

Lie algebra deformations in characteristic 2

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Of four types of Kaplansky algebras, type-2 and type-4 algebras have previously unobserved $\mathbb{Z}/2$ -gradings: nonlinear in roots. A method assigning a simple Lie superalgebra to every $\mathbb{Z}/2$ -graded simple Lie algebra in characteristic 2 is illustrated by seven new series. Type-2 algebras and one of the two type-4 algebras are demystified as nontrivial deformations (the results of deformations) of the alternate Hamiltonian algebras. The type-1 Kaplansky algebra is recognized as the derived of the nonalternate version of the Hamiltonian Lie algebra, the one that preserves a tensorial 2-form.

Deformations corresponding to nontrivial cohomology classes can be isomorphic to the initial algebra, e.g., we confirm Grishkov’s implicit claim and explicitly describe the Jurman algebra as such a “semitrivial” deformation of the derived of the alternate Hamiltonian Lie algebra. This paper helps to sharpen the formulation of a conjecture describing all simple finite-dimensional Lie algebras over any algebraically closed field of nonzero characteristic and supports a conjecture of Dzhumadil'daev and Kostrikin stating that all simple finite-dimensional modular Lie algebras are either of “standard” type or deformations thereof.

In characteristic 2, we give sufficient conditions for the known deformations to be semitrivial.

1. Introduction

Hereafter, \mathbb{K} is an algebraically closed field of characteristic $p > 0$ unless otherwise stated. The letter p also denotes “momenta” indeterminates but confusion is impossible.

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1.1. Preparatory information

Assuming that $p^\infty = \infty$ and $\mathbb{N} = \{1, 2, \dots\}$, we designate

$$(1) \quad \mathcal{O}(m; \underline{N}) := \mathbb{K}[u; \underline{N}] := \text{Span}_{\mathbb{K}} \left(u^{(r)} \mid r_i < p^{N_i} \right)$$

for $u = (u_1, \dots, u_m)$ and $r = (r_1, \dots, r_m)$ and any $\underline{N} = (N_1, \dots, N_m)$, where $N_i \in \mathbb{N} \cup \infty$ for any i , the addition is natural, and the product is given by

$$(2) \quad u^{(r)} \cdot u^{(s)} = \binom{r+s}{r} u^{(r+s)}, \quad \text{where } \binom{r+s}{r} := \prod_{i=1}^m \binom{r_i+s_i}{r_i}.$$

The elements of the algebra $\mathcal{O}(m; \underline{N})$ of divided powers serve as “functions” over \mathbb{K} . The shearing vector with smallest coordinates

$$(3) \quad \underline{N}_s = (1, \dots, 1)$$

is of particular interest (see [9, 46]). Only one of the algebras of divided powers $\mathcal{O}(n; \underline{N})$ is indeed generated by the indeterminates declared: the one for $\underline{N} = \underline{N}_s$. Otherwise, the list of generators consists of $u_i^{(p^{k_i})}$ for all i and k_i such that $1 \leq k_i < N_i$. We define *distinguished* partial derivatives by setting

$$\partial_i(u_j^{(k)}) = \delta_{ij} u_j^{(k-1)} \quad \text{for any } k < p^{N_j}.$$

Let $\mathbf{vect}(m; \underline{N}) := \mathbf{der}_{\text{dist}}(\mathcal{O}(m; \underline{N}))$ be the general *vectorial Lie algebra* spanned by all distinguished derivations $f_i \partial_i$, where $f_i \in \mathcal{O}(m; \underline{N})$; let $\mathbf{svect}(m; \underline{N})$ be its subalgebra of divergence-free derivations. Various vectorial Lie algebras are complete or partial *Cartan prolongs*, i.e., the results of generalized prolongation procedures.

1.1.1. Cartan prolongations. Let DS^k be the operation of raising to the k th divided symmetric power and $DS^\bullet := \bigoplus_{k \geq 0} DS^k$; we set (for further details, see [41])

$$(4) \quad \begin{aligned} i: DS^{k+1}(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_{-1} &\longrightarrow DS^k(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}, \\ j: DS^k(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_0 &\longrightarrow DS^k(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1} \end{aligned}$$

as the natural maps. Let the (k, \underline{N}) th *prolong of the pair* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ be

$$(5) \quad \mathfrak{g}_{k, \underline{N}} = (j(DS^\bullet(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_0) \cap i(DS^\bullet(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_{-1}))_{k, \underline{N}},$$

where the subscript k in the right-hand side singles out the component of degree k . It is easy to show that if the \mathfrak{g}_0 -module \mathfrak{g}_{-1} is faithful, then

$(\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*, \underline{N}} = \bigoplus_k \mathfrak{g}_{k, \underline{N}}$ is a Lie subalgebra in $\mathbf{vect}(\dim \mathfrak{g}_{-1}; \underline{N})$; it is called the *(complete) Cartan prolong* of the pair $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$. A *partial* prolong is a subalgebra of $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*, \underline{N}}$ generated by \mathfrak{g}_{-1} , \mathfrak{g}_0 , and a \mathfrak{g}_0 -submodule of \mathfrak{g}_1 .

1.1.2. Lie algebras of Hamiltonian series. For a detailed description of several types of Hamiltonian series, their divergence-free subalgebras, their central extensions—Poisson algebras, and simple derived in characteristic 2 of any of the above algebras, see [33]. Here, we briefly recall that a given symmetric bilinear form B on the space V is said to be *alternate* if $B(v, v) = 0$ for any $v \in V$ and *nonalternate* otherwise. The normal shapes of nondegenerate bilinear forms B , whose Gram matrices are also denoted by B , are denoted by $\Pi(n)$ and $I(n)$, where $n = \dim V$, respectively reduced to the side and main diagonal. The orthogonal Lie algebra that preserves B is denoted by $\mathfrak{o}_B(V)$. If n is odd, then there is only one equivalence class of nondegenerate symmetric bilinear forms, and we can drop the subscript B in $\mathfrak{o}_B(V)$; if $\dim V$ is even, then there are two equivalence classes of nondegenerate symmetric forms: one contains $\Pi(n)$ and the other one $I(n)$.

The Hamiltonian Lie algebra can be *alternate* $\mathfrak{h}_\Pi(V; \underline{N})$ or *nonalternate* $\mathfrak{h}_I(V; \underline{N})$ depending on the type of the differential 2-form¹ the algebra preserves by means of the Lie derivative. We often write $\mathfrak{h}_B(n; \underline{N})$ instead of $\mathfrak{h}_B(V; \underline{N})$. Both $\mathfrak{h}_\Pi(V; \underline{N})$ and $\mathfrak{h}_I(V; \underline{N})$ have divergence-free subalgebras described, together with a history of earlier partial discoveries, in [33].

In this paper, we describe $\mathfrak{h}_B(V; \underline{N})$ as the Cartan prolong $(V, \mathfrak{o}_B(V))_{*, \underline{N}}$ with the multiplication given by the *Poisson bracket* of generating functions

$$(6) \quad \{F, G\}_B = \sum_{1 \leq i, j \leq n} B_{ij} \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j} \quad \text{for any } F, G \in \mathcal{O}(n; \underline{N}),$$

where $(B_{ij})_{i, j=1}^n = B$. The elements of $\mathfrak{h}_B(V; \underline{N})$ can be realized by vector fields

$$(7) \quad H_F = \sum_{1 \leq i, j \leq n} B_{ij} \frac{\partial F}{\partial x_i} \frac{\partial}{\partial x_j} \quad \text{for any } F \in \mathcal{O}(n; \underline{N}).$$

The Lie algebra whose space is $\mathcal{O}(n; \underline{N})$ with the bracket (6) is called the *Poisson algebra* if $B \not\sim I$; it is a central extension of $\mathfrak{h}_B(n; \underline{N})$ for $B \sim \Pi$.

We note that although $\mathfrak{h}_I(n; \underline{N})$ is well defined, **there is no Lie algebra** $\mathfrak{po}_I(n; \underline{N})$ with the bracket (6). Indeed, the bracket should be antisymmetric, i.e., alternate, while $\{x_i, x_i\}_I = 1$, not 0. More on possible brackets

¹It was shown in [33] that this 2-form is the sum of products of 1-forms, the product being either exterior (alternate case) or tensor (nonalternate case).

corresponding to the alternate bilinear form B can be found in [33] and Subsection 3.1.

1.2. Overview of the situation

Even the incomplete stock of nonisomorphic species in the zoo of simple finite-dimensional Lie algebras for $p = 2$ was until recently considered uncomfortably numerous (see the introduction in [45]). It has many more exhibits than would have been considered “normal” if the classification in cases $p > 3$ were taken as a “norm.” A final touch in the proof of the classification can be found in [2].

The improved version of the Kostrikin-Shafarevich conjecture due to Dzumadildaev and Kostrikin [30] states that for any $p > 0$,

- (8) **all simple Lie algebras are either of “standard” type or deforms (the results of deformations) thereof.**

The improved conjecture definitely embraces $p \geq 5$ as proved in [30], where the Melikyan algebras are identified as deforms of the Poisson algebras. The claim in [32] identifying the simple Ermolaev algebras as deforms of the contact algebras supports the Conjecture (8) for $p = 3$. The Conjecture (8) seems plausible if, for $p = 2$ and 3, we enlarge the stock of examples analogous to those “standard” for $p \geq 5$ by exhibits from [3, 5, 7, 8, 18, 24, 33] found after (8) was formulated. Therefore, we must find out which simple Lie algebras are “standard” from the standpoint of (8), and solve a “small technical problem” of describing all nonisomorphic deforms.

A conjectural description of “standard”, hence, all (together with their deforms) simple finite-dimensional Lie algebras over \mathbb{K} for $p = 2$, although longer than that for $p > 3$, is possible to grasp. This conjecture stemmed from an idea that had already led to the classification of simple Lie superalgebras of polynomial vector fields over \mathbb{C} (see [37]). The new conjecture yielded new examples for $p = 3$ (see the arXiv version of [24]).

For $p = 2$, the new conjecture (see how it was being molded from [35] to [7]) gathers all examples known to us in describable groups and indicates ways to obtain new examples. Apart from these and several new examples given in [8], there are also known examples (due to Kaplansky, Shen, Skryabin, Brown, Jurman, Vaughan-Lee, and recently Eick) of mysterious nature. In this paper and [7, 8], we study which of these “mysterious” examples, if any, might qualify as “standard” from the standpoint of Conjecture (8), see also [22, 44]. We demystify the other examples by identifying

them either as deforms of or as isomorphic to some of the “standard” examples.

1.2.1. On limited information derived from cohomology in describing deforms of Lie algebras. In §2, we recall how deformations of Lie algebras are calculated. The *trivial deformation* of \mathfrak{g} corresponds to the change of the basis in \mathfrak{g} corresponding to a 2-coboundary, while the linear part of any global deformation is a cocycle; hence, deforms linear in the deformation parameter, also called infinitesimal deforms, correspond to cocycles representing classes of $H^2(\mathfrak{g}; \mathfrak{g})$.

If $\text{char } \mathbb{K} > 0$, there are “fake deformations”. This means something much worse than “some of linear deforms corresponding to cocycles representing classes of $H^2(\mathfrak{g}; \mathfrak{g})$ might be not extendable to a global deformation”. The textbooks and papers on Lie (super)algebra cohomology do not yet indicate the following important phenomenon:

- (9) **Let each cocycle representing a class of $H^2(\mathfrak{g}; \mathfrak{g}) \neq 0$ be extendable to a global deformation. This does not preclude some (or all) deforms of \mathfrak{g} from being isomorphic to \mathfrak{g} .**

Let *semitrivial* deformations (and their results, the deforms) be the ones whose linear parts are given by cocycles representing nontrivial cohomology classes but whose deforms are isomorphic to the initial Lie algebra. In addition to examples in [10], we show that the Jurman algebras are *semitrivial deforms*.

Examples of semitrivial deforms have been known to us since 1987 when we computed that² $\dim H^2(\mathfrak{o}'(3); \mathfrak{o}'(3)) = 2$ over \mathbb{K} for $p = 2$ while, up to an isomorphism, there is only one simple 3-dimensional Lie algebra: $\mathfrak{o}'(3)$. (Ten years earlier the phenomenon (9) was observed without any explanations of its origin in [17].) The first *explanation* of the cause of the phenomenon (9) was given in [10]. For $p = 2$, we give sufficient conditions for semitriviality of a deformation, see Subsection 2.3.

1.2.2. The Vaughan-Lee algebras are not new over \mathbb{K} . The table on p. 948 in [18] shows that simple algebras of Vaughan-Lee (all new over \mathbb{F}_2) are only new as forms of Lie algebras known over \mathbb{K} (or even over a Galois field extending \mathbb{F}_2).

²Hereafter, the prime ' denotes the first derived, also called the commutant.

1.2.3. The Eick algebras are new. Several simple Lie algebras were introduced in [18] and conjectured (e.g., because the list of “known” algebras Eick used for comparison was incomplete as compared with a wider list known to us) to be new. These algebras had to be interpreted and described in more detail than in [18]. With Eick’s help, we recently established that all the six tentatively new algebras in [18] are indeed new. All the six new Eick algebras are obtained in one of the ways predicted by the conjecture [35]: Eick algebras are *partial* Cartan prolongs (see Subsection 1.1.1), like Frank algebras for $p = 3$ (cf. [24]), or deforms of something “standard.”

1.2.4. One of the Shen algebras and its generalized Cartan prolong due to Brown are “standard”. In [42], Shen described several simple Lie algebras. One Shen algebra was rediscovered, together with several algebras new at that time, by Brown [11]. Brown’s examples, described only in components in [11], were interpreted in [7, 24] together with a clarification of their structure and related new simple Lie superalgebras. One remarkable exceptional simple Lie algebra (we call it $\mathfrak{gs}(2)$ in honor of Shen Guangyu, who discovered it; see [42]) is a true analog of the Lie algebra $\mathfrak{g}(2)$ in characteristic 2. Brown rediscovered this algebra (Eick called it $\text{Bro}_2(1, 1)$ in [18]) and considered its Cartan prolong. We called the derived of this prolong $\mathfrak{me}'(5; \underline{N})$, and interpreted it as a version of the Melikyan algebras in characteristic 2 (see [8]). These $\mathfrak{me}'(5; \underline{N})$, where $N_1 = N_2 = N_3 = 1$ are the only possible values, and its particular case $\mathfrak{gs}(2) = \mathfrak{me}'(5; \underline{N}_s)$ seem to be new “standard” examples. Several Shen algebras were interpreted in [4] as deforms of certain “standard” algebras, and several of Shen’s examples are either nonsimple or not new (see [39]). Moreover, the multiplication in several Shen algebras does not satisfy the Jacobi identity.

1.2.5. Jurman and Kaplansky algebras as deforms. We started this paper intending to prove that the Jurman and Kaplansky algebras are deforms of more “conventional” simple Lie algebras (such as the two nonisomorphic versions of the Lie algebra of Hamiltonian vector fields) and their divergence-free subalgebras (see [33]). While this paper was being written, Grishkov published a note³ [21] claiming that the Jurman algebra is **isomorphic** to the (derived of) a Hamiltonian Lie algebra. Grishkov’s paper is based on a difficult result due to Skryabin, and its claim is implicit. Consequently, we heard doubts that it is correct. It IS correct: we give an explicit isomorphism in Prop. 4.5. Amazingly, the existence of this isomorphism does not contradict the fact that the Jurman algebra is a deform corresponding

³A draft of this note was available on Grishkov’s home page since 2009.

to a cocycle personifying a nontrivial cohomology class of the (derived of the) Hamiltonian Lie algebra: Jurman algebras are examples of “semitrivial” deforms (see Subsections 1.2.1 and 2.3).

In §5, we identify type-1 Kaplansky algebras with certain known “standard” Lie algebras and prove that type-2 Kaplansky algebras are deforms of certain “standard” Lie algebras. Type-3 algebras were identified (in different terms) by Kaplansky himself as $\mathfrak{o}'_7(n)$. Type-4 algebras might be “standard”, see Subsection 5.1.

1.3. Main results

The three **most interesting parts** of our paper:

(1) The discovery of a $\mathbb{Z}/2$ -grading quadratic in roots. Among the Lie algebras known to us, the type-2 and type-4 Kaplansky algebras are the only ones with such gradings (cf. (88)). We present details on relations between gradings and derivations, in particular, an observation that gradings are not always defined by derivations, in Subsection 5.0.2.

(2) A method assigning a simple Lie superalgebra to **every** simple Lie algebra if $p = 2$. It is illustrated with seven new series, superizations of the Kaplansky algebras in Subsection 5.2. Modulo classification of Lie algebras, this had led us to a classification of simple Lie superalgebras, see [9].

(3) Thanks to the insistence of a referee, we managed to give sufficient conditions for deformations encountered so far for $p = 2$ to be semitrivial (see Theorem 2.3); but other types of semitrivial deformations are possible.

The **main bulk** of the paper is devoted to interpreting the simple Lie algebras discovered by Jurman and Kaplansky in terms of better known (“standard”) examples of Lie algebras of Hamiltonian vector fields or their simple derived. Voluminous computations are performed using Grozman’s *Mathematica*-based package *SuperLie*, cf. Lemmas 4.3, 6.3, 6.4, 6.5.

1.4. Open problems

1. We have described deforms of $\mathfrak{h}'_{\Pi}(n; \underline{N})$ for $n = 2$. Investigation of the isomorphism classes of the deforms for $n > 2$ and any \underline{N} is a must. The classification of the deforms of the more natural nonsimple relative of the simple algebra, i.e., of $\mathfrak{h}_{\Pi}(n; \underline{N})$, is also needed: it had led to an interpretation of previously mysterious type-2 Kaplansky algebras. The search for deforms of another relative, the Poisson Lie algebra $\mathfrak{po}_{\Pi}(n; \underline{N})$, is equally reasonable (answers to such problems for $p = 0$ have physical interpretations), see [29, 30].

2. A new way to construct simple Lie algebras in the absence of classification⁴ is provided in [18] if the structure constants belong to \mathbb{F}_2 ; the parametric families cannot be captured by Eick's method. Regrettably restricted to algebras of dimension ≤ 20 at the moment, Eick's computer-aided approach would promise (when its range will have been widened to dimension 250, or at least 80) to give a base for a conjecture making its future theoretical proof psychologically comfortable (no new examples would be expected).

3. Our results show that $\mathfrak{h}_{\Pi}(2; \underline{N})$ and $\mathfrak{h}'_{\Pi}(2; \underline{N})$ have different numbers of deforms and both types of deforms are important for classifying simple Lie algebras. The situation is similar to that in characteristic 0, where the Lie superalgebra $\mathfrak{h}(2n|m)$ has (only if $(2n|m) = (2|2)$) more deformations than $\mathfrak{po}(2n|m)$ (see [38]). A problem is to describe deformations of $\mathfrak{h}_{\Pi}(2n; \underline{N})$ and $\mathfrak{h}'_{\Pi}(2n; \underline{N})$.

4. In [43], Skryabin classified nonequivalent normal shapes of the exterior 2-forms preserved by the Hamiltonian Lie algebras $\mathfrak{h}_{\Pi}(n; \underline{N})$. It remains to classify à la [43] nonequivalent normal shapes of the tensorial 2-forms preserved by the Hamiltonian Lie algebras $\mathfrak{h}_I(n; \underline{N})$ and its divergence-free subalgebras.

5. Listing all nonisomorphic deforms of $\mathfrak{g} = \mathfrak{h}'_{\Pi}(2; (g, h + 1))$ requires considering orbits of the $\text{Aut}(\mathfrak{g})$ -action on the space $H^2(\mathfrak{g}; \mathfrak{g})$ following Kuznetsov and his students (see [12, 31]). If the algebraic group $\text{Aut}(\mathfrak{g})$ of automorphisms of \mathfrak{g} were computed, then the result in [19] could be extended to the simple Lie algebras without a Cartan matrix. So far, this has been done only in certain particular cases (see [40]).

6. Several identification problems remain (see Subsections. 4.3, 5.2.3, 6.2, 6.5, and 6.4.2 and Eq. (33)).

7. A tough problem, partly solved in §2.3, is to give sufficient conditions for any (other than those we considered) deformation to be semitrivial.

2. Deformations and cohomology ([10])

2.1. Lie algebras

A *multiparameter deformation* of a Lie algebra \mathfrak{g} over \mathbb{K} is a Lie algebra \mathfrak{g}_t , where $t = (t_1, \dots, t_r)$, given by a Lie algebra structure on the tensor product $\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[[t]]$ such that the Lie algebra \mathfrak{g}_0 , i.e., the algebra obtained for $t = 0$,

⁴Eick's search is random; to estimate the probability of a miss is very interesting.

is isomorphic to \mathfrak{g} . For any $x, y \in \mathfrak{g}$, the deformed bracket has the form

$$(10) \quad [x, y]_{t_1, \dots, t_r} = c^0(x, y) + t_1 c_1^1(x, y) + \dots + t_r c_r^1(x, y) + t_1^2 c_{1,1}^2(x, y) + t_1 t_2 c_{1,2}^2(x, y) + \dots + t_r^2 c_{r,r}^2(x, y) + \dots,$$

where $c^0(x, y) := [x, y]$.

The maps $c_i^1: \mathfrak{g} \otimes_{\mathbb{K}} \mathfrak{g} \rightarrow \mathfrak{g}$ must be 2-cocycles (say the degree-1 conditions), i.e., for all $i = 1, \dots, r$ and any $x, y, z \in \mathfrak{g}$, we have

$$dc_i^1(x, y, z) := c_i^1([x, y], z) + c_i^1([y, z], x) + c_i^1([z, x], y) - [x, c_i^1(y, z)] - [y, c_i^1(z, x)] - [z, c_i^1(x, y)] = 0.$$

For brevity, we recall properties of deformations for 1-parameter deformations; the multidimensional case is considered routinely. For example, Eq. (10) becomes

$$(11) \quad [x, y]_t = c^0(x, y) + tc^1(x, y) + t^2 c^2(x, y) + \dots.$$

Two 1-parameter deforms \mathfrak{g}_t and $\tilde{\mathfrak{g}}_t$ given by the collections $c = (c^1, c^2, \dots)$ and $\tilde{c} = (\tilde{c}^1, \tilde{c}^2, \dots)$ lead to equivalent deforms, i.e., \mathfrak{g}_t and $\tilde{\mathfrak{g}}_t$ are isomorphic as Lie algebras by an isomorphism of the form $\tau(x; t) = \sum_{i \geq 0} \tau_i(x) t^i$ for $\tau_0 = \text{id}$ and any $x \in \mathfrak{g}$, if and only if c and \tilde{c} are related as follows (for all $n > 0$, where $i, j, k \geq 0$):

$$(12) \quad \sum_{i+j=n} \tau_i(\tilde{c}^j(x, y)) = \sum_{i+j+k=n} c^i(\tau_j(x), \tau_k(y)).$$

Any change of basis of \mathfrak{g} can be included in a 1-parameter family of isomorphisms $\tau(\cdot; t): \mathfrak{g} \rightarrow \mathfrak{g}$ and regarded, naturally, as a trivial deformation. We see that the trivial deformation corresponds to $d\tau_1$ modulo t^2 ; the search for the most general multiparameter deformation of a given Lie algebra therefore begins with computing the space $H^2(\mathfrak{g}; \mathfrak{g})$. Its explicit basis given by 2-cocycles (representing the classes) determines infinitesimal deformations. We then try to prolong each infinitesimal deformation to higher degrees. The Jacobi identity imposes conditions on all terms in the deformed bracket, which must be satisfied in each degree. In particular, two 1-parameter degree-1 cocycles c^1 and \tilde{c}^1 are *infinitesimally equivalent* (i.e., $\tau = \text{id} + t\tau_1$ modulo t^2) if and only if $c^1 - \tilde{c}^1 = d\tau_1$.

Let \mathfrak{g}_t be a 1-parameter deformation of a Lie algebra \mathfrak{g} given by the collection $c = (c^1, c^2, \dots)$. The Jacobi identity yields that the coefficient of

t^n vanishes for each n :

$$(13) \quad \sum_{0 \leq i, j \leq n; i+j=n} (c^i(c^j(x, y), z) + \text{cyclic}(x, y, z)) = 0,$$

where $\text{cyclic}(x, y, z)$ denotes the sum of all cyclic permutations of the arguments of the expression written on the left of it. We set

$$(14) \quad \begin{aligned} [[c^i, c^j]](x, y, z) &:= c^i(c^j(x, y), z) + c^j(c^i(x, y), z) + \text{cyclic}(x, y, z), \\ c^k \circ c^k(x, y, z) &:= c^k(c^k(x, y), z) + \text{cyclic}(x, y, z). \end{aligned}$$

The brackets $[[c^i, c^j]]$ are called *Nijenhuis brackets* (in differential geometry) or *Massey brackets* (in deformation theory). The sum (13) can be expressed as a *Maurer-Cartan* equation:

$$(15) \quad dc^n = \sum_{0 < i < j \leq n; i+j=n} [[c^i, c^j]] + \begin{cases} 0 & \text{if } n = 2k + 1, \\ c^k \circ c^k & \text{if } n = 2k. \end{cases}$$

To prolong an infinitesimal deformation given by a cocycle c^1 , we first compute $[[c^1, c^1]]$. If $[[c^1, c^1]] = 0$, then the infinitesimal deformation satisfies the Jacobi identity and is a true deformation. If $[[c^1, c^1]] \in Z^3(\mathfrak{g}, \mathfrak{g})$ and $[[c^1, c^1]] \notin B^3(\mathfrak{g}, \mathfrak{g})$, then the infinitesimal deformation is obstructed and cannot be prolonged. If $[[c^1, c^1]] = dc^2$ with $c^2 \neq 0$, then $-t^2c^2$ is the 2nd degree term of the deformation. To prolong the deformation to degree 3, we compute the Massey product $[[c^1, c^2]]$. There are the three possibilities

- 1) $[[c^1, c^2]] = 0$,
- 2) $[[c^1, c^2]] = dc^3$ for some $c^3 \neq 0$,
- 3) $[[c^1, c^2]] \neq dc^3$ for any c^3 .

If $[[c^1, c^2]] = dc^3$, then $-t^3c^3$ gives the 3rd degree term of the deformation. To go up to degree 4, we must be able to compensate $c^2 \circ c^2 + [[c^1, c^3]]$ by a coboundary dc^4 , and so on. The main difficulty here is that the representatives of the cohomology classes and the cochains c^2, c^3 , etc., are not uniquely⁵ defined. A good choice of cochains may considerably facilitate computations. The following lemma is helpful.

⁵If c^2 is a solution of the equation $dc^2 = c^1 \circ c^1$, then $c^2 + \text{cocycle}$ is also a solution. The choice of a certain c^2 affects the expression of the c^3 terms. The problem is how to find a “nice” c^2 in order to have as few c^3 terms as possible and, more importantly, vanishing Massey products in degrees > 3 . If we fail to achieve this with c^2 , then we can try with c^3 , and so on.

2.1.1. Grozman's lemma ([10]). For any finite-dimensional Lie algebra \mathfrak{g} , all cochains with coefficients in the adjoint module can be expressed as sums of tensor products of the form $a \otimes \omega$, where $a \in \mathfrak{g}$ and $\omega \in \bigwedge^r(\mathfrak{g}^*)$.

Lemma 2.1. For any $c = a \otimes \omega$ with $a \in \mathfrak{g}$ and $\omega \in \bigwedge^r(\mathfrak{g}^*)$, let dc denote the coboundary of c in the complex with coefficients in the adjoint module, $d\omega$ be the coboundary in the complex with coefficients in the trivial module, and da be the coboundary of $a \in \mathfrak{g}$ regarded as a 0-cochain in the complex with coefficients in the adjoint module. Then $dc = a \otimes d\omega + da \wedge \omega$.

2.2. Lie superalgebras

In certain problems, if $p > 0$, then we must replace the (formulas of) conventional cohomology in the preceding subsection with divided power ones (see [6]).

2.3. Semitrivial deformations

In all examples we know, the deforms of \mathfrak{g} corresponding to semitrivial deformations for $p = \text{char } \mathbb{K}$ are isomorphic to the initial Lie algebra via an isomorphism $\tau(\cdot, t)$ given by an expression of the form (also see [10], where equality $\sqrt[p]{(1+t)^i} = 1 + (\sqrt[p]{t})^i$ should be used)

$$(16) \quad \tau(x; t) = x + \sum_{i \geq 1} \tau_i(x) (\sqrt[p]{t})^i \quad \text{with the } \tau_i \text{ satisfying conditions (12).}$$

Therefore, the semitrivial cocycle c , that would have been obtained for $p = 0$ as the differential of a 1-cochain C such that $C(x) = \frac{\partial \tau(x, t)}{\partial t}$ for any $x \in \mathfrak{g}$, cannot be thus obtained if $p > 0$, because the function $\sqrt[p]{t}$ is not differentiable for $p > 0$.

For any derivation D of the Lie algebra \mathfrak{g} , let $c_D \in C^2(\mathfrak{g}; \mathfrak{g})$ be defined by

$$(17) \quad c_D(x, y) = [Dx, Dy] \quad \text{for any } x, y \in \mathfrak{g}.$$

It is easy to verify that c_D is a cocycle, i.e., $dc_D(x, y, z) = 0$ for all $x, y, z \in \mathfrak{g}$. Moreover, if $p \neq 2$, then $c_D = -\frac{1}{2}d(D^2)$, i.e., c_D is trivial.

Here is a general description of semitrivial deformations for $p = 2$ in the case where an isomorphism between the algebra and its deform is a polynomial in \sqrt{t} . We do not know how to characterize arbitrary semitrivial deformations.

Lemma 2.2. *Let \mathfrak{g} be a Lie algebra in characteristic 2, and let $c \in Z^2(\mathfrak{g}; \mathfrak{g})$ generate a semitrivial deformation such that the isomorphism between the algebra and its deform is polynomial in \sqrt{t} . Then there is a derivation D of \mathfrak{g} such that c is equivalent to c_D , i.e., $c - c_D \in B^2(\mathfrak{g}; \mathfrak{g})$.*

Proof. Let the isomorphism be $F_t(x) = x + \sum_{i \geq 1} f_i(x)(\sqrt{t})^i$, where only a finite number of the $f_i \in C^1(\mathfrak{g}; \mathfrak{g})$ are nonzero. Then the deformed bracket has the form

$$(18) \quad [x, y]_t = [x, y] + (f_1([x, y]) + [f_1(x), y] + [x, f_1(y)])\sqrt{t} + ([f_1(x), f_1(y)] + df_2(x, y))t + \text{higher order terms in } \sqrt{t}.$$

Because the coefficient of \sqrt{t} is zero and the coefficient of t is $c(x, y)$, we see that f_1 is a derivation, and c is equivalent to c_{f_1} . □

We do not know whether every cocycle of the form (17) generates a global deformation, nor do we know whether all such deformations are semitrivial or trivial. We can prove it for two classes of derivations.

Theorem 2.3. *Let \mathfrak{g} be a Lie algebra over a perfect field \mathbb{K} of characteristic 2, and let D be a derivation of \mathfrak{g} satisfying at least one of the following two conditions:*

1. $D^2 = \alpha D$ for some $\alpha \in \mathbb{K}$ (if $\alpha = 1$, then such D corresponds to a $\mathbb{Z}/2$ -grading);
2. D is nilpotent.

Then the cocycle c_D (see Lemma 2.2) is either trivial or semitrivial.

Proof. To prove that c is either trivial or semitrivial, we show that there is a polynomial family of invertible maps $F_t(x) = x + \sum_{i \geq 1} f_i(x)t^i$, where only a finite number of the $f_i \in C^1(\mathfrak{g}; \mathfrak{g})$ are nonzero, such that

$$(19) \quad [F_t(x), F_t(y)] = F_t([x, y]) + \sum_{j \geq 1} c_j(x, y)t^{2j} \quad \text{for all } x, y \in \mathfrak{g},$$

where only a finite number of the $c_j \in C^2(\mathfrak{g}; \mathfrak{g})$ are nonzero, and $c_1 = c_D$. This would mean that there exists a global deformation of \mathfrak{g} given by the bracket

$$[x, y]_t = [x, y] + \sum_{j \geq 1} c_j(x, y)t^j,$$

all the deformed algebras being isomorphic to \mathfrak{g} with the isomorphism given by $F_{\sqrt{t}}$.

1. If $D^2 = \alpha D$, then we set $F_t(x) = x + tD(x)$; this map is invertible if $t\alpha \neq 1$. We also set $c_1 = c_D$ and $c_j = 0$ for $j > 1$. It is easy to see that (19) holds because $D[Dx, Dy] = 0$.

2. Now let D be nilpotent. We set

$$f_i = \begin{cases} D^i & \text{if } i = 2^k \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases}$$

We note that all the f_i are derivations. We set⁶

$$c_j(x, y) = \begin{cases} \sum_{1 \leq s \leq 2j-1} [D^s x, D^{2j-s} y] & \text{if } j = 2^k \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases}$$

Because D is nilpotent, the map F_t is invertible. We must prove that the coefficients of t^n in (19) are the same for any $n > 0$, i.e., that

$$\begin{aligned} (20) \quad & [f_n(x), y] + [x, f_n(y)] + \sum_{1 \leq i \leq n-1} [f_i(x), f_{n-i}(y)] \\ &= f_n([x, y]) + \begin{cases} c_m(x, y) + \sum_{1 \leq j \leq m-1} f_{n-2j}(c_j(x, y)) & \text{if } n = 2m, \\ \sum_{1 \leq j \leq m} f_{n-2j}(c_j(x, y)) & \text{if } n = 2m + 1. \end{cases} \end{aligned}$$

Because f_n is a derivation, Eq. (20) can be simplified to

$$\begin{aligned} (21) \quad & \sum_{1 \leq i \leq n-1} [f_i(x), f_{n-i}(y)] \\ &= \begin{cases} c_m(x, y) + \sum_{1 \leq j \leq m-1} f_{n-2j}(c_j(x, y)) & \text{if } n = 2m, \\ \sum_{1 \leq j \leq m} f_{n-2j}(c_j(x, y)) & \text{if } n = 2m + 1. \end{cases} \end{aligned}$$

If $n = 1$ or if n cannot be represented as a sum of at most two powers of 2, then both sides of (21) are equal to 0 because f_i and c_i are only nonzero

⁶It might be wondered, ‘‘Why does the deformed bracket satisfy the Jacobi identity? In particular, why do the c_j satisfy the Maurer-Cartan equations?’’ We forget for a moment that \mathfrak{g} is a Lie algebra and regard it just as an algebra. Then if (19) holds (and we show that it does), then the deformed algebra is isomorphic to \mathfrak{g} with the isomorphism given by F . Because \mathfrak{g} is a Lie algebra, the deformed algebra is also a Lie algebra, i.e., the deformed bracket satisfies Jacobi identity.

if i is a power of 2. Therefore, (21) has nonzero terms only if n is the sum of at most two powers of 2.

If $n = 2^k + 2^l$, where $k > l \geq 1$, then the right-hand side of (21) is

$$\begin{aligned} & f_{2^k}(c_{2^{l-1}}(x, y)) + f_{2^l}(c_{2^{k-1}}(x, y)) \\ &= \sum_{1 \leq s \leq 2^{l-1}-1} ([D^{2^k+s}x, D^{2^l-s}y] + [D^s x, D^{2^k+2^l-s}y]) \\ & \quad + \sum_{1 \leq s \leq 2^k-1} ([D^{2^l+s}x, D^{2^k-s}y] + [D^s x, D^{2^k+2^l-s}y]) \\ &= [D^{2^l}x, D^{2^k}y] + [D^{2^k}x, D^{2^l}y], \end{aligned}$$

and the last expression is exactly the left-hand side of (21).

If $n = 2^k + 1$, where $k \geq 1$ (this case is in fact a particular subcase of the above case for $l = 1$, but we consider it separately because there is no $c_{2^{l-1}}$ in this case), then the right-hand side of (21) is

$$\begin{aligned} f_1(c_{2^{k-1}}(x, y)) &= \sum_{1 \leq s \leq 2^k-1} ([D^{s+1}x, D^{2^k-s}y] + [D^s x, D^{2^k+1-s}y]) \\ &= [Dx, D^{2^k}y] + [D^{2^k}x, Dy], \end{aligned}$$

and the last expression is exactly the left-hand side of (21).

If $n = 2^k$, where $k \geq 2$, then the right-hand side of (21) is

$$\begin{aligned} & f_{2^{k-1}}(c_{2^{k-2}}(x, y)) + c_{2^{k-1}}(x, y) \\ &= \sum_{1 \leq s \leq 2^{k-1}-1} ([D^{2^{k-1}+s}x, D^{2^{k-1}-s}y] + [D^s x, D^{2^k-s}y]) \\ & \quad + \sum_{1 \leq s \leq 2^k-1} [D^s x, D^{2^k-s}y] \\ &= [D^{2^{k-1}}x, D^{2^{k-1}}y], \end{aligned}$$

and the last expression is exactly the left-hand side of (21).

Finally, if $n = 2$, then the left-hand side of (21) is $[Dx, Dy]$, same as its right side, $c_1(x, y)$. □

3. Modular vectorial Lie algebras as deforms of each other

Weisfeiler and Kac were the first to discover parametric families of simple finite-dimensional Lie algebras with a Cartan matrix over \mathbb{K} (see [47]). For

further examples of deforms of simple Lie algebras, see [10, 15, 17, 24, 30, 32, 33, 43].

In this section, we extend the list of such examples and also show that several nonisomorphic Poisson Lie algebras are deforms of one Lie algebra nonsimple over \mathbb{K} but simple over a ring and thus resemble forms over algebraically nonclosed fields of an algebra defined over an algebraically closed field. We regard expressions of the form $k \pmod p$, where $k \in \mathbb{Z}$, as integers from the segment $[0, p - 1]$ and not as elements of \mathbb{K} .

Lemma 3.1. *We consider a linear endomorphism Φ_α , where $\alpha \in \mathbb{K}$, of the algebra $\mathcal{O}(1; \underline{n})$, given by the formula*

$$(22) \quad \Phi_\alpha(x^{(k)}) = \alpha \binom{k}{p} x^{(k)},$$

where $\binom{k}{p}$ denotes the integer part of $\frac{k}{p}$ and $k < p^n$. If $\alpha \neq 0$, then Φ_α is an automorphism of $\mathcal{O}(1; \underline{n})$.

Proof. Clearly, Φ_α is a bijection, and we need only prove that

$$(23) \quad \Phi_\alpha(x^{(k)} \cdot x^{(l)}) = \Phi_\alpha(x^{(k)}) \cdot \Phi_\alpha(x^{(l)}),$$

i.e.,

$$(24) \quad \alpha \binom{k+l}{p} x^{(k+l)} = \alpha \binom{k}{p} + \binom{l}{p} \binom{k+l}{k} x^{(k+l)}.$$

One can see that⁷

$$(25) \quad \begin{aligned} \binom{k+l}{p} &= \binom{k}{p} + \binom{l}{p} && \text{if } (k \pmod p) + (l \pmod p) < p, \\ \binom{k+l}{k} &\equiv 0 \pmod p && \text{if } (k \pmod p) + (l \pmod p) \geq p, \end{aligned}$$

and the statement in the lemma hence holds in both cases. □

⁷The thing equal to 0 in the second line of (25) is not the same as the thing equal to $\binom{k}{p} + \binom{l}{p}$ in the first line. We also note that in the first line, the equality (involving integer parts used as power degrees) is over \mathbb{Z} ; in the second line (involving the binomial coefficient), the equality is over \mathbb{K} or modulo p . In both lines, the residues of k and l modulo p should be understood as integers from the segment $[0, p - 1]$; then the inequalities make sense.

We consider the endomorphism $D_\alpha = \Phi_\alpha^{-1} \circ \partial \circ \Phi_\alpha$ of $\mathcal{O}(1; \underline{n})$ given by

$$(26) \quad D_\alpha(x^{(k)}) = \begin{cases} \partial x^{(k)} & \text{if } p \nmid k; \\ \alpha \partial x^{(k)} & \text{if } p \mid k. \end{cases}$$

We define D_0 (i.e., D_α for $\alpha = 0$, when Φ_0 is not defined) using relation (26). Note that if we consider the isomorphism $\mathcal{O}(1; \underline{n}) \simeq \mathcal{O}(2; (1, n - 1))$ given by

$$(27) \quad x^{(k)} \longleftrightarrow y_1^{(k \bmod p)} y_2^{\left(\left[\frac{k}{p}\right]\right)},$$

then D_0 on $\mathcal{O}(1; \underline{n})$ corresponds to ∂_1 on $\mathcal{O}(2; (1, n - 1))$.

Similarly, in the algebra $\mathcal{O}(d; \underline{N})$ with indeterminates $x = (x_1, \dots, x_d)$, we can consider the map

$$(28) \quad \Phi_\alpha(x^{(r)}) = \alpha^{\sum_{1 \leq i \leq d} \left[\frac{r_i}{p}\right]} x^{(r)},$$

which is an isomorphism for $\alpha \neq 0$. The maps (here $\partial_i := \partial_{x_i}$)

$$(29) \quad D_{\alpha,i} = \Phi_\alpha^{-1} \circ \partial_i \circ \Phi_\alpha \quad \text{act as } D_{\alpha,i}(x^{(r)}) = \begin{cases} \partial_i x^{(r)} & \text{if } p \nmid r_i, \\ \alpha \partial_i x^{(r)} & \text{if } p \mid r_i. \end{cases}$$

We define $D_{0,i}$ using the relations (29).

3.1. Poisson Lie algebras

We consider the Lie algebra $\mathfrak{po}_B(d; \underline{N})$, where $B = (B_{ij})$ is an alternate (the analog of antisymmetric for $p = 2$) nondegenerate bilinear form on a d -dimensional space. The space of this algebra coincides with $\mathcal{O}(d; \underline{N})$, and the Poisson bracket is defined by Eq. (6).

We consider the deformed bracket of $\mathfrak{po}_B(d; \underline{N})$ determined by the map Φ_α on $\mathcal{O}(d; \underline{N})$ (observe that the deformation parameter is $\alpha - 1$, not α):

$$(30) \quad \begin{aligned} [F, G]_{B,\alpha} &:= \Phi_\alpha^{-1}([\Phi_\alpha(F), \Phi_\alpha(G)]) = \sum_{1 \leq i, j \leq d} B_{ij} \Phi_\alpha^{-1}(\partial_i \Phi_\alpha(F) \cdot \partial_j \Phi_\alpha(G)) \\ &= \sum_{1 \leq i, j \leq d} B_{ij} \Phi_\alpha^{-1}(\partial_i \Phi_\alpha(F)) \cdot \Phi_\alpha^{-1}(\partial_j \Phi_\alpha(G)) \\ &= \sum_{1 \leq i, j \leq d} B_{ij} D_{\alpha,i} F \cdot D_{\alpha,j} G, \end{aligned}$$

because the map Φ_α on $\mathcal{O}(d; \underline{N})$ for $\alpha \neq 0$ preserves the (associative and commutative) multiplication of functions.⁸

We now consider the Lie algebra with the bracket (30) for any α . Because we obtained this bracket from a trivial deformation (for $\alpha \neq 0$), the obtained Lie algebra is isomorphic to the initial Lie algebra $\mathfrak{po}_B(d; \underline{N})$. To what is the Lie algebra for $\alpha = 0$ isomorphic?

Under the isomorphism $\mathcal{O}(d; \underline{N}) \simeq \mathcal{O}(2d; (1, \dots, 1, N_1 - 1, \dots, N_d - 1))$ given by the formula

$$(31) \quad x_1^{(r_1)} \cdots x_d^{(r_d)} \longleftrightarrow y_1^{(r_1 \bmod p)} \cdots y_d^{(r_d \bmod p)} y_{d+1}^{\left(\left\lfloor \frac{r_1}{p} \right\rfloor\right)} \cdots y_{2d}^{\left(\left\lfloor \frac{r_d}{p} \right\rfloor\right)},$$

the operator ∂_i on $\mathcal{O}(2d; (1, \dots, 1, N_1 - 1, \dots, N_d - 1))$ is the image of the operator $D_{0,i}$ on $\mathcal{O}(d; \underline{N})$. Hence, the Lie algebra given by commutation relation (30) with $\alpha = 0$ is isomorphic to

$$(32) \quad \begin{aligned} & \mathfrak{po}_B(d; (1, \dots, 1)) \otimes \mathcal{O}(d; (N_1 - 1, \dots, N_d - 1)) \\ & \simeq \mathfrak{po}_B(d; (1, \dots, 1)) \otimes \mathcal{O}(1; (N_1 + \cdots + N_d - d)) \\ & \simeq \mathfrak{po}_B(d; (1, \dots, 1)) \otimes \mathcal{O}(1; (1))^{\otimes N_1 + \cdots + N_d - d}. \end{aligned}$$

We see from (32) that all Poisson algebras with the same number of indeterminates, the same $\sum N_i$, and bilinear forms B equivalent over the ground field are deforms of one Lie algebra.

Conjecturally, the same statement

$$(33) \quad \text{“the vectorial Lie algebra } X(k; \underline{N}) \text{ is a deform of the tensor product } X(k; \underline{N}_s) \otimes \mathcal{O}, \text{ where } \mathcal{O} := \mathcal{O}(u; \tilde{N}) \text{ with an appropriate } \tilde{N}\text{”}$$

holds whenever the space of the Lie algebra $X(k; \underline{N})$ can be identified with \mathcal{O} , or the direct sum of several copies of \mathcal{O} each endowed with its extra structures of (associative) multiplication, and the bracket can be defined using only distinguished derivatives, associative and commutative multiplication of functions, and linear operations, e.g., for \mathfrak{vect} and \mathfrak{k} (cf. the proof of Theorem 5 in [15]).

⁸Any automorphism of the space $\mathcal{O}(d; \underline{N})$ produces a deformed bracket, but the second equality in (30) is due the fact that

$$\Phi_\alpha^{-1}(\partial_i \Phi_\alpha(F) \cdot \partial_j \Phi_\alpha(G)) = \Phi_\alpha^{-1}(\partial_i \Phi_\alpha(F)) \cdot \Phi_\alpha^{-1}(\partial_j \Phi_\alpha(G)).$$

4. The Jurman algebra is a semitrivial deform

4.1. The Jurman algebra

Jurman introduced a Lie algebra over $\mathbb{F}_2 = \{0, 1\}$ that seemed to have no analog over fields \mathbb{K} of characteristic $p \neq 2$ (see [25]) until its interpretation in [21]. Jurman constructed this algebra by, in a sense, doubling the *Zassenhaus algebra*, which is the derived of the *Witt algebra* $\mathbf{vect}(1; \underline{N})$. Jurman therefore called his algebra the *Bi-Zassenhaus algebra* denoted by $B(g, h)$. But the letter B is overused, and we wish to emphasize properties of the Lie algebra $B(g, h)$ that differ from the properties in which Jurman was interested. We therefore designate this algebra $\mathfrak{j}(g, h)$ in honor of Jurman. The following description (see [25]) allows extending the ground field and considering $\mathfrak{j}(g, h)$ over \mathbb{K} .

Let $g \geq 2$ and $h \geq 1$ be integers. Let $\eta = 2^g - 1$ and $\varkappa = 2^{g+h} \geq 8$. Taking the elements

$$(34) \quad \{Y_j(t) \mid t \in \{0, 1\}, j \in \{-1, 0, \dots, \varkappa - 3\}\}$$

as a basis in $\mathfrak{j}(g, h)$, Jurman defined the bracket by setting

$$(35) \quad [Y_i(s), Y_j(t)] = b_{s,t}^{i,j} Y_{i+j+st(1-\eta)}(s+t),$$

where (in the next formula, the binomial coefficients and their sum are considered modulo 2 and meaningless expressions are taken to be 0; see Example 4.1 for further elucidations of the meaning of the binomial coefficient for $s = t = 1$)

$$(36) \quad b_{s,t}^{i,j} = \begin{cases} \binom{i+j+st(2-\eta)}{i+1} + \binom{i+j+st(2-\eta)}{j+1} & \text{if } -1 \leq i+j+st(2-\eta) \leq \varkappa-3, \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.1. Let $(g, h) = (2, 1)$. We have $b_{1,1}^{i,-1} = \binom{i-2}{i+1} + \binom{i-2}{0}$. The first summand is meaningless for any i and should be understood as a 0. The second summand makes sense for $i \geq 2$ when it is equal to 1.

We have $b_{1,1}^{i,0} = \binom{i-1}{i+1} + \binom{i-1}{1}$. The first summand makes no sense for any i , and the second summand makes no sense for $i = -1, 0, 1$. Each of these meaningless binomial coefficients should be understood as 0. If $i > 1$, then $\binom{i-1}{1} \equiv i-1 \pmod{2}$.

We have $b_{1,1}^{i,1} = \binom{i}{i+1} + \binom{i}{2}$ with the first summand always meaningless (hence equal to 0) and the second summand equal to 0 for $i < 2$.

4.2. The Jurman algebra $\mathfrak{j}(g, h)$ as a deform of $\mathfrak{h}'_{\Pi}(2; (g, h + 1))$

To interpret the Jurman algebra $\mathfrak{j}(g, h)$ somehow, we compare it with a known simple Lie algebra. The most plausible comparison candidates can be found in [33], where all possible versions of Poisson Lie algebras were described in characteristic 2, as well as Lie (sub)algebras of Hamiltonian vector fields. We realize the Poisson Lie algebra $\mathfrak{po}_{\Pi}(2; \underline{N})$ by generating functions (divided powers) in the two indeterminates p and q with the bracket

$$(37) \quad \{F, G\} = \frac{\partial F}{\partial p} \frac{\partial G}{\partial q} + \frac{\partial F}{\partial q} \frac{\partial G}{\partial p} \quad \text{for any } F, G \in \mathcal{O}(2; \underline{N}),$$

where ∂_p and ∂_q are distinguished partial derivatives.

We consider the Lie algebra $\mathfrak{h}_{\Pi}(2; \underline{N}) = \mathfrak{po}_{\Pi}(2; \underline{N})/\mathbb{K} \cdot 1$ and its derived $\mathfrak{h}'_{\Pi}(2; \underline{N})$. We keep expressing the elements of \mathfrak{h}_{Π} and \mathfrak{h}'_{Π} via generating functions having in mind, by abuse of notation, their classes modulo the center of \mathfrak{po}_{Π} .

We recall (see [37]) that the *Weisfeiler filtrations* were initially used to describe infinite-dimensional vectorial Lie (super)algebras \mathcal{L} by selecting a maximal subalgebra \mathcal{L}_0 of finite codimension. Dealing with finite-dimensional algebras, we can confine ourselves to maximal subalgebras of *least* or “almost least” codimension. Let \mathcal{L}_{-1} be the minimal \mathcal{L}_0 -invariant subspace strictly containing \mathcal{L}_0 . For $i \geq 1$, we set

$$(38) \quad \begin{aligned} \mathcal{L}_{-i-1} &= [\mathcal{L}_{-1}, \mathcal{L}_{-i}] + \mathcal{L}_{-i} \\ \text{and } \mathcal{L}_i &= \{D \in \mathcal{L}_{i-1} \mid [D, \mathcal{L}_{-1}] \subset \mathcal{L}_{i-1}\}. \end{aligned}$$

We thus obtain a filtration,

$$(39) \quad \mathcal{L} = \mathcal{L}_{-d} \supset \mathcal{L}_{-d+1} \supset \cdots \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset \cdots .$$

The d in (39) is called the *depth* of \mathcal{L} and of the associated graded Lie superalgebra $\mathfrak{g} = \bigoplus_{-d \leq i} \mathfrak{g}_i$, where $\mathfrak{g}_i = \mathcal{L}_i/\mathcal{L}_{i+1}$.

We let \mathcal{L} denote $\mathfrak{j}(g, h)$ when considered with a *Weisfeiler filtration*. Eqs. (35) and (36) imply that assuming $g, h = \infty$, we have only one maximal

subalgebra of finite codimension:

$$\mathcal{L}_0 = \text{Span}(Y_i(0), Y_j(1) \mid i, j \geq 0).$$

Its maximality follows from table (47). The Weisfeiler filtration corresponding to the pair $(\mathcal{L}, \mathcal{L}_0)$ is

$$(40) \quad \mathcal{L} = \mathcal{L}_{-1} \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset \mathcal{L}_2 \cdots, \quad \text{where } \mathcal{L}_{i+1} = \{X \in \mathcal{L}_i \mid [\mathcal{L}, X] \subset \mathcal{L}_i\}.$$

Let $\text{gr } \mathfrak{j}(g, h) = \bigoplus \mathfrak{g}_i$, where $\mathfrak{g}_i = \mathcal{L}_i / \mathcal{L}_{i+1}$ for $i \geq -1$.

Proposition 4.2. *We have $\text{gr } \mathfrak{j}(g, h) \cong \mathfrak{h}'_{\Pi}(2; (g, h + 1))$.*

Proof. For brevity, we set $\mathfrak{h} := \mathfrak{h}'_{\Pi}(2; (g, h + 1))$ and $\mathfrak{g} := \text{gr } \mathfrak{j}(g, h)$. Note that every element of the Cartan prolong is uniquely determined by its brackets with the elements of the (-1) st component. In particular, any $X = p^{(\beta)}q^{(\gamma)} \in \mathfrak{h}$ is uniquely determined by the conditions

$$(41) \quad \begin{aligned} &(\text{ad}_p)^\gamma (\text{ad}_q)^{\beta-1}(X) = p, & (\text{ad}_p)^{\gamma-1} (\text{ad}_q)^\beta(X) = q & \text{ for } \beta\gamma > 0, \\ &(\text{ad}_q)^{\beta-1}(X) = p, & \text{ad}_p(X) = 0 & \text{ for } \beta > 1, \gamma = 0, \\ &(\text{ad}_p)^{\gamma-1}(X) = q, & \text{ad}_q(X) = 0 & \text{ for } \beta = 0, \gamma > 1. \end{aligned}$$

Let \bar{X} be the image of an $X \in \mathfrak{j}(p, q)$ in \mathfrak{g} . The definition of a filtration implies that $\dim \mathfrak{g}_{-1} = 2$ and $\mathfrak{g}_{-1} = \text{Span}(\overline{Y_{-1}(0)}, \overline{Y_{-1}(1)})$. We identify

$$\overline{Y_{-1}(0)} \longleftrightarrow q, \quad \overline{Y_{-1}(1)} \longleftrightarrow p.$$

Let $0 \leq \alpha \leq 2^h - 1$ and $0 \leq \beta \leq \eta = 2^g - 1$. Our next goal is to establish the correspondence

$$(42) \quad \overline{Y_i(s)} \longleftrightarrow p^{(\beta)}q^{(2\alpha+1-s)}, \quad \text{where } s = 0, 1 \text{ and } i = \alpha(\eta + 1) - 1 - s + \beta.$$

For manual computations, it is more convenient to consider the two cases $s = 0, 1$ separately:

$$(43) \quad \begin{aligned} \overline{Y_a(1)} &\longleftrightarrow p^{(\beta)}q^{(2\alpha)} & \text{for } a = \alpha(\eta + 1) - 2 + \beta, \\ \overline{Y_b(0)} &\longleftrightarrow p^{(\beta)}q^{(2\alpha+1)} & \text{for } b = \alpha(\eta + 1) - 1 + \beta. \end{aligned}$$

Let $A_{\gamma,\delta} := (\text{ad}_{Y_{-1}(0)})^\gamma (\text{ad}_{Y_{-1}(1)})^\delta$. Clearly, the image \bar{X} of $X \in \mathfrak{j}(p, q)$ belongs to \mathfrak{g}_k if and only if there exist γ and δ such that $\gamma + \delta = k + 1$ and $A_{\gamma,\delta}(X) \notin \mathcal{L}_0$ while $A_{\gamma,\delta}(X) \in \mathcal{L}_0$ for γ and δ such that $\gamma + \delta < k + 1$.

We now consider the brackets in the Lie algebra $\mathfrak{j}(p, q)$:

$$(44) \quad \begin{aligned} [Y_{-1}(0), Y_j(s)] &= \begin{cases} Y_{j-1}(s) & \text{if } j \geq 0, \\ 0 & \text{if } j = -1, \end{cases} \\ [Y_{-1}(1), Y_j(0)] &= \begin{cases} Y_{j-1}(1) & \text{if } j \geq 0, \\ 0 & \text{if } j = -1, \end{cases} \\ [Y_{-1}(1), Y_j(1)] &= \begin{cases} 0 & \text{if } j < \eta - 1, \\ Y_{j-\eta}(0) & \text{if } j \geq \eta - 1. \end{cases} \end{aligned}$$

Equations (44) for the elements $X \in \mathfrak{j}(p, q)$ of the form $X = Y_{\beta-2}(1)$ for $2 \leq \beta \leq \eta$ imply that

$$(45) \quad A_{\beta-1,0}(X) = Y_{-1}(1) \longleftrightarrow p, \quad A_{0,1}(X) = 0.$$

Expressions (45) mean that $X \in \mathfrak{g}_{\beta-2}$, and the element \bar{X} corresponds to $p^{(\beta)} \in \mathfrak{h}$. We have thus obtained the first correspondence in (43) for $\alpha = 0$.

Similarly, for $X = Y_{\beta-1}(0)$, where $1 \leq \beta \leq \eta$, we have

$$\begin{aligned} A_{\beta,0}(X) &= Y_{-1}(0) \longleftrightarrow q \quad \text{and} \\ A_{\beta-1,1}(X) &= Y_{-1}(1) \implies X \in \mathfrak{g}_{\beta-1} \quad \text{and} \\ X &\longleftrightarrow p^\beta q \in \mathfrak{h}, \end{aligned}$$

implying the second correspondence in Eq. (43) for $\alpha = 0$.

Equations (44) also imply that

$$(46) \quad (\text{ad}_{Y_{-1}(1)})^2(Y_j(s)) = \begin{cases} Y_{j-\eta-1}(1) & \text{if } j \geq \eta, \\ 0 & \text{if } j < \eta. \end{cases}$$

Therefore, for any $\alpha > 0$ and $\beta > 0$ and $X = Y_{\alpha(\eta+1)+\beta-2}(1)$, we have

$$\begin{aligned} (\text{ad}_{Y_{-1}(0)})^{\beta-1} \left(\text{ad}_{Y_{-1}(1)}^2 \right)^\alpha (X) &= A_{\beta-1,2\alpha}(X) = Y_{-1}(1) \longleftrightarrow p, \\ (\text{ad}_{Y_{-1}(0)})^\beta \left(\text{ad}_{Y_{-1}(1)}^2 \right)^{\alpha-1} \text{ad}_{Y_{-1}(1)}(X) &= A_{\beta,2\alpha-1}(X) = Y_{-1}(0) \longleftrightarrow q, \end{aligned}$$

implying the correspondence $\bar{X} \longleftrightarrow p^{(\beta)}q^{(2\alpha)}$. This provides the first correspondence in Eq. (43) for the case where $\alpha > 0$ and $\beta > 0$. We obtain the second correspondence in Eq. (43) for the case where $\alpha > 0$ and $\beta > 0$ absolutely analogously.

It remains to consider the case where $\alpha > 0$ and $\beta = 0$. Let $X = Y_{\alpha(\eta+1)-1}(0)$. Then

$$A_{0,2\alpha}(X) = Y_{-1}(0) \longleftrightarrow q \implies \bar{X} \in \mathfrak{g}_{2\alpha-1}.$$

But

$$[Y_{-1}(0), X] = Y_{\alpha(\eta+1)-2}(0) = Y_{(\alpha-1)(\eta+1)+(\eta-1)}(0) \longleftrightarrow p^{(\eta)}q^{(2\alpha-1)} \in \mathfrak{g}_{\eta+2\alpha-3}.$$

Since $\eta \geq 3$, it follows that $2\alpha - 1 < \eta + 2\alpha - 3$, and hence $[\overline{Y_{-1}(0)}, \bar{X}] = 0$, i.e., the element X corresponds to $q^{(2\alpha+1)}$. This provides the second correspondence in Eq. (43) for $\alpha > 0$ and $\beta = 0$. The first correspondence in Eq. (43) for this case is similarly obtained.

We see that the maximal power of p is equal to $\eta = 2^g - 1$, and hence $\underline{N}(p) = g$. Because $2\alpha + 1 \leq 2^{h+1} - 1$, it follows that $\underline{N}(q) = h + 1$. \square

Accordingly, the basis elements of the components of the first five degrees are as follows:

	\mathfrak{g}_{-1}	\mathfrak{g}_0	\mathfrak{g}_1	\mathfrak{g}_2	\mathfrak{g}_3
(47)	$p \longleftrightarrow \overline{Y_{-1}(1)}$	$p^{(2)} \longleftrightarrow \overline{Y_0(1)}$	$p^{(3)} \longleftrightarrow \overline{Y_1(1)}$	$p^{(4)} \longleftrightarrow \overline{Y_2(1)}$	$p^{(5)} \longleftrightarrow \overline{Y_3(1)}$
	$q \longleftrightarrow \overline{Y_{-1}(0)}$	$pq \longleftrightarrow \overline{Y_0(0)}$	$p^{(2)}q \longleftrightarrow \overline{Y_1(0)}$	$p^{(3)}q \longleftrightarrow \overline{Y_2(0)}$	$p^{(4)}q \longleftrightarrow \overline{Y_3(0)}$
		$q^{(2)} \longleftrightarrow \overline{Y_{\eta-1}(1)}$	$pq^{(2)} \longleftrightarrow \overline{Y_{\eta}(1)}$	$p^{(2)}q^{(2)} \longleftrightarrow \overline{Y_{\eta+1}(1)}$	$p^{(3)}q^{(2)} \longleftrightarrow \overline{Y_{\eta+2}(1)}$
			$q^{(3)} \longleftrightarrow \overline{Y_{\eta}(0)}$	$pq^{(3)} \longleftrightarrow \overline{Y_{\eta+1}(0)}$	$p^{(2)}q^{(3)} \longleftrightarrow \overline{Y_{\eta+2}(0)}$
				$q^{(4)} \longleftrightarrow \overline{Y_{2\eta}(1)}$	$pq^{(4)} \longleftrightarrow \overline{Y_{2\eta+1}(1)}$
				$q^{(5)} \longleftrightarrow \overline{Y_{2\eta+1}(0)}$	

Let $\{ \cdot, \cdot \}$ and $[\cdot, \cdot]$ denote the respective brackets in $\mathfrak{g} = \mathfrak{h}'_{\Pi}(2; (g, h + 1))$ and $\mathfrak{j}(g, h)$. Expressing the $\overline{Y_i(s)}$ in terms of monomials in p and q , we see that for the simplest case $g = h + 1$, the *Jurman cocycle* c (which deforms $\mathfrak{h}'_{\Pi}(2; (g, g))$ into the Jurman algebra) is (for any $F, G \in \mathcal{O}(2; (g, g))$), as direct calculations show,

$$(48) \quad [F, G] = \{F, G\} + m_c(F, G),$$

where $c = \sum_{m < n} p^{(\eta)}q^{(m+n-3)} \otimes d(q^{(m)}) \wedge d(q^{(n)}),$

and $m_c(F, G)$ (see Eq. (55)) is the map corresponding to the cocycle c .

We call all other cocycles (which do not deform $\mathfrak{h}'_{\Pi}(2; (g, g))$ to the Jurman algebra) *non-Jurman cocycles*.

With respect to the pair of operators $(\deg_p(\cdot) - 1, \deg_q(\cdot) - 1)$ the weight of the cocycle $F \otimes d(G_1) \wedge \dots \wedge d(G_n)$ for any $F, G_1, \dots, G_n \in \mathfrak{h}'_{\Pi}(2; (g, g))$

is equal to

$$(49) \quad \left(\deg_p(F) - 1 - \sum (\deg_p(G_i) - 1), \deg_q(F) - 1 - \sum (\deg_q(G_i) - 1) \right).$$

So, the Jurman cocycle has the weight $(2^g, -2)$. By the symmetry $p \longleftrightarrow q$, there is *another Jurman cocycle* of weight $(-2, 2^g)$ leading to an isomorphic Jurman algebra.

If $g \neq h + 1$, then there is no symmetry $p \longleftrightarrow q$, but there still is *another Jurman cocycle* making $\mathfrak{h}'_{\Pi}(2; (g, h + 1))$ into $\mathfrak{j}(h + 1, g - 1)$. It is of the form

$$(50) \quad c = \sum_{m < n} q^{(\theta)} p^{(m+n-3)} \otimes d(p^{(m)}) \wedge d(p^{(n)}), \text{ where } \theta = 2^{h+1} - 1.$$

If $p > 2$, most of the cocycles representing classes of $H^2(\mathfrak{g}; \mathfrak{g})$ are not integrable for the simple vectorial Lie algebras \mathfrak{g} (see [15]). If $p = 2$, we do not know any simple Lie algebra \mathfrak{g} with a nonintegrable cocycle representing a class of $H^2(\mathfrak{g}; \mathfrak{g})$.

Lemma 4.3 $((g, h) = (2, 1), (2, 2);$ **conjecture for generic values of (g, h)**). *Any linear combination of cocycles representing classes of $H^2(\mathfrak{g}; \mathfrak{g})$ for $\mathfrak{g} = \mathfrak{h}'_{\Pi}(2; (g, h + 1))$ can be integrated to a global deform (Cf. Lemma 6.5 for $\mathfrak{g} = \mathfrak{h}_I(2; (g, h + 1))$).*

For $g + h = g' + h' = K$, the Jurman algebras $\mathfrak{j}(g, h)$ and $\mathfrak{j}(g', h')$ regarded as $\mathbb{Z}/2$ -graded Lie algebras $\mathfrak{j} = \mathfrak{j}_0 \oplus \mathfrak{j}_1$, with \mathfrak{j}_0 spanned by the $Y_i(0)$ for all i , have these even parts isomorphic, and the odd parts, as modules over the even part, are also isomorphic. This is clear from Eqs. (35) and (36). We note that the brackets of two odd elements given by Jurman’s cocycles can be united into one bracket depending on as many parameters as there are partitions $K = g + h$ with $g \geq 2$ and $h \geq 1$. To see this, we consider (a) the brackets of two “odd” elements and one “even” element and (b) the brackets of three “odd” elements; the statement is obvious in both cases. The bracket obtained linearly depends on all K parameters.

Proposition 4.4. *For $0 \leq k < 2^{h+1}$ and $0 \leq l < 2^{g-1}$, the isomorphism $\mathfrak{j}(g, h) \simeq \mathfrak{h} := \mathfrak{h}'_{\Pi}(2; (g, h + 1))$ is given by the maps*

$$(51) \quad Y_{2^{g-1}k+l-1}(0) \longleftrightarrow Y_{2^{g-1}k+l-1} = p^{(2^{g-1}+l)} q^{(k)} + (k + 1)p^{(l)} q^{(k+1)},$$

$$(52) \quad Y_{2^{g-1}k+l-2}(1) \longleftrightarrow Z_{2^{g-1}k+l-2} = \begin{cases} p^{(2^{g-1}+l)} q^{(k-1)} + (k + 1)p^{(l)} q^{(k)} & \text{if } k > 0, \\ p^{(l)} & \text{if } k = 0. \end{cases}$$

Assuming that $q^{(m)} = 0$ for $m < 0$, we can express Z in (52) uniformly:

$$(53) \quad Z_{2^{g-1}k+l-2} = p^{(2^{g-1}+l)}q^{(k-1)} + (k+1)p^{(l)}q^{(k)} \quad \text{for all values } 0 \leq k < 2^h.$$

Proof. Taking into account that $(k + 1) \bmod 2 = 0$ or 1 , we can unite correspondences (51) and (52) by considering the 1-parameter family of maps (we note that $\text{ad}_{p^{(\mu+1)}}$ is a derivation of \mathfrak{h} such that $\text{ad}_{p^{(\mu+1)}}^2 = 0$):

$$\Phi_t : \mathfrak{h} \longrightarrow \mathfrak{h}, \quad F \mapsto F + tp^{(\mu)} \frac{\partial F}{\partial q} = F + t \text{ad}_{p^{(\mu+1)}}(F), \quad \text{where } \mu = 2^{g-1}.$$

Let $m_c(F, G)$ be the map (54) corresponding to the cocycle c (see (48)). Direct computations show that $\Phi_{\sqrt{t}}$ is an isomorphism of Lie algebras because

$$\{\Phi_t(F), \Phi_t(G)\} = \Phi_t(\{F, G\}) + t^2 m_c(F, G). \quad \square$$

4.2.1. Deforms of $\mathfrak{h}'_{\Pi}(2; (g, h + 1))$ for the smallest values of (g, h) .

Clearly, if $g = h + 1$, it suffices to consider only cocycles of nonnegative weight because of the symmetry $p \longleftrightarrow q$.

Let $(g, h) = (2, 1)$. Here are cocycles representing a basis of the space $H^2(\mathfrak{g}; \mathfrak{g})$. The Jurman cocycle $c := c_{2^g, -2}$ in (48) is $c_{4, -2}$ from our list (59) below.

Proposition 4.5. *Here, $F, G \in \mathfrak{h}'_{\Pi}(2; (g, h + 1))$ are arbitrary, $\{\cdot, \cdot\}$ is the Poisson bracket of functions generating $\mathfrak{h}_{\Pi}(2m; \underline{N})$, and $\hbar \in \mathbb{K}$.*

The following cocycles are semitrivial:

1. *The Jurman cocycle $c_{2^g, -2}$ (see (48)) represents the map*

$$(54) \quad p^{(\eta)}(\partial_q F \cdot \partial_q^2 G + \partial_q^2 F \cdot \partial_q G), \quad \text{where } \eta = 2^g - 1.$$

For $(g, h) = (2, 1)$, the cocycle $c_{4, -2}$ (see (59)) represents the map

$$(55) \quad m_{4, -2}(F, G) = p^{(3)}(\partial_q F \cdot \partial_q^2 G + \partial_q^2 F \cdot \partial_q G).$$

2. *The cocycle $c_{0, -4}$ (see (59)) represents the map whose shape is independent of (g, h) :*

$$(56) \quad m_{0, -4}(F, G) = \partial_p \partial_q^2 F \cdot \partial_q^3 G + \partial_q^3 F \cdot \partial_p \partial_q^2 G = \{\partial_q^2 F, \partial_q^2 G\}.$$

3. *The cocycle $c_{0, -2}$ (see (59)) is equivalent to the cochain that represents the map whose shape is independent of (g, h) :*

$$(57) \quad m_{0, -2}(F, G) = \{\partial_q F, \partial_q G\}.$$

The following cocycles are nontrivial:

4. The cocycle $c_{2,0}$ (see (59)) is equivalent to the cochain representing the map

$$(58) \quad m_{2,0}(F, G) = p^{(\eta)}(\partial_q F \cdot \partial_p^2 G + \partial_p^2 F \cdot \partial_q G), \quad \text{where } \eta = 2^g - 1,$$

which yields one of the filtered deforms (see [15]).

5. The cocycle $c_{-2,-2}$ (see (59)) is inherited from the quantization of the Poisson Lie algebra $\mathfrak{po}_\Pi(2; (a, a))$ being the linear in the Planck constant part of the cocycle restricted to the subquotient \mathfrak{h}'_Π of \mathfrak{po}_Π . The deformation turns $\mathfrak{h}'_\Pi(2; (a, a))$ into $\mathfrak{psl}(2^a)$ for any a .

The index of each cocycle is equal to its weight (further on, to save trees, we give full expression of only the cocycles with short expressions; the lexicographic order of summands adding up to the cocycle makes it possible to distinguish cocycles by looking at the pieces displayed; if these pieces are insufficient to interpret them, then see the \TeX file in arXiv and uncomment the hidden terms):

$$(59) \quad \begin{aligned} c_{4,-2} &= p^{(3)} \otimes d(q) \wedge d(q^{(2)}) + p^{(3)} q \otimes d(q) \wedge d(q^{(3)}) + p^{(3)} q^{(2)} \otimes d(q^{(2)}) \wedge d(q^{(3)}), \\ c_{0,-4} &= p \otimes d(pq^{(2)}) \wedge d(pq^{(3)}) + p \otimes d(q^{(3)}) \wedge d(p^{(2)}q^{(2)}) + q \otimes d(q^{(3)}) \wedge d(pq^{(3)}) + \dots \\ c_{2,0} &= p^{(2)} \otimes d(p) \wedge d(q) + pq^{(2)} \otimes d(q) \wedge d(q^{(2)}) + p^{(3)} q \otimes d(q) \wedge d(p^{(2)}q) \\ &\quad + p^{(3)} q^{(2)} \otimes d(p) \wedge d(pq^{(3)}) + p^{(3)} q^{(2)} \otimes d(q^{(2)}) \wedge d(p^{(2)}q) + p^{(2)} q^{(3)} \otimes d(q) \wedge d(pq^{(3)}), \\ c_{0,-2} &= p \otimes d(p) \wedge d(pq^{(3)}) + p \otimes d(pq) \wedge d(pq^{(2)}) + p \otimes d(q^{(2)}) \wedge d(p^{(2)}q) \\ &\quad + q \otimes d(q) \wedge d(pq^{(3)}) + q \otimes d(pq) \wedge d(q^{(3)}) + q \otimes d(q^{(2)}) \wedge d(pq^{(2)}) + \dots \\ c_{-2,-2} &= p \otimes d(pq^{(2)}) \wedge d(p^{(3)}q) + q \otimes d(pq^{(2)}) \wedge d(p^{(2)}q^{(2)}) + q \otimes d(q^{(3)}) \wedge d(p^{(3)}q) \\ &\quad + p^{(2)} \otimes d(p^{(3)}q) \wedge d(p^{(2)}q^{(2)}) + pq \otimes d(pq^{(2)}) \wedge d(p^{(3)}q^{(2)}) + \dots \end{aligned}$$

Proof. The fact that the maps m_w , where w is a weight, do correspond to the cocycles c_w as claimed is subject to a direct verification. The idea is as follows. We see that the image of $c_{4,-2}$ is always divisible by $p^{(3)}$ and that this image is nonzero only if both arguments are polynomials (divided powers) in q . Taking into account the weight of the cocycle and its (anti)symmetry, we seek m_c in the form

$$p^{(3)}(A(F\partial_q^3 G + G\partial_q^3 F) + B(\partial_q F \cdot \partial_q^2 G + \partial_q^2 F \cdot \partial_q G)), \quad \text{where } A, B \in \mathbb{K}.$$

It turns out that for $A = 0$ and $B = 1$, we obtain the desired. For other cocycles, we seek the operators in the form $D_1 F \cdot D_2 G + D_2 F \cdot D_1 G$ or $p^{(3)}(D_1 F \cdot D_2 G + D_2 F \cdot D_1 G)$, where the D_i are compositions of some derivations. We must check if any of these operators m_c in fact matches c . It could

be that to have a nice expression for m_c , we must replace c with a \tilde{c} of the same cohomology class.

1. The semitriviality of the Jurman cocycle is explicitly proven for arbitrary (g, h) in Proposition 4.4.

2. We consider the maps $\Phi_{\hbar}(F) = F + \hbar DF$, where $D = \partial_q^2$. Because $D^2 = 0$, it follows that the corresponding deformed bracket produced by $c_{0,-4}$ is

$$(60) \quad \{F, G\}_{\hbar}^{c_{0,-4}} := \{F, G\}_{\sqrt{\hbar}}^{\Phi}.$$

3. In this case, although $D^2 \neq 0$ for $D = \partial_q$, the derivation D is still nilpotent, and Theorem 2.3 is hence applicable here.

4. In this case, the deformed bracket is equivalent to

$$(61) \quad \{F, G\}_{\hbar} = (\partial_p + \hbar p^{(3)} \partial_q^2) F \cdot \partial_q G + \partial_q F \cdot (\partial_p + \hbar p^{(3)} \partial_q^2) G.$$

5. Let the Poisson Lie algebra be realized by the Poisson bracket on the space of functions in $\vec{p} = (p_1, \dots, p_m)$ and $\vec{q} = (q_1, \dots, q_m)$. We consider the deformation (over the ground field $\mathbb{F} = \mathbb{C}$ or \mathbb{R} , physicists call it *quantization*) that turns the Poisson Lie algebra into the Lie algebra of differential operators with polynomial coefficients (see Sect. 1.4.7 in [20]). The cocycle that determines *quantization* corresponds to the map

$$(62) \quad \mathcal{Q}(F, G) = \sum_{1 \leq i \leq m} \frac{\partial^2 F}{\partial p_i^2} \frac{\partial^2 G}{\partial q_i^2} - \frac{\partial^2 F}{\partial q_i^2} \frac{\partial^2 G}{\partial p_i^2} \quad \text{for any } F, G \in \mathbb{F}[p, q].$$

Here, we encounter an analog of quantization over $\mathbb{F} = \mathbb{K}$ for $\text{char } \mathbb{K} = 2$. Let the coordinates of the shearing vector corresponding to \vec{p} be the same as those corresponding to \vec{q} . Let $\hat{\cdot} : F \mapsto \hat{F}$ be the map that to any monomial $F \in \mathcal{O}(\vec{p}, \vec{q}; (\underline{N}, \underline{N}))$ ordered such that each p_i is to the left of all the q_j for all i and j assigns a differential operator obtained by the replacement $q_i \mapsto \hbar \partial_{p_i}$, where $\hbar \in \mathbb{K}$, for each i . All linear operators in the finite-dimensional space $\mathcal{O}(\vec{p}; \underline{N})$ are differential, and the deformed Lie algebra is hence isomorphic to $\mathfrak{gl}(\mathcal{O}(\vec{p}; \underline{N})) \simeq \mathfrak{gl}(2^{|\underline{N}|})$, where $|\underline{N}| = \sum \underline{N}_j$. Clearly, the same cocycle induces a deformation of $\mathfrak{h}'(2m; \underline{N})$ into $\mathfrak{psl}(2^{|\underline{N}|})$. For any $\hbar \neq 0$, the deforms are obviously isomorphic (use rescaling, i.e., divide by \hbar), and the commutator of differential operators is related to the Poisson bracket as

$$(63) \quad [\hat{F}, \hat{G}] = \{F, G\}_{P.b.} + O(\hbar) \quad \text{for any } F, G \in \mathcal{O}(\vec{p}, \vec{q}; (\underline{N}, \underline{N})).$$

For $m = 1$, the weight of the cocycle linear in \hbar in the right side of Eq. (63) (up to a sign corresponding to the interchange $p \leftrightarrow q$) is precisely $(2, 2)$. \square

For $\mathfrak{g} = \mathfrak{h}'_{\Pi}(2; (g, h + 1))$, where $(g, h) = (2, 1)$, the cocycles considered in Proposition 4.5 represent a basis of $H^2(\mathfrak{g}; \mathfrak{g})$. Because $\dim H^2(\mathfrak{g}; \mathfrak{g})$ grows together with the coordinates of the shearing vector (g, h) , there are more deformations to be interpreted in the general case. It seems reasonable to switch attention from cocycles c to maps m_c . **Conjecturally**, all nonJurman cocycles correspond to the filtered deforms classified by Skryabin (see [43]) or to the quantization. This is so for $(g, h) = (2, 1)$.

5. What Kaplansky algebras are isomorphic to. Nonlinear superizations

In 1981, Kaplansky described four types (in fact, five: the dimensions of the two cases of the type-4 algebras differ significantly) of simple Lie algebras for $p = 2$ (see [26]). He described them only by means of the multiplication table. We interpret them in terms of familiar Lie algebras of vector fields.

Kaplansky defined the algebras in terms of *J-systems* resembling the notion of a root system. Over \mathbb{F}_2 , a *J-system* Γ in the space V with a symmetric inner product B is a set of nonzero vectors with the property that if $u, v \in \Gamma$ are distinct and satisfy $B(u, v) = 1$, then $u + v \in \Gamma$. Given any *J-system* Γ , we construct a Lie algebra \mathfrak{g}_{Γ} over \mathbb{F}_2 with basis elements e_u for every $u \in \Gamma$, and the multiplication given by the expressions

$$(64) \quad [e_u, e_v] = \begin{cases} B(u, v)e_{u+v} & \text{for } u, v \text{ distinct and } u + v \in \Gamma, \\ 0 & \text{for } u + v \notin \Gamma \text{ or } u = v. \end{cases}$$

We note that the second half of the lower property in (64) is automatically satisfied if the form B is alternating. Each Kaplansky algebra $\text{Kap}_i(n)$, where $i = 1, 2, 3, 4$, has the form \mathfrak{g}_{Γ} for some Γ .

Any algebra defined over \mathbb{F}_2 can obviously be defined over \mathbb{K} by extension of the ground field. In what follows, speaking of Kaplansky algebras, we assume that such an extension is performed unless otherwise specified.

Kap₁(n): For $n \geq 4$, let $\dim V = n$, and let V carry a nondegenerate and nonalternate inner product B . Let e^1, \dots, e^n be an orthonormal basis of V . For Γ , we take all vectors in V except 0 and $e = e^1 + \dots + e^n$, which is the unique element satisfying $B(e, y) = B(y, y)$ for all y . Clearly, $B \sim I$. We recall that the brackets in $\mathfrak{lh}_I(V; \underline{N}) := (V, \mathfrak{o}'_I(V))_{*, \underline{N}}$ (\mathfrak{l} is for “little”) and $\mathfrak{h}_I(V; \underline{N}) := (V, \mathfrak{o}_I(V))_{*, \underline{N}}$ are the same (see [33]):

$$\{F, G\}_I := \sum_{1 \leq i \leq n} \partial_{z_i} F \cdot \partial_{z_i} G, \quad \text{where } F, G \in \mathbb{K}[z; \underline{N}],$$

only the stocks of generating functions of these Lie algebras differ. We make the assignment $e_u \longleftrightarrow \prod_{1 \leq i \leq n} (1 + z_i)^{u_i}$. (We note that Kaplansky considered monomials in $X_i := 1 + z_i$ instead of monomials in z_i .) We have $\text{Kap}_1(n) \simeq \mathfrak{h}'_I(n)$ because $\text{Kap}_1(n)$ does not contain e . In particular, we have an interpretation of $\text{Kap}_1(4)$ sought, but not found, in [21, 25]. Eick proved the isomorphism for $n = 4$ (in different terms) in [18].

$\text{Kap}_2(2m)$: Let $\dim V = 2m$, and let V carry a nondegenerate and alternate inner product Π . We take all nonzero vectors in V . Kaplansky mentioned this algebra because it fits into the approach he suggested although this algebra has analogs for any characteristic⁹ $p > 0$, and we could hence have ignored it; it is a filtered deform of $\mathfrak{h}_\Pi(2m; \underline{N}_s)$. If we had ignored it, then we would not have discovered a nonlinear $\mathbb{Z}/2$ -grading.

$\text{Kap}_3(n) = \mathfrak{o}'_I(n)$, as Kaplansky observed (in different terms). Kaplansky wrote “the gaps (in the set of values of $n = 5, 7$, and ≥ 9) avoid duplication.”

$\text{Kap}_{4,a}(2m)$, where $a = 0$ or 1 , is a temporary notation, for lack of a better idea, for two similarly described and equally mysterious algebras of quite different dimensions. In their description, we need Arf invariants of quadratic forms. For a most lucid definition of an Arf invariant, see [14]. In Eq. (65), a is the value of the Arf invariant (here, 0 or 1), and B is short for “Big” and is reminiscent of the form B (see Eq. (66)).

Let $\dim V = 2m$, where $m \geq 3$, and let Q be a nondegenerate quadratic form on V . We set

$$(65) \quad \begin{aligned} \text{Kap}_{4,a}(2m) &:= \mathfrak{g}_{\Gamma_a}(2m) \quad \text{for } \Gamma_a = \{u \in V \mid Q(u) = 1\} \text{ and } a = \text{Arf}(Q), \\ \text{Kap}_{4,B}(2m) &:= \mathfrak{g}_{\Gamma_B}(2m) \quad \text{for } \Gamma_B = \{u \in V\}, \end{aligned}$$

where the alternating bilinear form B is given by the formula

$$(66) \quad B(u, v) = Q(u + v) + Q(u) + Q(v).$$

We note that several quadratic forms Q , nonequivalent and with different values of the Arf invariant, in Eq. (66) can produce the same bilinear form B . Observe that

$$\text{Kap}_{4,a}(2m) \subset \text{Kap}_2(2m) \subset \text{Kap}_{4,B}(2m).$$

⁹Kaplansky did not describe such algebras explicitly. Here is a description for any $p > 0$: Consider the polynomial algebra in $y_i := \exp(x_i)$ and set $\partial_{x_i} y_j = \delta_{ij} y_j$ and $(y_i)^p = \exp(px_i) = 1$. In the space $\mathbb{K}[y]$, we introduce the Poisson bracket. Then $\text{Kap}_2(2m)$ is isomorphic to the quotient of the Poisson algebra modulo the ideal of constants.

Proposition 5.1. 1. The Lie algebra $\text{Kap}_{4,B}(2m)$ is isomorphic to the algebra whose space is $\mathcal{O}(2m; \underline{N}_s)$ with indeterminates p_i and q_i for $1 \leq i \leq m$ and the bracket

$$(67) \quad [f, g] = \sum_{1 \leq i \leq m} (1 + p_i)(1 + q_i)(\partial_{p_i} f \cdot \partial_{q_i} g + \partial_{q_i} f \cdot \partial_{p_i} g).$$

2. The Lie algebra $\text{Kap}_{4,B}(2m)$ is isomorphic to a deform of the Poisson algebra $\mathfrak{po}_{\Pi}(2m; \underline{N}_s)$ with the deformed bracket

$$(68) \quad [f, g]_{\hbar} = \sum_{1 \leq i \leq m} (1 + \hbar p_i q_i)(\partial_{p_i} f \cdot \partial_{q_i} g + \partial_{q_i} f \cdot \partial_{p_i} g) \quad \text{for any } \hbar \neq 0$$

and

$$(69) \quad \text{Kap}_{4,B}(2m) \simeq \text{Kap}_2(2m) \oplus \mathfrak{c},$$

where the center \mathfrak{c} is generated by constant functions.

Proof. 1. The isomorphism is given as follows. We choose a symplectic basis for the inner product B in V . If (u_1, \dots, u_{2m}) are coordinates of a vector $u \in V$ in this basis, then

$$e_u \longleftrightarrow f_u = (1 + p_1)^{u_1} \cdots (1 + p_m)^{u_m} (1 + q_1)^{u_{m+1}} \cdots (1 + q_m)^{u_{2m}}.$$

2. Clearly, (67) is a particular case of the bracket

$$(70) \quad [f, g]_{\hbar} = \sum_{1 \leq i \leq m} (1 + \hbar' p_i)(1 + \hbar' q_i)(\partial_{p_i} f \cdot \partial_{q_i} g + \partial_{q_i} f \cdot \partial_{p_i} g) \quad \text{with } \hbar' = 1.$$

Here, the part linear in \hbar' describes a trivial deformation of $\mathfrak{po}_{\Pi}(2m; \underline{N}_s)$ (as can be verified), and the quadratic part corresponds to (68) with $\hbar = (\hbar')^2$. This cocycle is nontrivial, as a computer-aided study shows.

The center is a direct summand because all weight spaces in $\text{Kap}_{4,B}(2m)$ are 1-dimensional, and the weight of the space generated by constants is 0, but there are no two distinct weight vectors of the same weight. \square

5.0.1. Kaplansky algebras $\text{Kap}_{4,B}(2m)$ and $\text{Kap}_{4,a}(2m)$ in convenient indeterminates. Examples of forms Q_a with an Arf invariant equal

to a are

$$(71) \quad \begin{aligned} Q_0(u) &= \sum_{1 \leq i \leq m} u_i u_{m+i}, \\ Q_1(u) &= u_1^2 + u_{m+1}^2 + \sum_{1 \leq i \leq m} u_i u_{m+i}, \end{aligned}$$

We introduce operators L_i , where $i = 1, \dots, 2m$:

$$L_i = \begin{cases} (1 + p_i) \partial_{p_i} & \text{if } 1 \leq i \leq m, \\ (1 + q_{i-m}) \partial_{q_{i-m}} & \text{if } m + 1 \leq i \leq 2m. \end{cases}$$

Then $L_i f_u = u_i f_u$. We set $\Delta = \sum_{1 \leq i \leq m} L_i L_{i+m}$.

The nonzero elements f_u such that $Q_a(u) = 1$ span the subalgebras $\text{Kap}_{4,a}(2m) \subset \text{Kap}_{4,B}(2m)$ with bracket (67). The definition (65) yields the conditions that single out the subalgebras $\text{Kap}_{4,a}(2m)$ in $\text{Kap}_{4,B}(2m)$:

$$(72) \quad \begin{aligned} f + \Delta f &= 0 && \text{for } \text{Kap}_{4,0}(2m), \\ f + (1 + p_1) \partial_{p_1} f + (1 + q_1) \partial_{q_1} f + \Delta f &= 0 && \text{for } \text{Kap}_{4,1}(2m). \end{aligned}$$

The condition $\Delta f + f = 0$ in (72) singles out the eigenvectors of Δ with the eigenvalue 1. But

$$\Delta f_u = \sum_{1 \leq i \leq m} u_i u_{i+m} f_u = Q_0(u) f_u,$$

and this eigenspace is therefore spanned by all f_u such that $Q_0(u) = 1$, which is exactly the image of $\text{Kap}_{4,0}$.

The case of $\text{Kap}_{4,1}$ is similar. For simplicity, we replace $L_1^2 f$ with $L_1 f$ and $L_{m+1}^2 f$ with $L_{m+1} f$. This is possible because $L_i^2 = L_i$ and the u_i only take values 0 and 1. Indeed,

$$L_i^2 f_u = u_i^2 f_u = u_i f_u = L_i f_u.$$

Kaplansky claimed (and we see that the claim obviously follows from Eq. (71)) that

$$(73) \quad \dim \mathfrak{g}_{\Gamma_a} = 2^{m-1} (2^m - (-1)^a) = \begin{cases} 2^{m-1} (2^m - 1) & \text{if } \text{Arf}(Q) = 0, \\ 2^{m-1} (2^m + 1) & \text{if } \text{Arf}(Q) = 1. \end{cases}$$

We now study the structure of these algebras. It is more convenient to pass to the coordinates $x_i := (1 + p_i)$ and $y_i := (1 + q_i)$. The bracket (67)

and operators (72) become

$$(74) \quad [f, g] = \sum_{1 \leq i \leq m} x_i y_i (\partial_{x_i} f \cdot \partial_{y_i} g + \partial_{y_i} f \cdot \partial_{x_i} g)$$

and

$$(75) \quad \begin{aligned} & \left(1 + \sum_{1 \leq i \leq m} x_i y_i \partial_{x_i} \partial_{y_i}\right) f = 0 && \text{for Kap}_{4,0}(2m), \\ & \left(1 + x_1 \partial_{x_1} + y_1 \partial_{y_1} + \sum_{1 \leq i \leq m} x_i y_i \partial_{x_i} \partial_{y_i}\right) f = 0 && \text{for Kap}_{4,1}(2m). \end{aligned}$$

For example,

$$(76) \quad \begin{aligned} \text{Kap}_{4,0}(2) &= \text{Span}(x_1 y_1), \quad \text{Kap}_{4,1}(2) \simeq \mathfrak{o}'_{\Pi}(3) \simeq \mathbf{vect}'(1; (2)), \\ \text{Kap}_{4,0}(4) &\simeq \mathfrak{o}'_{\Pi}(3) \oplus \mathfrak{o}'_{\Pi}(3), \quad \text{Kap}_{4,1}(4) \simeq \text{Kap}_3(5) = \mathfrak{o}'_{\Pi}(5). \end{aligned}$$

5.0.2. Gradings and derivations. The commutative subalgebra \mathfrak{h} in the algebra $\mathfrak{der}(\mathfrak{g})$ of derivations of the type-2 or type-4 Kaplansky algebra \mathfrak{g} , i.e., the subalgebra \mathfrak{h} that determines the $(\mathbb{Z}/2)^{2m}$ -grading Kaplansky used to construct \mathfrak{g} , is not the maximal torus \mathfrak{t} in $\mathfrak{der}(\mathfrak{g})$. Clearly, the Kaplansky algebras of type 2 and type 4 are $(\mathbb{Z}/2)^{2m}$ -graded by degrees modulo 2 with respect to each indeterminate x_i and y_i ; hence,

$$\mathfrak{h} = \text{Span}(x_i \partial_{x_i}, y_i \partial_{y_i} \mid i = 1, \dots, m).$$

On the other hand, there exists an element $D \in \mathfrak{t}$ commuting with all elements of \mathfrak{h} but not belonging to \mathfrak{h} . Equivalently, there exists a basis of \mathfrak{g} simultaneously homogeneous with respect to both the $(\mathbb{Z}/2)^{2m}$ -grading Kaplansky used and an extra $\mathbb{Z}/2$ -grading given by D (which is a second-order operator; see (75)), and this extra grading cannot be linearly expressed via the $(\mathbb{Z}/2)^{2m}$ -grading. We explain why this situation is remarkable.

It might be thought that we should have taken the maximal torus \mathfrak{t} from the very beginning. The catch is that in all cases we know, except for these Kaplansky algebras, the extra grading operator “splits” some of the weight spaces of the previous grading. For each of these Kaplansky algebras, this is not the case: the weight spaces of the $(\mathbb{Z}/2)^{2m}$ -grading are already 1-dimensional (except the weight-0 space if we consider the 2-closure of the algebra, but this weight-0 space does not split, anyway). Therefore, the weight spaces cannot be split further. Hence, it seems there is nowhere the extra grading can appear from, but it does appear.

We note that the derivation might be given by a differential operator of order > 1 but the corresponding grading might still be “linear” in a sense. We consider the Witt Lie algebra W_n over $\mathbb{K} = \mathbb{F}_{2^n}$, where $n > 1$. For its basis, we take $\{e_\alpha\}_{\alpha \in \mathbb{K}}$ with the relations $[e_\alpha, e_\beta] = (\beta - \alpha)e_{\alpha+\beta}$. In fact, W_n is $\mathbf{vect}(1; (n))$ over \mathbb{K} . On W_n , there is a natural grading: $\deg(e_\alpha) = \alpha$.

Now, consider a new grading: $\deg_{new}(e_\alpha) = \alpha^2$, which resembles the “nonlinear” gradings of Kaplansky algebras. Indeed, all weight spaces are 1-dimensional with respect to the old grading, and the new grading is expressed nonlinearly in terms of the old grading if $n > 1$.

But if the new grading is regarded as $(\mathbb{Z}/2)^n$ -grading (recall that $\mathbb{K} = \mathbb{F}_{2^n} = (\mathbb{Z}/2)^n$ as a vector space), then the new weight is obtained from the old weight by a linear transformation. The function $f : \alpha \mapsto \alpha^2$ is linear in the sense that $f(\alpha + \beta) = f(\alpha) + f(\beta)$, and at the same time it is nonlinear in the sense that it is not true that

$$(77) \quad f(c\alpha) \neq cf(\alpha) \quad \text{for any } c \in \mathbb{K}.$$

The condition (77) holds only if $c = 0$ or 1 , i.e., for $n = 1$.

5.0.3. Gradings not given by derivations. We also note that over a field \mathbb{K} of characteristic $p > 0$, a (super)algebra \mathfrak{g} can have gradings not given by any derivations of \mathfrak{g} . Indeed, consider $\mathfrak{g} := \mathfrak{l} \otimes \mathbb{K}[x]/(x^3 - 1)$ for some algebra \mathfrak{l} , where the polynomials in x are usual ones, not divided powers. Obviously, \mathfrak{g} is $\mathbb{Z}/3$ -graded. Because $\text{Hom}(\mathbb{Z}/q, \mathbb{Z}/p) = \{0\}$ for primes $q \neq p$, there is no derivation of \mathfrak{g} that determines this $\mathbb{Z}/3$ -grading if $p \neq 3$. The same applies to $\mathbb{Z}/4$ -gradings of the simple Lie algebras for $p = 3$, see [24].

5.0.4. The invariant symmetric bilinear forms. Kaplansky claimed that each Kaplansky algebra of type 2, 3, or 4 has a nondegenerate invariant bilinear symmetric form (we call it K here) and several other interesting properties whose verification “is quite routine.” Unlike Kaplansky, we think that a lucid proof of these properties is also of interest. Here, we prove the existence of the invariant form K . The description of K in presence of the alternate form B is very simple:

$$(78) \quad K(e_u, e_v) = \delta_{u,v}.$$

The form K is invariant, i.e.,

$$K([e_u, e_z], e_v) = K(e_u, [e_z, e_v])$$

because

if $u + z \neq v$, then $u \neq z + v$, and both sides vanish, and
 if $u + z = v$ (and $u = z + v$), then the l.h.s. is $K(B(u, z)e_v, e_v) = B(u, z)$
 and the r.h.s. is $B(z, v) = B(z, u + z) = B(z, u)$ because B is alternate and hence
 $B(z, z) = 0$.

We cannot guess how Kaplansky reasoned in the case of the nonalternate form B . In the case of the alternate form B , our argument relies on the invariant form on the Poisson Lie algebra induced by (the “desuperization” of) the Berezin integral¹⁰

$$(79) \quad K(f, g) = \int fg := \text{the coefficient of the highest term of } fg$$

if the Poisson algebra $\mathfrak{po}_\Pi(n; \underline{N}_s)$ is regarded as a “desuperization” of the Lie superalgebra $\mathfrak{po}(0|n)$, i.e., if the space of $\mathfrak{po}(0|n)$, the Grassmann superalgebra, is identified with the algebra of truncated polynomials in even indeterminates.

5.1. The restricted closures of Kaplansky algebras

Over \mathbb{F}_2 , the 2-closures of $\mathfrak{g} = \text{Kap}_2(2m)$ and $\text{Kap}_{4,a}(2m)$, except¹¹ $\text{Kap}_{4,0}(2)$, can be described as follows. Let the space of the closure be $\mathfrak{g} \oplus V^*$. We set

$$(80) \quad [\alpha, \beta] = 0, \quad [\alpha, e_u] = \alpha(u)e_u \quad \text{for any } \alpha, \beta \in V^*, e_u \in \mathfrak{g}.$$

For a fixed $u \in V$, let $B_u \in V^*$ be the map

$$(81) \quad B_u : v \mapsto B(u, v) \quad \text{for any } v \in V.$$

We can then define squaring by setting

$$(82) \quad \alpha^{[2]} = \alpha, \quad e_u^{[2]} = B_u \in V^*.$$

¹⁰See [37] for a short summary of the basics of linear algebra and geometry in a super setting; for a textbook, see [36] or Bernstein’s lectures in [13].

¹¹This is a degenerate case: the algebra is 1-dimensional and its 2-closure is itself.

The squaring thus defined does indeed satisfy the required conditions:

$$\begin{aligned}
 [e_u, [e_u, e_v]] &= [e_u, B(u, v)e_{u+v}] \\
 &= B(u, u + v)B(u, v)e_v = B(u, v)e_v = [B_u, e_v]
 \end{aligned}$$

and

$$[\alpha, [\alpha, e_u]] = (\alpha(u))^2 e_u = \alpha(u)e_u.$$

Over an *arbitrary* field \mathbb{K} of characteristic 2, the space of the 2-closure is also $\mathfrak{g} \oplus V^*$, but \mathfrak{g} and V^* are considered over \mathbb{K} , and squaring is given by the formula

$$(83) \quad (a\alpha)^{[2]} = a^2\alpha, \quad (ae_u)^{[2]} = a^2B_u \in V^* \quad \text{for any } a \in \mathbb{K}.$$

This description of the 2-closure shows that none of the Lie algebras $\text{Kap}_{4,a}(2m)$ for $m > 2$ is isomorphic to the simple derived of the orthogonal Lie algebra of the same dimension. Indeed, the 2-closures of these algebras have different dimensions: the codimension of the simple derived of the orthogonal algebra in its 2-closure is much greater than $\dim V^*$. Since $\mathfrak{o}'_I(n)$ consists of zero-diagonal symmetric matrices, $\dim \mathfrak{o}'_I(n) = \frac{1}{2}n(n - 1)$.

Equation (73) implies that $\dim \text{Kap}_{4,a}(2m) = \dim \mathfrak{o}'_I(n)$ if $n = 2^m + 1$ for $a = 1$ or if $n = 2^m$ for $a = 0$. We therefore wonder if $\text{Kap}_{4,a}(2m)$ is a part of the $\mathfrak{o}'_I(n)$ family. If $n > 2$, then the 2-closure of $\mathfrak{o}'_I(n)$ is the algebra of symmetric traceless matrices, and the codimension of $\mathfrak{o}'_I(n)$ in its 2-closure is $n - 1$ (the dimension of the space of diagonal matrices of trace 0). From the above description, the codimension of $\mathfrak{o}'_I(n)$ in its 2-closure is equal to $\dim V^* = 2m$. Since $n - 1 > 2m$ (if $m > 2$), we see that the algebras $\text{Kap}_{4,a}(2m)$ and $\mathfrak{o}'_I(n)$ are nonisomorphic except for $\text{Kap}_{4,0}(2) \simeq \mathfrak{o}'_I(2)$, $\text{Kap}_{4,1}(2) \simeq \mathfrak{o}'_I(3)$, and $\text{Kap}_{4,1}(4) \simeq \mathfrak{o}'_I(5)$.

5.2. General remark on superizations of Lie algebras

For basics of Lie superalgebras for $p = 2$, see [3, 33]. If $p = 2$, there are two methods each of which assigns a simple Lie superalgebra to **every** simple Lie algebra (see [9], where it is proved that every simple Lie superalgebra is obtained by one of these two methods from a simple Lie algebra). Here, we apply one of these methods to Kaplansky algebras. Let gr be a $\mathbb{Z}/2$ -grading of $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and $(\mathfrak{g}, \text{gr})$ be the minimal subalgebra of $\bar{\mathfrak{g}}$ containing \mathfrak{g} and all the elements $x^{[2]}$, where $x \in \mathfrak{g}_1$. Clearly, there is just one way to extend the grading gr to $(\mathfrak{g}, \text{gr})$. We define squaring by $x^2 := x^{[2]}$ for any $x \in \mathfrak{g}_1$ and let $S(\mathfrak{g}, \text{gr})$ denote the simple Lie superalgebra obtained.

5.2.1. Nonlinear $\mathbb{Z}/2$ -gradings of Kaplansky algebras. The only known way (until this paper) to obtain a $\mathbb{Z}/2$ -grading on a Lie algebra amounts to the following. We take an arbitrary linear function of the weights, more precisely, a homomorphism from the grading group to $\mathbb{Z}/2$. Examples of Lie superalgebras $S(\mathfrak{g}, \text{gr})$ obtained from these gradings: $\mathfrak{gl}(n)$ produces $\mathfrak{gl}(k|n-k)$; $\mathfrak{e}(6)$, $\mathfrak{e}(7)$, and $\mathfrak{e}(8)$ produce their superizations; $\mathfrak{o}_{\text{II}}(2(n+m))$ produces $\mathfrak{o}_{\text{III}}(2n|2m)$ and $\mathfrak{pe}(2n)$ for $n = m$, whereas $\mathfrak{h}_{\text{II}}(2\tilde{N}; \underline{N})$ produces $\mathfrak{h}_{\text{II}}(2k; \tilde{N}|2n-2k)$ and $\mathfrak{le}(n; \tilde{N})$, if the coordinates of $\underline{N} = (\tilde{N}, 1, \dots, 1)$ corresponding to odd indeterminates are equal to 1 (see [3, 33]).

The space V^* (more precisely, $\mathbb{K} \otimes_{\mathbb{F}_2} V^*$, where V^* is considered over \mathbb{F}_2) is a torus in the 2-closure of $\text{Kap}_2(2m)$ or $\text{Kap}_{4,a}(2m)$, and $u \in V$ is precisely a weight with respect to this torus. That is how we obtain what we call *linear* superizations of the 2-closures of $\text{Kap}_2(2m)$ and $\text{Kap}_{4,a}(2m)$ (see below).

The Lie algebras $\text{Kap}_2(2m)$ give the first (and **conjecturally** unique) examples of how to introduce a $\mathbb{Z}/2$ -grading *nonlinearly*, and there are even two nonequivalent ways to do this.

Under any superization (linear or not), the even part of the superized Lie algebra is a Lie subalgebra of the initial Lie algebra. Hence, there is nothing extraordinary in the fact that the even part of the superized $\text{Kap}_2(2m) \oplus V^*$ is $\text{Kap}_{4,a}(2m) \oplus V^*$.

The whole $\text{Kap}_2(2m)$ cannot enter the even part of the superized Lie algebra, because the odd part would otherwise be zero. If $\mathfrak{g} = \text{Kap}_2 \oplus V^*$, then V^* cannot be a part of $\mathfrak{g}_{\bar{1}}$ because $\alpha^2 = \alpha$ for any $\alpha \in V^*$. Therefore, V^* must be a part of $\mathfrak{g}_{\bar{0}}$. Hence, if the whole Kap_2 goes into $\mathfrak{g}_{\bar{0}}$, there is nothing left for $\mathfrak{g}_{\bar{1}}$.

5.2.2. Linear superizations of $\text{Kap}_2(2m)$ and $\text{Kap}_{4,a}(2m)$. Here, we say “linear” in the sense that every e_u is homogenous and its parity is a linear function of $u \in V$ considered over \mathbb{F}_2 .

We define the parity by any element $\varphi \in V^*$ by setting $p(e_u) = \varphi(u)$. Because the form B is nondegenerate, there is a unique $v \in V$ such that

$$(84) \quad \varphi = B_v \quad (\text{see (81)}, \text{ i.e., } \varphi(u) = B(v, u) \text{ for all } u \in V).$$

We let φ_v denote this φ .

To show that two such superizations induced by distinct nonzero vectors v and v' are isomorphic, it suffices to find a linear map $M : V \rightarrow V$ such

that

- 1₂. M preserves B for $\text{Kap}_2(2m)$,
- (85) 1₄. M preserves Q and hence also preserves B for $\text{Kap}_{4,a}(2m)$, and
- 2. $Mv = v'$.

The induced maps

$$(86) \quad \tilde{M} : e_u \mapsto e_{Mu}, \quad M^* : \varphi \mapsto \varphi \circ M^{-1} \quad \text{for any } \varphi \in V^*$$

then determine an isomorphism between superizations. Indeed, for the first one,

$$[\tilde{M}e_u, \tilde{M}e_v] = [e_{Mu}, e_{Mv}] = B(Mu, Mv)e_{Mu+Mv} = B(u, v)\tilde{M}e_{u+v},$$

and if we also define $P'(e_u) = B(v', u)$, then

$$P'(e_{Mu}) = B(v', Mu) = B(Mv, Mu) = B(v, u) = P(e_u).$$

Lemma 5.2. *For $\text{Kap}_2(2m)$, an operator M with properties (85) exists for any two nonzero vectors v and v' (considered over \mathbb{F}_2).*

Proof. If B is an alternate bilinear form on a vector space V of dimension $2m$ and B is nondegenerate, then there is an “alternate basis” for B , i.e., a basis e^1, \dots, e^{2m} of V such that (this is true over any field of any characteristic; see [1])

$$(87) \quad B(e^i, e^j) = \begin{cases} 1 & \text{if } j = i + m, \\ -1 & \text{if } i = j + m, \\ 0 & \text{in all other cases,} \end{cases}$$

i.e., the Gram matrix of B in this basis is $\begin{pmatrix} 0_m & 1_m \\ -1_m & 0_m \end{pmatrix}$. □

Lemma 5.3. *Let B and V be as in Lemma 5.2 and $v \in V$ a nonzero vector. Then there is a basis e^1, \dots, e^{2m} of V satisfying (87) such that $e^1 = v$.*

Proof. We choose any vector $w \in V$ such that $B(v, w) = 1$ and set $e^{m+1} = w$. We set

$$V_\perp = \{x \in V \mid B(x, v) = B(x, w) = 0\}.$$

Then $\dim V_\perp = 2m - 2$, and the restriction B_\perp of B on V_\perp is nondegenerate. We choose $e^2, \dots, e^m, e^{m+2}, \dots, e^{2m}$ as an alternate basis of B_\perp . □

Now let e^1, e^2, \dots, e^{2m} and $\tilde{e}^1, \tilde{e}^2, \dots, \tilde{e}^{2m}$ be two alternate bases of V such that $e^1 = v$ and $\tilde{e}^1 = v'$. We set $Me^i = \tilde{e}^i$. Because

$$B(Me^i, Me^j) = B(\tilde{e}^i, \tilde{e}^j) = B(e^i, e^j),$$

it follows that M preserves B , and $Mv = v'$.

For $\text{Kap}_{4,a}(2m)$, such an M exists for two nonzero vectors v and v' if and only if $Q(v) = Q(v')$ (we recall that v and v' are considered over \mathbb{F}_2). Hence, there are two linear superizations for each $\text{Kap}_{4,a}(2m) \oplus V^*$ with the exception of $\text{Kap}_{4,a}(2)$, where $Q(u) = 1$ for any nonzero u , which has only one superization (it is $\mathfrak{so}'_{II}(1|2)$).¹²

5.2.3. Nonlinear superizations $\text{KapS}_{2,a}(2m)$, and $\text{KapS}_{4,a,\varepsilon}(2m)$.

We note that the superization (88) is *nonlinear*, which means that the parity is not a linear function of u because it is equal to $Q(u) + \bar{1}$.

Let all spaces defined over \mathbb{F}_2 be considered over \mathbb{K} by extension of the ground field. We set¹³

$$(88) \quad \begin{aligned} (\text{KapS}_{2,a}(2m))_{\bar{0}} &:= \text{Kap}_{4,a}(2m) \oplus V^*, \\ (\text{KapS}_{2,a}(2m))_{\bar{1}} &:= \text{Span}(e_u \mid u \in V, u \neq 0, Q(u) = 0) \end{aligned}$$

and define the bracket of even elements with any element and squaring of the odd elements by Eqs. (64), (80), and

$$(ae_u)^2 := (ae_u)^{[2]} = a^2 B_u \in V^* \quad (\text{see (83)}).$$

Let $v \in V$ be such that $Q(v) = \varepsilon$ and $\text{KapS}_{4,a,\varepsilon}(2m)$ the corresponding nonlinear superization of $\text{Kap}_{4,a}(2m) \oplus V^*$.

To describe these Lie superalgebras, we recall the definition of the parity φ_v (see (84)), $\varphi_v(u) = B(v, u)$, and consider the following vectors $v = v_{a,\varepsilon} \in$

¹²In fact, the argument with the map (86) does not prove that the two superizations of $\text{Kap}_{4,a}(2m)$ are nonisomorphic but only that there is no isomorphism of the form (86) between them. **Conjecturally**, they are nonisomorphic.

¹³We are not sure which notation to use here. The Q_a are just *examples* of quadratic forms with the Arf invariant a , while the Q in (88) can be any quadratic form with the Arf invariant a .

V assuming that the quadratic forms Q_a are as in Eq. (71):

$$\begin{aligned}
 v_{0,0} &= v_{1,1} = (1, 0, \dots, 0), \\
 v_{0,1} &= (1, 0, \dots, 0, 1, 0, \dots, 0) \quad \text{(the second 1 is in} \\
 &\quad \text{the } (m + 1)\text{th position),} \\
 (89) \quad v_{1,0} &= (0, 1, 0, \dots, 0) \text{ for } m > 1 \quad \text{(if } m = 1, \text{ then } Q_1(v) = 1 \\
 &\quad \text{for any nonzero } v \in V, \text{ and } v_{0,1} \\
 &\quad \text{hence cannot be chosen).}
 \end{aligned}$$

We set

$$\begin{aligned}
 (90) \quad \text{KapS}_{4,a,\varepsilon}(2m)_{\bar{0}} &:= \text{Span}(e_u \mid u \neq 0, Q_a(u) = 1, B(v_{a,\varepsilon}, u) = 0) \oplus V^*, \\
 \text{KapS}_{4,a,\varepsilon}(2m)_{\bar{1}} &:= \text{Span}(e_u \mid u \neq 0, Q_a(u) = 1, B(v_{a,\varepsilon}, u) = 1).
 \end{aligned}$$

(Here, as usual, $B(u, v) = Q(u + v) - Q(u) - Q(v)$, and $Q = Q_a$.)

5.2.4. There are no nonlinear superizations of $\text{Kap}_{4,a}(2m)$ induced by nonlinear superizations of $\text{Kap}_2(2m)$. In $\text{KapS}_{2,a}(2m)$ corresponding to a form Q , can we take the part corresponding to $\text{Kap}_{4,a}(2m)$ with another form Q' ? We can, but this superization fortunately (the classification would otherwise certainly be a nightmare) coincides with a linear one. This subsuperalgebra is singled out by the condition $Q'(u) = 1$ while its even part is singled out by this condition together with the extra condition $Q(u) = 1$, which can be replaced with $Q(u) + Q'(u) = 0$; because both Q and Q' should yield the same bilinear form B , the quadratic form $Q + Q'$ degenerates into a linear function. Therefore, this superization is equivalent to a linear one.

Therefore, up to an isomorphism, there is one linear superization of $\text{Kap}_2(2m)$, this superization¹⁴ is here denoted by $\text{KapLS}_2(2m)$. The three Lie superalgebras $\text{KapLS}_2(2m)$ and $\text{KapS}_{2,a}(2m)$ for $a = 0, 1$ are nonisomorphic.

¹⁴It would be interesting to find out if $\text{KapS}_2(2m)$ is a deform of a superization of \mathfrak{h}_Π . This is clearly not so for $\mathfrak{h}_\Pi(2k|2m - 2k)$ because their dimensions differ (we recall that $\text{KapS}_2(2m)$ contains V^*). But it might be a deform of a larger algebra. **Conjecturally**, it is not.

6. D'inachevé

6.1. Generalizations of the Jurman construction

We consider $\mathfrak{a}(2; (g, h))$, the Lie algebra whose space is $\mathcal{O}(2; (g + h, 1))$, and the bracket of any $F, G \in \mathcal{O}(2; (g + h, 1))$ is given by the formula (we write x and y to avoid confusion with p and q in the preceding sections)

$$(91) \quad \begin{aligned} [F, G] &= \partial_x F \cdot (\partial_y + y\partial_x^{2g})G + (\partial_y + y\partial_x^{2g})F \cdot \partial_x G \\ &= [F, G]_{P.b.} + y(\partial_x F \cdot \partial_x^{2g} G + \partial_x^{2g} F \cdot \partial_x G). \end{aligned}$$

Both ∂_x and $\partial_y + y\partial_x^{2g}$ are derivations of $\mathcal{O}(2; (g + h, 1))$ and they mutually commute and therefore the Jacobi identity holds. (We note that the fact that the conventional Poisson bracket satisfies the Jacobi identity is a corollary of the similar properties of ∂_x and ∂_y .) The first derived $\mathfrak{a}'(2; (g, h))$ of $\mathfrak{a}(2; (g, h))$ is spanned by all monomials except the highest-degree element $x^{(2g+h-1)}y$.

Lemma 6.1. *We have $\mathfrak{a}'(2; (g, h))/\mathfrak{c} \simeq \mathfrak{j}(g, h)$ with an isomorphism realized by the expressions*

$$Y_i(0) = x^{(i+1)}y, \quad Y_i(1) = x^{(i+2)}.$$

Proof. We directly verify the commutation relations. We first note that the brackets of $Y_i(0)$ with anything do not contain additional terms because these terms contain not ∂_y but multiplication by y , and $Y_i(0)$ already contains y while $y \cdot y = 0$. We also note that $[Y_i(1), Y_j(1)]_{P.b.} = 0$. Taking this into account, we see that

$$(92) \quad \begin{aligned} [Y_i(0), Y_j(0)] &= x^{(i)}y \cdot x^{(j+1)} + x^{(i+1)} \cdot x^{(j)}y \\ &= \left(\binom{i+j+1}{j+1} + \binom{i+j+1}{i+1} \right) x^{(i+j+1)}y \\ &= \left(\binom{i+j+1}{i+1} + \binom{i+j+1}{j+1} \right) Y_{i+j}(0), \end{aligned}$$

$$(93) \quad \begin{aligned} [Y_i(0), Y_j(1)] &= x^{(i+1)} \cdot xp^{(j+1)} = \binom{i+j+2}{i+1} x^{(i+j+2)} \\ &= \left(\binom{i+j+1}{i+1} + \binom{i+j+1}{j+1} \right) Y_{i+j}(1). \end{aligned}$$

The statement of Eq. (92) is clear; that of Eq. (93) holds because if $i + j + 1 \geq 0$, then

$$\binom{i+j+1}{i+1} + \binom{i+j+1}{j+1} = \binom{i+j+1}{i+1} + \binom{i+j+1}{i} = \binom{i+j+2}{i+1},$$

while if $i + j + 1 < 0$, then $i = j = -1$, and $\binom{i+j+2}{i+1} x^{(i+j+2)} = 1$, i.e., is a constant, which generates the center \mathfrak{c} . Therefore, it is equal to 0 in the quotient $\mathfrak{a}'(2; (g, h))/\mathfrak{c}$. Hence, in this case, we also have

$$\left(\binom{i+j+1}{i+1} + \binom{i+j+1}{j+1} \right) Y_{i+j}(1) = 0.$$

We now have

$$\begin{aligned} (94) \quad [Y_i(1), Y_j(1)] &= y \left(x^{(i+1)} \cdot x^{(j+1-\eta)} + x^{(i+1-\eta)} \cdot x^{(j+1)} \right) \\ &= \left(\binom{i+j+2-\eta}{i+1} + \binom{i+j+2-\eta}{j+1} \right) x^{i+j+2-\eta} y \\ &= \left(\binom{i+j+2-\eta}{i+1} + \binom{i+j+2-\eta}{j+1} \right) Y_{i+j+1-\eta}(0). \end{aligned}$$

We hence see that the commutation relations are the same as in $\mathfrak{j}(g, h)$ in all cases. □

6.2. Comparison with known Lie algebras

The direct analog of bracket (91) exists in any characteristic p and has the form

$$\begin{aligned} (95) \quad [F, G] &= \partial_x F \cdot (\partial_y + y^{p-1} \partial_x^{p^g}) G + (\partial_y + y^{p-1} \partial_x^{p^g}) F \cdot \partial_x G \\ &= [F, G]_{P.b.} + y^{p-1} (\partial_x F \cdot \partial_x^{p^g} G + \partial_x^{p^g} F \cdot \partial_x G). \end{aligned}$$

For $p > 3$, all finite-dimensional simple Lie algebras are classified, and this bracket is therefore the bracket of a known Lie algebra.

Question 6.2. To which of the filtered deforms of Lie algebras of Hamiltonian vector fields (see [33]) is the Lie algebra with the bracket (95) isomorphic?

6.3. On further generalizations

We can replace $\partial_y + y\partial_x^{2^g}$ with $\partial_y + R(y)\partial_x^{2^g}$, where R is any polynomial of a divided degree $\leq \underline{N}(y)$. Conjecturally, the only R of interest is the monomial of highest possible degree; the other shapes of R can be reduced to this or a constant. But it seems that for any $\underline{N}(y) > 1$, the result is $\mathfrak{j}(g + N - 1, h)$: the cocycles that make Jurman algebras from $\mathfrak{h}'_{\Pi}(2; (2, 2))$ and $\mathfrak{h}'_{\Pi}(2; (3, 2))$ change the bracket in precisely this way.

We can consider any number k of pairs of indeterminates with the bracket

$$(96) \quad [F, G] = \sum_{1 \leq i \leq k} \partial_{x_i} F \cdot (\partial_{y_i} + y_i \partial_{x_i}^{2^{g_i}}) G + (\partial_{y_i} + y_i \partial_{x_i}^{2^{g_i}}) F \cdot \partial_{x_i} G.$$

We note that the g_i can differ for different i .

Lemma 6.3. *The Lie algebra $\mathfrak{a}'_{\Pi}(2k; (g_1, h_1), \dots, (g_k, h_k))$ has no center and no homogenous ideals for $k = 2$ and $(g_1, h_1) = (g_2, h_2) = (2, 1)$. (Conjecturally, it is simple.)*

6.4. $\mathfrak{a}_I(2; (g, h))$

The Lie algebra $\mathfrak{a}_I(2; (g, h))$ based on $\mathfrak{h}_I(2; (g + h, 1))$ can also be generalized in the above way by beginning with the bracket

$$(97) \quad [F, G] = \partial_x F \cdot \partial_x G + (\partial_y + y\partial_x^{2^g}) F \cdot (\partial_y + y\partial_x^{2^g}) G$$

and generalizing further as indicated above.

Lemma 6.4. *The Lie algebra $\mathfrak{a}_I(2k; (g_1, h_1), \dots, (g_k, h_k))$ has no center and no homogenous ideals for $k = 2$ and $(g_1, h_1) = (g_2, h_2) = (2, 1)$. (Conjecturally, it is simple.)*

Lemma 6.5. *For $\mathfrak{g} := \mathfrak{h}_I(2; (2, 2))$, each cocycle (98) representing the weight elements of $H^2(\mathfrak{g}; \mathfrak{g})$ is integrable, and all except c^3_{-2} (see Eq. (99)) are **linearly** integrable.*

$$\begin{aligned}
(98) \quad & c_{-4}^1 = p \otimes \left(d(pq) \wedge d(p^{(2)} q^{(3)}) + d(pq^{(2)}) \wedge d(p^{(2)} q^{(2)}) + d(pq^{(3)}) \wedge d(p^{(2)} q) \right) + \dots, \\
& c_{-4}^2 = p \otimes d(p^{(2)} q) \wedge d(p^{(3)} q) + q \otimes d(p^{(3)} q) \wedge d(p^{(3)} q) + q \otimes d(p^{(2)} q) \wedge d(p^{(2)} q^{(2)}) + \dots, \\
& c_{-4}^3 = p \otimes d(pq) \wedge d(p^{(2)} q^{(3)}) + p \otimes d(pq^{(2)}) \wedge d(p^{(2)} q^{(2)}) + p \otimes d(pq^{(3)}) \wedge d(p^{(2)} q) + \dots, \\
& c_{-2}^1 = p \otimes d(p^{(2)}) \wedge d(p^{(3)}) + q \otimes d(p^{(2)}) \wedge d(p^{(2)} q) + q^{(2)} \otimes d(p^{(2)}) \wedge d(p^{(2)} q^{(2)}) + \dots, \\
& c_{-2}^2 = p \otimes d(q^{(2)}) \wedge d(pq^{(2)}) + q \otimes d(q^{(2)}) \wedge d(q^{(3)}) + p^{(2)} \otimes d(q^{(2)}) \wedge d(p^{(2)} q^{(2)}) + \dots, \\
& c_{-2}^3 = p \otimes d(p^{(2)}) \wedge d(p^{(3)}) + q \otimes d(p) \wedge d(p^{(3)} q) + q \otimes d(p^{(2)}) \wedge d(p^{(2)} q) + \dots, \\
& c_{-2}^4 = p \otimes d(p^{(2)}) \wedge d(pq^{(2)}) + p \otimes d(p^{(3)}) \wedge d(q^{(2)}) + q \otimes d(p^{(2)}) \wedge d(q^{(3)}) + \dots, \\
& c_0 = p \otimes d(q) \wedge d(pq) + p^{(2)} \otimes d(q) \wedge d(p^{(2)} q) + p^{(3)} \otimes d(q) \wedge d(p^{(3)} q) + \dots, \\
& c_2^1 = q^{(3)} \otimes d(q) \wedge d(p^{(2)}) + p q^{(3)} \otimes d(q) \wedge d(p^{(3)}) + p q^{(3)} \otimes d(p^{(2)}) \wedge d(pq) + \dots, \\
& c_2^2 = p^{(3)} \otimes d(p) \wedge d(q^{(2)}) + p^{(3)} q \otimes d(p) \wedge d(q^{(3)}) + p^{(3)} q \otimes d(q^{(2)}) \wedge d(pq) + \dots, \\
& c_2^3 = q^{(3)} \otimes d(q) \wedge d(q^{(2)}) + p q^{(3)} \otimes d(q) \wedge d(pq^{(2)}) + p q^{(3)} \otimes d(q^{(2)}) \wedge d(pq) + \dots, \\
& c_2^4 = p^{(3)} \otimes d(p) \wedge d(p^{(2)}) + p^{(3)} q \otimes d(p) \wedge d(p^{(2)} q) + p^{(3)} q \otimes d(p^{(2)}) \wedge d(pq) + \dots, \\
& c_6 = p^{(3)} q^{(3)} \otimes d(p) \wedge d(q).
\end{aligned}$$

Proof. Computer-aided. The nonlinear deform (cf. [10]) is

$$[\cdot, \cdot]_{\hbar} = [\cdot, \cdot] + c_{-2}^3 \hbar + A \hbar^2 + B \hbar^3,$$

where

$$\begin{aligned}
(99) \quad & A = p q^{(3)} \otimes d(p^{(2)} q^{(2)}) \wedge d(p^{(3)} q^{(3)}) + p q^{(3)} \otimes d(p^{(2)} q^{(3)}) \wedge d(p^{(3)} q^{(2)}) \\
& \quad + p^{(2)} q^{(2)} \otimes d(p^{(3)} q) \wedge d(p^{(3)} q^{(3)}) + q^{(3)} \otimes d(pq^{(2)}) \wedge d(p^{(3)} q^{(3)}) \\
& \quad + q^{(3)} \otimes d(pq^{(3)}) \wedge d(p^{(3)} q^{(2)}) + p q^{(2)} \otimes d(p^{(2)} q) \wedge d(p^{(3)} q^{(3)}) \\
& \quad + p^{(2)} q \otimes d(p^{(3)} q) \wedge d(p^{(3)} q^{(2)}) + q^{(2)} \otimes d(pq) \wedge d(p^{(3)} q^{(3)}) \\
& \quad + q \otimes d(p^{(3)}) \wedge d(p^{(3)} q) + q \otimes d(pq) \wedge d(p^{(3)} q^{(2)}), \\
& B = q^{(3)} \otimes d(p^{(3)} q^{(2)}) \wedge d(p^{(3)} q^{(3)}).
\end{aligned}$$

□

6.4.1. The Lie algebra $\mathfrak{a}_I(2; (g, h))$ is a deform of $\mathfrak{h}_I(2; (g + h, 1))$.

To prove this claim for the smallest values of (g, h) , we list all infinitesimal deforms of $\mathfrak{h}_I(2; (2, 2))$. For the cochain $F \otimes (dG_1 \wedge \dots \wedge dG_n)$, where $F, G_1, \dots, G_n \in \mathfrak{h}_I(2; (g + h, 1))$, its weight is equal to

$$(100) \quad \left((\deg_p(F) - \sum_{1 \leq i \leq n} \deg_p(G_i)) \bmod 2, (\deg_q(F) - \sum_{1 \leq i \leq n} \deg_q(G_i)) \bmod 2 \right).$$

We note that this grading is induced by elements of a maximal torus, more specifically, by $p^{(2)}$ and $q^{(2)}$. This grading is therefore modulo 2 and is not a \mathbb{Z} -grading. This algebra also has the outer grading deg_{out} given by

$$(101) \quad \text{deg}(p) = \text{deg}(q) = 1, \text{deg}_{\text{out}}(F) = \text{deg}(F) - 2, \text{deg}_{\text{out}}(dF) = 2 - \text{deg}(f).$$

The cocycles (102) are all of weight $\{0, 0\}$. They are indexed in accordance with deg_{out} .

$$(102) \quad \begin{aligned} c_{0,-8} &= p \otimes d(pq^{(4)}) \wedge d(pq^{(5)}) + p \otimes d(q^{(5)}) \wedge d(p^{(2)}q^{(4)}) + q \otimes d(pq^{(4)}) \wedge d(q^{(6)}) + \dots, \\ c_{1,-7} &= p \otimes d(q^{(4)}) \wedge d(pq^{(4)}) + q \otimes d(q^{(4)}) \wedge d(q^{(5)}) + p^{(2)} \otimes d(q^{(4)}) \wedge d(p^{(2)}q^{(4)}) + \dots, \\ c_{4,-4} &= p^{(3)} \otimes d(q) \wedge d(q^{(4)}) + p^{(3)}q \otimes d(q) \wedge d(q^{(5)}) + p^{(3)}q \otimes d(q^{(2)}) \wedge d(q^{(4)}) + \dots, \\ c_{4,-2} &= p^{(3)} \otimes d(q) \wedge d(q^{(2)}) + p^{(3)}q \otimes d(q) \wedge d(q^{(3)}) + p^{(3)}q^{(2)} \otimes d(q) \wedge d(q^{(4)}) + \dots, \\ c_{1,-5} &= p \otimes d(q^{(2)}) \wedge d(pq^{(4)}) + p \otimes d(pq^{(2)}) \wedge d(q^{(4)}) + \dots, \\ c_{0,-4} &= p \otimes d(pq^{(2)}) \wedge d(pq^{(3)}) + p \otimes d(q^{(3)}) \wedge d(p^{(2)}q^{(2)}) + \dots, \\ c_{-1,-5} &= p \otimes d(p^{(2)}) \wedge d(pq^{(6)}) + p \otimes d(p^{(3)}) \wedge d(q^{(6)}) + \dots, \\ c_{-2,-6} &= p \otimes d(pq^{(4)}) \wedge d(p^{(3)}q^{(3)}) + q \otimes d(pq^{(4)}) \wedge d(p^{(2)}q^{(4)}) + \dots, \\ c_{-2,-4} &= p \otimes d(pq^{(2)}) \wedge d(p^{(3)}q^{(3)}) + p \otimes d(p^{(3)}q) \wedge d(pq^{(4)}) + \dots, \\ c_{-1,-3} &= p \otimes d(q^{(2)}) \wedge d(p^{(3)}q^{(2)}) + p \otimes d(p^{(2)}q) \wedge d(pq^{(3)}) + \dots, \\ c_{0,-2} &= p \otimes d(pq) \wedge d(pq^{(2)}) + p \otimes d(q^{(2)}) \wedge d(p^{(2)}q) + \dots, \\ c_{2,0} &= p^{(2)} \otimes d(p) \wedge d(q) + pq^{(2)} \otimes d(q) \wedge d(q^{(2)}) + \dots, \\ c_{-2,-2} &= p \otimes d(pq^{(2)}) \wedge d(p^{(3)}q) + q \otimes d(q) \wedge d(p^{(3)}q^{(3)}) + \dots, \\ c_{-2,0} &= p \otimes d(p^{(2)}) \wedge d(p^{(2)}q) + p \otimes d(pq) \wedge d(p^{(3)}) + \dots, \\ c_{-4,-2} &= p \otimes d(p^{(3)}) \wedge d(p^{(3)}q^{(3)}) + q \otimes d(p^{(3)}) \wedge d(p^{(2)}q^{(4)}) + \dots, \\ c_{-4,0} &= p \otimes d(p^{(3)}) \wedge d(p^{(3)}q) + q \otimes d(p^{(3)}) \wedge d(p^{(2)}q^{(2)}) + \dots, \\ c_{0,4} &= (q^{(4)} \otimes d(p) \wedge d(q) + (p^{(2)}q^{(3)} \otimes d(p) \wedge d(p^{(2)})) + \dots, \\ c_{0,6} &= q^{(6)} \otimes d(p) \wedge d(q) + p^{(2)}q^{(5)} \otimes d(p) \wedge d(p^{(2)}) + \dots, \\ c_{-2,8} &= q^{(7)} \otimes d(p) \wedge d(p^{(2)}) + pq^{(7)} \otimes d(p) \wedge d(p^{(3)}) + p^{(2)}q^{(7)} \otimes d(p^{(2)}) \wedge d(p^{(3)}). \end{aligned}$$

6.4.2. Question. How to interpret the non-Jurman cocycles à la Proposition 4.5 for the other values of (g, h) ? For example, for $(g, h) = (3, 1)$ and $(2, 2)$, i.e., for the deformations of $\mathfrak{h}'_{\Pi}(2; (3, 2)) \simeq \mathfrak{h}'_{\Pi}(2; (2, 3))$, the Jurman cocycle deforming $\mathfrak{h}'_{\Pi}(2; (3, 2))$ into $\mathfrak{j}(3, 1)$ is $c_{-2,8}$, and the Jurman cocycle deforming $\mathfrak{h}'_{\Pi}(2; (2, 3))$ into $\mathfrak{j}(2, 2)$ is $c_{4,-2}$, see (102).

Conjecture 6.6. We have

$$\text{Kap}_{4,B}(2m) \not\cong \mathfrak{po}_{\Pi}(2m; \underline{N}_s), \text{ and } \text{Kap}_2(2m) \not\cong \mathfrak{h}_{\Pi}(2m; \underline{N}).$$

We verified this for small m . For $m = 1$, $\text{Kap}_{4,B}(2)$ is isomorphic to $\mathfrak{o}'(3) \oplus \mathfrak{c}$, where \mathfrak{c} is the 1-dimensional trivial center, and hence is not isomorphic to $\mathfrak{po}_{\Pi}(2; \underline{N}_s)$, which is solvable. For $m = 2$, computer-aided computations show that the infinitesimal deformation corresponding to (68) is a nontrivial cocycle. To prove the conjecture, we must show that the cocycle is not semitrivial either. Of course, what we really need to know is what $\text{Kap}_{4,B}(2m)$ and its subalgebras $\text{Kap}_{4,a}(2m)$ are isomorphic to. We present some plausible conjectures.

Conjecture 6.7. 1. The Lie algebra $\text{Kap}_{4,1}(2m)$ is a deform of the subalgebra in the Poisson algebra $\mathfrak{po}(2m; \underline{N}_s)$ generated by functions $f \in \mathcal{O}(2m; \underline{N}_s)$ satisfying $\sum_{1 \leq i \leq 3} \frac{\partial^2 f}{\partial p_i \partial q_i} = 0$. (The quotient of this subalgebra modulo center is isomorphic to $\mathfrak{sl}\mathfrak{h}(2m)$; see [33].)

2. The Lie algebra $\text{Kap}_{4,1}(2m)$ is a deform of $\mathfrak{o}'_I(2m + 1; \underline{N}_s)$ while $\text{Kap}_{4,0}(2m)$ is a deform of a subalgebra in $\mathfrak{o}'_I(2m; \underline{N}_s)$ (see [33]).

The dimension of $H^2(\mathfrak{g}; \mathfrak{g})$ is big and grows quickly with m . How can we select the needed deform? The Poisson algebra and its subalgebra consisting of harmonic functions have a center generated by constants, while $\text{Kap}_{4,1}(2m)$ is simple. Therefore, in the huge space of cocycles representing infinitesimal deformations, we need only select cocycles of the form

$$(103) \quad f \otimes d(1) \wedge d(g) + \dots$$

and compare the global deforms corresponding to them with $\text{Kap}_{4,1}(2m)$. For small m , $\dim H^2(\mathfrak{g}; \mathfrak{g})$ does not explode yet. For $m = 2$ and $m = 3$, we have $\dim H^2(\mathfrak{g}; \mathfrak{g}) = 34$; all cocycles are integrable and all global deforms corresponding to them (if a representative is chosen carefully by means of coboundaries) are linear in the deformation parameter. For $m = 2$ and $m = 3$, there is only one cocycle of the form (103) (up to coboundaries). These cocycles are of degree 2. In degree 2, there is only one cocycle for $m = 3$, and there are five cocycles for $m = 2$. Further investigations show that Conjecture 1 only holds for $m = 2$; for $m = 3$, the two algebras to be compared have different numbers of central extensions.

6.5. How to establish nonisomorphy?

Skryabin [43] classified the filtered deforms of Hamiltonian Lie algebras $\mathfrak{h}_{\Pi}(2m; \underline{N})$. It remains to select which of them is the simple Lie algebra $\text{Kap}_{4,B}(2m)/\mathfrak{c} \simeq \text{Kap}_2(2m)$. We have not yet performed such an identification.

To find out if two given Lie algebras of the same dimension are isomorphic, Eick considered the following invariants in [18]:¹⁵ $\dim H^1(\mathfrak{g}; \mathfrak{g})$ or rather $\dim \mathfrak{der}(\mathfrak{g})$, the order of the group $\text{Aut}(\mathfrak{g})$, the number of elements in $\text{Ann}(\mathfrak{g})$, and the order of $\text{Exp}(\mathfrak{g})$.

Speaking of deforms, we can consider the action of $\text{Aut}(\mathfrak{g})$ on the space of infinitesimal deformations, as in [12, 31].

For algebras of small dimension, there is still another approach, at least theoretically. We can compare identities that the algebras satisfy. A. A. Kirillov formulated the following analog of the Amitsur-Levitzki theorem, whose proof was only preprinted in the Keldysh Institute of Applied Mathematics in the 1980s (see [28] for a translation of one such preprint; the other preprints with related results by Kirillov, Kontsevich, and Molev have not yet been translated, but they were at least reviewed by Molev).

Theorem 6.8 ([27]). *Let \mathfrak{g} be a simple Lie algebra of vector fields over a field of characteristic 0. Let*

$$(104) \quad a_k(X_1, \dots, X_k) = \sum_{\sigma \in S_k} (-1)^{\text{sign } \sigma} \text{ad}_{X_{\sigma(1)}} \cdots \text{ad}_{X_{\sigma(k)}}.$$

The identity $a_k(X_1, \dots, X_k) \equiv 0$ for any $X_1, \dots, X_k \in \mathfrak{g}$ holds

- a. for $k \geq (n+1)^2$ if $\mathfrak{g} = \mathfrak{vect}(n)$,*
- b. for $k \geq n(2n+5)$ if $\mathfrak{g} = \mathfrak{h}(2n)$, and*
- c. for $k \geq 2n^2 + 5n + 5$ if $\mathfrak{g} = \mathfrak{k}(2n+1)$.*

Dzhumadil'daev suggested an interesting modification of emphasis in this train of thought, finding a hidden supersymmetry for an analog of antisymmetrizers with x instead of ad_x in (104). He also showed a relation to strongly homotopy algebras (for further details, see [16] and [34]).

¹⁵An almost exact quotation from [18]: “We say that a derivation $d \in \mathfrak{der}(\mathfrak{g})$ is p -nilpotent if $d^p = 0$ holds. For a p -nilpotent derivation d , we define its exponential matrix $\exp d := \sum_{0 \leq i \leq p-1} \frac{d^i}{i!}$. We call a p -nilpotent derivation d an *annihilator* if $d^i(X)d^j(Y) = 0$ for all $X, Y \in \mathfrak{g}$ and $i, j \geq 0$ with $i+j \geq p$. Let $\text{Ann}(\mathfrak{g}) \subset \mathfrak{der}(\mathfrak{g})$ denote the subset of annihilators. We define $\text{Exp}(\mathfrak{g})$ to be the subgroup of $\text{Aut}(\mathfrak{g})$ generated by $\{\exp(d) \mid d \in \text{Ann}(\mathfrak{g})\}$. We note that the order of every element $\exp(d)$ is equal to either p or 1. Hence, $\text{Exp}(\mathfrak{g})$ is a subgroup of $\text{Aut}(\mathfrak{g})$ generated by automorphisms of order p .”

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