Math. Res. Lett.
Volume 22, Number 4, 989–1003, 2015

The Chow ring of the classifying space of $GO(2n)$

Saurav Bhaumik

Let $GO(2n)$ be the general orthogonal group scheme (the group of orthogonal similitudes). In the topological category, Y. Holla and N. Nitsure determined the singular cohomology ring $H^*_{\text{sing}}(BGO(2n, \mathbb{C}), \mathbb{F}_2)$ of the classifying space $BGO(2n, \mathbb{C})$ of the corresponding complex Lie group $GO(2n, \mathbb{C})$ in terms of explicit generators and relations. The author of the present note showed that over any algebraically closed field of characteristic not equal to 2, the smooth-étale cohomology ring $H^*_{\text{sm-ét}}(BGO(2n), \mathbb{F}_2)$ of the classifying algebraic stack $BGO(2n)$ has the same description in terms of generators and relations as the singular cohomology ring $H^*_{\text{sing}}(BGO(2n, \mathbb{C}), \mathbb{F}_2)$. Totaro defined for any reductive group G over a field, the Chow ring A^*_G, which is canonically identified with the ring of characteristic classes in the sense of intersection theory, for principal G-bundles, locally trivial in étale topology. In this paper, we calculate the Chow group $A^*_{GO(2n)}$ over any field of characteristic different from 2 in terms of generators and relations.

1. Introduction

The Chow ring of the classifying space of a reductive group was introduced by Totaro in [10], where he calculated the Chow rings of the classifying spaces of several finite groups and algebraic groups including $O(n)$, $Sp(2n)$, etc. Edidin and Graham in [3] introduced the equivariant Chow ring. Molina Rojas and Vistoli in [7], using the techniques of equivariant Chow groups, calculated the Chow ring $A^*_{SO(n)}$ in case n is even (the odd case was already addressed in Pandharipande [9] and Totaro [10]).

Holla and Nitsure in [5] considered the general orthogonal group $GO(n, \mathbb{C})$, which is also called the group of orthogonal similitudes, and calculated the singular cohomology ring of its classifying space $H^*_{\text{sing}}(BGO(n, \mathbb{C}); \mathbb{F}_2)$. In [1], the author of the present paper considered the algebraic version of the above Lie group, namely the general orthogonal group scheme $GO(n)$ over an algebraically closed field of characteristic different from 2, and showed that the smooth-étale cohomology ring $H^*_{\text{sm-et}}(BGO(n);\mathbb{F}_2)$ of the algebraic stack $BGO(n)$ has the same description in terms of generators and relations over $\mathbb{F}_2$ as the singular cohomology ring computed by Holla and Nitsure in [5]. In this present note, we calculate the Chow ring of the classifying space of $GO(n)$ over a field of characteristic different from 2 in the sense of Totaro [10], using the methods of equivariant Chow groups. By the results of Totaro [10], this ring can be canonically identified with the ring of characteristic classes for principal $GO(n)$-bundles on smooth, quasi-projective schemes. In other words, the Chow ring of the classifying space of $GO(n)$ is the ring of all intersection theoretical invariants for families of line bundle valued nondegenerate quadratic forms.

Henceforth, all schemes and morphisms are over a fixed field k (not necessarily algebraically closed) with characteristic different from 2. We recall the definition of the algebraic group $GO(n)$ over k. Let $V = k^n$, and let $q : V \to k$ be the quadratic form, defined by

$$q(x_1, \dots, x_{2m}) = x_1x_{m+1} + \cdots + x_mx_{2m},$$

for the even case $n = 2m$, and by

$$q(x_1, \dots, x_{2m+1}) = x_1x_{m+1} + \cdots + x_mx_{2m} + x_{2m+1}^2,$$

for the odd case $n = 2m + 1$. Let $GO(n)$ be the affine algebraic group scheme of invertible linear automorphisms of V that preserve the quadratic form q up to a scalar. In terms of matrices, let J denote the nonsingular symmetric matrix of the bilinear form corresponding to q. Then as a functor of points, $GO(n)$ attaches to each k-algebra S the group

$$GO(n)(S) = \{A \in GL_n(S) \colon \exists\, a \in S^\times,\ {}^tAJA = aJ\}.$$

The algebraic group $GO(n)$ is reductive, since its defining representation on k^n is irreducible. Note that if k'/k is a field extension such that the quadratic form q extended to $V \otimes_k k' = k'^n$ is equivalent to the quadratic form $\sum_i x_i^2$, given by the identity matrix I_n, then over k', the algebraic group $GO(n)$ defined above is isomorphic to the algebraic group $GO(n)$ defined in [1].

The scalar a in the definition determines the character $\sigma : GO(n) \to \mathbb{G}_m$ that satisfies ${}^tAJA = \sigma(A)J$. Given a scheme X, and a principal $GO(n)$-bundle P on X (locally trivial in the étale topology), consider the rank n vector bundle E associated to the defining representation $GO(n) \subset GL_n$, and the line bundle L determined by the character σ. The nondegenerate symmetric bilinear form corresponding to q induces a nondegenerate symmetric bilinear form $b : E \otimes_{\mathcal{O}_X} E \to L$. Conversely, given a *nondegenerate quadratic triple of rank n* (E, L, b), which is a triple consisting of a vector bundle E of rank n, a line bundle L and a nondegenerate symmetric bilinear $b : E \otimes_{\mathcal{O}_X} E \to L$, we can reduce the structure group of E to GO_n by applying Gram-Schmidt orthonormalization étale locally on X.

Let A^*_G denote the Chow ring of the classifying space of a reductive group G in the sense of Totaro [10]. Note that for any $n \geq 1$, there is a canonical isomorphism

$$SO(2n+1) \times \mathbb{G}_m \xrightarrow{\sim} GO(2n+1).$$

Note that $B\mathbb{G}_m$ is approximated by the projective spaces $\mathbb{P}^m_k$ in the sense of Totaro [10], and we have for any smooth scheme X the following natural isomorphisms.

$$A^*(X) \otimes A^*(\mathbb{P}^m_k) \xrightarrow{\sim} A^*(X \times \mathbb{P}^m_k)$$

Since $BSO(2n+1)$ is approximated by smooth schemes, there is the Künneth isomorphism

$$A^*_{SO(2n+1)} \otimes A^*_{\mathbb{G}_m} \xrightarrow{\sim} A^*_{GO(2n+1)}.$$

This determines the Chow ring for $GO(2n+1)$, because $A^*_{\mathbb{G}_m} \cong \mathbb{Z}[\lambda]$, and the Chow ring for $SO(2n+1)$ is given by [9] and [10]. Therefore we are left with the task of calculating the Chow ring only in the even case $GO(2n)$. The rest of this note is devoted to the calculation of $A^*_{GO(2n)}$.

In Section 2, we recall Totaro's definition of the Chow ring of a classifying space from [10], and the basic notions of equivariant Chow groups from Molina Rojas and Vistoli [7]. In Section 3, we calculate $A^*_{GO(2n)}$ in terms of explicit generators and relations over $\mathbb{Z}$. We show, in terms of quadratic triples (E, L, b), $A^*_{GO(2n)}$ is generated by the Chern classes $c_i(E)$ and the Chern class λ of L. The nondegenerate symmetric bilinear form determines an isomorphism of vector bundles $E \xrightarrow{\sim} E^\vee \otimes L$. This isomorphism gives the following relations among Chern classes of E and the class λ.

$$c_p = \sum_{i=0}^{p} (-1)^i \binom{2n-i}{p-i} c_i \lambda^{p-i}, \quad p = 1, \ldots, 2n$$

The main theorem of this note is Theorem 3.1, which asserts that the ideal of relations in $A^*_{GO(2n)}$ is generated by the above relations.

The invariants in $A^*_{GO(2n)}$ transform under the cycle map to the corresponding classes in $H^*_{\text{sm-ét}}(BGO(2n), \mathbb{F}_2)$ (see Remark 3.2).

To prove the main theorem (Theorem 3.1), we first observe that λ and the even Chern classes are algebraically independent in $A^*_{GO(2n)}$. Eventually we focus on the torsion subgroup of $A^*_{GO(2n)}$ (which is an ideal) in order to carry out the rest of the proof. The inclusion $O(2n) \subset GO(2n)$, where $O(2n)$ is looked upon as the algebraic group of linear automorphisms of k^{2n} that preserve the quadratic form $\sum_i x_i x_{i+n}$, gives rise to a ring homomorphism $A^*_{GO(2n)} \to A^*_{O(2n)}$. A biproduct of the proof of the main theorem is that this homomorphism determines an isomorphism of the corresponding torsion subgroups.

Acknowledgements

The author would like to thank Yogish Holla and Nitin Nitsure for useful discussions and suggestions. Thanks are also due to the referee, who informed the author that the cycle map in étale cohomology with torsion coefficients factors through rational equivalence.

2. Basic notions recalled

2.1. Chow ring of the classifying space

Totaro [10] defined the Chow groups of the classifying space of a linear algebraic group G as follows. Let $N > 0$ be an integer. Then there is a finite dimensional linear representation V of G, with a G-equivariant closed subset S of codimension $\geq N$ such that the action of G on $(V - S)$ is free, such that the quotient $(V - S)/G$ exists in the category of schemes and $(V - S) \to (V - S)/G$ is a principal bundle, locally trivial in étale topology. For any such other pair (V', S'), there are canonical isomorphisms for $i < N$

$$A^i\left(\frac{(V-S)}{G}\right) \xrightarrow{\sim} A^i\left(\frac{(V'-S')}{G}\right).$$

The i-th Chow group A^i_G of the classifying space of G is defined to be $A^i\left(\frac{V-S}{G}\right)$, where (V, S) satisfies the above condition. The isomorphism above shows that A^i_G is independent of the choice of the particular pair (V, S). The graded group A^*_G naturally has the structure of a graded ring under intersection product.

Note that in the above approach, one defines the Chow ring of the classifying space without referring to a possible classifying space BG whether in the category of algebraic stacks or simplicial spaces. But Morel and Voevodsky defined the classifying space of a linear algebraic group as an object in the $\mathbb{A}^1$-homotopy category, and their construction gives the same Chow ring (Chapter 4, Proposition 2.6, [8]).

2.2. Equivariant Chow groups

Let X be a scheme on which a reductive algebraic group G acts. Suppose V is a finite dimensional linear representation of G, and $S \subset V$ is a G-equivariant closed subset such that the induced action of G on $(V - S)$ is free, with a principal bundle quotient $(V - S) \to (V - S)/G$. If the codimension of S in V is N, then for $i < N$, the equivariant Chow groups of X are defined as

$$A_G^i(X) := A^i\left(\frac{X \times (V - S)}{G}\right).$$

This definition is independent of the particular choices of V and S.

The Chow ring of the classifying space is recovered by taking X to be the point Spec k, as $A_G^i(\text{Spec } k) = A_G^i$.

Suppose G fits into an exact sequence of reductive algebraic groups as follows

$$1 \to H \to G \xrightarrow{\chi} \mathbb{G}_m \to 1.$$

Consider the action of G on $\mathbb{A}^1$ by the character χ. The localisation sequence for $\mathbb{G}_m \subset \mathbb{A}^1$ is

$$A_G^*(\text{Spec } k) \to A_G^*(\mathbb{A}^1) \to A_G^*(\mathbb{G}_m) \to 0.$$

The quotient $(\mathbb{G}_m \times (\mathbb{A}^1 - \{0\}))/G$ can be naturally identified with $(\mathbb{A}^1 - \{0\})/H$. Therefore the above localization sequence gives an exact sequence

$$A_G^* \xrightarrow{c} A_G^* \to A_H^* \to 0,$$

where c is the Chern class of the line bundle given by the character χ.

3. Calculation of $A^*_{GO(2n)}$

Generators for $A^*_{GO(2n)}$. Recall that we have a short exact sequence of reductive algebraic groups

$$1 \to O(2n) \to GO(2n) \xrightarrow{\sigma} \mathbb{G}_m \to 1,$$

which, by the results of the last section, gives an exact sequence

$$A^*_{GO(2n)} \xrightarrow{\lambda} A^*_{GO(2n)} \to A^*_{O(2n)} \to 0 \tag{3.1}$$

where λ is the Chern class corresponding to the character σ.

Totaro [10] (assuming the base field to be the field of complex numbers) and later Molina Rojas and Vistoli [7] (over a general field of characteristic different from 2) showed that

$$A^*_{O(2n)} \cong \frac{\mathbb{Z}[c_1, \ldots, c_{2n}]}{(2c_{\mathrm{odd}})},$$

hence $A^*_{O(2n)}$ is generated by the Chern classes $c_1, \ldots, c_{2n}$. The exact sequence (3.1) shows that $A^*_{GO(2n)}$ is generated by $c_1, \ldots, c_{2n}, \lambda$.

Relations among $\boldsymbol{c_1, \ldots, c_{2n}, \lambda}$. Recall that to give a principal $GO(2n)$-bundle is to give a triple (E, L, b) consisting of a vector bundle E of rank $2n$, a line bundle L and a nondegenerate symmetric form $b : E \otimes E \to L$. The form $b : E \otimes E \to L$ determines an isomorphism of vector bundles $E \xrightarrow{\sim} E^\vee \otimes L$, which gives the following relations among λ and the Chern classes of E.

$$c_p = \sum_{i=0}^{p} (-1)^i \binom{2n-i}{p-i} c_i \lambda^{p-i}, \quad p = 1, \ldots, 2n \tag{3.2}$$

Let R be the quotient of the polynomial ring $\mathbb{Z}[\lambda, c_1, \ldots, c_{2n}]$ by the ideal generated by the above $2n$ relations. Let $q : R \to A^*_{GO(2n)}$ be the ring homomorphism that sends λ to the Chern class of the line bundle defined by the character σ, and c_i to the ith Chern class of the defining representation $GO(2n) \subset GL_{2n}$.

Theorem 3.1. *The map $q : R \to A^*_{GO(2n)}$ is an isomorphism.*

Before giving the proof of this theorem, we make a remark.

Remark 3.2. For a nonsingular variety X over the field of complex numbers, the cycle map is a homomorphism of graded rings $cl^X : A^*(X) \to H^{2*}_{\mathrm{sing}}(X, \mathbb{Z})$, functorial in X (see Fulton [4], Chapter 19). By composing with the change of coefficients map $H^{2*}_{\mathrm{sing}}(X, \mathbb{Z}) \to H^{2*}_{\mathrm{sing}}(X, \mathbb{F}_2)$, we get a homomorphism of graded rings $A^*(X) \to H^{2*}_{\mathrm{sing}}(X, \mathbb{F}_2)$, which we will denote by $\bar{cl}^X$. If E is a vector bundle on X, then $\bar{cl}^X(c_i(E)) = \bar{c}_i(E) \in H^{2i}_{\mathrm{sing}}(X, \mathbb{F}_2)$, the mod-2 reduced Chern classes. For a reductive group G, this gives rise to a homomorphism of graded rings (see [10]), which is functorial in the group G

$$\bar{cl}^G : A^*_G \to H^{2*}_{\mathrm{sing}}(BG, \mathbb{F}_2).$$

For $G = GO(2n, \mathbb{C})$, the Chern classes $c_i \in A^*_{GO(2n.\mathbb{C})}$ transform under the above cycle map to the corresponding classes in $H^{2*}(BGO(2n, \mathbb{C}), \mathbb{F}_2)$ described by Holla and Nitsure (see Section 3 of [6]).

Also in case of étale cohomology, if X is a smooth scheme over a base field of characteristic different from 2, the cycle map $Z^i(X) \to H^{2i}_{\text{ét}}(X, \mathbb{F}_2)$ passes through rational equivalence. Indeed, let $f : X \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection to the second factor. Let $W \subset X \times \mathbb{P}^1$ be an irreducible closed subset of pure codimension i and mapping dominantly to $\mathbb{P}^1$. By the relative theory of cycles developed in Chapter 4 Section 2.3 of SGA $4\frac{1}{2}$ [2] (see Remark 2.3.10 in particular), W determines a section of the sheaf $R^{2i}f_*\mathbb{F}_2$ over $\mathbb{P}^1$. Since $R^{2i}f_*\mathbb{F}_2$ is the constant sheaf $H^{2i}(X, \mathbb{F}_2)$ and $\mathbb{P}^1$ is connected, the images of the fibres of W over 0 and ∞ are equal in $H^{2i}(X, \mathbb{F}_2)$. In Remark 2.3.10 referred to above, the base field is assumed to be algebraically closed so that $R^{2i}f_*\mathbb{Z}/n(i)$ is a constant sheaf. But $R^{2i}f_*\mathbb{Z}/2(i)$ is a constant sheaf for any base field of characteristic different from 2, so the argument works in our case. This shows that we have a well defined graded homomorphism for a reductive group G as before,

$$\bar{cl}^G : A^*_G \to H^{2*}_{\text{sm-ét}}(BG, \mathbb{F}_2).$$

Similarly, for $G = GO(2n)$, the Chern classes $c_i \in A^*_{GO(2n)}$ transform under the above cycle map to the corresponding classes in $H^{2*}_{\text{sm-ét}}(BGO(2n), \mathbb{F}_2)$ (see Bhaumik [1] Section 5).

Question. Are the resulting maps $A^*_{GO(2n,\mathbb{C})} \otimes \mathbb{F}_2 \to H^{2*}_{\mathrm{sing}}(BGO(2n, \mathbb{C}), \mathbb{F}_2)$ and $A^*_{GO(2n)} \otimes \mathbb{F}_2 \to H^{2*}_{\text{sm-ét}}(BGO(2n), \mathbb{F}_2)$ injective?

The rest of this article is devoted to the proof of the Theorem 3.1, which is the main result of this article. Let us begin by recalling that the map q is surjective. We will prove that it is injective. The plan of the proof is as follows. We will first show that the elements $\lambda, c_2, c_4, \ldots, c_{2n}$ are algebraically independent in $A^*_{GO(2n)}$ (Corollary 3.5). Then we prove Lemma 3.6 and Corollary 3.7, which will imply that it is enough to prove the injectivity only for the torsion part. We will complete the proof of injectivity for the torsion part in a few steps. As a concluding remark, we observe (Corollary 3.10) that the torsion subgroup is in fact an $\mathbb{F}_2$-vector space.

Remark 3.3. Note that for each *odd* p, we have the following identity in R

$$2c_p = \sum_{i=0}^{p-1} (-1)^i c_i \binom{2n-i}{p-i} \lambda^{p-i} \tag{3.3}$$

In particular, $(2c_{odd})R \subset \lambda R$.

Lemma 3.4. *Let the free polynomial algebra $B = \mathbb{Z}[\lambda, c_2, c_4, \ldots, c_{2n}]$ be given the grading where λ has homogeneous degree 1 and each c_{2i} has homogeneous degree $2i$. Let D be a graded domain, and let $\phi : B \to D$ be a graded homomorphism such that*

1. *the restriction of ϕ to $\mathbb{Z}[c_2, c_4, \ldots, c_{2n}]$ is an injection,*
2. *so is the composite $\mathbb{Z}[c_2, c_4, \ldots, c_{2n}] \to D \to D/\phi(\lambda)D$, and*
3. *$\phi(\lambda) \in D$ is non-zero.*

Then ϕ is an injection.

Proof. For a polynomial in B with degree ≤ 1, the image is always non-zero in D, by (1) and (3) of the hypothesis. We will prove the injectivity of ϕ by induction on the degree, as we know that the injectivity holds in degree ≤ 1.

Suppose the injectivity holds for degree $< r$, and suppose $f \in B$ is a homogeneous polynomial of degree r with $\phi(f) = 0$. We can write $f = \lambda \cdot g + h$, where $g \in B$ in homogeneous of degree $(r-1)$, and $h \in \mathbb{Z}[c_2, c_4, \ldots, c_{2n}]$ is of degree r. By (2), $h = 0$. Therefore, $f = \lambda \cdot g$. Suppose $g \neq 0$. Since $\deg(g) < r$, by induction hypothesis, $\phi(g) \neq 0$. But since D is a domain, and as by (3) $\phi(\lambda) \neq 0$, we see that $\phi(f) = \phi(\lambda)\phi(g) \neq 0$, a contradiction, so that $g = 0$. □

Let us consider the invertible $2n \times 2n$ matrix $J = (0, I_n\,; I_n, 0)$. By definition, $GO(2n)$ is the group scheme of invertible matrices A such that ${}^tAJA = aJ$ for some scalar a. Consider the closed subgroup scheme $\Gamma \subset GO(2n)$, consisting of matrices of the following form

$$\begin{pmatrix} \lambda t_1 & & & & & \\ & \ddots & & & & \\ & & \lambda t_n & & & \\ & & & t_1^{-1} & & \\ & & & & \ddots & \\ & & & & & t_n^{-1} \end{pmatrix}$$

Then Γ is a maximal torus in $GO(2n)$. The inclusion $\Gamma \subset GO(2n)$ induces a homomorphism of the corresponding Chow rings

$$A^*_{GO(2n)} \to A^*_\Gamma = \mathbb{Z}[\lambda, t_1, \ldots, t_n],$$

sending λ to λ. Applying the above lemma to $D = A^*_\Gamma$ and the graded homomorphism which is the composite $B \to A^*_{GO(2n)} \to D$, and using the fact that $c_2, \ldots, c_{2n}$ are algebraically independent in the ring $A^*_{O(2n)} \cong A^*_{GO(2n)}/(\lambda)$, we get the following.

Corollary 3.5. *The elements $\lambda, c_2, c_4, \ldots, c_{2n}$ are algebraically independent in the ring $A^*_{GO(2n)}$.*

Notation. Let us recall that we adopted the notation R for $\mathbb{Z}[\lambda, c_1, \ldots, c_{2n}]/I$, where I is generated by the relations coming from $E \xrightarrow{\sim} E^\vee \otimes L$ enlisted in Equation (3.2). We have the surjective map $q : R \to A^*_{GO(2n)}$. Let A denote $A^*_{GO(2n)}$ and let B denote the polynomial ring $\mathbb{Z}[\lambda, c_{even}]$. Since the composite $B \to R \to A$ is injective by the corollary, the map $B \to R$ is also injective. Hence we might consider $B \subset R$ as well as $B \subset A$. We will denote the torsion ideals of R and A by T_R and T_A, respectively.

Lemma 3.6. *For any $a \in R$, there is some $s > 0$ such that $2^s a \in B$. Similarly, for any $a \in A$, there is some $s > 0$ such that $2^s a \in B$.*

Proof. We show this by induction on the degree of a. It is enough to prove the lemma for all odd Chern classes i.e. for $a = c_{2i+1}$. By Equation (3.3), $2c_{2i+1} = g_i(\lambda, c_1, \ldots, c_{2i})$, a polynomial in λ and lower degree Chern classes. Now by the induction hypothesis, for each $j < i$, there is some $m_j > 0$ such that $2^{m_j} c_{2j+1} \in A$. If n_j is the greatest index with which c_{2j+1} occurs in g_i, then $s = \sum_j n_j m_j$ serves the purpose. □

We make the following observations as corollaries.

Corollary 3.7.

(a) *The groups* T_R *and* T_A *consist of* 2*-primary elements only.*

(b) *We have* $\ker(q) \subset T_R$.

(c) *We have* $q^{-1}(T_A) = T_R$. *In particular, we have the induced isomorphism of free abelian group*

$$R/T_R \cong A/T_A.$$

(d) *If* $a \in R$ *such that* $\lambda a \in T_R$, *then* $a \in T_R$. *Similarly, if* $a \in R$ *such that* $\lambda a \in T_R$, *then* $a \in T_R$.

(e) $\ker(q) \subset T_R \cap \lambda R$.

(f) *We have* $T_R = \ker(R \to A^*_\Gamma))$ *and* $T_A = \ker(A \to A^*_\Gamma)$.

(g) *There are isomorphisms* $R/(\lambda) \xrightarrow{\sim} A/(\lambda) \xrightarrow{\sim} A^*_{O(2n)}$.

Proof. (a) Follows from the lemma.

(b) Suppose $a \in R$ is such that $q(a) = 0$. Then there is some $m \geq 0$ such that $0 = 2^m q(a) = q(2^m a)$, and $2^m a \in B$. Hence $2^m a = 0$. This shows that the kernel of q has only 2-primary elements.

(c) Follows from (b). For, if $a \in q^{-1}(T_A)$, then $2^s a \in \ker(q) \subset T_R$, so $a \in T_R$.

(d) Otherwise, by the lemma, we get $2^s a \in B$ and $2^s a \neq 0$. This means λa is never a torsion, because $2^s \lambda a \in B - \{0\}$.

(e) To see this, first note that any element a of $\mathbb{Z}[\lambda, c_1, \ldots, c_{2n}]$ is written as $\alpha + \lambda\beta + \gamma$, where $a \in \mathbb{Z}[c_{even}]$, and $\gamma \in (c_{odd})\mathbb{Z}[c_1, \ldots, c_{2n}]$. If $\alpha \neq 0$, then see that the image of $2a$ in $A^*_{O(2n)}$ is equal to the image of 2α, which is nonzero. Therefore if $a \in \ker(q)$, then $\alpha = 0$ and $a = \lambda\beta + \gamma$. Now, the image of a in $A^*_{O(2n)}$ vanishes, so $\gamma \in (2c_{odd})\mathbb{Z}[c_1, \ldots, c_{2n}]$. But by Equation (3.3), we have $(2c_{odd})R \subset \lambda R$. So the assertion follows.

(f) That $T_R \subset \ker(R \to A^*_\Gamma)$ is obvious, because A^*_Γ has no torsions. To see the other inclusion, let $a \in \ker(R \to A^*_\Gamma)$. If a was not a torsion, then by lemma, there is $s > 0$ such that $2^s a \in B - \{0\}$. But $B \to A^*_\Gamma$ is injective as we have already seen. Similarly for T_A.

(g) The map $R/(\lambda) \to A^*_{O(2n)}$ is an isomorphism by the definition of R. On the other hand, Equation (3.1) shows that $A/(\lambda) \to A^*_{O(2n)}$ is an isomorphism. □

Proof of the main theorem. We only have to prove that $T_R \cap \lambda R = 0$, since $\ker(q) \subset T_R \cap \lambda R$.

Step 1. Let λ_A denote the multiplication by $\lambda : A \to A$. In what follows, $\ker \lambda$ will denote the kernel of the multiplication $\lambda : R \to R$, while $\ker \lambda_A$ will denote the kernel of $\lambda_A : A \to A$. We have the following short exact sequences, where the right side maps are multiplications by λ

$$0 \to \ker \lambda \cap T_R \to T_R \to \lambda R \cap T_R \to 0,$$
$$0 \to \ker \lambda_A \cap T_A \to T_A \to \lambda A \cap T_A \to 0.$$

Indeed, we need only see that the right side map is surjective. But if $x\lambda \in T_R$, then x has to be a torsion element by (d) of Corollary 3.7. Similar reasons apply to the latter sequence.

Step 2. If C denotes the image in A^*_Γ of the composite $R \xrightarrow{q} A \to A^*_\Gamma$ (which we will call π), then we have the following short exact sequence by (f) of Corollary 3.7

$$0 \to T_R \to R \xrightarrow{\pi} C \to 0.$$

Now, $C \subset A^*_\Gamma$, so that multiplication by λ is injective on C. Therefore, if $x \in R$ such that $\lambda x = 0$, then $\pi(x)$ cannot be nonzero in C. Therefore, we get the following two inclusions, of which the latter follows by a similar argument

$$\ker \lambda \subset T_R$$
$$\ker \lambda_A \subset T_A.$$

Step 3. Let T_O denote the subgroup of all torsion elements in $A^*_{O(2n)}$. Under the composite $R \to R/\lambda R \xrightarrow{\simeq} A^*_{O(2n)}$, torsion elements map inside T_O. This gives a map $T_R \to T_O$. We will shortly show that this is surjective. As a consequence, we will have a short exact sequence

$$0 \to \lambda R \cap T_R \to T_R \to T_O \to 0.$$

Again, the surjectivity of the composite $T_R \to T_A \to T_O$ will imply the surjectivity of $T_A \to T_O$, which, together with the isomorphism $A/\lambda \xrightarrow{\simeq} A^*_{O(2n)}$ will give another short exact sequence

$$0 \to \lambda A \cap T_A \to T_A \to T_O \to 0.$$

Now let us go back to the proof of the surjectivity of $T_R \to T_O$. Since we have $T_O \xrightarrow{\simeq} (c_{odd})A^*_{O(2n)}$ as graded groups, it is sufficient to show that for each odd $p < 2n$, there is a torsion element β_p in R such that $\beta_p \mapsto c_p$

under $T_R \to T_O$. From Equation (3.2) that in R, for each odd p, we have the following equality

$$c_{p+1} = c_{p+1} - (2n-p)c_p\lambda + \lambda^2\alpha'_p \tag{3.4}$$

for some $\alpha'_p \in R$, so that $\lambda((2n-p)c_p - \lambda\alpha'_p) = 0$. But since $\ker\lambda \subset T_R$, we see that the element $\beta_p = (2n-p)c_p - \lambda\alpha'_p$ is torsion. Since $(2n-p)$ is odd, and since $2c_p = 0$ in T_O, we see that $\beta_p \mapsto c_p$ under $T_R \to T_O$, as desired.

Step 4. We have these two short exact sequences, which come from Step 1, by the substitutions $\ker\lambda = \ker\lambda \cap T_R$ and $\ker\lambda_A = \ker\lambda_A \cap T_A$.

$$0 \to \ker\lambda \to T_R \to \lambda R \cap T_R \to 0$$
$$0 \to \ker\lambda_A \to T_A \to \lambda A \cap T_A \to 0$$

Step 5. Note that both R and A are noetherian rings, and their ideals T_R and T_A are finitely generated graded ideals. By (a) of 3.7 torsion elements are 2-primary. So there is some $N > 0$ such that $2^N T_R = 0$, and $2^N T_A = 0$. For each m, the graded pieces R_m and therefore A_m are finitely generated abelian groups. Therefore for each m, the graded pieces $(T_R)_m$ and $(T_A)_m$, which are finitely generated abelian group and hence finitely generated $\mathbb{Z}/2^N\mathbb{Z}$-modules, are finite sets. As $\ker\lambda \subset T_R$ and $\ker\lambda_A \subset T_A$, the sets $(\ker\lambda)_m$ and $(\ker\lambda_A)_m$ are finite as well. The group T_O is actually an $\mathbb{F}_2$-vector space and, for similar reasons as above, each $(T_O)_m$ is a finite set.

Therefore, from the two short exact sequences listed in Step 3, we get

$$\#(T_R)_m - \#(\lambda R \cap T_R)_m = \#(T_O)_m = \#(T_A)_m - \#(\lambda A \cap T_A).$$

From those listed in Step 4, we get

$$\#(T_R)_m - \#(\lambda R \cap T_R)_m = \#(\ker\lambda)_m,$$
$$\#(T_A)_m - \#(\lambda A \cap T_A)_m = \#(\ker\lambda_A)_m.$$

Therefore for each m,

$$\#(\ker\lambda)_m = \#(\ker\lambda_A)_m = \#(T_O)_m.$$

Step 6. We finally proceed to prove that $R \to A$ is injective (therefore bijective). By induction on degree, we will prove that $R_m \to A_m$ is injective (therefore bijective) for each m.

To begin the induction, note that this is true for $m = 0$ and $m = 1$, by 3.5. Now, suppose this is true for $1, \ldots, m$, and we will prove it for $m + 1$.

Since $R_m \to A_m$ is bijective by assumption, $(\ker \lambda)_m \to (\ker \lambda_A)_m$ is injective. But they have the same number of elements by Step 5. Hence $(\ker \lambda)_m \to (\ker \lambda_A)_m$ is surjective (therefore bijective).

With this, and the fact that $R/\lambda R \xrightarrow{\sim} A/\lambda A$, we have the following commutative diagram, whose left and right side vertical maps are isomorphisms, and whose rows are exact.

$$\begin{tikzcd}
0 \arrow[r] & (\ker \lambda)_m \arrow[r] \arrow[d, "\cong"'] & R_m \arrow[r, "\lambda"] \arrow[d, "q"', "\cong"] & R_{m+1} \arrow[r] \arrow[d, "q"'] & (R/\lambda R)_m \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & (\ker \lambda_A)_m \arrow[r] & A_m \arrow[r, "\lambda"'] & A_{m+1} \arrow[r] & (A/\lambda A)_m \arrow[r] & 0
\end{tikzcd}$$

By five lemma, $q : R_{m+1} \to A_{m+1}$ is an isomorphism. □

Remark 3.8. For each odd number $2p+1$, there is an odd number n_{2p+1} such that $n_{2p+1}\lambda c_{2p+1} = \lambda f_p(c_{2p}, c_{2p-1}, \ldots, c_1, \lambda)$, where f_p is a polynomial. Indeed, in Equation (3.4) in Step 3 of the proof of the main theorem, the term α'_p is a polynomial in lower dimensional Chern classes and λ, so one can do induction on p. Now, similarly, $n_{2p-1}n_{2p+1}\lambda c_p = \lambda f'_p$, where f'_p is a polynomial of $c_{2p}, c_{2p-2}, c_{2p-3}, \ldots, c_1, \lambda$. In this way, there is an odd number N_{2p+1} such that $N_{2p+1}\lambda c_{2p+1} = \lambda g_{2p+1}$, where $g_{2p+1} \in B$. Therefore we can say that if $\gamma = \lambda\gamma' \in A$, then there is some odd N'' such that $N''\gamma \in B$.

Lemma 3.9. *Given any $\gamma \in A$, there is an odd number N such that $2N\gamma \in B$.*

Proof. To prove our lemma, is enough to assume that γ can be given by a monomial, involving some odd Chern classes. So $\gamma = c_{2i_1+1}^{m_1} \cdots c_{2i_s+1}^{m_s} g$, where g does not involve any c_{odd}. It is also enough to assume that $g \equiv 1$. Now,

$$2\gamma = f_{i_1}(c_{2i_1}, \ldots, c_1, \lambda)\lambda c_{2i_1+1}^{m_1-1} \cdots c_{2i_s+1}^{m_s},$$

where f is a polynomial. Hence there is some odd number N' such that

$$2N'\gamma = \lambda f_{i_1}(c_{2i_1}, \ldots, c_1, \lambda)g',$$

where $g' \in B$. Now, by the remark preceding our lemma, there is some odd number N'' such that $2N'N''\gamma \in B$. □

Corollary 3.10. *The torsion subgroup $T_A \subset A^*_{GO(2n)}$ is an $\mathbb{F}_2$-vector space.*

Proof. This follows from the last lemma and the fact that each element in T_A is 2-primary. $\square$

Remark 3.11. Remark 3.8 also shows that $\lambda T_A = \lambda R \cap T_A = 0$. Indeed, from 3.8, if $\gamma \in \lambda T_A$, then there is an odd number N such that $N\gamma = 0$. But $\gamma \in T_A$ also, hence is a 2-torsion. So $\gamma = 0$, as desired. Consequently, from Step 3 and Step 4 of the proof of the main theorem,

$$\ker \lambda = T_A \xrightarrow{\sim} T_O.$$

References

[1] Saurav Bhaumik, *Characteristic classes for GO(2n) in étale cohomology.* Proc. Indian Acad. Sci. (Math. Sci.), Volume 123, No. 2 (May 2013), 225–233

[2] Pierre Deligne, *Cohomologie étale.* (With the collaboration of J. F. Boutot, A. Grothendieck, L. Illusie and J.-L. Verdier.) LNM 569, Springer 1977.

[3] Dan Edidin and William Graham, *Equivariant intersection theory.* Invent. Math., **131** (1998), no. 3, 595–634.

[4] William Fulton, *Intersection Theory.* Second Edition, Springer, 1998.

[5] Yogish I Holla and Nitin Nitsure, *Characteristic classes for* $GO(2n, \mathbb{C})$. Asian J. Math., **5** (2001), 169–182.

[6] Yogish I Holla and Nitin Nitsure, *Topology of quadric bundles.* Internat. J. Math., **12** (2001), 1005–1047.

[7] Luis Alberto Molina Rojas and Angelo Vistoli, *On the Chow rings of classifying spaces for classical groups.* Rend. Sem. Mat. Univ. Padova, Vol. 116 (2006).

[8] Fabien Morel and Vladimir Voevodsky, $\mathbb{A}^1$*-homotopy theory of schemes.* Publications mathématiques de l'I.H.É.S, tome **90** (1999), 45–143.

[9] Rahul Pandharipande, *Equivariant Chow rings of* $O(k)$, $SO(2k+1)$, *and* $SO(4)$. J. Reine Angew. Math., **496** (1998), 131–148.

[10] Burt Totaro, *Chow ring of a classifying space.* Algebraic K-Theory, Proc. Sympos. Pure Math., **67**, Amer. Math. Soc. (1999).

Max Planck Institute for Mathematics
Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: bhaumiksaurav@gmail.com

Received July 2, 2014