potential I_α is defined by

$$I_\alpha f(x) = \int_{R^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

We also consider the related fractional maximal operator M_α given by

$$M_\alpha f(x) = \sup_{Q\ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)|\, dy.$$

It is well known (see [18]) that these operators are bounded from $L^p(w^p)$ to $L^q(w^q)$ if and only if the exponents p and q are related by the equation $1/q - 1/p = \alpha/n$ and w satisfies the so called $A_{p,q}$ condition. More precisely, $w \in A_{p,q}$ if

$$[w]_{A_{p,q}} := \sup_Q \left(\frac{1}{|Q|}\int_Q w^q\ dx\right)\left(\frac{1}{|Q|}\int_Q w^{-p'}\ dx\right)^{q/p'} < \infty.$$

We first note that

$$\|M_\alpha\|^q_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} \gtrsim \frac{1}{q-\frac{n}{n-\alpha}}.$$

This can be seen again by considering the indicator of the unit cube. Now we can use an off-diagonal version of the extrapolation theorem for $A_{p,q}$ classes from [6, Theorem 5.1]. Then we obtain, by the same line of ideas from Theorem 1.2, that the following inequality is sharp:

$$\|M_\alpha\|_{L^p(w^p)\to L^q(w^q)} \leq c\,[w]_{A_{p,q}}^{\frac{p'}{q}(1-\frac{\alpha}{n})} \tag{3.17}$$

for $0\leq \alpha < n$, $1<p<n/\alpha$ and q is defined by the relationship $1/q = 1/p - \alpha/n$ and $w \in A_{p,q}$.

For the case of the fractional integral, we can easily compute that

$$\|I_\alpha\|^q_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} \geq \frac{1}{q-\frac{n}{n-\alpha}}.$$

Then, arguing as above we conclude that the following weighted inequality is also sharp:

$$\|I_\alpha\|_{L^p(w^p)\to L^q(w^q)} \leq c\,[w]_{A_{p,q}}^{(1-\frac{\alpha}{n})\max\{1,\frac{p'}{q}\}}. \tag{3.18}$$

The proof of inequalities (3.17) and (3.18) can be found in [14].

4. Muckenhoupt bases

In this section, we address the problem of finding optimal exponents for maximal operators defined over Muckenhoupt bases. Recall that given a family $\mathcal{B}$ of open sets, we can define the maximal operator $M_{\mathcal{B}}$ as

$$M_{\mathcal{B}}f(x) = \sup_{x\in B\in\mathcal{B}} \fint_B |f(y)|\,dy,$$

if x belongs to some set $b\in\mathcal{B}$ and $M_{\mathcal{B}}f(x)=0$ otherwise. The natural classes of weights associated with this operator are defined in the same way as the classical Muckenhoupt classes: $w\in A_{p,\mathcal{B}}$ if

$$[w]_{A_{p,\mathcal{B}}} := \sup_{B\in\mathcal{B}} \left(\frac{1}{|B|}\int_B w(y)\,dy\right)\left(\frac{1}{|B|}\int_B w(y)^{1-p'}\,dy\right)^{p-1} < \infty.$$

We say that a basis $\mathcal{B}$ is a Muckenhoupt basis if $M_{\mathcal{B}}$ is bounded on $L^p(w)$ whenever $w\in A_{p,\mathcal{B}}$ (see [20]).

Proof of Theorem 1.3. The idea is to perform the iteration technique from Theorem 1.2 but with $M_{\mathcal{B}}$ instead of the standard H–L maximal operator. Then we obtain, for $1<p<p_0$, that

$$\|M_{\mathcal{B}}\|_{L^p(\mathbb{R}^n)} \leq c\,\|M_{\mathcal{B}}\|^{\beta(p_0-p)}_{L^p(\mathbb{R}^n)} \leq c\,\|M_{\mathcal{B}}\|^{\beta(p_0-1)}_{L^p(\mathbb{R}^n)}. \tag{4.1}$$

The last inequality holds since $\|M_{\mathcal{B}}\|_{L^p(\mathbb{R}^n)}\geq 1$. We remark here that, since we are comparing $M_{\mathcal{B}}$ to itself, it is irrelevant to know the precise quantitative behaviour of its L^p for p close to 1. In fact, we cannot use any estimate like (2.2) since we are dealing with a generic basis. Just knowing that the L^p norm blows up when p goes to 1, allows us to conclude that $\beta\geq\frac{1}{p_0-1}$. □

As an example of this result, we can show that the result for Calderón weights from [7] is sharp. Precisely, for the basis $\mathcal{B}_0$ of open sets in $\mathbb{R}$ of the form $(0,b)$, $b>0$, the authors prove that the associated maximal operator N defined as

$$Nf(t) = \sup_{b>t} \frac{1}{b} \int_0^b |f(x)|\, dx$$

is bounded on $L^p(w)$ if and only if $w \in A_{p,\mathcal{B}_0}$ and, moreover, that

$$\|N\|_{L^p(w)} \leq c\,[w]_{A_{p,\mathcal{B}_0}}^{\frac{1}{p-1}}.$$

By the preceding result, this inequality is sharp with respect to the exponent on the characteristic of the weight.

As another example of a Muckenhoupt basis we can consider the basis $\mathcal{R}$ of rectangles with edges parallel to the axis. The corresponding maximal operator $M_{\mathcal{R}}$ is bounded in $L^p(\mathbb{R}^n)$. Indeed,

$$\|M_{\mathcal{R}}\|_{L^p(\mathbb{R}^n)} \sim (p')^n, \tag{4.2}$$

where $1<p<\infty$. In addition, it is not difficult to see that

$$\|M_{\mathcal{R}}\|_{L^p(w)} \leq c\,[w]_{A_{p,\mathcal{R}}}^{\frac{n}{p-1}}, \quad w \in A_{p,\mathcal{R}}. \tag{4.3}$$

From our Theorem 1.3, we can only deduce that the exponent on the weight must be greater or equal to $1/(p-1)$ as it is already known. Therefore, the problem of finding the sharp dependence for $M_{\mathcal{R}}$ is still open.

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