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Spherical varieties with the A_k-property

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An algebraic variety is said to have the A_k-property if any k points are contained in some common affine open neighbourhood. A theorem of Włodarczyk states that a normal variety has the A_2-property if and only if it admits a closed embedding into a toric variety. Spherical varieties can be regarded as a generalization of toric varieties, but they do not have the A_2-property in general. We provide a combinatorial criterion for the A_k-property of spherical varieties by combining the theory of bunched rings with the Luna-Vust theory of spherical embeddings.

1. Introduction

Throughout the paper, we work with algebraic varieties and algebraic groups over an algebraically closed field $\mathbb{K}$ of characteristic zero.

Definition 1.1 (see, for instance, [8, 20, 22]). A variety X is said to have the A_k*-property* if any k points $x_1, \ldots, x_k \in X$ are contained in some common affine open neighbourhood.

Clearly, any quasi-projective variety has the A_k-property for every k. According to the generalized Kleiman-Chevalley criterion for quasi-projectivity (see [2, 11, 23]), the converse is true for normal varieties.

There exist toric varieties of dimension 3 and greater which are not quasi-projective (see [15, after 2.16]), but they always have the A_2-property. In fact, Włodarczyk has shown in [22] that a normal variety has the A_2-property if and only if it admits a closed embedding into a toric variety.

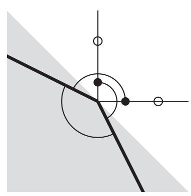
In this paper, we consider spherical varieties, which may be regarded as a generalization of toric varieties. Fix a connected reductive group G and a Borel subgroup $B \subseteq G$. A closed subgroup $H \subseteq G$ is called *spherical* if G/H contains an open B-orbit, and then G/H is called a *spherical homogeneous space*. A G-equivariant open embedding $G/H \hookrightarrow X$ into a normal irreducible G-variety X is called a *spherical embedding*, and then X is called a *spherical variety*.

According to the Luna-Vust theory (see [12, 14]), we can associate to any spherical embedding $G/H \hookrightarrow X$ a combinatorial object called a *colored fan*. We denote by $\mathcal{M}$ the weight lattice of B-semi-invariants in the function field $\mathbb{K}(G/H)$ and by $\mathcal{N}_\mathbb{Q}$ the vector space dual to $\mathcal{M}_\mathbb{Q} \coloneqq \mathcal{M} \otimes_\mathbb{Z} \mathbb{Q}$. We denote by Δ the set of B-invariant prime divisors in X. The subset $\mathcal{D} \subseteq \Delta$ of B-invariant prime divisors in G/H is called the set of *colors*. Moreover, there is a natural map $\rho\colon \Delta \to \mathcal{N}_\mathbb{Q}$. The colored fan associated to the spherical embedding $G/H \hookrightarrow X$ is then given by

$$\Sigma \coloneqq \Big\{(\operatorname{cone}(\rho(I_Y)), I_Y \cap \mathcal{D}) : Y \subseteq X \text{ is a } G\text{-orbit}, \ I_Y = \{D \in \Delta : Y \subseteq \overline{D}\}\Big\},$$

which means that the colored fan Σ contains a pair $(\mathcal{C}, \mathcal{F})$, called a *colored cone*, for every G-orbit $Y \subseteq X$. For details, we refer to Section 2.

It was noticed by Huruguen that, in contrast to toric varieties, spherical varieties do not have the A_2-property in general. In fact, according to [10, Remark 2.38], the example of a non-projective spherical variety with $\dim \mathcal{N}_\mathbb{Q} = 2$ considered in [16, Remarque 3.11] and [21, Example 17.7] fails to have the A_2-property. It is a spherical embedding of $\mathrm{SL}_3(\mathbb{K})/\mathrm{SL}_2(\mathbb{K})$, whose colored fan is shown in the following picture. This example can be generalized to higher dimensions (see [7, Example 4.2]).



Note that two of the colored cones in this colored fan do not intersect in a common face (we refer to Section 2 for the precise definition of "face"), which is allowed by the Luna-Vust theory as long as this does not happen inside a certain valuation cone $\mathcal{V} \subseteq \mathcal{N}_\mathbb{Q}$, which is shown in grey.

In order to give a precise characterization of the A_k-property, we define

$$\Sigma^\sharp \coloneqq \{\operatorname{cone}([\Delta \setminus I]) : I \subseteq \Delta, (\operatorname{cone}(\rho(I)), I \cap \mathcal{D}) \in \Sigma\},$$

which is a set of cones inside the vector space $\mathrm{Cl}(X)_\mathbb{Q} \coloneqq \mathrm{Cl}(X) \otimes_\mathbb{Z} \mathbb{Q}$. The divisor class group $\mathrm{Cl}(X)$ is generated by the divisor classes $[D]$ for $D \in \Delta$, and the relations are given in [6, Proposition 4.1.1]. For details, we refer to Sections 3 and 5.

If X has the A_2-property and $\Gamma(X, \mathcal{O}_X^*) = \mathbb{K}^*$, then we will see in Section 5 that $\Sigma^\sharp$ is the bunch of cones associated to X by the theory of bunched rings, but our main result also holds without these assumptions.

Theorem 1.2. *Let $G/H \hookrightarrow X$ be a spherical embedding with associated colored fan Σ. Then X has the A_k-property if and only if for any k cones $\tau_1, \dots, \tau_k \in \Sigma^\sharp$ we have $\tau_1^\circ \cap \dots \cap \tau_k^\circ \neq \emptyset$ (where τ_i° denotes the relative interior of τ_i).*

For the A_2-property, we obtain the following characterization, which can be verified on the colored fan Σ itself.

Definition 1.3. The *intersection* of two colored cones $(\mathcal{C}_1, \mathcal{F}_1)$ and $(\mathcal{C}_2, \mathcal{F}_2)$ is defined to be the colored cone $(\mathcal{C}_1 \cap \mathcal{C}_2, \mathcal{F}_1 \cap \mathcal{F}_2)$.

Theorem 1.4. *Let $G/H \hookrightarrow X$ be a spherical embedding with associated colored fan Σ. Then X has the A_2-property if and only if any two colored cones in Σ intersect in a common face.*

A spherical variety is called *horospherical* if $\mathcal{V} = \mathcal{N}_{\mathbb{Q}}$.

Corollary 1.5. *Every horospherical variety has the A_2-property.*

2. Spherical embeddings and colored fans

We give a brief overview over the parts of the Luna-Vust theory of spherical embeddings which are relevant for us. For details, we refer to [12, 14]. A survey can also be found in [21].

Let G/H be a spherical homogeneous space. We denote by $\mathcal{M}$ the weight lattice of B-semi-invariants in the function field $\mathbb{K}(G/H)$ and by $\mathcal{N} \coloneqq \operatorname{Hom}(\mathcal{M}, \mathbb{Z})$ the dual lattice, together with the natural pairing

$$\langle \cdot, \cdot \rangle \colon \mathcal{N} \times \mathcal{M} \to \mathbb{Z}.$$

We denote by $\mathcal{D}$ the set of B-invariant prime divisors in G/H. The elements in $\mathcal{D}$ are called the *colors*. Moreover, we denote by $\rho : \mathcal{D} \to \mathcal{N}$ the map given by $\langle \rho(D), \chi \rangle \coloneqq \nu_D(f_\chi)$ for $D \in \mathcal{D}$ where ν_D is the discrete valuation on $\mathbb{K}(G/H)$ induced by the prime divisor D and $f_\chi \in \mathbb{K}(G/H)$ is a B-semi-invariant rational function of weight $\chi \in \mathcal{M}$ (such a rational function f_χ is uniquely determined up to a constant factor).

In the same way, we define a map $\mathcal{V} \to \mathcal{N}_{\mathbb{Q}}$ from the set $\mathcal{V}$ of G-invariant discrete valuations on $\mathbb{K}(G/H)$. This map is injective, so that we may consider $\mathcal{V}$ as a subset of $\mathcal{N}_{\mathbb{Q}}$. It is known from [5] that $\mathcal{V}$ is a cosimplicial (in particular full-dimensional) cone, called the *valuation cone* of G/H. The objects

$$\rho\colon \mathcal{D} \to \mathcal{N}_{\mathbb{Q}} \supseteq \mathcal{V}$$

are called a *colored vector space.*

Definition 2.1. A *colored cone* is a pair $(\mathcal{C}, \mathcal{F})$ such that $\mathcal{F} \subseteq \mathcal{D}$ is a subset and $\mathcal{C} \subseteq \mathcal{N}_{\mathbb{Q}}$ is a cone generated by $\rho(\mathcal{F})$ and finitely many elements of $\mathcal{V}$. It is called

(i) *supported* if $\mathcal{C}^\circ \cap \mathcal{V} \neq \emptyset$, where $\mathcal{C}^\circ$ denotes the relative interior of $\mathcal{C}$,

(ii) *pointed* if $\mathcal{C}$ is pointed and $0 \notin \rho(\mathcal{F})$,

(iii) *simplicial* if $\mathcal{C}$ is spanned by a part of a $\mathbb{Q}$-basis of $\mathcal{N}_{\mathbb{Q}}$ which contains $\rho(\mathcal{F})$ and $\rho|_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{N}_{\mathbb{Q}}$ is injective.

A *face* of a colored cone $(\mathcal{C}, \mathcal{F})$ is a colored cone $(\mathcal{C}_0, \mathcal{F}_0)$ such that $\mathcal{C}_0$ is a face of $\mathcal{C}$ and $\mathcal{F}_0 = \mathcal{F} \cap \rho^{-1}(\mathcal{C}_0)$. A *colored fan* is a nonempty set Σ of pointed supported colored cones such that every supported face of a colored cone in Σ also belongs to Σ and for every $u \in \mathcal{V}$ there is at most one $(\mathcal{C}, \mathcal{F}) \in \Sigma$ with $u \in \mathcal{C}^\circ$.

Remark 2.2. In contrast to some of the literature, we do not require colored cones and their faces to be supported.

For a spherical embedding $G/H \hookrightarrow X$, we denote by Γ the set of G-invariant prime divisors in X. Then $\Delta \coloneqq \mathcal{D} \cup \Gamma$ is the set of all B-invariant prime divisors, and the definition of $\rho\colon \mathcal{D} \to \mathcal{N}_{\mathbb{Q}}$ extends to $\rho\colon \Delta \to \mathcal{N}_{\mathbb{Q}}$.

For any G-orbit $Y \subseteq X$, we denote by $I_Y \subseteq \Delta$ the set of B-invariant prime divisors containing Y in their closure and we set

$$(\mathcal{C}_Y, \mathcal{F}_Y) \coloneqq (\operatorname{cone}(\rho(I_Y)), I_Y \cap \mathcal{D}).$$

Theorem 2.3 ([12, Theorem 3.3]). *The map*

$$(G/H \hookrightarrow X) \mapsto \Sigma \coloneqq \{(\mathcal{C}_Y, \mathcal{F}_Y) : Y \subseteq X \text{ is a } G\text{-orbit}\}$$

defines a bijection between isomorphism classes of spherical embeddings of G/H and colored fans.

Moreover, the assignment

$$\{G\text{-orbits in } X\} \to \Sigma$$
$$Y \mapsto (\mathcal{C}_Y, \mathcal{F}_Y)$$

is a bijection such that for two G-orbits $Y_1, Y_2 \subseteq X$ we have $Y_1 \subseteq \overline{Y_2}$ if and only if $(\mathcal{C}_{Y_2}, \mathcal{F}_{Y_2})$ is a face of $(\mathcal{C}_{Y_1}, \mathcal{F}_{Y_1})$.

Remark 2.4. Note that $\rho|_{\mathcal{D}}$ need not be injective (see, for instance, Example 3.11). On the other hand, Theorem 2.3 implies that the elements of $\rho(\Gamma)$ generate pairwise different nonzero rays. It also implies $\rho(\Gamma) \subseteq \mathcal{V}$. Moreover, for $D' \in \mathcal{D}$ with $\rho(D') \in \mathcal{V}$ it is possible that there exists $D'' \in \Gamma$ such that $\rho(D')$ and $\rho(D'')$ generate the same ray.

It follows from [12, Theorem 6.6] that, under the orbit-cone correspondence of Theorem 2.3, the G-orbits in X of codimension 1 correspond to the colored cones in Σ of the form $(\mathrm{cone}(u), \emptyset)$ for $u \in \mathcal{N}_{\mathbb{Q}}$ (which implies $u \in \mathcal{V}$). These are exactly the (pairwise distinct) colored cones $(\mathrm{cone}(\rho(D)), \emptyset)$ for $D \in \Gamma$.

Remark 2.5 ([4, Proposition 3.1]). Let $G/H \hookrightarrow X$ be a spherical embedding with associated colored fan Σ. Then X is $\mathbb{Q}$-factorial if and only if every colored cone in Σ is simplicial.

Remark 2.6. Two colored cones $(\mathcal{C}_1, \mathcal{F}_1)$ and $(\mathcal{C}_2, \mathcal{F}_2)$ intersect in a common face if and only if there exists $e \in \mathcal{M}_{\mathbb{Q}}$ with $e|_{\mathcal{C}_1} \geq 0$, $e|_{\mathcal{C}_2} \leq 0$,

$$\mathcal{C}_1 \cap e^{\perp} = \mathcal{C}_1 \cap \mathcal{C}_2 = e^{\perp} \cap \mathcal{C}_2, \quad \text{and} \quad \mathcal{F}_1 \cap \rho|_{\mathcal{D}}^{-1}(e^{\perp}) = \rho|_{\mathcal{D}}^{-1}(e^{\perp}) \cap \mathcal{F}_2.$$

Remark 2.7. According to [13, Proposition 1.3(ii)], the units in $\Gamma(X, \mathcal{O}_X)$ are G-semi-invariant. In particular, they are B-semi-invariant, so that we have $\Gamma(X, \mathcal{O}_X^*) = \mathbb{K}^*$ if and only if the set $\rho(\Delta)$ generates $\mathcal{N}_{\mathbb{Q}}$ as a vector space.

The following result will be used in Section 5.

Proposition 2.8. *Let $G/H \hookrightarrow X$ be a spherical embedding with at most one closed G-orbit of codimension at least 2. Then X is quasi-projective.*

Proof. Let $Y \subseteq X$ be a G-orbit of maximal codimension (this is the unique closed G-orbit of codimension at least 2 if such an orbit exists). We denote by $X_Y \subseteq X$ the open G-stable subvariety obtained by removing from X all G-orbits not containing Y in their closure. As Y is the unique closed G-orbit in X_Y, it follows from [18, Lemma 8] that X_Y is quasi-projective.

According to [19, Theorem 4.9], there exists a G-equivariant surjective morpishm $q\colon X' \to X$ where X' is a quasi-projective G-variety and q is an isomorphism over X_Y. We may moreover assume X' to be normal. Let Σ be the colored fan associated to $G/H \hookrightarrow X$, and let Σ' be the colored fan associated to $G/H \hookrightarrow X'$. Since the G-orbits in $X \setminus X_Y$ are all of codimension 1, they correspond to the colored cones $(\operatorname{cone}(\rho(D)), \emptyset) \in \Sigma$ with $D \in \Gamma_{X\setminus X_Y}$ for some subset $\Gamma_{X\setminus X_Y} \subseteq \Gamma$. Then [12, Theorem 4.1] implies that we have $(\operatorname{cone}(\rho(D)), \emptyset) \in \Sigma'$ for every $D \in \Gamma_{X\setminus X_Y}$, hence we obtain $X \subseteq X'$. □

3. Gale duality for spherical embeddings

The aim of this section is to generalize some results on Gale duality for toric varieties, as presented in [1, 2.2.1], to the setting of spherical embeddings.

There are two main differences between the toric and the spherical case. The first difference is that there could exist distinct $D, D' \in \Delta$ such that $\rho(D)$ and $\rho(D')$ generate the same ray in $\mathcal{N}_\mathbb{Q}$ when colors are involved. The reason why this is not a problem is that colored cones by definition keep track of the colors. The second difference is that in the case $\mathcal{V} \neq \mathcal{N}_\mathbb{Q}$ some cones will be ignored when they are not supported according to the definitions below.

Let $\rho\colon \mathcal{D} \to \mathcal{N}_\mathbb{Q} \supseteq \mathcal{V}$ be a colored vector space, and let Γ be a finite set equipped with another map $\rho\colon \Gamma \to \mathcal{V}$. We define $\Delta \coloneqq \mathcal{D} \cup \Gamma$ to be a disjoint union and obtain a map $\rho\colon \Delta \to \mathcal{N}_\mathbb{Q}$. We assume $\operatorname{span}_\mathbb{Q} \rho(\Delta) = \mathcal{N}_\mathbb{Q}$. Let $\mathbb{Q}^\Delta$ and $(\mathbb{Q}^\Delta)^*$ be dual vector spaces with respective standard bases $\{e_D : D \in \Delta\}$ and $\{e_D^* : D \in \Delta\}$, which are dual to each other. The map $\mathcal{P}\colon (\mathbb{Q}^\Delta)^* \to \mathcal{N}_\mathbb{Q}$ with $e_D^* \mapsto \rho(D)$ induces the following pair of mutually dual exact sequences of vector spaces.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_\mathbb{Q} & \longrightarrow & (\mathbb{Q}^\Delta)^* & \xrightarrow{\mathcal{P}} & \mathcal{N}_\mathbb{Q} & \longrightarrow & 0 \\ 0 & \longleftarrow & K_\mathbb{Q} & \xleftarrow{\mathcal{Q}} & \mathbb{Q}^\Delta & \longleftarrow & \mathcal{M}_\mathbb{Q} & \longleftarrow & 0 \end{array}$$

For $D \in \Delta$, we also write $\mathcal{P}(D)$ for $\mathcal{P}(e_D^*)$ and $\mathcal{Q}(D)$ for $\mathcal{Q}(e_D)$.

Definition 3.1. A $\mathcal{P}$-*cone* is a pair $(\operatorname{cone}(\mathcal{P}(I)), I \cap \mathcal{D})$ where $I \subseteq \Delta$.

Remark 3.2. Note that every $\mathcal{P}$-cone is a colored cone. In particular, the definitions of “supported”, intersections, faces, “pointed”, and “simplicial” from Section 2 are applicable.

Definition 3.3. A $\mathcal{Q}$-*cone* is a cone in $K_\mathbb{Q}$ generated by a subset of $\mathcal{Q}(\Delta)$.

Definition 3.4. For any set Σ of $\mathcal{P}$-cones and any set Θ of $\mathcal{Q}$-cones we define

$$\Sigma^\natural := \{\operatorname{cone}(\mathcal{Q}(\Delta \setminus I)) : I \subseteq \Delta, (\operatorname{cone}(\mathcal{P}(I)), I \cap \mathcal{D}) \in \Sigma\},$$
$$\Theta^\natural := \{(\operatorname{cone}(\mathcal{P}(\Delta \setminus J)), (\Delta \setminus J) \cap \mathcal{D}) : J \subseteq \Delta, \operatorname{cone}(\mathcal{Q}(J)) \in \Theta\}.$$

Definition 3.5. A $\mathcal{Q}$-cone τ is called *supported* if $\{\tau\}^\natural$ contains a supported $\mathcal{P}$-cone. For any set Σ of $\mathcal{P}$-cones we define

$$\overline{\Sigma} := \{(\mathcal{C}, \mathcal{F}) \in \Sigma : (\mathcal{C}, \mathcal{F}) \text{ is supported}\},$$

i. e. the $\mathcal{P}$-cones which are not supported are removed from Σ.

Remark 3.6. A $\mathcal{Q}$-cone τ is supported if and only if $\overline{\{\tau\}^\natural}$ is not empty.

Definition 3.7. A set Σ of $\mathcal{P}$-cones is called a $\mathcal{P}$*-quasifan* if it is nonempty and

(i) every $(\mathcal{C}, \mathcal{F}) \in \Sigma$ is supported,

(ii) any two $(\mathcal{C}_1, \mathcal{F}_1), (\mathcal{C}_2, \mathcal{F}_2) \in \Sigma$ intersect in a common face,

(iii) for any $(\mathcal{C}, \mathcal{F}) \in \Sigma$ every supported face of $(\mathcal{C}, \mathcal{F})$ also belongs to Σ.

A $\mathcal{P}$*-fan* is a $\mathcal{P}$-quasifan consisting of pointed $\mathcal{P}$-cones. A $\mathcal{P}$-(quasi)fan is called *maximal* if it cannot be extended by adding supported $\mathcal{P}$-cones. It is called *true* if it contains the $\mathcal{P}$-cone $(0, \emptyset)$ in the case $\mathcal{D} \neq \emptyset$ as well as the $\mathcal{P}$-cones $(\operatorname{cone}(\rho(D)), \emptyset)$ for $D \in \Gamma$.

Definition 3.8. A set Θ of $\mathcal{Q}$-cones is called a $\mathcal{Q}$*-bunch* if it is nonempty and

(i) every $\tau \in \Theta$ is supported,

(ii) for any $\tau_1, \tau_2 \in \Theta$ we have $\tau_1^\circ \cap \tau_2^\circ \neq \emptyset$,

(iii) for any $\tau \in \Theta$ every supported $\mathcal{Q}$-cone τ_0 with $\tau^\circ \subseteq \tau_0^\circ$ also belongs to Θ.

A $\mathcal{Q}$-bunch Θ is called *maximal* if it cannot be extended by adding supported $\mathcal{Q}$-cones. It is called *true* if it contains the $\mathcal{Q}$-cone $\operatorname{cone}(\mathcal{Q}(\Delta))$ in the case $\mathcal{D} \neq \emptyset$ as well as the $\mathcal{Q}$-cones $\operatorname{cone}(\mathcal{Q}(\Delta \setminus \{D\}))$ for $D \in \Gamma$.

We can now state our generalization of [1, Theorem 2.2.1.14] to the spherical situation. The proof will be given in Section 4.

Theorem 3.9. *We have an order reversing map*

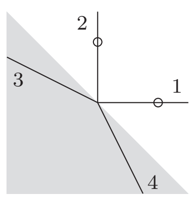
$$\{\mathcal{Q}\text{-}bunches\} \to \{\mathcal{P}\text{-}quasifans\}, \qquad \Theta \mapsto \Theta^\sharp := \overline{\Theta^\natural}.$$

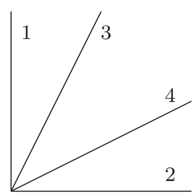
Now assume that the elements in $\rho(\Gamma)$ generate pairwise different rays. Then there are mutually inverse order reversing bijections

$$\{true\ maximal\ \mathcal{Q}\text{-}bunches\} \leftrightarrow \{true\ maximal\ \mathcal{P}\text{-}fans\},$$
$$\Theta \mapsto \Theta^\sharp,$$
$$\Sigma^\natural =: \Sigma^\sharp \leftarrow\!\shortmid \Sigma.$$

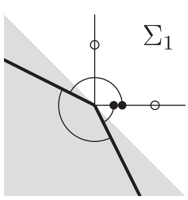
Under these bijections, the true maximal $\mathcal{P}$-fans consisting of simplicial cones correspond to the true maximal $\mathcal{Q}$-bunches consisting of full-dimensional cones.

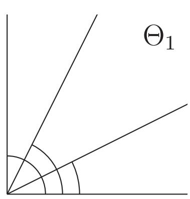
Example 3.10. We consider the spherical homogeneous space $\mathrm{SL}_3(\mathbb{K})/\mathrm{SL}_2(\mathbb{K})$. The left-hand picture below shows the vector space $\mathcal{N}_\mathbb{Q}$ with the valuation cone $\mathcal{V}$ in grey. There are two colors, $\mathcal{D} = \{D_1, D_2\}$, where the elements $\rho(D_i) \in \mathcal{N}_\mathbb{Q}$ are represented by a circle. For details, we refer to [16], [21, Example 17.7], or [7, Example 4.2]. We have added two $\mathrm{SL}_3(\mathbb{K})$-invariant prime divisors, $\Gamma = \{D_3, D_4\}$. In the picture, the number i is written near the ray generated by $\mathcal{P}(D_i)$. The right-hand picture shows the vector space $K_\mathbb{Q}$ and the number i is written near the ray generated by $\mathcal{Q}(D_i)$.

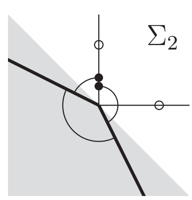


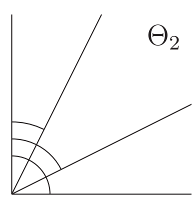


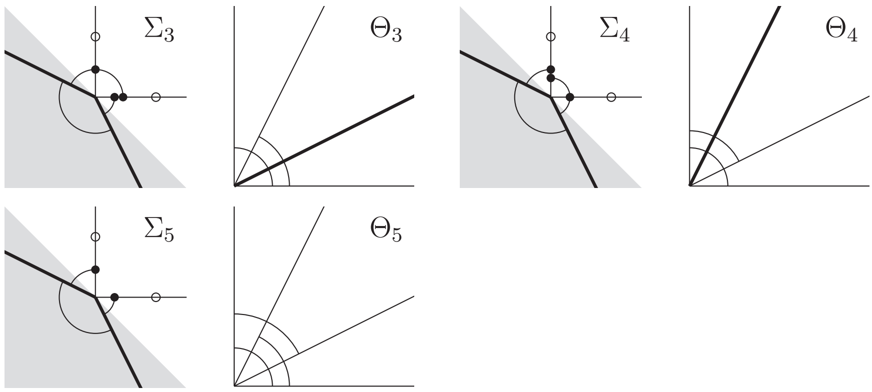
The following pictures show all possible true maximal $\mathcal{P}$-fans Σ_j and their corresponding true maximal $\mathcal{Q}$-bunches Θ_j. In the pictures, the included cones of dimension 2 are represented by arcs while the included cones of dimension 1 are represented by thick rays. For colored cones, the colors which are included are indicated by large black dots.



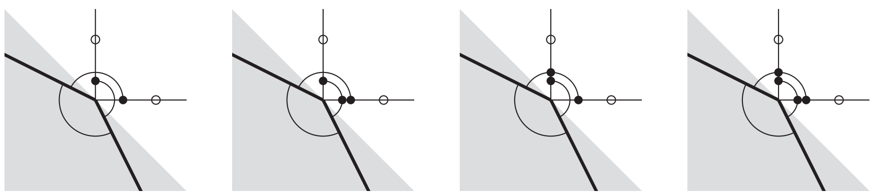








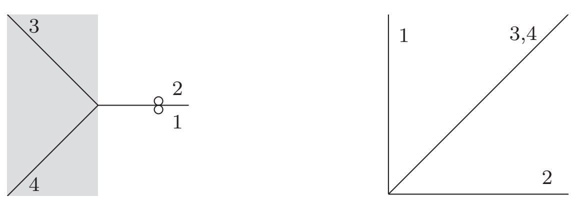
Moreover, we have the following colored fans which are not $\mathcal{P}$-fans. According to Theorem 1.4, these correspond to embeddings which do not have the A_2-property.



Example 3.11. We consider the spherical homogeneous space

$$(\mathrm{SL}_2(\mathbb{K}) \times \mathbb{K}^*)/(T \times \{1\})$$

where T is a maximal torus in $\mathrm{SL}_2(\mathbb{K})$. For details, we refer, for instance, to [17, Example 2.5.1]. The following pictures show the vector spaces $\mathcal{N}_\mathbb{Q}$ and $K_\mathbb{Q}$ (with the same notation as in Example 3.10). For the two colors, $\mathcal{D} = \{D_1, D_2\}$, we have $\rho(D_1) = \rho(D_2)$, so that in order to be able to distinguish them, we have put one circle together with the number 1 below and one circle together with the number 2 above the ray. We have added two $\mathrm{SL}_2(\mathbb{K}) \times \mathbb{K}^*$-invariant prime divisors, $\Gamma = \{D_3, D_4\}$. Note that we have $\mathcal{Q}(D_3) = \mathcal{Q}(D_4)$, see the paragraph before Remark 5.4.



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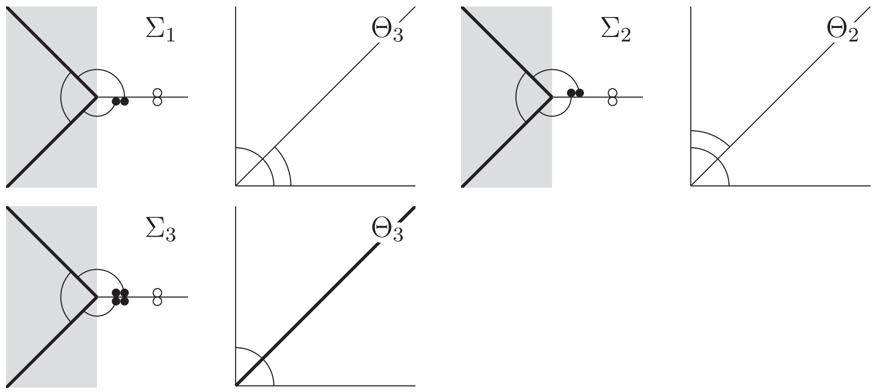

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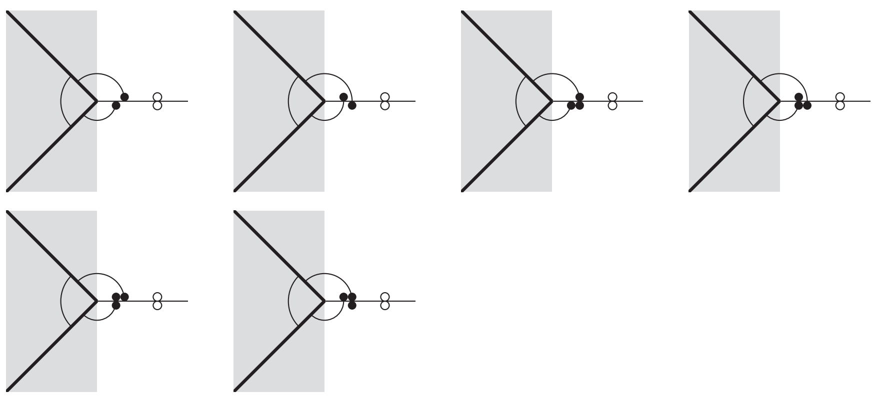

4. Proof of Theorem 3.9

This section is a shortened and suitably modified version of [1, 2.2.3]. We define

$$\delta \coloneqq \operatorname{cone}(e_D^* : D \in \Delta), \qquad \gamma \coloneqq \operatorname{cone}(e_D : D \in \Delta).$$

These cones are dual to each other, and we have the face correspondence, i. e. mutually inverse bijections

$$\begin{aligned} \mathrm{faces}(\delta) &\leftrightarrow \mathrm{faces}(\gamma), \\ \delta_0 &\mapsto \delta_0^* := \delta_0^\perp \cap \gamma, \\ \gamma_0^\perp \cap \delta =: \gamma_0^* &\leftarrow\!\shortmid \gamma_0. \end{aligned}$$

Definition 4.1. A face $\delta_0 \preceq \delta$ is called *supported* if $\mathcal{P}(\delta_0)^\circ \cap \mathcal{V} \neq \emptyset$. A face $\gamma_0 \preceq \gamma$ is called *supported* if $\gamma_0^* \preceq \delta$ is supported. A *δ-collection* (resp. a *γ-collection*) is a set of supported faces of δ (resp. of γ). A *$\mathcal{P}$-collection* (resp. a *$\mathcal{Q}$-collection*) is a set of supported $\mathcal{P}$-cones (resp. of supported $\mathcal{Q}$-cones).

Remark 4.2. A $\mathcal{Q}$-cone τ is supported if and only if there exists a supported face $\gamma_0 \preceq \gamma$ with $\mathcal{Q}(\gamma_0) = \tau$.

The idea of the proof is to decompose the $\sharp$-operation between $\mathcal{Q}$-collections and $\mathcal{P}$-collections according to the following scheme of further operations.

$$\begin{array}{ccc} \{\gamma\text{-collections}\} & \xleftrightarrow{\;*\;} & \{\delta\text{-collections}\} \\ {\scriptstyle \mathcal{Q}^\uparrow}\big\uparrow\big\downarrow{\scriptstyle \mathcal{Q}_\downarrow} & & {\scriptstyle \mathcal{P}_\downarrow}\big\downarrow\big\uparrow{\scriptstyle \mathcal{P}^\uparrow} \\ \{\mathcal{Q}\text{-collections}\} & \underset{\sharp}{\longleftrightarrow} & \{\mathcal{P}\text{-collections}\} \end{array}$$

Definition 4.3. An *$L_{\mathbb{Q}}$-invariant separating linear form* for two faces $\delta_1, \delta_2 \preceq \delta$ is an element $e \in \mathbb{Q}^\Delta$ such that

$$e|_{L_{\mathbb{Q}}} = 0, \qquad e|_{\delta_1} \geq 0, \qquad e|_{\delta_2} \leq 0, \qquad \delta_1 \cap e^\perp = \delta_1 \cap \delta_2 = e^\perp \cap \delta_2.$$

Definition 4.4. A δ-collection $\mathfrak{A}$ is called

(i) *separated* if any two $\delta_1, \delta_2 \in \mathfrak{A}$ admit an $L_{\mathbb{Q}}$-invariant separating linear form,

(ii) *saturated* if for any $\delta_1 \in \mathfrak{A}$ every supported $\delta_2 \preceq \delta_1$ which is $L_{\mathbb{Q}}$-invariantly separable from δ_1 also belongs to $\mathfrak{A}$,

(iii) *true* if we have $0 \in \mathfrak{A}$ for $\mathcal{D} \neq \emptyset$ and $\mathrm{cone}(e_D^*) \in \mathfrak{A}$ for every $D \in \Gamma$,

(iv) *maximal* if it is maximal among the separated δ-collections.

Definition 4.5. A γ-collection $\mathfrak{B}$ is called

(i) *connected* if for any $\gamma_1, \gamma_2 \in \mathfrak{B}$ we have $\mathcal{Q}(\gamma_1)^\circ \cap \mathcal{Q}(\gamma_2)^\circ \neq \emptyset$,

(ii) *saturated* if for any $\gamma_1 \in \mathfrak{B}$ every supported γ_2 with $\gamma \succeq \gamma_2 \succeq \gamma_1$ and $\mathcal{Q}(\gamma_1)^\circ \subseteq \mathcal{Q}(\gamma_2)^\circ$ also belongs to $\mathfrak{B}$,

(iii) *true* if we have $\gamma \in \mathfrak{B}$ for $\mathcal{D} \neq \emptyset$ and $\operatorname{cone}(e_D^*)^* \in \mathfrak{B}$ for every $D \in \Gamma$,

(iv) *maximal* if it is maximal among the connected γ-collections.

Proposition 4.6. *We have mutually inverse bijections sending separated (saturated, true, maximal) collections to connected (saturated, true, maximal) collections:*

$$\begin{aligned}\{\textit{separated } \delta\textit{-collections}\} &\leftrightarrow \{\textit{connected } \gamma\textit{-collections}\},\\ \mathfrak{A} &\mapsto \mathfrak{A}^* := \{\delta_0^* : \delta_0 \in \mathfrak{A}\},\\ \{\gamma_0^* : \gamma_0 \in \mathfrak{B}\} =: \mathfrak{B}^* &\leftarrow\!\shortmid \mathfrak{B}.\end{aligned}$$

Proof. See [1, Proposition 2.2.3.5]. □

Definition 4.7. A $\mathcal{Q}$-collection Θ is called

(i) *connected* if for any $\tau_1, \tau_2 \in \Theta$ we have $\tau_1^\circ \cap \tau_2^\circ \neq \emptyset$,

(ii) *saturated* if for any $\tau \in \Theta$ every supported $\mathcal{Q}$-cone τ_0 with $\tau^\circ \subseteq \tau_0^\circ$ also belongs to Θ,

(iii) *true* if we have $\operatorname{cone}(\mathcal{Q}(\Delta)) \in \Theta$ for $\mathcal{D} \neq \emptyset$ and $\operatorname{cone}(\mathcal{Q}(\Delta \setminus \{D\})) \in \Theta$ for every $D \in \Gamma$,

(iv) *maximal* if it is maximal among the connected $\mathcal{Q}$-collections.

Definition 4.8. We define the $\mathcal{Q}$-lift and the $\mathcal{Q}$-drop to be the maps

$$\begin{aligned}\mathcal{Q}^\uparrow \colon \{\mathcal{Q}\text{-collections}\} &\to \{\gamma\text{-collections}\},\\ \Theta &\mapsto \mathcal{Q}^\uparrow\Theta := \{\gamma_0 \preceq \gamma : \gamma_0 \text{ is supported and } \mathcal{Q}(\gamma_0) \in \Theta\},\\ \mathcal{Q}_\downarrow \colon \{\gamma\text{-collections}\} &\to \{\mathcal{Q}\text{-collections}\},\\ \mathfrak{B} &\mapsto \mathcal{Q}_\downarrow\mathfrak{B} := \{\mathcal{Q}(\gamma_0) : \gamma_0 \in \mathfrak{B}\}.\end{aligned}$$

Proposition 4.9. *The $\mathcal{Q}$-lift is injective and sends connected (saturated, true, maximal) $\mathcal{Q}$-collections to connected (saturated, true, maximal) γ-collections. Moreover, we have mutually inverse bijections sending true collections to true* *collections:*

$$\{\textit{maximal } \mathcal{Q}\textit{-collections}\} \leftrightarrow \{\textit{maximal } \gamma\textit{-collections}\},$$
$$\Theta \mapsto \mathcal{Q}^{\uparrow}\Theta,$$
$$\mathcal{Q}_{\downarrow}\mathfrak{B} \leftarrow\!\shortmid \mathfrak{B}.$$

Proof. See [1, Proposition 2.2.3.8]. □

Definition 4.10. A $\mathcal{P}$-collection Σ is called

(i) *separated* if any two $(\mathcal{C}_1, \mathcal{F}_1), (\mathcal{C}_2, \mathcal{F}_2) \in \Sigma$ intersect in a common face,

(ii) *saturated* if for any $(\mathcal{C}, \mathcal{F}) \in \Sigma$ every supported face of $(\mathcal{C}, \mathcal{F})$ also is in Σ,

(iii) *true* if we have $(0, \emptyset) \in \Sigma$ for $\mathcal{D} \neq \emptyset$ and $(\mathrm{cone}(\mathcal{P}(D)), \emptyset) \in \Sigma$ for every $D \in \Gamma$,

(iv) *maximal* if it is maximal among the separated $\mathcal{P}$-collections.

Definition 4.11. We define the $\mathcal{P}$-lift and the $\mathcal{P}$-drop to be the maps

$$\begin{aligned}
\mathcal{P}^{\uparrow}\colon \{\mathcal{P}\text{-collections}\} &\to \{\delta\text{-collections}\},\\
&\Sigma \mapsto \mathcal{P}^{\uparrow}\Sigma := \{\delta_0 \preceq \delta : (\mathcal{P}(\delta_0), \{D \in \mathcal{D} : e_D^* \in \delta_0\}) \in \Sigma\},\\
\mathcal{P}_{\downarrow}\colon \{\delta\text{-collections}\} &\to \{\mathcal{P}\text{-collections}\},\\
&\mathfrak{A} \mapsto \mathcal{P}_{\downarrow}\mathfrak{A} := \{(\mathcal{P}(\delta_0), \{D \in \mathcal{D} : e_D^* \in \delta_0\}) : \delta_0 \in \mathfrak{A}\}.
\end{aligned}$$

Proposition 4.12. *The $\mathcal{P}$-drop is surjective and sends separated (saturated, true, maximal) δ-collections to separated (saturated, true, maximal) $\mathcal{P}$-collections. If the elements in $\mathcal{P}(\Gamma)$ generate pairwise different rays, then we have mutually inverse bijections sending saturated (maximal) collections to saturated (maximal) collections:*

$$\{\textit{true separated } \delta\textit{-collections}\} \leftrightarrow \{\textit{true separated } \mathcal{P}\textit{-collections}\},$$
$$\mathfrak{A} \mapsto \mathcal{P}_{\downarrow}\mathfrak{A},$$
$$\mathcal{P}^{\uparrow}\Sigma \leftarrow\!\shortmid \Sigma.$$

Proof. For every $\mathcal{P}$-collection Σ we have $\Sigma = \mathcal{P}_{\downarrow}\mathcal{P}^{\uparrow}\Sigma$. In particular, $\mathcal{P}_{\downarrow}$ is surjective. Let $\delta_1, \delta_2 \preceq \delta$ be two faces admitting an $L_{\mathbb{Q}}$-invariant separating linear form $e \in \mathbb{Q}^{\Delta}$, and define

$$(\mathcal{C}_1, \mathcal{F}_1) \coloneqq (\mathcal{P}(\delta_1), \{D \in \mathcal{D} : e_D^* \in \delta_1\}),$$
$$(\mathcal{C}_2, \mathcal{F}_2) \coloneqq (\mathcal{P}(\delta_2), \{D \in \mathcal{D} : e_D^* \in \delta_2\}).$$

Then e can be interpreted as $e \in \mathcal{M}_{\mathbb{Q}}$ with $e|_{\mathcal{C}_1} \geq 0$, $e|_{\mathcal{C}_2} \leq 0$,

$$\mathcal{C}_1 \cap e^{\perp} = \mathcal{C}_1 \cap \mathcal{C}_2 = e^{\perp} \cap \mathcal{C}_2, \quad \text{and} \quad \mathcal{F}_1 \cap \rho|_{\mathcal{D}}^{-1}(e^{\perp}) = \rho|_{\mathcal{D}}^{-1}(e^{\perp}) \cap \mathcal{F}_2.$$

It now follows from Remark 2.6 that $\mathcal{P}_{\downarrow}$ preserves separatedness and saturatedness. The fact that $\mathcal{P}_{\downarrow}$ preserves the properties true and maximal is obvious.

Now assume that the elements in $\mathcal{P}(\Gamma)$ generate pairwise different rays. Consider a true separated $\mathcal{P}$-collection Σ. Then, for every $(\mathcal{C}, \mathcal{F}) \in \Sigma$ and every $D \in \Gamma$ we have $\mathcal{P}(D) \in \mathcal{C}$ if and only if $\mathbb{Q}_{\geq 0} \cdot \mathcal{P}(D)$ is an extremal ray of $\mathcal{C}$. Moreover, for every $D' \in \mathcal{D}$ such that there exists $D'' \in \Gamma$ with $\mathbb{Q}_{\geq 0} \cdot \mathcal{P}(D') = \mathbb{Q}_{\geq 0} \cdot \mathcal{P}(D'')$ we have $D' \notin \mathcal{F}$. Consequently, for every $(\mathcal{C}, \mathcal{F}) \in \Sigma$ there is a unique $\delta_0 \preceq \delta$ with $(\mathcal{P}(\delta_0), \{D \in \mathcal{D} : e_D^* \in \delta_0\}) = (\mathcal{C}, \mathcal{F})$. It follows that $\mathcal{P}^{\uparrow}\Sigma$ is true and separated, and, if Σ is saturated (maximal), then $\mathcal{P}^{\uparrow}\Sigma$ is also saturated (maximal). Moreover, we conclude that $\mathcal{P}_{\downarrow}$ restricted to the true separated δ-collections is injective. □

Proof of Theorem 3.9. First, observe that the (true, maximal) $\mathcal{Q}$-bunches are precisely the (true, maximal) connected saturated $\mathcal{Q}$-collections and the (true, maximal) $\mathcal{P}$-quasifans are precisely the (true, maximal) separated saturated $\mathcal{P}$-collections. Next observe that we have

$$\Theta^{\sharp} = \mathcal{P}_{\downarrow}((\mathcal{Q}^{\uparrow}\Theta)^*), \qquad \Sigma^{\sharp} = \mathcal{Q}_{\downarrow}((\mathcal{P}^{\uparrow}\Sigma)^*).$$

The first part of Theorem 3.9 now follows from Propositions 4.9, 4.6, and 4.12.

It remains to show that $\mathcal{P}$-fans consisting of simplicial $\mathcal{P}$-cones correspond to $\mathcal{Q}$-bunches consisting of full-dimensional $\mathcal{Q}$-cones. A true $\mathcal{P}$-fan Σ is simplicial exactly when for every $(\operatorname{cone}(I), I \cap \mathcal{D}) \in \Sigma$ with $I \subseteq \Delta$ and any subset $I_0 \subseteq I$ we have that $(\operatorname{cone}(I_0), I_0 \cap \mathcal{D})$ is a face of $(\operatorname{cone}(I), I \cap \mathcal{D})$. This means that for every $\tau = \operatorname{cone}(\mathcal{Q}(J))$ with $J \subseteq \Delta$ in the corresponding true $\mathcal{Q}$-bunch Θ and every J_1 with $J \subseteq J_1 \subseteq \Delta$ we have $\tau^{\circ} \subseteq \operatorname{cone}(\mathcal{Q}(J_1))^{\circ}$. Because the vectors $\{\mathcal{Q}(D) : D \in \Delta\}$ generate $K_{\mathbb{Q}}$, this is exactly the case when all cones in Θ are of full dimension. □

5. Bunched rings

Let X_0 be a normal irreducible variety with $\Gamma(X_0, \mathcal{O}^*_{X_0}) = \mathbb{K}^*$, finitely generated divisor class group $K \coloneqq \mathrm{Cl}(X_0)$, and finitely generated *Cox ring*

$$R \coloneqq \mathcal{R}(X_0) \coloneqq \bigoplus_{[D]\in K} \Gamma(X_0, \mathcal{O}_{X_0}(D)),$$

where some care has to be taken in order to define the multiplication law. The Cox ring R is factorially K-graded. This means that every homogeneous nonzero nonunit in R can be written as a product of K-primes, where a K-prime is a homogeneous nonzero nonunit $f \in R$ such that $f \mid gh$ with homogeneous $g, h \in R$ always implies $f \mid g$ or $f \mid h$. For details, we refer to [1, 1.4 and 1.6].

With $\overline{X} \coloneqq \operatorname{Spec} R$ the K-grading on R corresponds to an S-action on $\overline{X}$ where $S \coloneqq \operatorname{Spec} \mathbb{K}[K]$ is a quasitorus (i. e. a diagonalizable group) with character group K. There exists an open S-stable subvariety $\widehat{X}_0 \subseteq \overline{X}$ with complement of codimension at least 2 such that we obtain a good quotient $\pi\colon \widehat{X}_0 \to X_0$ for the S-action.

If X is any other normal irreducible variety with the same graded Cox ring as X_0, we obtain X as a good quotient $\pi\colon \widehat{X} \to X$ for some other open subvariety $\widehat{X} \subseteq \overline{X}$ with complement of codimension at least 2 (the varieties X_0 and X also differ at most in codimension 2). The theory of bunched rings (which first appeared in [3, 9]) can be used to find such $\widehat{X} \subseteq \overline{X}$ provided that the quotient X has the A_2-property. We recall some definitions and results on bunched rings from [1, 3.2].

Definition 5.1 ([1, Definitions 3.2.1.1 and 3.2.1.2]). Let $\mathfrak{F}$ be a finite system of pairwise nonassociated K-prime generators for R.

(i) The K-grading is said to be *almost free* if for every $f_0 \in \mathfrak{F}$ the set

$$\{\deg f : f \in \mathfrak{F} \setminus \{f_0\}\}$$

generates K as an abelian group.

(ii) The set of *projected* $\mathfrak{F}$*-faces* is the set

$$\Omega_{\mathfrak{F}} \coloneqq \left\{ \mathrm{cone}(\deg f : f \in J) : J \subseteq \mathfrak{F}, \bigcap_{f \notin J} \mathbb{V}(f) \setminus \bigcup_{f\in J} \mathbb{V}(f) \neq \emptyset \right\}$$

of cones in $K_{\mathbb{Q}}$ where the $\mathbb{V}(f)$ are considered as subsets of $\overline{X}$.

(iii) An $\mathfrak{F}$-*bunch* is a nonempty subset $\Theta \subseteq \Omega_{\mathfrak{F}}$ such that
  a) for any $\tau_1, \tau_2 \in \Theta$ we have $\tau_1^\circ \cap \tau_2^\circ = \emptyset$,
  b) for any $\tau \in \Theta$ every $\tau_0 \in \Omega_{\mathfrak{F}}$ with $\tau^\circ \subseteq \tau_0^\circ$ also belongs to Θ.

(iv) An $\mathfrak{F}$-bunch Θ is called *true* if it contains

$$\operatorname{cone}(\deg f : f \in \mathfrak{F} \setminus \{f_0\})$$

for every $f_0 \in \mathfrak{F}$.

If Θ is an $\mathfrak{F}$-bunch, the triple $(R, \mathfrak{F}, \Theta)$ is called a *bunched ring*.

As in [1, 3.2.1], for any $\mathfrak{F}$-bunch Θ we set

$$\widehat{X} := \bigcup_{\tau \in \Theta} \bigcup_{\substack{J \subseteq \mathfrak{F} \\ \operatorname{cone}(\deg f : f \in J) = \tau}} \left(\overline{X} \setminus \bigcup_{f \in J} \mathbb{V}(f) \right).$$

Then there exists a good quotient $\pi\colon \widehat{X} \to X$ and we say that the variety X arises from the bunched ring $(R, \mathfrak{F}, \Theta)$. As in [1, 3.3.1], for any $\tau \in \Omega_{\mathfrak{F}}$ we define

$$X(\tau) := \bigcup_{\substack{J \subseteq \mathfrak{F} \\ \operatorname{cone}(\deg f : f \in J) = \tau}} \left(\bigcap_{f \notin J} \pi(\mathbb{V}(f)) \Bigg\backslash \bigcup_{f \in J} \pi(\mathbb{V}(f)) \right) \subseteq X$$

where the $\mathbb{V}(f)$ are now considered as subsets of $\widehat{X}$. We obtain a disjoint union

$$X = \bigcup_{\tau \in \Theta} X(\tau).$$

Note that we have $X(\tau) \neq \emptyset$ if and only if $\tau \in \Theta$.

Proposition 5.2 ([1, Proposition 3.2.1.9]). *Let X be a variety with graded Cox ring R. Then X has the A_2-property if and only if it is an open subvariety of a variety arising from a bunched ring $(R, \mathfrak{F}, \Theta)$.*

This can be generalized to take into account the A_k-property for any $k \geq 2$ in a straightforward manner.

Proposition 5.3 ([1, Exercise 3.5(4)]). *Let X be a variety with graded Cox ring R, and let $k \geq 2$. Then X has the A_k-property if and only if it is an* *open subvariety of a variety arising from a bunched ring* $(R, \mathfrak{F}, \Theta)$ *such that for any* k *cones* $\tau_1, \ldots, \tau_k \in \Theta$ *we have* $\tau_1^\circ \cap \cdots \cap \tau_k^\circ \neq \emptyset$.

From now on, we further assume that $G/H \hookrightarrow X_0$ is a spherical embedding with associated colored fan Σ_0 such that X_0 only contains G-orbits of codimensions 0 and 1, i. e. we have

$$\Sigma_0 = \{(0, \emptyset)\} \cup \{(\mathrm{cone}(\rho(D)), \emptyset) : D \in \Gamma\}.$$

We continue to denote by X a (normal) variety with the same (graded) Cox ring R as X_0. In particular, if the G-action extends to X, then the variety X is spherical and X_0 is obtained from X by removing the G-orbits of codimension at least 2.

The description of the divisor class group of a spherical variety from [6, Proposition 4.1.1] shows that in the exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_{\mathbb{Q}} & \longrightarrow & (\mathbb{Q}^{\Delta})^* & \xrightarrow{\mathcal{P}} & \mathcal{N}_{\mathbb{Q}} & \longrightarrow & 0 \\ 0 & \longleftarrow & K_{\mathbb{Q}} & \xleftarrow{\mathcal{Q}} & \mathbb{Q}^{\Delta} & \longleftarrow & \mathcal{M}_{\mathbb{Q}} & \longleftarrow & 0 \end{array}$$

of Section 3 we have $K_{\mathbb{Q}} = \mathrm{Cl}(X_0)_{\mathbb{Q}}$ and $\mathcal{Q}(D) = [D]$. Moreover, Brion has shown that spherical varieties have finitely generated Cox rings. The following Remark 5.4 summarizes the properties of the Cox ring which we are going to use.

Remark 5.4. It follows from [6, Theorem 4.3.2], [7, Theorem 3.6], or [1, Theorem 4.5.4.6] and [7, Proposition 2.4] that there exist positive integers n_D and elements $f_{D,1}, \ldots, f_{D,n_D} \in R$ with the following properties:

(i) For every $D \in \Delta$ and every $1 \leq \ell \leq n_D$ we have $\deg f_{D,\ell} = [D]$.

(ii) The system $\mathfrak{F} := \{f_{D,\ell} : D \in \Delta, 1 \leq \ell \leq n_D\}$ consists of pairwise nonassociated K-prime generators for R such that the K-grading is almost free.

(iii) We have $n_D = 1$ for $D \in \Gamma$ and $n_D \geq 2$ for $D \in \mathcal{D}$.

(iv) Assume that the G-action on X_0 extends to X. Then, for every $D \in \Delta$ the (possibly empty) closed subset $\pi(\mathbb{V}(f_{D,1}, \ldots, f_{D,n_D})) \subseteq X$ is G-stable. Moreover, for every G-orbit $Y \subseteq X$ we have $Y \subseteq \overline{D}$ if and only if $Y \subseteq \pi(\mathbb{V}(f_{D,1}, \ldots, f_{D,n_D}))$.

According to [1, Corollary 3.1.4.6], the G-action on X_0 extends to any X arising from a bunched ring $(R, \mathfrak{F}, \Theta)$ with a maximal bunch Θ. Then, the following Remark 5.5 together with Remark 5.4(iv) shows that the G-action on X_0 also extends to X when the bunch is not maximal.

Remark 5.5. Let X arise from the bunched ring $(R, \mathfrak{F}, \Theta)$. Then we have

$$\begin{aligned} X(\tau) &= \bigcup_{\substack{J \subseteq \mathfrak{F} \\ \operatorname{cone}(\deg f_{D,\ell} : f_{D,\ell} \in J) = \tau}} \left(\bigcap_{f_{D,\ell} \notin J} \pi(\mathbb{V}(f_{D,\ell})) \setminus \bigcup_{f_{D,\ell} \in J} \pi(\mathbb{V}(f_{D,\ell})) \right) \\ &= \bigcup_{\substack{J \subseteq \Delta \\ \operatorname{cone}(\mathcal{Q}(J)) = \tau}} \left(\bigcap_{D \notin J} \pi(\mathbb{V}(f_{D,1}, \ldots, f_{D,n_D})) \setminus \bigcup_{D \in J} \pi(\mathbb{V}(f_{D,1}, \ldots, f_{D,n_D})) \right), \end{aligned}$$

where the last equality follows from the fact that for every $1 \leq \ell \leq n_D$ we have $\deg f_{D,\ell} = \mathcal{Q}(D)$.

We now explain the relation between the $\mathfrak{F}$-bunches and the $\mathcal{Q}$-bunches from Section 3. Note that every projected $\mathfrak{F}$-face is a $\mathcal{Q}$-cone.

Lemma 5.6. *Let X arise from the bunched ring $(R, \mathfrak{F}, \Theta)$ and denote by Σ the colored fan associated to the spherical embedding $G/H \hookrightarrow X$. Then for any $I \subseteq \Delta$ such that $(\operatorname{cone}(\mathcal{P}(I)), I \cap \mathcal{D})$ is supported we have $(\operatorname{cone}(\mathcal{P}(I)), I \cap \mathcal{D}) \in \Sigma$ if and only if $\operatorname{cone}(\mathcal{Q}(\Delta \setminus I)) \in \Theta$.*

Proof. Let $(\mathcal{C}, \mathcal{F}) \coloneqq (\operatorname{cone}(\mathcal{P}(I)), I \cap \mathcal{D}) \in \Sigma$ and $\tau \coloneqq \operatorname{cone}(\mathcal{Q}(\Delta \setminus I))$. Then the G-orbit corresponding to $(\mathcal{C}, \mathcal{F})$ is contained in $X(\tau)$ by Remarks 5.4(iv) and 5.5. It follows from $X(\tau) \neq \emptyset$ that we have $\tau \in \Theta$.

Let $\tau \coloneqq \operatorname{cone}(\mathcal{Q}(\Delta \setminus I)) \in \Theta$ and assume that $(\mathcal{C}, \mathcal{F}) \coloneqq (\operatorname{cone}(\mathcal{P}(I)), I \cap \mathcal{D})$ is supported. According to Proposition 2.8, there exists a quasi-projective spherical embedding $G/H \hookrightarrow X'$ with associated colored fan Σ' such that $\Sigma_0 \subseteq \Sigma'$, $(\mathcal{C}, \mathcal{F}) \in \Sigma'$, and $X' \setminus X_0$ is of codimension at least 2. According to Proposition 5.2, then X' is an open subvariety of a variety arising from a bunched ring $(R, \mathfrak{F}, \Theta')$, which we call again X'. The G-orbit corresponding to $(\mathcal{C}, \mathcal{F})$ is then contained in $X'(\tau)$ by the first part of the proof. As $X'(\tau) = X(\tau)$, we obtain $(\mathcal{C}, \mathcal{F}) \in \Sigma$. ☐

Lemma 5.7. *The projected $\mathfrak{F}$-faces are exactly the supported $\mathcal{Q}$-cones.*

Proof. It follows from Lemma 5.6 that every projected $\mathfrak{F}$-face is a supported $\mathcal{Q}$-cone. On the other hand, if τ is a supported $\mathcal{Q}$-cone, there exists $(\mathcal{C}, \mathcal{F}) \in \{\tau\}^\sharp$. As in the second part of the proof of Lemma 5.6, the G-orbit corresponding to $(\mathcal{C}, \mathcal{F})$ is contained in some spherical embedding arising from a bunched ring $(R, \mathfrak{F}, \Theta)$ with $\tau \in \Theta$. Hence τ is a projected $\mathfrak{F}$-face. □

Lemma 5.8. *The definitions of "true" for $\mathfrak{F}$-bunches and $\mathcal{Q}$-bunches coincide.*

Proof. For every $D \in \mathcal{D}$ we have $n_D \geq 2$ and $\deg f_{D,\ell} = \mathcal{Q}(D)$ for every $1 \leq \ell \leq n_D$. It follows that we have

$$\operatorname{cone}(\mathcal{Q}(\Delta)) = \operatorname{cone}(\deg f_{D,\ell} : f_{D,\ell} \in \mathfrak{F} \setminus \{f_{D_0,\ell_0}\})$$

for every $D_0 \in \mathcal{D}$ and every $1 \leq \ell_0 \leq n_{D_0}$, from which the claim follows. □

Theorem 5.9. *The true $\mathfrak{F}$-bunches are exactly the true $\mathcal{Q}$-bunches. Moreover, for a true $\mathfrak{F}$-bunch Θ the variety X associated to the bunched ring $(R, \mathfrak{F}, \Theta)$ is the spherical embedding $G/H \hookrightarrow X$ associated to the colored fan $\Theta^\sharp$.*

Proof. This follows immediately from Lemmas 5.6, 5.7, and 5.8. □

We can now prove Theorem 1.4 under the condition $\Gamma(X, \mathcal{O}_X^*) = \mathbb{K}^*$.

Proposition 5.10. *Let $G/H \hookrightarrow X$ be a spherical embedding with $\Gamma(X, \mathcal{O}_X^*) = \mathbb{K}^*$ and associated colored fan Σ. Then X has the A_2-property if and only if any two colored cones in Σ intersect in a common face.*

Proof. According to Proposition 5.2, if X has the A_2-property, it is an open subvariety of a variety arising from a bunched ring, hence any two colored cones in Σ intersect in a common face by Theorems 5.9 and 3.9

On the other hand, if any two colored cones in Σ intersect in a common face, then Σ is a true $\mathcal{P}$-fan and can be extended to a true maximal $\mathcal{P}$-fan. It then follows from Theorems 3.9 and 5.9 that X is an open subvariety of a variety arising from a bunched ring. Therefore X has the A_2-property. □

Under the condition $\Gamma(X, \mathcal{O}_X^*) = \mathbb{K}^*$, this proves Theorem 1.2 in the case $k = 2$ since the $\sharp$-operation sends colored cones in Σ which do not intersect in a common face to cones in $\Sigma^\sharp$ whose relative interiors do not intersect, while the case $k \geq 3$ follows from Proposition 5.3. The following result shows that the condition $\Gamma(X, \mathcal{O}_X^*) = \mathbb{K}^*$ can be removed and completes the proof of Theorems 1.2 and 1.4.

Proposition 5.11. *Let $G/H \hookrightarrow X$ be a spherical embedding with associated colored fan Σ. Then there exist linearly independent rays $\rho_1, \ldots, \rho_d \subseteq \mathcal{V}$ such* *that*

$$\mathcal{N}_{\mathbb{Q}} = \operatorname{span}_{\mathbb{Q}} \{\rho_1, \ldots, \rho_d\} \oplus \operatorname{span}_{\mathbb{Q}} \{\mathcal{C} : (\mathcal{C}, \mathcal{F}) \in \Sigma\}.$$

Moreover, for the spherical embedding $G/H \hookrightarrow X'$ associated to the colored fan $\Sigma' := \Sigma \cup \{(\rho_i, \emptyset) : 1 \leq i \leq d\}$, where we denote by D_i the G-invariant prime divisor in X' corresponding to the colored cone $(\rho_i, \emptyset)$, the following statements hold:

(i) We have $\Gamma(X', \mathcal{O}^*_{X'}) = \mathbb{K}^*$.

(ii) We have $\mathrm{Cl}(X')_{\mathbb{Q}} = \mathrm{Cl}(X)_{\mathbb{Q}}$ with $[D_i] = 0 \in \mathrm{Cl}(X')_{\mathbb{Q}}$ for every $1 \leq i \leq d$.

(iii) We have $(\Sigma')^\sharp = \Sigma^\sharp$.

(iv) X' has the A_k-property if and only if X has the A_k-property.

Proof. The existence of the rays $\rho_1, \ldots, \rho_d$ follows from the fact that the valuation cone $\mathcal{V}$ is of full dimension in $\mathcal{N}_{\mathbb{Q}}$. Then, (i) follows from Remark 2.7 and (ii) follows from [6, Proposition 4.1.1]. From $\Sigma \subseteq \Sigma'$, we obtain $\Sigma^\sharp \subseteq (\Sigma')^\sharp$. On the other hand, it follows from (ii) that we have $\{(\rho_i, \emptyset)\}^\sharp = \{(0, \emptyset)\}^\sharp \subseteq \Sigma^\sharp$, so that we also obtain $(\Sigma')^\sharp \subseteq \Sigma^\sharp$. This proves (iii).

If X' has the A_k-property, then the open subvariety $X \subseteq X'$ also has the A_k-property. Now assume that X' does not have the A_k-property. Then there exist $x_1, \ldots, x_k \in X'$ which are not contained in any common affine open neighbourhood. Since we have proven Theorem 1.2 in the case $\Gamma(X', \mathcal{O}^*_{X'}) = \mathbb{K}^*$, for every $1 \leq i \leq k$ we have $x_i \in X'(\tau_i)$ for some $\tau_i \in (\Sigma')^\sharp$ such that $\tau_1^\circ \cap \cdots \cap \tau_k^\circ = \emptyset$. Now let τ_0 be the unique element in $\{(0, \emptyset)\}^\sharp$. From $\{(\rho_i, \emptyset)\}^\sharp = \{(0, \emptyset)\}^\sharp$, we obtain $X' \setminus X \subseteq X'(\tau_0)$. Since $(0, \emptyset)$ is a face of every colored cone in Σ', we have $\tau^\circ \subseteq \tau_0^\circ$ for every $\tau \in (\Sigma')^\sharp$. By induction, we may assume that X' does have the A_{k-1}-property, hence it follows from $\tau_1^\circ \cap \cdots \cap \tau_k^\circ = \emptyset$ that we have $\tau_i \neq \tau_0$ for every $1 \leq i \leq k$. Therefore we have $x_1, \ldots, x_k \in X$, so that X does not have the A_k-property. □

Using Theorem 5.9, it is possible to apply results on bunched rings to spherical varieties. We give two examples.

Remark 5.12 (The criterion for $\mathbb{Q}$-factoriality). We recover Remark 2.5 by combining [1, Corollary 3.3.1.9], Theorem 5.9, and the last statement of Theorem 3.9.

Remark 5.13 (The canonical toric embedding; see [1, 3.2.5]). Let X arise from the bunched ring $(R, \mathfrak{F}, \Theta)$. Consider the K-graded polynomial ring

$\mathbb{K}[\mathfrak{F}]$ where the elements $f_{D,\ell} \in \mathfrak{F}$ are interpreted as homogeneous variables of degree $[D] \in K$. Let $\mathbb{Q}^{\mathfrak{F}}$ and $(\mathbb{Q}^{\mathfrak{F}})^*$ be dual vector spaces with respective standard bases $\{e_{f_{D,\ell}} : f_{D,\ell} \in \mathfrak{F}\}$ and $\{e^*_{f_{D,\ell}} : f_{D,\ell} \in \mathfrak{F}\}$, which are dual to each other. The map $Q\colon \mathbb{Q}^{\mathfrak{F}} \to K_{\mathbb{Q}}$ with $e_{f_{D,\ell}} \mapsto \deg f_{D,\ell} = [D]$ induces the following pair of mutually dual exact sequences of vector spaces.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_{\mathbb{Q}} & \longrightarrow & (\mathbb{Q}^{\mathfrak{F}})^* & \xrightarrow{P} & N_{\mathbb{Q}} & \longrightarrow & 0 \\ 0 & \longleftarrow & K_{\mathbb{Q}} & \xleftarrow{Q} & \mathbb{Q}^{\mathfrak{F}} & \longleftarrow & M_{\mathbb{Q}} & \longleftarrow & 0 \end{array}$$

Let Z be the (toric) variety arising from the bunched ring $(\mathbb{K}[\mathfrak{F}], \mathfrak{F}, \Theta)$. Its fan can be obtained as $\Theta^\sharp$, where now $\sharp$ denotes the $\sharp$-operation with respect to the exact sequences given here, i. e. yielding a fan in $N_{\mathbb{Q}}$. The surjective homomorphism of graded (Cox) rings $\mathbb{K}[\mathfrak{F}] \to R$ induces a closed embedding $X \hookrightarrow Z$.

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