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Good reduction and canonical heights of subvarieties

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We bound the length of the periodic part of the orbit of a preperiodic rational subvariety via good reduction information. This bound depends only on the degree of the map, the degree of the subvariety, the dimension of the projective space, the degree of the number field, and the prime of good reduction. As part of the proof, we extend the corresponding good reduction bound for points proven by the author for non-singular varieties to all projective varieties. Toward proving an absolute bound on the period for a given map, we study the canonical height of a subvariety via Chow forms and compute the bound between the height and canonical height of a subvariety. This gives the existence of a bound on the number of preperiodic rational subvarieties of bounded degree for a given map. An explicit bound is given for hypersurfaces.

1. Introduction

Let K be a number field of degree $\nu = [K:\mathbb{Q}]$ and $f : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree d defined over K. Let $X \subseteq \mathbb{P}^N$ be an irreducible projective subvariety of degree D defined over K. We say X is *periodic* if there is an integer $n \geq 1$ such that $f^n(X) = X$, and X is *preperiodic* if f^m is periodic for some integer $m \geq 0$.

In the case that the dimension of X is 0, i.e., X is a rational point, Morton-Silverman [21] conjectured the existence of a constant $C(d, \nu, N)$ bounding the number of rational preperiodic points depending only on the degree of f, the degree of K, and the dimension. While little is known about this conjecture in general, adding an additional hypothesis about primes of good reduction yields the existence of a constant $C(d, \nu, \mathfrak{p}, N)$ bounding the number of rational periodic points, where $\mathfrak{p}$ is a prime of K where f has good reduction; for $N = 1$ see [2, 21, 37], for $N > 1$ see [17, 26].

When the dimension of $X \geq 1$, much less is known. By restricting to coordinate-wise univariate polynomial maps, Medvedev-Scanlon are able to classify the fixed subvarieties [20]. Studying étale maps where X has at least one smooth rational point, Bell-Ghioca-Tucker are able to bound the size of the periodic part of the orbit in terms of $\nu, N, \mathfrak{p}$ using p-adic methods [1]. It should be noted that this is not a bound similar to the one conjectured by Morton and Silverman because it does not bound the number of rational preperiodic varieties; it only provides a bound on the period. However, such a bound on the period of points is a key step in obtaining the overall bound on the number of preperiodic points [19, 31]. Furthermore, it is not at all clear that a bound on the number of rational preperiodic subvarieties is possible. If such a bound exists, it would need to at least depend on the degree of the subvariety because for a homogeneous bivariate polynomial f of degree d, a map $F : \mathbb{P}^2 \to \mathbb{P}^2$ of the form $F = (f(x,z), f(y,z), z^d)$ has infinitely many fixed curves of the form $(f^n(x,z), xz^{d^n-1}, z^{d^n})$.

In this article we prove a bound on the periodic part of the orbit that depends only on $(d, b, N, D, \mathfrak{p})$, where b is the codimension of X. Note that for a subvariety X, we denote $\overline{X}$ as the reduction of the subvariety modulo a prime.

Theorem. *(Theorem 3.8) Let K be a number field. Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree $d \geq 2$ defined over K. Let $X \subset \mathbb{P}^N$ defined over K be an irreducible subvariety of degree D and codimension b that is periodic for f with minimal period n. Let $\mathfrak{p} \in K$ be a prime of good reduction for f. If $\deg(f^\ell(X)) = \deg(\overline{f^\ell(X))}$ for $0 \leq \ell \leq n$, then there exists a constant C depending only on $d, D, N, b, \mathfrak{p}$ such that*

$$n \leq C(d, D, N, b, \mathfrak{p}).$$

The basic idea of the method is to move the problem to an endomorphism of a component of the Chow variety and appeal to the similar bound for points from the author's previous work [17] extended to singular varieties (Theorem 3.6 and Theorem 3.7). The condition on the degrees of the reductions $\overline{f^\ell(X))}$ can be thought of as a notion of primes of good reduction for X. Furthermore, for a given periodic subvariety, there are only finitely many primes which do not satisfy this degree condition. However, this is probably not quite enough for application to problems such as the dynamical Mordell-Lang conjecture; for a general statement of the conjecture and additional references see [11]. To apply this kind of good reduction bound to the dynamical Mordell-Lang problem, one probably needs a bound on the period obtained independently of properties of X other than its degree.

The second part of this article examines the canonical height of a subvariety in terms of its Chow form. Starting with work of Nesterenko [24] and

Philippon [27], heights of elimination forms, a version of the more general Chow form, was introduced in transcendence theory. Philippon continued to study the heights of subvarieties via the heights of associated Chow forms and proved many properties of heights as well as canonical heights of abelian varieties [28–30]. Alternatively, using the arithmetic intersection theory of Gillet-Soulé [12], Faltings defined the height of a subvariety X as the intersection of the fundamental class of X with the first Chern class of the canonical Hermitian line bundle on $\mathbb{P}^N$ raised to the power $\dim(X)$ [8]. Faltings' height is the arithmetic analog of the degree in algebraic geometry. Bost-Gillet-Soulè defined the height as the intersection of the fundamental class of X with the d-th Chern class of the canonical quotient Hermitian bundle on $\mathbb{P}^N$ [4]. The arithmetic intersection theory machinery is quite powerful, and they prove many results on the heights of subvarieties. Furthermore, Zhang proved the Bogomolov conjecture for abelian varieties [36] with this framework. Zhang also studied heights of semistable varieties, in the sense of arithmetic geometric invariant theory from the Chow perspective [35]. However, less is done with canonical heights. Gubler studied local heights and canonical heights in this framework [13, 14]. In this article, we extend Philippon's approach for abelian varieties to general projective morphisms and prove a number of basic results about the canonical height of subvarieties [13]. We define

$$\hat{h}(X) = \lim_{n\to\infty} \frac{\deg(X)}{\deg(f^n(X))}\frac{h(f^n(X))}{\deg(f^n)},$$

where $h(X)$ is the height of the associated Chow form. We summarize our main results in the following theorem.

Theorem 1.1. *Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree $d \geq 2$ defined over a number field K. Let X be an irreducible subvariety of degree D and codimension b defined over K.*

1) *(Theorem 4.6) If X is a hypersurface, then*

$$\left| h(f(X)) - d\frac{\deg(f(X))}{\deg(X)}h(X)\right| \leq C(f, N, D)$$

for an explicitly computed constant C.

2) *(Theorem 4.10) The canonical height converges and satisfies the functional equation*

$$\hat{h}(f(X)) = \frac{\deg(f)\deg(f(X))}{\deg(X)}\hat{h}(X).$$

3) *(Theorem 4.12) If X is a hypersurface, then*

$$\left|\hat{h}(X) - h(X)\right| \leq \frac{CD}{(d-1)d^{N-b-1}},$$

where C is from part (1).

These results immediately imply that there are only finitely many rational preperiodic subvarieties of bounded degree (Corollary 4.13).

In both parts we restrict to number fields, although the theorems concerning the period bound for points on a singular variety should hold as stated for global fields. The issue with non-perfect fields is that a K-rational Chow form may not correspond to a K-rational subvariety [32, I.§9, p.47].

2. Preliminaries

Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree d. Since f is also proper, the forward image $f(X)$ is a subvariety. Furthermore, if X has pure codimension b and degree D, then $f(X)$ has pure codimension b and degree $d^{N-b}D$. Note that if X is irreducible, then so is $f(X)$. We can explicitly compute forward images via elimination theory.

Proposition 2.1. *Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree d and $X = V(g_1, \ldots, g_k)$ be a subvariety. Let $I \subset K[x_0, \ldots, x_N, y_0, \ldots, y_N]$ be the ideal*

$$I = (g_1(\bar{x}), \ldots, g_k(\bar{x}), y_0 - f_0(\bar{x}), \ldots, y_N - f_N(\bar{x})),$$

where $\bar{x} = (x_0, \ldots, x_N)$. Then the elimination ideal $I_{N+1} = I \cap K[y_0, \ldots, y_N]$ is a homogeneous ideal and

$$f(X) = V(I_{N+1}).$$

Proof. Adapted from [5, Theorem 8.5.12] which proves the similar result when $X = \mathbb{P}^N$.

We first prove that I_{N+1} is homogeneous. Since the polynomials $y_i - f_i(\bar{x})$ are not homogeneous, we introduce weights: each x_i has weight 1 and

y_i has weight d. Then I is a weighted homogeneous ideal with $\deg(g_i) = d_i$ and $\deg(y_i - f_i(\bar{x})) = d$. A reduced Groebner basis G with respect to any monomial ordering consists of weighted homogeneous polynomials ([5, Theorem 8.3.2]). For an appropriate lexicographic ordering, $G \cap K[y_0, \ldots, y_N]$ is a basis for $I_{N+1} = I \cap K[y_0, \ldots, y_N]$. Thus, I has a weighted homogeneous basis. Since the basis is in $K[y_0, \ldots, y_N]$, it must also be homogeneous.

Now we consider the image and work in the product $\mathbb{P}^N \times \mathbb{P}^N$. The ideal I is not generated by bi-homogeneous polynomials, so we need to consider the ideal:

$$J = (g_1(\bar{x}), \ldots, g_k(\bar{x}), y_i f_j(\bar{x}) - y_j f_i(\bar{x})).$$

We show that $V(J)$ is the graph of $f(X)$. Let $p \in X$, then $(p, f(p)) \in V(J)$. Conversely, suppose that $(p, q) \in V(J)$. Then we must have $p \in X$ and $q_i f(p_j) = q_j f(p_i)$. There is a j with $q_j \neq 0$ and, since f is a morphism, an i with $f(p_i) \neq 0$ so that $q_i f_j(p) = q_j f_i(p) \neq 0$ implying $q_i \neq 0$. Let $\lambda = q_i / f_i(p)$. We have $q = \lambda f(p)$ so that (p, q) is on the graph of $f(X)$. Let $\pi_2 : \mathbb{P}^N \times \mathbb{P}^N \to \mathbb{P}^N$ be the projection to the second component. Then

$$f(X) = \pi_2(V(J)).$$

Now we show $\pi_2(V(J)) = V(I_{N+1})$.

It suffices to work in the affine cone $\mathbb{A}^{N+1} \times \mathbb{A}^{N+1}$ and show that $\pi_2(V(J)) = V(I_{N+1}) \subset \mathbb{A}^{N+1}$. Once we exclude the origin, $q \in \pi_2(V(J))$ if and only if there is some $p \in \mathbb{A}^{N+1}$ such that $q = f(p)$ in $\mathbb{P}^N$. In other words, if and only if there is some $\lambda \neq 0$ such that $q = \lambda f(p)$ in $\mathbb{A}^{N+1}$. If we set $\lambda' = \sqrt[d]{\lambda}$, then $q = f(\lambda' p)$, which is equivalent to $q \in \pi(V(I))$. Since $V(I_{N+1}) \subset \mathbb{A}^{N+1}$ is the smallest variety containing $\pi_2(V(J)) \subset \mathbb{A}^{N+1}$, we have $V(I_{N+1}) = \pi_2(V(J))$. $\square$

Since f is flat, the preimage of a subvariety of degree D with pure codimension b is again a subvariety with pure codimension b. We compute the preimage of $X = V(g_1, \ldots, g_k)$ as $f^{-1}(X) = V(g_1 \circ f, \ldots, g_k \circ f)$ and it has degree $d^b D$. Note that the preimage of an irreducible variety is not necessarily irreducible.

2.1. Chow forms

We will define heights and canonical heights in Section 4 in terms of the height of the associated Chow form. See [10, Chapter 3] for more details on Chow forms.

Definition 2.2. Let X be an k-dimensional subvariety of $\mathbb{P}^N$ of degree D. The $(N-k-1)$-dimensional projective subspaces of $\mathbb{P}^N$ meeting X form a hypersurface in the Grassmannian $G(N-k-1,N)$. The homogeneous form defining this hypersurface in Plücker coordinates is called the *Chow form* of X denoted $Ch(X)$.

The polynomial $Ch(X)$ has degree D. If X is a cycle $\sum n_Y Y$, its Chow form is the hypersurface $\prod Ch(Y)^{n_Y}$. Note that when X is irreducible, its Chow form is the "forme éliminante" from Philippon [27]. There are explicit algorithms to compute the Chow form of a variety, see for example Dalbec [6].

3. Bounding the period

In this section we show that there is a bound on the period of a subvariety that depends only on the degree of the morphism, the degree of the subvariety, the codimension of the subvariety, the dimension of the space, the degree of the number field, and a prime of good reduction. The idea is that for a periodic subvariety we can restrict the function to an endomorphism of a subvariety of the Chow variety. Then the problem is about periodic points. Since this restriction may be to a singular variety, we need to extend the periodic point theorem from the author's previous work [17] to singular varieties. An important tool is the ability to extend morphisms of varieties to morphisms of projective space from Bhatnagar-Szpiro [3] and Fakhruddin [7].

To be able to restrict to a subvariety of the Chow variety, we need to have an upper bound on the degree of an element of the orbit. There are a couple issues that arise here. One is that a given orbit can have elements of different degree within the same orbit.

Example 3.1. For

$$\begin{aligned} f : \mathbb{P}^2(\mathbb{F}_2) &\to \mathbb{P}^2(\mathbb{F}_2) \\ (x,y,z) &\mapsto (z^2, y^2+xz+z^2, x^2) \end{aligned}$$

we have the orbit

$$\begin{aligned} V(y+z) &\to V(y^2+xz) \to V(x+y) \\ &\to V(x^2+y^2+xz+z^2) \to V(y+z) \to \cdots \end{aligned}$$

Secondly, there may be finitely many primes where the degree of the reduced subvariety is not the degree of the rational subvariety. If we exclude these primes and have a bound on the period modulo $\mathfrak{p}$, then we have an upper bound on the degree of an element in the orbit.

Although it is not explicit, seeing that there is a bound on the reduced period depending only on $\mathfrak{p}, d, D, N, b$ is quite simple. Since there are only finitely many residue classes of a map of degree d modulo $\mathfrak{p}$ and only finitely many subvarieties of a given degree D and codimension b modulo $\mathfrak{p}$, there are only finitely many possible cycles. So there is, in fact, a bound on the period of a subvariety modulo $\mathfrak{p}$ that depends only on $\mathfrak{p}, d, D, N, b$.

Lemma 3.2. *Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree d defined over a number field K. Let $X \subset \mathbb{P}^N$ be a periodic irreducible subvariety of degree D and codimension b defined over K. Let $\mathfrak{p} \in K$ be a prime of good reduction for f. Then there exists a constant m depending on $d, N, D, \mathfrak{p}, b$ such that the period of X modulo $\mathfrak{p}$ is bounded by m.*

Proof. Given $D, b, N, \mathfrak{p}$, there are only finitely many subvarieties defined over the residue field of degree at most D and codimension b (count, for example, the coefficients of the Chow form). Also, given $d, N, \mathfrak{p}$, there are only finitely many residue classes of morphisms of degree d defined over the residue field. Since X is assumed periodic, its reduction modulo $\mathfrak{p}$ must be in one of finitely many possible cycles. $\square$

Example 3.3. As an example, consider $d = 2$, $K = \mathbb{Q}$, $p = 2$, and $D = 1$. To make the computational feasible, we assume that the degree of an element in the orbit of a degree 1 subvariety is ≤ 8. Then $m = 7$.

If we assume that $\deg(f^n(X)) = \deg(\overline{f^n(X)})$, then Lemma 3.2 also gives a bound on the degree of the subvarieties in the orbit of X as $d^{m(N-b)}D$. In particular, we can embed every element of the orbit of X into the same Chow variety and look at the induced morphism. This restriction requires only excluding finitely many additional primes (those which divide the discriminant of the Chow forms of the varieties in the orbit of X).

We will need to restrict to a particular subvariety of the Chow variety, but first we extend the bound on the period for points using good reduction information to morphisms of possibly singular varieties. We do this in two different ways to arrive at two different bounds. First we use a result of Bhatnagar-Szpiro [3].

Proposition 3.4. *[3, Theorem 1] Let X be a projective variety defined over an infinite field K, L a very ample line bundle on X and $f : X \to X$ a morphism such that $f^*L = L^{\otimes d}$ for some integer $d \geq 2$. Then there exists a positive integer s and a morphism $\phi : \mathbb{P}^N \to \mathbb{P}^N$ extending f^s, where $N + 1 = \dim_K H^0(X, L)$. Moreover, if the linear system $H^0(X, L)$ is complete, then f extends to $\mathbb{P}^N$.*

However, to get a bound independent of f and X, we need to show that s can be bounded in terms of more general information.

Lemma 3.5. *Let X be a projective variety of dimension u defined over an infinite field K, L a very ample line bundle on X and $f : X \to X$ a morphism such that $f^*L = L^{\otimes d}$ for some integer $d \geq 2$. Consider the embedding $X \subset \mathbb{P}^N$ induced by L. Let D be the degree of X under this embedding. There exists a constant $C(u, d, D)$ such that the integer s in Proposition 3.4 satisfies*

$$s \leq C(u, d, D).$$

Proof. Following the proof of Bhatnagar-Szpiro [3], we need a uniform bound on n_0 from Serre's Vanishing Theorem such that for all $n \geq n_0$ we have $H^1(\mathbb{P}^N, I_X(n)) = 0$, where $I_X(n)$ is the twist of the sheaf of ideals $I(n)$ of X. Following Harsthorne [15, Proposition 24.4], n_0 is bounded over flat families. To apply this to our situation, we consider the Chow variety $CV(D)$ parameterizing varieties of degree D and dimension u in $\mathbb{P}^N$. We take a flattening decomposition of this Chow variety; see [22, Lecture 8] or [25, Theorem 4.3]. Let C_0 be the open set of points of $CV(D)$ where the universal subvariety X of $\mathbb{P}^N$ is flat. Hartshorne gives the existence of a uniform n_0 for C_0, [15, Proposition 24.4]. Now consider the family C_1 defined as the open set of points of $CV(D) - C_0$ where X restricted is flat. On this family there is an n_1. Continue this process for the finitely many possible families ([25, Theorem 4.3]) until we have $H^1(\mathbb{P}^N, I_Z(n)) = 0$ for all $Z \in CV(D)$. Then the desired n_0 is the max of these n_i.

Given this uniform n_0, s is chosen so that

$$d^s > \max(D, n_0) \geq \max(\deg(h_i), n_0)$$

where $X = V(h_1, \ldots, h_v)$ for some set of homogeneous generating polynomials. The second inequality comes from [16, Proposition 3] on the degree of generators. ∎

We can now prove the extended theorem.

Theorem 3.6. *Let X be an irreducible projective variety of degree D defined over a number field K equipped with a line bundle L that gives an embedding $X \hookrightarrow \mathbb{P}^N$. Let $f : X \to X$ be a morphism such that $f^*L \cong L^{\otimes d}$, $d \geq 2$. Let $\mathfrak{p}$ be a prime of K with residue field k for which f has good reduction.*

Let P be a periodic point of primitive period n for f whose reduction to the residue field k has primitive period m. Then there is a constant C depending on $d, D, N, \dim(X), \mathfrak{p}$ such that

$$n \leq C(d, D, N, \dim(X), \mathfrak{p}),$$

where

$$C \leq s \cdot \#X(k) \cdot \#GL_{N+1}(k) \cdot p^e,$$

where s is from Proposition 3.4, $p = \mathfrak{p} \cap \mathbb{Q}$, and

$$e \leq \begin{cases} 1 + \log_2(v(p)) & p \neq 2 \\ 1 + \log_\alpha\left(\frac{\sqrt{5}v(2)+\sqrt{5(v(2))^2+4}}{2}\right) & p = 2, \end{cases}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$.

Proof. By Proposition 3.4 and Lemma 3.5, there is an integer $s \geq 1$ bounded by a constant depending on $d, D, N, \dim(X)$ such that f^s extends to a morphism on $\mathbb{P}^N$. In other words, there is a map $\phi : \mathbb{P}^N \to \mathbb{P}^N$ that makes the following diagram commute

$$\begin{tikzcd} X \arrow[r, "f^s"] \arrow[d] & X \arrow[d] \\ \mathbb{P}^N \arrow[r, "\phi"] & \mathbb{P}^N. \end{tikzcd}$$

Let $g = \gcd(s, n)$. Applying [17, Theorem 1] to ϕ, we see that

$$\frac{n}{g} = m'rp^e,$$

where m' is the minimal period of $f^s(P)$ over k, r is the multiplicative order of the multiplier of $f^s(P)$ for ϕ over k, and e is bounded as in the statement of [17, Theorem 2]. In particular,

$$n \leq gm'rp^e \leq smrp^e.$$

□

Note that even when s is 1, this bound is slightly weaker than [17, Theorem 1] in the multiplier term, but it has the advantage of applying to singular varieties. Note that [7, Proposition 2.1] gives an equivalent set of conditions for $H^0(X,L)$ to be complete, i.e., for when $s=1$.

Since Lemma 3.5 is not effective, we give a second proof that can provide an effective bound by modifying Fakhruddin's construction in [7, Proposition 2.1]. Essentially, by results of Mumford [23, Theorem 1 and 3], we are able to provide an embedding where $s=1$ and just need to keep track of the degrees and dimensions. Note that this version also removes the dependency on the dimension of X.

Theorem 3.7. *Let X be a projective variety of degree D defined over a number field K equipped with a line bundle L giving an embedding $X \hookrightarrow \mathbb{P}^N$. Let $f: X \to X$ be a morphism such that $f^*L \cong L^{\otimes d}$, $d \geq 2$. Let $\mathfrak{p}$ be a prime of K with residue field k for which f has good reduction.*

Let P be a periodic point of primitive period n for f whose reduction to the residue field k has primitive period m. Then

$$n \leq \#X(k) \cdot \#\operatorname{GL}_{M+1}(k) \cdot p^e \leq \#\mathbb{P}^N(k) \cdot \#\operatorname{GL}_{M+1}(k) \cdot p^e$$

where $M = \binom{N+D}{N} - 1$, $p = \mathfrak{p} \cap \mathbb{Q}$, and

$$e \leq \begin{cases} 1 + \log_2(v(p)) & p \neq 2 \\ 1 + \log_\alpha\left(\frac{\sqrt{5}v(2) + \sqrt{5(v(2))^2+4}}{2}\right) & p = 2, \end{cases}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$.

Proof. Let $V = H^0(X,L)$ and let $i: X \hookrightarrow \mathbb{P}(V)$ be the embedding induced by L. For f to extend, we need to satisfy the following conditions from [7, Proposition 2.1]:

1) The maps $i^*: H^0(\mathbb{P}(V), \mathcal{O}(n)) \to H^0(X, L^{\otimes n})$ are surjective for all $n \geq 0$.

2) $i(X)$ is cut out set-theoretically by homogeneous forms of degree $\leq d$.

From Mumford [23, Theorem 1 and 3], embedding X in a larger projective space with the $\deg(X)$-uple Veronese embedding and replacing L by a power at most $\dim(X)+1$ results in satisfying both properties. We are embedding X in projective space of dimension $M = \binom{N+D}{N} - 1$. We apply [17, Theorem 1] to the resulting map $\psi: \mathbb{P}^M \to \mathbb{P}^M$, which extends f to have any rational periodic points with minimal period n, satisfying

$$n \leq \#X(k) \cdot \#\,\mathrm{GL}_{M+1}(k) \cdot p^e.$$

□

We can now prove the existence of a bound for the minimal period of subvarieties based on good reduction information.

Theorem 3.8. *Let K be a number field. Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree d defined over K. Let $X \subset \mathbb{P}^N$ defined over K be an irreducible subvariety of degree D and codimension b that is periodic for f with minimal period n. Let $\mathfrak{p}$ be a prime of good reduction for f. If $\deg(f^\ell(X)) = \deg(\overline{f^\ell(X)})$ for $0 \leq \ell \leq n$, then there is a constant C depending on $d, D, N, b, \mathfrak{p}$ such that*

$$n \leq C(d, D, N, b, \mathfrak{p}).$$

Proof. Let m be the period of $\overline{X}$ over the residue field k (which is bounded in Lemma 3.2). The morphism f^m induces a morphism ϕ from the Chow variety of degree D (i.e., $\mathbb{P}^M$ for $M = \binom{N+D}{N}$) to the Chow variety of degree $d^{m(N-b)}D$. We have $\deg(\phi) = d^{mN}$. However, there is a closed subvariety of the Chow variety on which f^m preserves the degree (which contains X). This is the subvariety where the image Chow form is a power. In particular, if we take the image of a symbolic Chow form T of degree D, then the discriminant of the resulting Chow form of degree $d^{m(N-b)}D$ gives the closed subvariety where the image degree is strictly less than $d^{m(N-b)}D$. We can compute this discriminant as [10, Prop 1.7, p434]:

$$Z_1(\phi) = \Delta = \mathrm{Res}\left(\frac{\partial^{d-1} Ch(\phi(T))}{\partial x_0^{d-1}}, \ldots, \frac{\partial^{d-1} Ch(\phi(T))}{\partial x_M^{d-1}}\right).$$

We want the subvariety where the image variety is of degree D. In particular, we need the image to be a $d^{m(N-b)}$-th power. So we take the subvariety defined by

$$Z_{m(N-b)}(\phi) = \mathrm{Res}\left(\frac{\partial^{d^{m(N-b)}-1} Ch(\phi(T))}{\partial x_0^{d^{m(N-b)}-1}}, \ldots, \frac{\partial^{d^{m(N-b)}-1} Ch(\phi(T))}{\partial x_M^{d^{m(N-b)}-1}}\right).$$

Note that we have [10, Prop 1.1, p427]

$$\deg(Z_{m(N-b)}(\phi)) = (M+1)(d^{m(N-b)}(D-1)+1)^M.$$

On each irreducible component $Y_1, \ldots, Y_v$ of this subvariety, the map ϕ induces a map $\psi_i : Y_i \to \mathbb{P}^M$ of degree $\frac{d^{mN}}{d^{m(N-b)}} = d^{mb}$. Note that the Y_i may be singular. Assume, after possibly renaming, that $Ch(X) \in Y_1$. This map is defined by a single set of homogeneous polynomials, but it is not yet a self-map of varieties. Consider the following construction. Since Y_1 is irreducible, the intersection $\psi_1(Y_1) \cap Y_1$ is the intersection of two irreducible subvarieties, so either is Y_1 or has dimension strictly less than Y_1. If it is all of Y_1, then we have our self-map. If the dimension is strictly less, consider the component $Y_{1,1}$ of $\psi_1(Y_1)$ which contains X. We again consider $\psi_1(Y_{1,1}) \cap Y_1$. Again, we get either a self-map or a strictly smaller dimension that $Y_{1,1}$. This process will terminate after finitely many steps in an irreducible variety $\tilde{Y_1}$. Since $Ch(X) \in \tilde{Y_1} \subset \mathbb{P}^M$, this intersection is nonempty and we have that

$$\psi_1 : \tilde{Y_1} \to \tilde{Y_1}$$

is a morphism of varieties.

To get our final constant depending only on $d, D, N, b, \mathfrak{p}$, it remains to prove that the degree of $\tilde{Y_1}$ can be bounded only in terms of $d, D, N, b, \mathfrak{p}$. We prove this bound by applying Bźout's theorem on the degree of the intersection of subvarieties as follows.

The number of steps in the restriction process is at most $\dim(Y_1) = M-1$ and $\deg(Y_1) \leq \deg(Z_{m(N-b)}(\phi))$. If it takes j steps to get a self-map, the degree of each irreducible component containing X is bounded above by

$$\begin{aligned} \deg(Y_{1,i}) &\leq \deg(\psi_1)^i \deg(Y_1) \\ &\leq (d^{mb})^i (M+1)(d^{m(N-b)}(D-1)+1)^M \quad \text{for } 0 \leq i \leq j-1, \end{aligned}$$

where we let $Y_{1,0} = Y_1$. The degree at each stage depends on the degree of Y_1 and ψ_1. Since for each step we are intersecting an irreducible variety with a hypersurface and the resulting dimension is one less, we can apply Bézout's Theorem on the degree of intersections. Since we are taking only one irreducible component, this provides an upper bound:

$$\deg(\tilde{Y_1}) \leq \prod_{i=0}^{j-1} (d^{mb})^i (M+1)(d^{m(N-b)}(D-1)+1)^M.$$

Most importantly, the number of steps is bounded by $M-1$, and this degree bound depends only on $d, D, N, b, \mathfrak{p}$.

We apply Theorem 3.6 to ψ_1 on $\tilde{Y_1}$ to get a constant such that

$$n \leq C(d, D, N, b, \mathfrak{p}).$$

□

The following example constructs an explicit instance of the endomorphism $\psi_1 : \tilde{Y_1} \to \tilde{Y_1}$ induced on the component of the Chow variety needed in the proof of Theorem 3.8.

Example 3.9. Consider the map

$$f(x, y, z) = (x^2, y^2 + z^2, z^2).$$

It is clear that $V(z) = \{(x, y, z) \in \mathbb{P}^2 : x = 0\}$ is fixed, so we will consider linear subvarieties

$$T = V(a_0 x + a_1 y + a_2 z).$$

We have

$$\begin{aligned} f(T) = V(&a_0^4 x^2 - 2a_0^2 a_1^2 xy + a_1^4 y^2 + (2a_0^2 a_1^2 - 2a_0^2 a_2^2) xz \\ &+ (-2a_1^4 - 2a_1^2 a_2^2) yz + (a_1^4 + 2a_1^2 a_2^2 + a_2^4) z^2). \end{aligned}$$

So we have the map

$$\begin{aligned} \phi &: \mathbb{P}^2 \to \mathbb{P}^5 \\ (a_0, a_1, a_2) &\mapsto (a_0^4, -2a_0^2 a_1^2, a_1^4, 2a_0^2 a_1^2 - 2a_0^2 a_2^2, -2a_1^4 - 2a_1^2 a_2^2, a_1^4 + 2a_1^2 a_2^2 + a_2^4). \end{aligned}$$

We want to know the subvarieties that remain linear under f, so we compute

$$\begin{aligned} Z_1 &= \operatorname{Res}\left(\frac{\partial \phi(T)}{\partial x}, \frac{\partial \phi(T)}{\partial y}, \frac{\partial \phi(T)}{\partial z}\right) \\ &= -32 a_0^4 a_1^4 a_2^4 \end{aligned}$$

and we see that Z_1 has three components:

$$Y_1 = V(a_0), \qquad Y_2 = V(a_1), \qquad Y_3 = V(a_2).$$

On $Y_1 \supset X$ we have

$$\begin{aligned} \psi &: Y_1 \to \mathbb{P}^2 \\ (0, a_1, a_2) &\mapsto (0, a_1^2, -a_1^2 - a_2^2), \end{aligned}$$

which we can extend to

$$\begin{aligned} \psi_1 &: Y_1 \to \mathbb{P}^2 \\ (a_0, a_1, a_2) &\mapsto (a_0^2, a_1^2, -a_1^2 - a_2^2). \end{aligned}$$

Notice that ψ_1 is an endomorphism of Y_1.

We give an example showing that the constant in Theorem 3.8 is explicitly computable.

Example 3.10. Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a morphism of degree 2 and $X \subset \mathbb{P}^2$ be an irreducible hyperplane. Assume that the prime $p = 2$ satisfies the necessary good reduction hypotheses. Further assume the bound on m is 7, as computed experimentally in Example 3.3. We need to keep track of the dimensions and degrees as we construct the induced map on the Chow variety. We have

$$N = 2, D = 1, b = 1, p = 1, d = 2 \quad \text{so that} \quad M = 3, m \le 7.$$

Following the proof of Theorem 3.8, we have the map $\psi_1 : \tilde{Y_1} \to \tilde{Y_1}$ with $\deg(\psi_1) = d^{mb} \le 2^7 = 128$. We compute the bound on the degree of $\tilde{Y_1}$ as

$$\deg(\tilde{Y_1}) \le \prod_{i=0}^{1} (2^7)^i (4)(2^7 \cdot 0 + 1)^3 = 4 \cdot (2^7 \cdot 4) = 2048.$$

Now we apply Theorem 3.7 with $N = 3, D = 2048, p = 2, d = 128$ to get

$$n \le \#\mathbb{P}^3(\mathbb{F}_2) \cdot \#\,\mathrm{GL}_{2049}(\mathbb{F}_2) \cdot 2^3 \approx 10^{10^6}.$$

4. Explicit heights and canonical heights

In this section we prove basic properties of heights and canonical heights of subvarieties as defined by the height of the associated Chow form. Recall that the height of a polynomial is the maximum of the heights of its coefficients.

Definition 4.1. For a map $f : \mathbb{P}^N \to \mathbb{P}^N$ consisting of homogeneous polynomials $f = [f_0, \ldots, f_N]$, we define

$$h(f) = \max_i h(f_i).$$

Definition 4.2. Given a subvariety $X \subset \mathbb{P}^N$, we define the height of X as

$$h(X) = h(Ch(X)),$$

where $Ch(X)$ is the associated Chow form defined in Section 2.

The height satisfies the following properties.

Proposition 4.3. *Let $X \subset \mathbb{P}^N$ be a subvariety of degree D.*

1) $h(X) \geq 0$.
2) *There are only finitely many subvarieties of bounded height and bounded degree over a number field of bounded degree.*
3) *Philippon's height [28], Faltings' height [9], Bost-Gillet-Soulé's height [4], and h are all equivalent.*

Proof. The first two properties are obvious from the definition of $h(X)$ since the height of $Ch(X)$ is the max of the height of its coefficients. The third property can be found in Bost-Gillet-Soulé [4]; see, for example, Proposition 4.1.2, the remark after Theorem 4.2.3, and the remarks starting with 4.3.12. □

4.1. Height bounds of forward images

It is well known that for a morphism f and a point P

$$|h(f(P)) - \deg(f)h(P)| \leq C$$

for an explicitly computable constant C [33, Theorem 3.11]. Similarly [4, Prop 3.2.2], for a subvariety X of codimension b, there exists a (non-explicit) constant C such that

$$\left| h(f(X)) - \deg(f)\frac{\deg(f(X))}{\deg(X)}h(X)\right| \leq C\deg(X). \tag{1}$$

We first recall a lemma on the size of coefficients occurring in resultants.

Lemma 4.4. *[34, Proposition 7] For $N+1$ multi-homogeneous polynomials in $N+1$ variables of degrees $D, d, \ldots, d$,*

$$h(Res_{D,d,\ldots,d}) \leq \left(\binom{D+Nd^N+1}{N}\right) + \log\left(\binom{D+Nd^N+1}{N}!\right).$$

Lemma 4.5. *The map induced on the Chow coordinates in codimension b is a morphism of degree d^{N-b+1}.*

Proof. A flat proper map induces a homomorphism between the Chow varieties. In particular, a codimension b subvariety maps to a codimension b subvariety. We can compute the image subvariety via Proposition 2.1. We think of this map as acting on a projective point representing the coefficients of the Chow form (i.e., the Chow coordinates). The image coordinates will be polynomials in the original coordinates. Since we know each image is a codimension b subvariety, this map is a morphism.

To compute its degree, we simply count the number of inverse images under the pullback map. If $\deg(X) = D$, then $\deg(f^{-1}(X))$ is dD. Generically, a subvariety of degree $\frac{D}{d^{N-b}}$ maps to a subvariety of degree D, so the number of inverse images is

$$dD\frac{d^{N-b}}{D} = d^{N-b+1}.$$

$\square$

To help keep the expressions readable: for positive integers N and D, define the number of monomials of degree D in $N+1$ variables as

$$\tau(D) = \binom{N+D}{N}.$$

For a third positive integer d, define

$$e(D) = \tau(D)d^N - (\tau(D)-1) + 1 = (\tau(D)-1)d^N + 2$$

which is used in bounds on the degree of the resultant from Wustholz [34, §10].

Theorem 4.6. *Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree $d \geq 2$. Let $X \subset \mathbb{P}^N$ be a hypersurface of degree D. Then there exists an explicitly computable constant such that*

$$\left| h(f(X)) - d\frac{\deg(f(X))}{\deg(X)}h(X) \right| \leq C(f, N, D).$$

Moreover,

$$C = \frac{\deg(f(X))/\deg(X)}{d^{N-1}}\Bigg(\log(\tau(\deg(f(X))))$$
$$+\log\left(\binom{\tau(D)-1+e(D)-d^N}{e(D)-d^N}\right) + \Big(4\tau(D)(\tau(D)+1)(d^N)^{\tau(D)}(2d^{N-1}Dh(f)$$
$$+\log(N) + \log(C) + \log(\tau(\deg(f(X)))) + (\tau(D)+7)\log(\tau(D)+1)d^N)\Big)\Bigg).$$

Proof. Since X is a hypersurface of degree D, it is the vanishing locus of a homogeneous polynomial of degree D, g. Let $f(\overline{x}) = (f_0(\overline{x}), \ldots, f_N(\overline{x}))$ be a tuple of homogenous polynomials in the variables $\overline{x} = (x_0, \ldots, x_N)$. For a second set of variables $\overline{y} = (y_0, \ldots, y_N)$, define the tuple

$$J = (y_j f_i(\overline{x}) - y_i f_j(\overline{x}), g(\overline{x})) \text{ for } 0 \leq i < j \leq N.$$

The equations $y_j f_i(\overline{x}) - y_i f_j(\overline{x})$ are of degree d in $\overline{x}$, and $g(\overline{x})$ has degree D. The ideal generated by the generalized resultant $\mathrm{Res}_{D,d,\ldots,d}(J)$ in terms of x is the forward image of X by f. Recall that the generalized resultant Res_{d_i} is homogeneous of degree $\frac{\prod d_i}{d_j}$ in the coefficients of the equation corresponding to d_j for each j. In this particular case, the resultant has degree d^N in the coefficients of g and degree $d^{n-1}D$ in the coefficients of each of the $y_j f_i(\overline{x}) - f_j(\overline{x}) y_i$.

It is possible that $\frac{\deg(f(X))}{\deg(X)} < d^{N-1}$, i.e., that the resultant is a power. In particular, the $\frac{d^{N-1}}{\deg(X)/\deg(f(X))}$-th root of the bound on the resultant is a bound on $h(f(X))$. We see that an upper bound is given by

$$h(\mathrm{Res}) \leq [d^N h(X) + d^{N-1} D(2h(f)) + \log(N)] + \log(C),$$

where C is the max coefficient in the resultant. From Lemma 4.4, we have

$$\log(C) \leq \binom{D + Nd^N + 1}{N} + \log\left(\binom{D + Nd^N + 1}{N}!\right).$$

Therefore,

$$\begin{aligned} h(f(X)) &= \frac{\deg(f(X))/\deg(X)}{d^{N-1}} h(\mathrm{Res}) \\ &\leq \frac{\deg(f(X))/\deg(X)}{d^{N-1}} \Bigg(\Big(d^N h(X) + 2d^{N-1} D h(f) + \log(N) \Big) \\ &\quad + \binom{D + Nd^N + 1}{N} + \log\left(\binom{D + Nd^N + 1}{N}!\right) \Bigg). \end{aligned}$$

In particular,

$$\begin{aligned} h(f(X)) - d\frac{\deg(f(X))}{\deg(X)} h(X) \leq \frac{\deg(f(X))/\deg(X)}{d^{N-1}} \Bigg(2d^{N-1} D h(f) + \log(N) \\ + \binom{D + Nd^N + 1}{N} + \log\left(\binom{D + Nd^N + 1}{N}!\right) \Bigg). \end{aligned}$$

The lower bound is somewhat more complicated. The morphism f induces a map on Chow varieties. These Chow varieties can be thought of as points with their coefficients as coordinates in some projective space. For ease of notation, denote $\deg(f(X)) = D'$. This induced map is a morphism of degree d^N (Lemma 4.5):

$$\phi : \mathbb{P}^{\tau(D)-1} \to \mathbb{P}^{\tau(D')-1}.$$

For a point P we know that

$$\left| h(\phi(P)) - d^N h(P) \right| \leq C_2$$

for an explicitly computable (Nullstellensatz) constant C_2 [33, Theorem 3.11]. As with the upper bound, if the degree of $f(X)$ is not the full $d^{N-1}\deg(X)$, we get the actual bound from taking a root. We next bound the constant.

From [33, Theorem 3.11] and [18, Theorem 1], we have

$$\begin{aligned}
H(P)^{d^N} &\leq \tau(D')\binom{\tau(D)-1+e(D)-d^N}{e(D)-d^N}\max(H(g_i))H(\phi(P)) \\
h(g) &\leq 4\tau(D)(\tau(D)+1)(d^N)^{\tau(D)}\Big(h(\phi)+\log(\tau(D')) \\
&\quad + (\tau(D)+7)\log(\tau(D)+1)d^N\Big) \\
d^N h(P) &\leq \log(\tau(D')) + \log\left(\binom{\tau(D)-1+e(D)-d^N}{e(D)-d^N}\right) \\
&\quad + \Big(4\tau(D)(\tau(D)+1)(d^N)^{\tau(D)}\big(h(\phi)+\log(\tau(D')) \\
&\quad + (\tau(D)+7)\log(\tau(D)+1)d^N\big)\Big) + h(\phi(P)).
\end{aligned}$$

So we need an upper bound on $h(\phi)$. We can get this from an upper bound on the resultant from above (without the $h(V)$ part) as

$$h(\phi) \leq 2d^{N-1}Dh(f) + \log(N) + \log(C),$$

where C is the Wustholz upper bound on the max coefficient in the resultant (Lemma 4.4).

So we have

$$\begin{aligned}
d^N h(P) \leq \log(\tau(D')) &+ \log\left(\binom{\tau(D)-1+e(D)-d^N}{e(D)-d^N}\right) \\
&+ \Big(4\tau(D)(\tau(D)+1)(d^N)^{\tau(D)}\big(2d^{N-1}Dh(f)+\log(N)+\log(C) \\
&+ \log(\tau(D')) + (\tau(D)+7)\log(\tau(D)+1)d^N\big)\Big) + h(\phi(P)).
\end{aligned}$$

Now we take the $\frac{d^{N-1}}{\deg(f(X))/\deg(X)}$-th root of both sides (except for the $h(\phi(P))$ term on the right) to get

$$\begin{aligned}
&d\frac{\deg(f(X))}{\deg(X)}h(P) \\
&\leq \frac{\deg(f(X))/\deg(X)}{d^{N-1}}\Bigg(\log(\tau(D')) + \log\left(\binom{\tau(D)-1+e(D)-d^N}{e(D)-d^N}\right) \\
&+ \Big(4\tau(D)(\tau(D)+1)(d^N)^{\tau(D)}\big(2d^{N-1}Dh(f)+\log(N)+\log(C) \\
&+ \log(\tau(D')) + (\tau(D)+7)\log(\tau(D)+1)d^N\big)\Big)\Bigg) + h(\phi(P)).
\end{aligned}$$

This difference is clearly larger than the upper bound, so this is the desired constant. $\square$

We now want to simplify the form of the constant to get a rough estimate of growth. We use two main tools:

$$\binom{n+k}{n} \leq \frac{(n+k)^n}{n!} < (n+k)^n \quad \text{for } n \geq 2, k \geq 1$$

and

$$n! \leq n^n \quad \text{so that} \quad \log(n!) \leq n\log(n).$$

Corollary 4.7. *Fix positive integers N, d. Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree $d \geq 2$. Let X be a hypersurface of degree D. We have the following bound in terms of D*

$$\left| h(f(X)) - \deg(f)\frac{\deg(f(X))}{\deg(X)}h(X) \right| \leq O(D^{3N}\log(D)(d^N)^{D^N})$$

where the constant depends on f, d, N.

Proof. We see that an upper bound is given by

$$h(f(X)) - \deg(f)\frac{\deg(f(X))}{\deg(X)}h(X) \leq \big(d^{N-1}D(2h(f)) + \log(N)\big) + \log(C)$$

where C is bounded by

$$\begin{aligned}\log(C) &\leq \binom{D + Nd^N + 1}{N} + \log\left(\binom{D + Nd^N + 1}{N}!\right)\\ &\leq D^N + D^N N\log(D) = O(D^N\log(D)).\end{aligned}$$

So we have

$$h(f(X)) - \deg(f)\frac{\deg(f(X))}{\deg(X)}h(X) \leq C_2(d,N,f)D + C_3(d,N)D^N\log(D).$$

For the lower bound,

$$\tau(D) = \binom{N+D}{N} \leq (D+N)^N = O(D^N).$$

The size of the codomain

$$\tau(\deg(f(X))) \leq \binom{d^{N-1}D + N}{N} \leq (d^{N-1}D + N)^N = O(D^N)$$

and constant

$$e(D) = \left(\binom{N+D}{N} + 1\right)(d^N - 1) + 1 = O(D^N)$$

give

$$\begin{aligned}&\deg(f)\frac{\deg(f(X))}{\deg(X)}h(P)\\ &\leq \frac{\deg(f(X))/\deg(X)}{d^{N-1}}\Bigg(\log(\tau(D')) + \log\left(\binom{\tau(D) - 1 + e(D) - d^N}{e(D) - d^N}\right)\\ &\quad + \Big(4\tau(D)(\tau(D)+1)(d^N)^{\tau(D)}\big(2d^{N-1}Dh(f) + \log(N) + \log(C) + \log(\tau(D'))\\ &\quad + (\tau(D)+7)\log(\tau(D)+1)d^N\big)\Big)\Bigg) + h(\phi(P)).\end{aligned}$$

Estimating in the generic case ($\deg(f(X)) = d^{N-1}D$),

$$d^N h(P) \leq O(\log(D^N) + \log((D^N)^{D^N}) + O(D^{3N}\log(D)(d^N)^{D^N}) + h(\phi(P)).$$

□

We can use the above explicit bound to get an explicit bound on the height of a preperiodic subvariety of degree D by taking an upper bound of the D that can occur in the cycle.

Example 4.8. We compute C for $N = 2$, $d = 2$, and $D = 1$. Further, we assume that the degrees of all the iterates are also 1. The upper bound on $h(f(X))$ gives a smaller difference with $dh(X)$ than the lower bound, so we compute using the lower bound. We compute

$$\tau(D) = 6, \quad e(D) = 22, \quad \text{and} \quad \log(C_1) \leq \binom{10}{2} + \log\left(\binom{10}{2}!\right) \leq 175.$$

This gives the bound

$$\begin{aligned}
&|h(f(X)) - dh(X)| \\
&\leq \frac{1}{2}\Bigg(\log(\tau(dD)) + \log\left(\binom{\tau(D) - 1 + e(D) - d}{e(D) - d}\right) \\
&\quad + \Big(4\tau(D)(\tau(D)+1)(d)^{\tau(D)}\big(2d^{N-1}Dh(f) + \log(N) + \log(C) \\
&\quad + \log(\tau(dD)) + (\tau(D)+7)\log(\tau(D)+1)d^N\big)\Big)\Bigg) \\
&= \frac{1}{2}\Bigg(\log(2) + \log\left(\binom{18-2}{12-2}\right) + \Big(24(7)(2)^6\big(4h(f) + \log(2) \\
&\quad + 175 + \log(2) + (13)\log(7)4\big)\Big)\Bigg) \\
&= \frac{1}{2}\Big(\log(2) + \log(8008) + 10752\big(4h(f) + 2\log(2) + 175 + 52\log(7)\big)\Big) \\
&\leq 21504h(f) + 1492241.
\end{aligned}$$

4.2. Canonical heights

Gubler proves the existence of a canonical height and local canonical height in the language of arithmetic intersection theory [13, 14]. We take the more direct approach with Chow forms in order to obtain a height difference bound and, thus, an upper bound on the height of a preperiodic subvariety.

We define the canonical height as follows.

Definition 4.9. Let $X \subset \mathbb{P}^N$ be a subvariety with codimension b. Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism. Define

$$\hat{h}(X) = \lim_{n\to\infty} \frac{\deg(X)}{\deg(f^n(X))} \frac{h(f^n(X))}{\deg(f^n)}.$$

Theorem 4.10. *The canonical height converges and satisfies the functional equation*

$$\hat{h}(f(X)) = \frac{\deg(f)\deg(f(X))}{\deg(X)}\hat{h}(X).$$

Proof. For convergence, we show that the sequence is Cauchy. Assume that $n > m \geq 0$. We will use the fact that

$$\deg(f^{n+1}(X)) \leq d^{N-b}\deg(f^n(X))$$

and the existence of a constant C such that

$$\left| h(f(X)) - \deg(f)\frac{\deg(f(X))}{\deg(X)}h(X)\right| \leq C\deg(X).$$

$$\begin{aligned}
&\left|\frac{\deg(X)}{\deg(f^n(X))}\frac{h(f^n(X))}{d^n} - \frac{\deg(X)}{\deg(f^m(X))}\frac{h(f^m(X))}{d^m}\right| \\
&= \left|\sum_{i=m+1}^{n} \frac{\deg(X)h(f^i(X))}{\deg(f^i(X))d^i} - \frac{\deg(X)h(f^{i-1}(X))}{\deg(f^{i-1}(X))d^{i-1}}\right| \\
&= \left|\sum_{i=m+1}^{n} \frac{\deg(X)}{d^i}\left(\frac{h(f^i(X))}{\deg(f^i(X))} - d\frac{h(f^{i-1}(X))}{\deg(f^{i-1}(X))}\right)\right| \\
&\leq \left|\sum_{i=m+1}^{n} \frac{\deg(X)}{\deg(f^{i-1}(X))d^i}\left(\frac{h(f^i(X))}{d^{N-b}} - dh(f^{i-1}(X))\right)\right| \\
&= \left|\sum_{i=m+1}^{n} \frac{\deg(X)}{\deg(f^{i-1}(X))d^{N-b+i}}\left(h(f^i(X)) - d^{N-b+1}h(f^{i-1}(X))\right)\right|
\end{aligned}$$

$$\leq \sum_{i=m+1}^{n} \frac{C \deg(X)^2}{\deg(f^{i-1}(X)) d^i d^{N-b}} \leq \sum_{i=m+1}^{n} \frac{C \deg(X)^2}{d^i}$$
$$\leq C \deg(X)^2 \sum_{i=m+1}^{\infty} \frac{1}{d^i} = \frac{C \deg(X)^2}{d^m(1-1/d)},$$

which goes to 0 as $m \to \infty$.

For the functional equation, we compute

$$\begin{aligned}
\hat{h}(f(X)) &= \lim_{n\to\infty} \frac{\deg(f(X))}{\deg(f^n(f(X)))} \frac{h(f^n(f(X)))}{\deg(f^n)} \\
&= \lim_{n\to\infty} \frac{\deg(f(X))}{\deg(f^n(f(X)))} \frac{h(f^n(f(X)))}{\deg(f^n)} \frac{\deg(f)\deg(X)}{\deg(f)\deg(X)} \\
&= \lim_{n\to\infty} \deg(f) \frac{\deg(f(X))}{\deg(X)} \frac{\deg(X)}{\deg(f^{n+1}(X))} \frac{h(f^{n+1}(X))}{\deg(f^{n+1})} \\
&= \deg(f) \frac{\deg(f(X))}{\deg(X)} \lim_{n\to\infty} \frac{\deg(X)}{\deg(f^{n+1}(X))} \frac{h(f^{n+1}(X))}{\deg(f^{n+1})} \\
&= \deg(f) \frac{\deg(f(X))}{\deg(X)} \hat{h}(X).
\end{aligned}$$

□

Remark. If we normalize the canonical height by the degree of X, we have the more visually appealing

$$\frac{\hat{h}(f(X))}{\deg(f(X))} = d \frac{\hat{h}(X)}{\deg(X)}.$$

An immediate corollary of the functional equation is the following.

Corollary 4.11. *Preperiodic subvarieties have canonical height* 0.

With the constant C from Equation 1 we can have a bound between the height and canonical height of a subvariety.

Theorem 4.12. *With the constant C from Equation 1, we have*

$$\left| \hat{h}(X) - h(X) \right| \leq \frac{CD}{(d-1)d^{N-b-1}}.$$

Proof. We have

$$h(f^n(X)) \leq d^n \frac{D_n}{D_0} h(X) + d^{n-1} C \frac{D_n D_0}{D_1} + d^{n-2} C \frac{D_n D_1}{D_2} + \cdots + dC \frac{D_n D_{n-2}}{D_{n-1}} + C D_n$$

so that

$$\begin{aligned}
\frac{Dh(f^n(X))}{d^n D_n} &\leq h(X) + \frac{C}{d}\frac{D_0^2}{D_1} + \frac{C}{d^2}\frac{D_0 D_1}{D_2} + \cdots + \frac{C}{d^{n-1}}\frac{D_0 D_{n-2}}{D_{n-1}} + \frac{C}{d^n} D_0 \\
&\leq h(X) + CD_0 \left(\frac{D_0}{dD_1} + \frac{D_1}{d^2 D_2} + \cdots + \frac{D_{n-2}}{d^{n-1} D_{n-1}} + \frac{1}{d^n} \right) \\
&\leq h(X) + CD_0 \left(\frac{1}{dd^{N-b}} + \frac{1}{d^2 d^{N-b}} + \cdots + \frac{1}{d^{n-1} d^{N-b}} + \frac{1}{d^n} \right) \\
&\leq h(X) + CD_0 \frac{1}{d^{N-b}} \left(\frac{1}{d} + \frac{1}{d^2} + \cdots + \frac{1}{d^{n-1}} + \frac{1}{d^n} \right) \\
&\leq h(X) + CD_0 \frac{1}{d^{N-b}} \frac{1}{1 - 1/d} \\
&\leq h(X) + \frac{CD}{(d-1)d^{N-b-1}}.
\end{aligned}$$

□

Corollary 4.13. *Given $f : \mathbb{P}^N \to \mathbb{P}^N$, a morphism of degree d defined over a number field K, there are only finitely many preperiodic rational subvarieties of degree at most D defined over K.*

Proof. A preperiodic subvariety has canonical height 0, so there is a height bound on preperiodic subvarieties of degree at most D. There are only finitely many rational subvarieties of bounded degree and height. □

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