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# Higher dimensional black hole initial data with prescribed boundary metric

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We obtain higher dimensional analogues of the results of Mantoulidis and Schoen in [8]. More precisely, we show that (i) any metric $g$ with positive scalar curvature on the 3-sphere $S^3$ can be realized as the induced metric on the outermost apparent horizon of a 4-dimensional asymptotically flat manifold with non-negative scalar curvature, whose ADM mass can be arranged to be arbitrarily close to the optimal value specified by the Riemannian Penrose inequality; (ii) any metric $g$ with positive scalar curvature on the $n$-sphere $S^n$, with $n \geq 4$, such that $(S^n, g)$ isometrically embeds into $\mathbb{R}^{n+1}$ as a star-shaped hypersurface, can be realized as the induced metric on the outermost apparent horizon of an $(n+1)$-dimensional asymptotically flat manifold with non-negative scalar curvature, whose ADM mass can be made to be arbitrarily close to the optimal value.

## 1. Introduction and statement of results

Recently, Mantoulidis and Schoen [8] gave an elegant construction of 3-dimensional asymptotically flat manifolds with non-negative scalar curvature, whose ADM mass [1] is arbitrarily close to the optimal value determined by the Riemannian Penrose inequality [2, 7], while the intrinsic geometry on the outermost apparent horizon is "far away" from being rotationally symmetric. Their result can be interpreted as a statement demonstrating the instability of the Riemannian Penrose inequality. The construction in [8] is geometric and can be outlined as a two-step process:

1) Consider the set $\mathscr{M}^+$ consisting of metrics on the 2-sphere $S^2$ satisfying $\lambda_1(-\Delta + K) > 0$, where $K$ is the Gaussian curvature. Given any $g \in \mathscr{M}^+$, construct a "collar extension" of $g$, which is a metric of positive scalar curvature on the product $[0, 1] \times S^2$ such that the

bottom boundary $\{0\} \times S^2$, having $g$ as the induced metric, is outer-minimizing while the top boundary $\{1\} \times S^2$ is metrically a round sphere

2) Pick any $m > 0$ arbitrarily close to $(A/16\pi)^{1/2}$, where $A$ is the area of $(S^2, g)$. Consider a 3-dimensional spatial Schwarzschild manifold of mass $m$ (which is scalar flat), deform it to have positive scalar curvature in a small region near the horizon, and then glue it to the above collar extension by making use of the positivity of the scalar curvature.

In this way, Mantoulidis and Schoen in [8] constructed asymptotically flat extensions of $(S^2, g)$ which have non-negative scalar curvature and, outside a compact set, coincide with a spatial Schwarzschild manifold whose mass can be arranged to be arbitrarily close to the optimal value $(A/16\pi)^{1/2}$.

In recent years, there has been a growing interest in black hole geometry in higher dimensions. Galloway and Schoen in [4] obtained a generalization of Hawking's black hole theorem [6] to higher dimensions. Their result shows that, in a spacetime satisfying the dominant energy condition, cross sections of the event horizon are of positive Yamabe type, i.e., they admit metrics of positive scalar curvature. Bray and Lee in [3] proved the Riemannian Penrose inequality for dimensions less than eight. The inequality asserts that the ADM mass $m_{\text{ADM}}$ of an $(n+1)$-dimensional $(n < 7)$, complete asymptotically flat manifold with non-negative scalar curvature, with boundary consisting of closed outer-minimizing minimal hypersurfaces, satisfies

$$m_{\text{ADM}} \geq \frac{1}{2} \left( \frac{A}{\omega_n} \right)^{(n-1)/n}, \tag{1.1}$$

where $A$ is the volume of its boundary and $\omega_n$ denotes the volume of the standard unit $n$-sphere.

Motivated by the above results, in this work we give some higher dimensional analogues of the Mantoulidis-Schoen theorem in [8]. Given an integer $n \geq 3$, denote the $n$-dimensional sphere by $S^n$. For simplicity, all metrics on $S^n$ below are assumed to be smooth. Our main results are

**Theorem 1.1.** *Let $g$ be a metric with positive scalar curvature on $S^3$. Denote the volume of $(S^3, g)$ by $\operatorname{vol}(g)$. Given any $m > 0$ such that $\omega_3 (2m)^{\frac{3}{2}} >$ $\operatorname{vol}(g)$, there exists an asymptotically flat 4-dimensional manifold $M^4$ with non-negative curvature such that*

*(i) $\partial M^4$ is isometric to $(S^3, g)$ and is minimal,*

*(ii)* $M^4$, *outside a compact set, is isometric to a spatial Schwarzschild manifold of mass* $m$, *and*

*(iii)* $M^4$ *is foliated by mean convex* 3*-spheres which eventually coincide with the rotationally symmetric* 3*-spheres in the spatial Schwarzschild manifold.*

**Theorem 1.2.** *Given any* $n \geq 4$, *let* $g$ *be a metric with positive scalar curvature on* $S^n$. *Suppose* $(S^n, g)$ *isometrically embeds into the Euclidean space* $\mathbb{R}^{n+1}$ *as a star-shaped hypersurface. Denote the volume of* $(S^n, g)$ *by* $\mathrm{vol}(g)$. *Given any* $m > 0$ *such that* $\omega_n (2m)^{n/(n-1)} > \mathrm{vol}(g)$, *there exists an asymptotically flat* $(n+1)$*-dimensional manifold* $M^{n+1}$ *with non-negative curvature such that*

*(i)* $\partial M^{n+1}$ *is isometric to* $(S^n, g)$ *and is minimal,*

*(ii)* $M^{n+1}$, *outside a compact set, is isometric to a spatial Schwarzschild manifold of mass* $m$, *and*

*(iii)* $M^{n+1}$ *is foliated by mean convex* $n$*-spheres which eventually coincide with the rotationally symmetric* $n$*-spheres in the spatial Schwarzschild manifold.*

We prove Theorems 1.1 and 1.2 by following the two-step process in Mantoulidis and Schoen's construction mentioned earlier. The key ingredient of our proof lies in the first step, in which we make use of results of Marques [9], and of Gerhardt [5] and Urbas [13], respectively, to construct the corresponding "collar extensions".

This paper is organized as follows. In Section 2, we apply a fundamental result of Marques [9] on deforming three-manifolds of positive scalar curvature to join an initial metric $g$ of positive scalar curvature on $S^3$ to a round metric via a smooth path of metrics of positive scalar curvature. In Section 3, we apply one type of inverse curvature flow in $\mathbb{R}^{n+1}$, studied by Gerhardt [5] and also by Urbas [13], to connect the metric $g$ on $S^n$ satisfying the condition in Theorem 1.2 to a round metric, via a smooth path of metrics of positive scalar curvature. In Section 4, we carry out Mantoulidis and Schoen's construction in higher dimensions $n \geq 3$. In Section 5, we prove Theorems 1.1 and 1.2.

## 2. Smooth paths in $\mathrm{Scal}^+(S^3)$

Let $\mathrm{Scal}^+(S^3)$ denote the set of smooth metrics with positive scalar curvature on $S^3$. Given $g \in \mathrm{Scal}^+(S^3)$, the first step to perform a collar extension of $g$, needed in the proof of Theorem 1.1, is to connect $g$ to a round metric on $S^3$ via a smooth path in $\mathrm{Scal}^+(S^3)$. We will achieve this by first applying the result of Marques [9] to obtain a continuous path, and then by mollifying this continuous path to obtain a smooth path.

We begin with a general path-smoothing procedure, suggested to us by Marques [10]. Let $M$ be an $n$-dimensional, $n \geq 2$, smooth closed manifold. Let $\mathcal{S}^k(M)$ denote the space of $C^k$ symmetric $(0,2)$ tensors on $M$ endowed with the $C^k$ topology. Here $k \geq 0$ is either an integer or $k = \infty$. Let $\mathcal{M}^k(M)$ be the open set in $\mathcal{S}^k(M)$ consisting of Riemannian metrics. Given any $g \in \mathcal{M}^k(M)$ with $k \geq 2$, let $R(g)$ denote the scalar curvature of $g$.

**Lemma 2.1.** *Let $\{g(t)\}_{t\in[0,1]}$ be a continuous path in $\mathcal{M}^k(M)$, $k \geq 2$. Suppose $R(g(t)) > 0$ for each $t$. Then there exists a constant $\epsilon > 0$ such that, for any $g \in \mathcal{M}^k(M)$, if $||g - g(t)||_{C^2} < \epsilon$ for some $t \in [0,1]$, then $R(g) > 0$.*

*Proof.* Suppose the claim is not true. Then for any integer $j > 0$, there exists a metric $g_j \in \mathcal{M}^k(M)$ and some $t_j \in [0,1]$ such that $||g_j - g(t_j)||_{C^2} < \frac{1}{j}$ while $R(g_j) \leq 0$ somewhere on $M$. Passing to a subsequence, we may assume $\lim_{j\to\infty} t_j = t_*$ for some point $t_* \in [0,1]$. Since $R(g(t_*)) > 0$, there exists $\epsilon_0 > 0$ such that if $g \in \mathcal{M}^k(M)$ and $||g - g(t_*)||_{C^2} < \epsilon_0$, then $R(g) > 0$. For large $j$, by the continuity of $\{g(t)\}$ in $\mathcal{M}^k(M)$, we now have $||g_j - g(t_*)||_{C^2} < \epsilon_0$, hence $R(g_j) > 0$ which is a contradiction. $\square$

**Proposition 2.1.** *Let $\{g(t)\}_{t\in[0,1]}$ be a continuous path in $\mathcal{M}^k(M)$, $k \geq 2$. Suppose $R(g(t)) > 0$ for each $t$. Then there exists a smooth path $\{h(t)\}_{t\in[0,1]}$ in $\mathcal{M}^k(M)$ satisfying $h(0) = g(0)$, $h(1) = g(1)$ and $R(h(t)) > 0$ for all $t$.*

*Proof.* Let $\epsilon > 0$ be the constant given by Lemma 2.1. Since the map $t \mapsto g_t \in \mathcal{M}^k(M)$ is continuous on $[0,1]$, there exists $\delta > 0$ such that, if $t$, $t' \in [0,1]$ with $|t - t'| < \delta$, then $||g(t) - g(t')||_{C^2} < \epsilon$.

Let $t_0 = 0 < t_1 < \cdots < t_{m-1} < t_m = 1$ be a sequence of points such that $|t_{i-1} - t_i| < \delta$, $\forall\ i = 1, \ldots, m$. On each $[t_{i-1}, t_i]$, define

$$h^{(i)}(t) = \frac{t - t_{i-1}}{t_i - t_{i-1}} g(t_i) + \frac{t_i - t}{t_i - t_{i-1}} g(t_{i-1}). \tag{2.1}$$

Clearly, $h^{(i)}(t) \in \mathcal{M}^k(M)$ and $\{h^{(i)}(t)\}_{t\in[t_{i-1},t_i]}$ is a smooth path in $\mathcal{M}^k(M)$. Moreover,

$$
\begin{aligned}
||h^{(i)}(t) - g(t_{i-1})||_{C^2} &= \frac{t-t_{i-1}}{t_i - t_{i-1}}||g(t_i) - g(t_{i-1})||_{C^2} < \epsilon, \\
||h^{(i)}(t) - g(t_i)||_{C^2} &= \frac{t_i - t}{t_i - t_{i-1}}||g(t_i) - g(t_{i-1})||_{C^2} < \epsilon.
\end{aligned}
\tag{2.2}
$$

In particular, $R(h^{(i)}(t)) > 0$ by Lemma 2.1. Let $\{\hat{h}(t)\}_{t\in[0,1]}$ be the path of metrics obtained by replacing $\{g(t)\}$ by $\{h^{(i)}(t)\}$ on each $[t_{i-1}, t_i]$. Then $\{\hat{h}(t)\}_{[0,1]}$ satisfy all the properties desired for $\{h(t)\}_{[0,1]}$ except that it is not smooth at the points $t_1, \dots, t_{m-1}$.

To complete the proof, we will mollify $\{\hat{h}(t)\}_{[0,1]}$ near each "corner" $t_i$, $1 \le i \le m-1$. We demonstrate the construction on $\left(\frac{t_0+t_1}{2}, \frac{t_1+t_2}{2}\right)$ as follows. Let $\phi = \phi(s)$ be a smooth function with compact support in $(-1, 1)$ such that $0 \le \phi \le 1$, $\int_{-\infty}^{\infty} \phi(s) ds = 1$ and

$$
\phi(s) = \phi(-s). \tag{2.3}
$$

Let $\sigma > 0$ be a fixed constant such that $\sigma < \min\left\{\frac{t_i - t_{i-1}}{4} \mid i = 1, \dots, m\right\}$. Let $\phi_\sigma(s) = \sigma^{-1}\phi(\frac{s}{\sigma})$. For each $t \in \left(\frac{t_0+t_1}{2}, \frac{t_1+t_2}{2}\right)$, define

$$
\begin{aligned}
h_\sigma^{(1)}(t) &= \int_{-\sigma}^{\sigma} \hat{h}(t-s)\phi_\sigma(s) ds \\
&= \int_0^1 \hat{h}(u)\phi_\sigma(t-u) du.
\end{aligned}
\tag{2.4}
$$

Evidently, $h_\sigma^{(1)}(t)$ lies in $\mathcal{S}^k(M)$ and is smooth in $t$. By the convexity of $\mathcal{M}^k(M)$ in $\mathcal{S}^k(M)$, $h_\sigma^{(1)}(t)$ indeed lies in $\mathcal{M}^k(M)$. Moreover,

$$
h_\sigma^{(1)}(t) - g(t_1) = \int_{-\sigma}^{\sigma} \left[\hat{h}(t-s) - g(t_1)\right] \phi_\sigma(s) ds, \tag{2.5}
$$

which combined with (2.2) implies

$$
||h_\sigma^{(1)}(t) - g(t_1)||_{C^2} < \epsilon. \tag{2.6}
$$

Hence, $R(h^{(i)}_\sigma(t)) > 0$ by Lemma 2.1. Now suppose $t \in \left(\frac{t_0+t_1}{2}, \frac{t_0+3t_1}{4}\right)$. Then $(t-\sigma, t+\sigma) \subset (t_0, t_1)$. Therefore, by (2.1) and (2.3),

$$
\begin{aligned}
h^{(1)}_\sigma(t) &= \int_{\mathbb{R}^1} \left[ \frac{(t-s)-t_0}{t_1-t_0} g_{t_1} + \frac{t_1-(t-s)}{t_1-t_0} g_{t_0} \right] \phi_\sigma(s) ds \\
&= \int_{\mathbb{R}^1} \left[ h^{(1)}(t) + \frac{s}{t_1-t_0}(g_{t_0} - g_{t_1}) \right] \phi_\sigma(s) ds \\
&= h^{(1)}(t).
\end{aligned} \tag{2.7}
$$

Similarly, for $t \in \left(\frac{t_1+3t_2}{4}, \frac{t_1+t_2}{2}\right)$, we have $h^{(1)}_\sigma(t) = h^{(2)}(t)$. In other words, the path $\{h^{(1)}_\sigma(t)\}$ coincides with $\{\hat{h}(t)\}$ near $\frac{t_0+t_1}{2}$ and $\frac{t_1+t_2}{2}$.

Applying the above construction on each $I_i = \left(\frac{t_{i-1}+t_i}{2}, \frac{t_i+t_{i+1}}{2}\right)$ to obtain $h^{(i)}_\sigma(t)$ and then replacing $\hat{h}(t)$ by $h^{(i)}_\sigma(t)$ on $I_i$, $i = 1, \dots, m-1$, we obtain a smooth path $\{h(t)\}_{t\in[0,1]}$ meeting all conditions required. This completes the proof. $\square$

Now we state the result of Marques [9, Corollary 1.1], asserting the path connectedness of the space $\mathrm{Scal}^+(S^3) \subset \mathcal{M}^\infty(S^3)$.

**Theorem 2.1 ([9]).** *Given any metric $g \in \mathrm{Scal}^+(S^3)$, there exists a continuous path $\{g(t)\}_{t\in[0,1]}$ in $\mathrm{Scal}^+(S^3)$ connecting $g$ to a round metric on $S^3$.*

The following corollary follows directly from Marques' theorem, Theorem 2.1, and Proposition 2.1.

**Corollary 2.1.** *Given any $g \in \mathrm{Scal}^+(S^3)$, there exists a smooth path $\{h(t)\}_{t\in[0,1]}$ in $\mathrm{Scal}^+(S^3)$ connecting $g$ to a round metric on $S^3$.*

## 3. Smooth paths in $\mathrm{Scal}^+_*(S^n)$

In this section, we make preparations for the proof of Theorem 1.2. For $n \geq 2$, let $\mathrm{Scal}^+_*(S^n)$ denote the set of smooth metrics $g$ with positive scalar curvature on $S^n$ such that $(S^n, g)$ isometrically embeds in $\mathbb{R}^{n+1}$ as a star-shaped hypersurface. When $n = 2$, by the results in [11, 12], $\mathrm{Scal}^+_*(S^2)$ agrees with the set of metrics on $S^2$ with positive Gaussian curvature.

Below, we focus on $n \geq 3$. Given $g \in \mathrm{Scal}^+_*(S^n)$, by applying the work of Gerhardt [5] and Urbas [13], we verify that $g$ can be connected to a round metric on $S^n$ via a smooth path in $\mathrm{Scal}^+_*(S^n)$. We begin with a lemma that ensures the positivity of the mean curvature of the embedding.

**Lemma 3.1.** *Let $\Sigma$ be a closed hypersurface in $\mathbb{R}^{n+1}$. Suppose the induced metric on $\Sigma$ has positive scalar curvature. Then, the mean curvature of $\Sigma$ with respect to the outward normal is everywhere positive.*

*Proof.* Let $R$ and $H$ be the scalar curvature and the mean curvature of $\Sigma$, respectively. By the Gauss equation, we have $R = H^2 - |\Pi|^2$, where $\Pi$ is the 2nd fundamental form of $\Sigma$ in $\mathbb{R}^{n+1}$. Hence, $R > 0$ implies $H^2 > 0$. Since $\Sigma$ is closed, there exists a point on $\Sigma$ at which $H \geq 0$, and hence, we conclude that $H > 0$ everywhere on $\Sigma$. $\square$

Given a closed hypersurface $\Sigma \subset \mathbb{R}^{n+1}$, let $\kappa_1, \ldots, \kappa_n$ denote the principal curvatures of $\Sigma$ with respect to the outward normal at each point. For $1 \leq k \leq n$, define the normalized $k$-th mean curvature $\sigma_k$ of $\Sigma$ by

$$\sigma_k = \frac{1}{\binom{n}{k}} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \kappa_{i_1} \cdots \kappa_{i_k}. \tag{3.1}$$

Clearly, $\sigma_1$ and $\sigma_2$ are related to the usual mean curvature $H$ and the scalar curvature $R$ of $\Sigma$, respectively, by

$$H = n\sigma_1, \quad R = n(n-1)\sigma_2. \tag{3.2}$$

We say that a smooth map $X : \Sigma \times [0, \infty) \longrightarrow \mathbb{R}^{n+1}$ is a solution to the $\sigma_1/\sigma_2$ flow if $X$ satisfies

$$\frac{\partial X}{\partial t} = \frac{\sigma_1}{\sigma_2}\nu = \frac{(n-1)H}{R}\nu, \tag{3.3}$$

where $\nu$ is the outward unit normal to $\Sigma_t = X(\Sigma, t)$ and $H$ and $R$ are the mean curvature and the scalar curvature of $\Sigma_t$, respectively. By definition, if the $\{\Sigma_t\}$ arise from a smooth solution to (3.3), $R$ does not vanish along $\Sigma_t$, hence must be positive. Consequently, by Lemma 3.1, $H$ must be positive along $\Sigma_t$. Hence, the surfaces $\Sigma_t$ are moving outward. The $\sigma_1/\sigma_2$ flow is one type of the inverse curvature flows in $\mathbb{R}^{n+1}$ studied by Gerhardt in [5] and independently by Urbas in [13]. In particular, the following theorem is a special case of the general result proved in [5] and [13].

**Theorem 3.1 ([5, 13]).** *Let $X_0 : S^n \to \mathbb{R}^{n+1}$ be a smooth embedding such that $\Sigma_0 = X_0(S^n)$ is star-shaped with respect to a point $P_0 \in \mathbb{R}^{n+1}$. If $\Sigma_0$ has* *positive scalar curvature and positive mean curvature, then the $\sigma_1/\sigma_2$ flow*

$$\frac{\partial X}{\partial t} = \frac{(n-1)H}{R}\nu, \tag{3.4}$$

*with the initial condition $X(\cdot,0) = X_0(\cdot)$ has a unique smooth solution $X : S^n \times [0,\infty) \to \mathbb{R}^{n+1}$. In particular, each $\Sigma_t = X(S^n,t)$ has positive scalar curvature $R$ and positive mean curvature $H$. Moreover, the rescaled surface $e^{-t}X(S^n,t)$ converges to a round sphere centered at $P_0$ in the $C^\infty$ topology as $t \to \infty$.*

In what follows, we argue that the proof of Theorem 3.1 in [5, 13] indeed provides a smooth path of metrics, with positive scalar curvature, connecting any $g \in \mathrm{Scal}_*^+(S^n)$ to a round metric on $S^n$. We follow the notations used in [13]. Identifying $S^n$ with the unit sphere $\{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ and assuming $X_0 : (S^n, g) \to \mathbb{R}^{n+1}$ is an isometric embedding such that $X_0(S^n)$ is star-shaped with respect to the origin (modulo a diffeomorphism on $S^n$), we can write $X_0$ as

$$X_0(x) = \rho_0(x)x, \ \ x \in S^n. \tag{3.5}$$

Here $\rho_0 : S^n \to \mathbb{R}^+$ is a smooth positive function on $S^n$, referred to as the *radial function* representing $\Sigma_0$ in [13]. For each $t > 0$, the surface $\Sigma_t$ in Theorem 3.1 is then given by the graph of a function $\rho(\cdot,t)$ over $S^n$, where

$$\rho(\cdot,\cdot) : S^n \times [0,\infty) \longrightarrow \mathbb{R}^+ \tag{3.6}$$

is a smooth function solving the parabolic equation (2.8) in [13], i.e.,

$$\frac{\partial \rho}{\partial t} = \frac{\left(\rho^2 + |\nabla \rho|^2\right)^{\frac{1}{2}}}{\rho F(a_{ij})}, \tag{3.7}$$

with the initial condition $\rho(\cdot,0) = \rho_0$. Here, $\nabla$ denotes the gradient on $S^n$ with respect to the standard metric and $F(a_{ij})$ is given by equation (2.9) in [13], which is simply the expression of $\sigma_2/\sigma_1$ in terms of $\rho(\cdot,t)$. For each $t$, one can rescale $\rho$ to define $\tilde\rho(\cdot,t) = e^{-t}\rho(\cdot,t)$. Then $\tilde\rho(\cdot,t)$ satisfies

$$\frac{\partial \tilde\rho}{\partial t} = \frac{\left(\tilde\rho^2 + |\nabla \tilde\rho|^2\right)^{\frac{1}{2}}}{\tilde\rho F(\tilde a_{ij})} - \tilde\rho, \tag{3.8}$$

where $F(\tilde a_{ij})$ is the expression of $\sigma_2/\sigma_1$ associated to the graph of $\tilde\rho(\cdot,t)$ over $S^n$ (see equation (3.28) in [13]). The following estimates on $\tilde\rho(\cdot,t)$ and the convergence of $\tilde\rho(\cdot,t)$ as $t \to \infty$ are given by (3.38), (3.39) and (3.40) in [13]:

a) There exist positive constants $C$ and $\gamma$ such that

$$\max_{S^n} |\tilde{\rho}(\cdot, t) - \rho^*| \leq Ce^{-\gamma t}. \tag{3.9}$$

Here, $\rho^* > 0$ is some constant.

b) For any positive integer $k$ and any constant $\tilde{\gamma} \in (0, \gamma)$, there exists a positive constant $C_k = C_k(\gamma, \tilde{\gamma})$ such that

$$\int_{S^n} |\nabla^k \tilde{\rho}(\cdot, t)|^2 \leq C_k e^{-\tilde{\gamma} t}. \tag{3.10}$$

c) Given any two integers $l \geq 0$ and $k > l + \frac{n}{2}$, there exists a positive $C = C(k, l)$ such that

$$||\tilde{\rho}(\cdot, t) - \rho^*||_{C^l(S^n)} \leq C \left[ \int_{S^n} |\nabla^k \tilde{\rho}(\cdot, t)|^2 + \int_{S^n} |\tilde{\rho}(\cdot, t) - \rho^*|^2 \right]^{\frac{1}{2}}. \tag{3.11}$$

It follows directly from a), b) and c) that there exists a constant $\delta > 0$ (say $\delta = \frac{1}{2}\gamma$) such that, for any integer $l \geq 0$,

$$||\tilde{\rho}(\cdot, t) - \rho^*||_{C^l(S^n)} \leq Ce^{-\delta t}. \tag{3.12}$$

This, combined with the PDE (3.8), in turn implies, for any integer $k \geq 1$,

$$\left\| \frac{\partial^k \tilde{\rho}}{\partial t^k} \right\|_{C^l(S^n)} \leq Ce^{-\delta t}, \tag{3.13}$$

for some constants $C$.

Now, we can define a path of metrics in $\mathrm{Scal}_*^+(S^n)$ connecting $g$ to a round metric $g^*$ that corresponds to a round sphere in $\mathbb{R}^{n+1}$ of radius $\rho^*$ as follows. Define $\Phi_t : S^n \to \mathbb{R}^{n+1}$ by $\Phi_t(x) = \tilde{\rho}(x, t)x$ and let

$$g(t) = \Phi_t^*(g_E), \tag{3.14}$$

where $g_E$ is the Euclidean metric on $\mathbb{R}^{n+1}$. Theorem 3.1 guarantees that $g(t)$ has positive scalar curvature. Moreover, given any integers $l$ and $k$, it follows from (3.12) and (3.13) that

$$\|g(t) - g^*\|_{C^l(S^n)} \le Ce^{-\delta t} \tag{3.15}$$

and

$$\left\| \frac{\partial^k}{\partial t^k} g(t) \right\|_{C^l(S^n)} \le Ce^{-\delta t}. \tag{3.16}$$

We make a change of variable $t = t(s)$ to view the metrics $\{g(t)\}$ as a new family of metrics $\{h(s)\}$ defined on the finite interval $[0, 1]$. Specifically, let

$$t(s) = \frac{1}{(s-1)^2} - 1, \quad \left(\text{then } s = 1 - \frac{1}{\sqrt{1+t}}\right) \tag{3.17}$$

and define

$$h(s) = \begin{cases} g(t(s)) & \text{when } s \in [0, 1) \\ g^* & \text{when } s = 1, \end{cases} \tag{3.18}$$

which is continuous by (3.15). Using the exponential decay estimate of the derivatives in (3.16), one concludes that the metric defined by

$$H = ds^2 + h(s), \tag{3.19}$$

is smooth on $I \times S^n$ and it satisfies that $h(0) = g$, $h(s)$ is a metric of positive scalar curvature on $S^n$ for all $s \in [0, 1]$, and $h(1)$ is a round metric.

## 4. Mantoulidis-Schoen construction in higher dimensions

In this section, we recall the construction of Mantoulidis and Schoen from Section 1 and 2 in [8]. Though stated for dimension $n = 2$, many of their arguments apply in a straightforward manner to higher dimensions $n \ge 3$. For readers' convenience, we include the proof of all lemmas stated below.

### 4.1. Deformations on $I \times S^n$

**Lemma 4.1.** *Suppose $\{h(t)\}_{0 \le t \le 1}$ is a family of metrics of positive scalar curvature on $S^n$ such that*

- *$h(1)$ is a round metric.*

- *$H = dt^2 + h(t)$ is a smooth metric on $I \times S^n$, where $I = [0, 1]$.*

*Also, suppose that $\mathrm{vol}\,(h(t))$, the volume of $(S^n, h(t))$, is a constant independent of $t$. Then, there exists a smooth metric $G = dt^2 + g(t)$ on $I \times S^n$ satisfying*

*(i) $g(0)$ is isometric to $h(0)$ on $S^n$,*

*(ii) $g(t)$ has positive scalar curvature $\forall\ t \in I$,*

*(iii) $g(1)$ is round, $g(t) = g(1)\ \forall\ t \in [1/2, 1]$, and*

*(iv) $\frac{d}{dt} dV_{g(t)} = 0$ for all $t \in [0, 1]$. Here, $dV_{g(t)}$ is the volume form of $g(t)$ on $S^n$.*

*Proof.* Choose a smooth monotone function $\zeta$ on $[0, 1]$ such that $\zeta(0) = 0$ and $\zeta(t) = 1$, $t \in [1/2, 1]$. Consider $h(\zeta(t))$, $t \in [0, 1]$. This new path $\{h(\zeta(t))\}_{0 \le t \le 1}$ satisfies the first three conditions and is volume preserving. Relabel $h(\zeta(t))$ as $h(t)$. To achieve condition (iv), we make use of diffeomorphisms on $S^n$. Let $\{\phi_t\}_{0 \le t \le 1}$ be a 1-parameter family of diffeomorphisms on $S^n$ generated by a $t$-dependent, smooth vector field $X_t = X(\cdot, t)$ to be chosen later. Define $g(t) = \phi_t^*(h(t))$. Let $\dot{g} = \frac{d}{dt} g$, then

$$\frac{d}{dt} dV_{g(t)} = \frac{1}{2} \mathrm{tr}_g \dot{g}\, dV_{g(t)} \tag{4.1}$$

and

$$\dot{g} = \frac{d}{dt} \phi_t^*(h(t)) = \phi_t^* \left( \frac{d}{dt} h(t) \right) + \phi_t^*(\mathcal{L}_{X_t} h(t)). \tag{4.2}$$

Hence,

$$\begin{aligned} \mathrm{tr}_g \dot{g} = &\ \mathrm{tr}_{\phi_t^*(h(t))} \left( \phi_t^* \left( \frac{d}{dt} h \right) + \phi_t^*(\mathcal{L}_{X_t} h(t)) \right) \\ &= \phi_t^* \left( \mathrm{tr}_h \dot{h} + \mathrm{tr}_h \mathcal{L}_{X_t} h(t) \right) \\ &= \phi_t^* \left( \mathrm{tr}_h \dot{h} + 2 \mathrm{div}_h X_t \right). \end{aligned} \tag{4.3}$$

Now let $\psi(t, x)$ be a smooth function on $I \times S^n$ obtained by solving the elliptic equation on $S^n$,

$$\Delta_h \psi(t, \cdot) = -\frac{1}{2} \mathrm{tr}_h \dot{h}, \tag{4.4}$$

for each $t$. (4.4) is solvable since $\int_{S^n} \frac{1}{2}\mathrm{tr}_h \dot{h}\, dV_{h(t)} = \frac{d}{dt}\int_{S^n} dV_{h(t)} = 0$. Furthermore, the solution $\psi(t,\cdot)$ depends smoothly on $t$. Let $X_t = \nabla^{h(t)}\psi$, where $\nabla^{h(t)}$ is the gradient on $(S^n, h(t))$. Clearly, $\mathrm{tr}_g \dot{g} = 0$ by (4.3) and (4.4). Condition (iv) is thus satisfied. □

Next, given a fixed choice of $\{h(t)\}$, we continue to denote the path provided in Lemma 4.1 by $\{g(t)\}$. The following lemma deforms the metric $dt^2 + g(t)$ on $I \times S^n$ to a metric of positive scalar curvature.

**Lemma 4.2.** *There exists $A_0 > 0$ such that for all $\varepsilon \in [0,1]$ and $A \geq A_0$, the metric on $[0,1] \times S^n$ given by*

$$\gamma_\varepsilon = A^2 dt^2 + (1+\varepsilon t^2) g(t), \tag{4.5}$$

*has positive scalar curvature on $I \times S^n$, $\{0\} \times S^n$ is minimal, and the spheres $\{t\} \times S^n$ for $t \in (0,1]$ are mean convex with respect to the normal direction $\partial_t$.*

*Proof.* Consider a metric of the form

$$\gamma = A^2 dt^2 + h(t), \tag{4.6}$$

where $h(t) = (1+\varepsilon t^2)g(t)$, $A > 0$ and $\varepsilon > 0$, to be determined later (here we are abusing notation by using $h(t)$ again). Direct calculations give

$$\begin{aligned} R(\gamma) &= R(h) + A^{-2}\left[-\mathrm{tr}_h \ddot{h} - \frac{1}{4}(\mathrm{tr}_h \dot{h})^2 + \frac{3}{4}|\dot{h}|_h^2\right] \\ &\geq \inf_{t,x} R(h) + A^{-2}\left[-\frac{2n\varepsilon}{(1+\varepsilon t^2)^{-1}} - \sup_{t,x}|\mathrm{tr}_g \ddot{g}| - \frac{n^2\varepsilon^2 t^2}{(1+\varepsilon t^2)^2}\right]. \end{aligned} \tag{4.7}$$

Hence, by picking $A_0 \gg 1$ sufficiently large and $A \geq A_0$ , the metric

$$\gamma_\varepsilon = A^2 dt^2 + (1+\varepsilon t^2) g(t) \tag{4.8}$$

has positive scalar curvature for all $\varepsilon \in [0,1]$. Note that the mean curvature of any slice $\{t\} \times S^n$, is given by

$$H_t = \frac{n\varepsilon t}{A(1+\varepsilon t^2)}. \tag{4.9}$$

Therefore, $H = 0$ when $t = 0$ and $H > 0$ when $t > 0$. □

**Remark 4.1.** *Since we have the stronger condition $R(g(t)) > 0$, we do not need to use the first positive eigenfunction of the operator $-\Delta_g + \frac{1}{2}R(g)$ as a warping factor in* (4.5) *as opposed to that being used in [8].*

### 4.2. Bending the Schwarzschild metric

We recall that the $(n+1)$-dimensional spatial Schwarzschild manifold (outside its horizon) is given by

$$(M^{n+1}, g_m) = \left( [r_0, \infty) \times S^n, \frac{1}{1 - \frac{2m}{r^{n-1}}} dr^2 + r^2 g_* \right),$$

where $g_*$ denotes the standard metric on $S^n$ with constant sectional curvature 1 and $r_0 = (2m)^{\frac{1}{n-1}}$. Replacing $r$ by $s$, which is the distance function to the horizon $\{r_0\} \times S^n$, one can re-write $g_m$ as

$$g_m = ds^2 + u_m(s)^2 g_*, \tag{4.10}$$

defined on $[0, \infty) \times S^n$. Here the horizon $\{r = r_0\}$ corresponds to $s = 0$. The function $u_m(s)$ satisfies

(a) $u_m(0) = (2m)^{\frac{1}{n-1}}$,

(b) $u_m'(0) = 0$,

(c) $u_m'(s) = \left(1 - \dfrac{2m}{u_m(s)^{n-1}}\right)^{1/2}$ for $s > 0$, and

(d) $u_m''(s) = (n-1)\dfrac{m}{u_m^n}$ for $s > 0$.

In particular, when $n = 3$,

$$u_m(0) = \sqrt{2m}, \quad u_m'(0) = 0, \tag{4.11}$$

and

$$u_m'(s) = \left(1 - \frac{2m}{u_m(s)^2}\right)^{1/2}, \quad u_m''(s) = \frac{2m}{u_m^3(s)}, \qquad \text{for } s > 0. \tag{4.12}$$

The next Lemma "bends" the metric $g_m$ near the horizon $\{s = 0\}$ so that the resulting metric has strictly positive scalar curvature near $\{s = 0\}$.

**Lemma 4.3.** *Let $s_0 > 0$. There exist a small $\delta > 0$ and a smooth function $\sigma : [s_0 - \delta, \infty) \to (0, \infty)$ satisfying*

*1)* $\sigma(s) = s$ *for all* $s \geq s_0$,

*2)* $\sigma$ *is monotonically increasing, and*

*3) the metric* $ds^2 + u_m(\sigma(s))^2 g_*$ *has positive scalar curvature for* $s_0 - \delta \leq s < s_0$ *and vanishing scalar curvature for* $s \geq s_0$.

*Proof.* Recall that for a metric $\tilde{g} = ds^2 + f(s)^2 g_*$, its scalar curvature is given by

$$\tilde{R} = nf^{-2}\left[(n-1) - (n-1)\dot{f}^2 - 2f\ddot{f}\right]. \tag{4.13}$$

Hence for $ds^2 + u_m(\sigma(s))^2 g_*$, we need to have

$$\tilde{R} = nu_m^{-2}\left[(n-1) - (n-1)\left(\frac{d}{ds}u_m\right)^2 - 2u_m\frac{d^2}{ds^2}u_m\right] > 0, \tag{4.14}$$

on $[s_0 - \delta, s_0)$. Thus, it is sufficient to require

$$(n-1) - (n-1)\left(\frac{d}{ds}u_m(\sigma)\right)^2 - 2u_m\frac{d^2}{ds^2}u_m((\sigma)) > 0, \tag{4.15}$$

on $[s_0 - \delta, s_0)$. By the fact that Schwarzschild is scalar flat, this is equivalent to

$$(n-1) - (n-1)(\sigma')^2 - 2u_m(\sigma)u_m'(\sigma)\sigma'' > 0, \tag{4.16}$$

on $[s_0 - \delta, s_0)$. Now define $\theta(s) = 1 + e^{-\frac{1}{(s-s_0)^2}}$ and $\theta(s_0) = 1$. For sufficiently small $\delta$, let

$$\sigma(s) = \int_{s_0-\delta}^{s} \theta(s)\, ds + K_\delta, \tag{4.17}$$

where $K_\delta$ is chosen so that $\sigma(s_0) = s_0$ and thus can be extended to be equal to $s$ for $s \geq s_0$. With such a choice of $\sigma(s)$, (4.16) becomes

(4.18)

$$\begin{aligned}&(n-1) - (n-1)[1 + 2e^{-\frac{1}{(s-s_0)^2}} + e^{-\frac{2}{(s-s_0)^2}}] - 4u_m(\sigma)u_m'(\sigma)\frac{e^{-\frac{1}{(s-s_0)^2}}}{(s-s_0)^3} \\ &= e^{-\frac{1}{(s-s_0)^2}}\left(-2(n-1) - (n-1)e^{-\frac{1}{(s-s_0)^2}} - 4u_m(\sigma)u_m'(\sigma)\frac{1}{(s-s_0)^3}\right).\end{aligned}$$

By taking $\delta$ sufficiently small, this last quantity is positive. $\square$

### 4.3. Gluing lemma

The following lemma allows one to glue a collar extension $(I \times S^n, \gamma_\epsilon)$ from Lemma 4.2 to the "bending" of the Schwarzschild metric in Lemma 4.3.

**Lemma 4.4.** *Let $g_*$ be the standard metric (of constant sectional curvature 1) on $S^n$. Let $f_i : [a_i, b_i] \to \mathbb{R}$, $i = 1, 2$, be two smooth functions satisfying*

*(I) $f_i > 0$, $f_i' > 0$ and $f_i'' > 0$ on $[a_i, b_i]$,*

*(II) the metric $dt^2 + f_i(t)^2 g_*$ on $[a_i, b_i] \times S^n$ has positive scalar curvature,*

*(III) $f_1(b_1) < f_2(a_2)$ and $f_1'(b_1) = f_2'(a_2)$.*

*Then, after translating the intervals so that $a_2 - b_1 = (f_2(a_2) - f_1(b_1))/f_1'(b_1)$, there exists a smooth function $f : [a_1, b_2] \to \mathbb{R}$ satisfying*

*(i) $f > 0$ and $f' > 0$ on $[a_1, b_2]$,*

*(ii) $f = f_1$ on $[a_1, \frac{a_1+b_1}{2}]$,*

*(iii) $f = f_2$ on $[\frac{a_2+b_2}{2}, b_2]$, and*

*(iv) the metric $dt^2 + f(t)^2 g_*$ on $[a_1, b_2] \times S^n$ has positive scalar curvature.*

*Proof.* Define $\tilde{f}$ on $[a_1, b_2]$ so that $\tilde{f}$ agrees with $f_1$ and $f_2$ on $[a_1, b_1]$ and $[a_2, b_2]$, respectively, and whose graph on $[b_1, a_2]$ is the line segment connecting $(b_1, f_1(b_1))$ and $(a_2, f_2(a_2))$. Clearly, $\tilde{f} \in C^{1,1}([a_1, b_2])$ and $\tilde{f}$ is smooth except at $b_1$ and $a_2$.

Define $m_i = (a_i + b_i)/2$, $i = 1, 2$. Let $\delta > 0$ be such that $m_1 < b_1 - \delta$ and $a_2 + \delta < m_2$. Let $\eta_\delta$ be a smooth cut-off function such that $\eta_\delta(t) = 1$ on $[b_1 - \delta, a_2 + \delta]$ and $\eta_\delta(t) = 0$ on $[a_1, m_1] \cup [m_2, b_2]$. Define the following mollification of $\tilde{f}$:

$$f_\nu(t) = \int_{\mathbb{R}} \tilde{f}(t - \nu\eta_\delta(t)s)\phi(s)\, ds. \tag{4.19}$$

This mollification fixes $\tilde{f}$ on $[a_1, m_1] \cup [m_2, b_2]$ and coincides with the standard mollification on an interval properly containing $[b_1, a_2]$; on the remaining part, its value is given by a standard mollification of $f$ with radius $\nu\eta_\delta(x) \le \nu$. It can be easily checked that both $f_\nu \to f$ and $f_\nu' \to f'$ in $C^0([a_1, b_2])$, as $\nu \to 0$.

A direct calculation shows that for $f > 0$, the metric $\tilde{g} = dt^2 + f(t)^2 g_*$ has positive scalar curvature if and only if

$$f''(t) < \frac{(n-1)}{2f(t)} \left(1 - f'(t)^2\right). \tag{4.20}$$

Thus, by assumption $(II)$,

$$f_i''(t) < \frac{(n-1)}{2f_i(t)} \left(1 - f_i'(t)^2\right), \text{ on } [a_i, b_i]. \tag{4.21}$$

The condition $f_i'' > 0$ on $[a_i, b_i]$ ensures that the graph of

$$\Omega[\tilde{f}](x) = \frac{(n-1)}{2\tilde{f}(t)} \left(1 - \tilde{f}'(t)^2\right) \tag{4.22}$$

lies strictly above the graph of $\tilde{f}''$ (when defined) and the graphs of $f_1''$ and $f_2''$. Clearly, $\Omega[f_\nu] \to \Omega[\tilde{f}]$ in $C^0([a_1, b_2])$ as $\nu \to 0$. Let $3d$ be the smallest vertical distance from the graph of $\Omega[\tilde{f}]$ to the graphs of $f_1''$ and $f_2''$. The uniform convergence imply that we can take a small $\nu$ so that the graph of $\Omega[f_\nu]$ lies exactly within a distance $d$ from the graph of $\Omega[\tilde{f}]$. Since $\Omega[\tilde{f}]$ is uniformly continuous, there exists a number $\nu > 0$ such that $\Omega[\tilde{f}](s) \leq \Omega[\tilde{f}](t) + d$ on $[t-\nu, t+\nu]$. For simplicity, abusing notation, set $\tilde{f}''(b_1) = f_1''(b_1)$ and $\tilde{f}''(a_2) = f_2''(a_2)$. Then it follows that for a sufficiently small $\nu$,

$$f_\nu''(t) \leq \sup_{[t-\nu, t+\nu]} \tilde{f}''(s) + d \leq \sup_{[t-\nu, t+\nu]} \Omega[\tilde{f}](s) - 3d + d \leq \Omega[\tilde{f}](t) - d, \tag{4.23}$$

and hence $f_\nu''(t) < \Omega[f_\nu](t)$ on $[a_1, b_2]$. It follows that the metric $dt^2 + f_\nu(t)^2 g_*$ has positive scalar curvature. □

## 5. Proofs of Theorem 1.1 and Theorem 1.2

With the paths of metrics $\{h(t)\}_{0 \leq t \leq 1}$ in $\mathrm{Scal}^+(S^3)$ and $\mathrm{Scal}^+_*(S^n)$ given in Sections 2 and 3, respectively, one can prove Theorem 1.1 and 1.2 in the same way that Theorem 2.1 was proved in [8].

*Proof of Theorem 1.1.* Let $g \in \mathrm{Scal}^+(S^3)$ and let $\{h(t)\}_{0 \leq t \leq 1}$ be given by Corollary 2.1. Normalize this path so that it is volume preserving by considering

$$\tilde{h}(t) = \psi(t) h(t), \quad \text{with } \psi(t) = \left(\frac{\mathrm{vol}\,(g)}{\mathrm{vol}\,(h(t))}\right)^{\frac{2}{3}}. \tag{5.1}$$

Apply Lemma 4.1 to $\{\tilde{h}(t)\}_{0\leq t\leq 1}$ to obtain $\{g(t)\}_{0\leq t\leq 1}$. Let $m > 0$ be a constant such that $\omega_3(2m)^{3/2} > \mathrm{vol}\,(g)$.

In what follows, we set $n = 3$, though we will keep using the notation $n$ to emphasize that this part of the proof holds in general dimensions. Consider the family of collar extensions obtained in Lemma 4.2, i.e., the metrics

$$\gamma_\varepsilon = A^2 dt^2 + (1 + \varepsilon t^2) g(t) \tag{5.2}$$

with positive scalar curvature on $[0, 1] \times S^n$ for $\varepsilon \in [0, 1]$. Let $g_*$ denote the round metric on $S^n$. Then $g(t) = \rho^2 g_*$ for some $\rho > 0$ on $[1/2, 1]$ (recall that $g(1)$ is round and $g(t) = g(1)$ for $t \in [1/2, 1]$). Make the change of variables $s = At$ on $[1/2, 1]$, obtaining

$$\gamma_\varepsilon = ds^2 + (1 + \varepsilon A^{-2} s^2)\rho^2 g_*, \tag{5.3}$$

for $s \in [A/2, A]$.

Define $f_\varepsilon(s) = (1 + \varepsilon A^{-2} s^2)^{1/2}\rho$. Then,

$$f'_\varepsilon(s) = \frac{\rho\varepsilon s}{A^2(1 + \varepsilon A^{-2} s^2)^{1/2}} > 0, \tag{5.4}$$

$$f''_\varepsilon(s) = \frac{\rho\varepsilon}{A^2(1 + \varepsilon A^{-2} s)^{3/2}} > 0. \tag{5.5}$$

This function will play the role of $f_1$ in Lemma 4.4. The role of $f_2$ will be played by the function $u_m(\sigma(s))$ in the Schwarzschild bending $ds^2 + u_m(\sigma(s))^2 g_*$ from Lemma 4.3. To be able to apply Lemma 4.4 we need $f_\varepsilon(A) < u_m(\sigma(s_0 - \delta))$ and $f'_\varepsilon(A) = u'_m(\sigma(s_0 - \delta))$; to achieve this condition we will choose $\varepsilon$ and $\delta$ accordingly. Consider the curves $\Gamma(\varepsilon) = (f_\varepsilon(A), f'_\varepsilon(A))$ and $\Delta(s) = (u_m(s), u'_m(s))$.

Notice that as $\varepsilon \to 0$

$$\Gamma(\varepsilon) \to (\rho, 0) = \left(\left(\frac{\mathrm{vol}\,(g)}{\omega_n}\right)^{\frac{1}{n}}, 0\right). \tag{5.6}$$

Moreover, the slope $f'_\varepsilon(A)/f_\varepsilon(A)$ is strictly decreasing.

When $s \to 0$, $\Delta(s) \to ((2m)^{1/(n-1)}, 0)$. Since $m$ is chosen so that

$$m > \frac{1}{2}\left(\frac{\mathrm{vol}\,(g)}{\omega_n}\right)^{(n-1)/n}, \tag{5.7}$$

$\Delta(0)$ lies to the right of $\Gamma(0)$. Using continuity, pick $s_0$ so the segment of the curve $\Delta(s)$ from 0 to $s_0$ lies strictly to the right and below the curve

$\Gamma(\varepsilon)$. Apply Lemma 4.3 with $\delta$ sufficiently small so that the curve $u_m(\sigma(s)) : [s_0 - \delta, s_0] \to \mathbb{R}$ has positive second derivative and the curve

$$\tilde{\Delta}(s) = \left( u_m(\sigma(s)), \frac{d}{ds}(u_m(\sigma(s))) \right)$$

still lies to the right and below $\Gamma(\varepsilon)$. Now pick $\varepsilon < 1$ so that $\Gamma(\varepsilon) = \tilde{\Delta}(s_0 - \delta)$, and apply Lemma 4.4 to construct a positive scalar bridge between the collar extensions and the bending of Schwarzschild. The result follows. $\square$

*Proof of Theorem 1.2.* Given $g \in \mathrm{Scal}_*^+(S^n)$, let $\{h(t)\}_{0 \le t \le 1}$ be the path of metrics on $S^n$ constructed in Section 3. As above, to apply Lemma 4.1 we normalize this path so that it is volume preserving by setting

$$\tilde{h}(t) = \psi(t) h(t), \text{ with } \psi(t) = \left( \frac{\mathrm{vol}\,(g)}{\mathrm{vol}\,(h(t))} \right)^{\frac{2}{n}}. \tag{5.8}$$

Apply Lemma 4.1 to this new path $\{\tilde{h}(t)\}_{0 \le t \le 1}$ to obtain $\{g(t)\}_{0 \le t \le 1}$. Let $m > 0$ such that $\omega_n (2m)^{n/(n-1)} > \mathrm{vol}(g)$. The rest of the proof now is the same as that of Theorem 1.1 above. $\square$

We finish this paper by pointing out a more elementary case in which the conclusion of Theorem 1.2 also holds. If $g$ is a metric of positive scalar curvature on $S^n$ ($n \ge 3$) that is conformal to the standard metric, say $g = u^{\frac{4}{n-2}} g_*$ for some smooth positive function $u$, then it is straightforward to check that the metric $h(t) = [(1-t)u + t]^{\frac{4}{n-2}} g_*$, $t \in [0,1]$, has positive scalar curvature for each $t$. Hence, by applying the proof of Theorem 1.2 to this path $\{h(t)\}_{0 \le t \le 1}$, one knows that Theorem 1.2 holds for such metrics in the standard conformal class on $S^n$.

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