Abstract integrable systems on hyperkähler manifolds arising from Slodowy slices

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We study holomorphic integrable systems on the hyperkähler manifold $G \times S_{\mathrm{reg}}$, where G is a complex semisimple Lie group and S_{reg} is the Slodowy slice determined by a regular $\mathfrak{sl}_2(\mathbb{C})$-triple. Our main result is that this manifold carries a canonical *abstract integrable system*, a foliation-theoretic notion recently introduced by Fernandes, Laurent-Gengoux, and Vanhaecke. We also construct traditional integrable systems on $G \times S_{\mathrm{reg}}$, some of which are completely integrable and fundamentally based on Mishchenko and Fomenko's argument shift approach.

1. Introduction

Generalizing the completely integrable systems of mathematical physics are ones described variously as *noncommutative*, *degenerately integrable*, *superintegrable*, or *non-Liouville*. Such integrable systems have Liouville fibres whose dimensions may be less than half of that of the total space. They arise in a variety of contexts [6, 12, 18, 21, 25, 27]. In this paper, we adopt the terminology *integrable system of fixed rank*. More precisely, we use the following definition of integrable system in the holomorphic category (cf. [11, Def. 2.1]):

Definition 1. Let X be an n-dimensional holomorphic symplectic manifold and denote by $\{\cdot, \cdot\}$ the induced Poisson bracket on its sheaf of holomorphic functions.

An *integrable system of fixed rank* r on X consists of holomorphic functions

$$f_1, f_2, \ldots, f_{n-r} : X \to \mathbb{C}$$

such that:

- $r \leq n/2$,

- $\{f_i, f_j\} = 0$ for all $i \in \{1, 2, \ldots, r\}$ and $j \in \{1, 2, \ldots, n - r\}$, and

- $f_1, f_2, \ldots, f_{n-r}$ are functionally independent, i.e. the holomorphic 1-forms $df_1, df_2, \ldots, df_{n-r}$ are linearly independent at each point in an open dense subset of X.

Note that we recover the usual notion of a completely integrable system by taking $r = n/2$.

Now, assume that $df_1, df_2, \ldots, df_{n-r}$ are linearly independent at each point of X. It follows that

$$
\begin{aligned}
F : X &\rightarrow \mathbb{C}^{n-r} \\
x &\mapsto (f_1(x), f_2(x), \ldots, f_{n-r}(x))
\end{aligned}
$$

is a holomorphic submersion, whose image is then necessarily open in $\mathbb{C}^{n-r}$. Each fibre is an r-dimensional complex submanifold of X. The connected components of all such submanifolds are the leaves of a holomorphic foliation $\mathcal{F}$ of X. Note that $f_1, f_2, \ldots, f_{n-r}$ are first integrals of $\mathcal{F}$, meaning that they are constant on the leaves of $\mathcal{F}$, while one can verify that the Hamiltonian vector fields of $f_1, f_2, \ldots, f_r$ span $T\mathcal{F} \subseteq TX$ (see [11, Prop. 2.5], for example). These observations motivate a useful definition, introduced by Fernandes, Laurent-Gengoux, and Vanhaecke [11] to study integrable systems by way of the foliations that they induce. The following is Definition 2.6 from [11], adapted to the holomorphic setting and with the term "non-commutative" omitted.

Definition 2. An *abstract integrable system of rank r* is a pair, $(X, \mathcal{F})$, consisting of a holomorphic symplectic manifold X with an r-dimensional holomorphic foliation $\mathcal{F}$, satisfying the following condition: each point $x \in X$ admits an open neighbourhood $U \subseteq X$, together with holomorphic first integrals defined on U whose Hamiltonian vector fields span $T\mathcal{F}$ on U.

The main goal of this work is to construct a hyperkähler variety and an abstract integrable system on it, for which we now give some context and motivation. Recall that a hyperkähler manifold carries a distinguished holomorphic symplectic form, so that one may study integrable systems on hyperkähler manifolds. Moreover, it is from gauge theory that there have emerged deep connections between completely integrable systems and hyperkähler geometry. A particular manifestation of this is the Hitchin system, originating in [15]. In its typical form, the Hitchin system is an algebraically completely integrable Hamiltonian system defined on the moduli space of

stable G-Higgs bundles over a fixed algebraic curve or Riemann surface, where G is a reductive complex Lie group. The moduli space is a noncompact hyperkähler manifold which fits into the picture of Strominger-Yau-Zaslow mirror symmetry through the invariant tori of the dynamical system [14]. This proper torus fibration is generally known as the Hitchin map.

Recognizing the rich interplay between completely integrable systems and hyperkähler geometry, as well as the recent emergence of abstract integrable systems, we are motivated to think about possible connections between abstract integrable systems and hyperkähler geometry. A natural first step is to construct non-trivial examples of abstract integrable systems carrying hyperkähler structures. It is to the construction of these examples that we devote most of our paper. We also include a few brief digressions on related subjects, such as connections between moment maps and abstract integrable systems, as well as the construction of traditional integrable systems.

While this helps to explain our motivations, we do believe that the abstract system constructed below is of independent interest as a canonical abstract integrable system associated to purely Lie-theoretic data.

1.1. Structure and statement of results

Our paper is organized as follows: Section 2 introduces the manifold

$$M := G \times S_{\text{reg}},$$

where G is a connected, simply-connected, complex semisimple linear algebraic group with Lie algebra $\mathfrak{g}$, and $S_{\text{reg}} \subseteq \mathfrak{g}$ is the Slodowy slice determined by a regular $\mathfrak{sl}_2(\mathbb{C})$-triple (defined formally in Section 2.2). Following Bielawski in [4], M has a canonical hyperkähler structure (see Theorem 4).

In Section 3, we consider the holomorphic map

$$\begin{aligned} \Phi : M &\to \mathfrak{g} \\ (g, x) &\mapsto -\text{Ad}_{g^{-1}}(x), \end{aligned}$$

where $\text{Ad} : G \to \text{GL}(\mathfrak{g})$ is the adjoint representation. We show that Φ is a holomorphic submersion with image the set of regular elements $\mathfrak{g}_{\text{reg}} \subseteq \mathfrak{g}$, and that each fibre $\Phi^{-1}(x) \subseteq M$ is an isotropic subvariety of dimension $\text{rank}(G)$ that is isomorphic to the G-stabilizer of x. These results are contained in Proposition 6, Corollary 7, and Proposition 10. Furthermore, Φ is a map of Poisson varieties for the holomorphic symplectic structure on M and

the Poisson structure on $\mathfrak{g}$ coming from the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ induced by the Killing form. This is Proposition 8. Next, we foliate M into the connected components of the fibres of Φ, and argue in Theorem 13 that this constitutes an abstract integrable system of rank equal to $\mathrm{rank}(G)$. Section 3 concludes with some consideration of Theorem 13 in a more general context. We give conditions under which a moment map will, analogously to Φ in Theorem 13, induce an abstract integrable system. These results are contained in Theorem 14.

Section 4 applies the results of Section 3 to construct traditional integrable systems on $G \times S_{\mathrm{reg}}$. The first such system appears in Section 4.1, and is an integrable system of rank equal to $\mathrm{rank}(G)$ (see Theorem 15). In Section 4.2, we use Mishchenko and Fomenko's argument shift approach [24] to construct a family of completely integrable systems on $G \times S_{\mathrm{reg}}$. This results in Theorem 17.

A few informal remarks are in order. First, it is worth comparing aspects of our work to the *imploded cross-section* construction introduced by Guillemin, Jeffrey, and Sjamaar in [13]. In this construction, one fixes a compact connected Lie group K, a maximal torus $T \subseteq K$, and a positive Weyl chamber $\mathfrak{t}_+^* \subseteq \mathfrak{t}^*$, where $\mathfrak{t} := \mathrm{Lie}(T)$. Every Hamiltonian K-manifold N then determines an *imploded cross-section* N_{impl}, a stratified symplectic space carrying a Hamiltonian action of T. It is known that N_{impl} is isomorphic to a symplectic quotient of $N \times (T^*K)_{\mathrm{impl}}$ by K, in recognition of which one calls $(T^*K)_{\mathrm{impl}}$ the *universal imploded cross-section*. It is also true that $(T^*K)_{\mathrm{impl}}$ contains $K \times \mathrm{int}(\mathfrak{t}_+^*)$ as an open dense symplectic stratum, where $\mathrm{int}(\mathfrak{t}_+^*)$ is the interior of the positive Weyl chamber. At the same time, $\mathrm{int}(\mathfrak{t}_+^*)$ parametrizes the set of regular coadjoint orbits of K, while in our case S_{reg} parametrizes the set of regular (co-)adjoint orbits of G (see Theorem 3). In this context, it is natural to compare the real symplectic manifold $K \times \mathrm{int}(\mathfrak{t}_+^*)$ to the holomorphic symplectic manifold $G \times S_{\mathrm{reg}}$. In particular, one might imagine that the open dense inclusion $K \times \mathrm{int}(\mathfrak{t}_+^*) \subseteq (T^*K)_{\mathrm{impl}}$ has the following counterpart: an inclusion of $G \times S_{\mathrm{reg}}$ as an open dense symplectic stratum in some appropriate universal imploded cross-section for complex group actions. While largely speculative, we believe that this point merits further study.

Our second set of remarks concerns analogies with the Hitchin system: unlike the Hitchin fibration, which is necessarily proper, the fibration given by Φ is one whose generic fibres are noncompact complex tori (see Section 3.2). Interestingly, this mirrors certain Hitchin-type examples in Section 3 of [31]. Moreover, our abstract integrable system is akin to the Hitchin-type systems studied by Beauville, Markman, Biswas-Ramanan, and Bottacin in

[3, 5, 7, 22], respectively, in that the dimension of the base is allowed to exceed that of the fibres. We also recognize that there is a well-known passage between Hitchin systems and Mishchenko-Fomenko flows: [1] makes a connection between Adler-Kostant-Symes flows and the flows of [24], while [10] explains the route from the AKS picture to Hitchin systems.

Acknowledgements

2. Background

2.1. Lie-theoretic preliminaries

Let G be a connected, simply-connected, complex semisimple linear algebraic group and denote its rank by $\mathrm{rk}(G)$. We shall use $\mathfrak{g}$ to denote the Lie algebra of G, on which one has a Killing form $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ and exponential map $\exp : \mathfrak{g} \to G$. One also has the adjoint and coadjoint representations of G, denoted

$$
\begin{aligned}
\mathrm{Ad} \ : \ G &\ \to\ \mathrm{GL}(\mathfrak{g}) \\
g &\ \mapsto\ \mathrm{Ad}_g
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Ad}^* \ : \ G &\ \to\ \mathrm{GL}(\mathfrak{g}^*) \\
g &\ \mapsto\ \mathrm{Ad}_g^*,
\end{aligned}
$$

respectively. Since the Killing form is non-degenerate and invariant under the first representation, the following is an isomorphism between the adjoint

and coadjoint representations:

$$\text{(1)} \qquad \mathfrak{g} \xrightarrow{\cong} \mathfrak{g}^*, \quad x \mapsto x^\vee := \langle x, \cdot \rangle.$$

The canonical Lie-Poisson structure on $\mathfrak{g}^*$ (see [8, Prop. 1.3.18]) thereby corresponds to a holomorphic Poisson structure on $\mathfrak{g}$, whose symplectic leaves turn out to be the adjoint orbits of G. We shall let $\mathcal{O}(x)$ denote the adjoint orbit containing $x \in \mathfrak{g}$, i.e.

$$\mathcal{O}(x) := \{\mathrm{Ad}_g(x) : g \in G\} \subseteq \mathfrak{g}.$$

Now let

$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\to \mathfrak{gl}(\mathfrak{g}) \\ x &\mapsto \mathrm{ad}_x \end{aligned}$$

denote the adjoint representation of $\mathfrak{g}$. Recall that ad is the derivative of Ad at the identity $e \in G$ and satisfies $\mathrm{ad}_x(y) = [x,y]$ for all $x,y \in \mathfrak{g}$. Furthermore, recall that $x \in \mathfrak{g}$ is called *semisimple* (resp. *nilpotent*) if the endomorphism $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ is semisimple (resp. nilpotent). Denote by $\mathfrak{g}_{ss}$ and $\mathcal{N}$ the sets of semisimple and nilpotent elements in $\mathfrak{g}$, respectively. The former is an open dense subvariety of $\mathfrak{g}$ while the latter is a closed subvariety called the *nilpotent cone*. Each is readily seen to be invariant under the adjoint representation of G, and one calls an adjoint orbit $\mathcal{O} \subseteq \mathfrak{g}$ *semisimple* (resp. *nilpotent*) if $\mathcal{O} \subseteq \mathfrak{g}_{ss}$ (resp. $\mathcal{O} \subseteq \mathcal{N}$).

We shall use $Z_G(x)$ to denote the G-stabilizer of $x \in \mathfrak{g}$, i.e.

$$Z_G(x) := \{g \in G : \mathrm{Ad}_g(x) = x\}.$$

Its Lie algebra is the $\mathfrak{g}$-stabilizer of x, namely

$$Z_\mathfrak{g}(x) := \{y \in \mathfrak{g} : [x,y] = 0\}.$$

Now recall that the dimension of $Z_\mathfrak{g}(x)$ (equivalently, the dimension of $Z_G(x)$) is at least $\mathrm{rk}(G)$ for all $x \in \mathfrak{g}$, and that x is called *regular* when $\dim(Z_\mathfrak{g}(x)) = \mathrm{rk}(G)$. Note that while some authors also require regular elements to be semisimple, we are not imposing this extra condition.

Let us consider the set of regular elements,

$$\mathfrak{g}_{\text{reg}} := \{x \in \mathfrak{g} : x \text{ is regular}\},$$

which is known to be a G-invariant open dense subset of $\mathfrak{g}$ (see [19, Lemma 6.51]). We shall call an adjoint orbit $\mathcal{O} \subseteq \mathfrak{g}$ *regular* if $\mathcal{O} \subseteq \mathfrak{g}_{\text{reg}}$. Note that while there exist infinitely many distinct regular semisimple orbits, there exists exactly one regular nilpotent orbit, to be denoted $\mathcal{O}_{\text{reg}}$ (see [8, Prop. 3.2.10], for example).

2.2. Regular $\mathfrak{sl}_2(\mathbb{C})$-triples and Slodowy slices

Given three vectors $\xi, h, \eta \in \mathfrak{g}$, recall that (ξ, h, η) is called an $\mathfrak{sl}_2(\mathbb{C})$-triple if $[\xi, \eta] = h$, $[h, \xi] = 2\xi$, and $[h, \eta] = -2\eta$. The elements ξ and η are then necessarily nilpotent and belong to the same nilpotent orbit. With this in mind, we shall call our $\mathfrak{sl}_2(\mathbb{C})$-triple (ξ, h, η) *regular* if $\xi, \eta \in \mathcal{O}_{\text{reg}}$. We will often restrict our attention to triples of this form.

Now fix an $\mathfrak{sl}_2(\mathbb{C})$-triple (ξ, h, η). One can associate a *Slodowy slice* to this triple, namely the affine-linear subspace $S(\xi, h, \eta) \subseteq \mathfrak{g}$ defined as follows:

$$S(\xi, h, \eta) := \xi + Z_{\mathfrak{g}}(\eta) := \{\xi + x : x \in Z_{\mathfrak{g}}(\eta)\}.$$

Named in recognition of Slodowy's work [29], this variety is well-studied and enjoys a number of remarkable properties. To formulate one of these properties, recall that two smooth subvarieties $Y, Z \subseteq \mathfrak{g}$ are called *transverse* if for all $x \in Y \cap Z$, we have $T_x Y + T_x Z = \mathfrak{g}$ as vector spaces. It turns out that if (ξ, h, η) is an $\mathfrak{sl}_2(\mathbb{C})$-triple, then $S(\xi, h, \eta)$ and $\mathcal{O}(\xi)$ are transverse and intersect only at ξ.

It is desirable to have a more precise understanding of orbit-slice intersections when the underlying $\mathfrak{sl}_2(\mathbb{C})$-triple is regular. The following are parts of Theorem 8 and Lemma 13 from Kostant's paper [20], rephrased to suit our purposes.

Theorem 3 (Kostant). *If (ξ, h, η) is a regular $\mathfrak{sl}_2(\mathbb{C})$-triple, then $S(\xi, h, \eta) \subseteq \mathfrak{g}_{\text{reg}}$ and each regular adjoint orbit intersects $S(\xi, h, \eta)$ in a single point. Moreover, $S(\xi, h, \eta)$ is transverse to each regular adjoint orbit.*

2.3. The hyperkähler varieties of interest

Recall that the cotangent bundle T^*G carries a distinguished holomorphic symplectic form, to be denoted ω_{T^*G}. It will be convenient to identify T^*G

with $G \times \mathfrak{g}$, using an isomorphism between the two to transport the holomorphic symplectic structure on the former to one on the latter. We define this isomorphism as follows:

$$(2) \qquad \Psi : G \times \mathfrak{g} \xrightarrow{\cong} T^*G$$
$$(g,x) \mapsto (g, x^\vee \circ (d_e R_g)^{-1}),$$

where $R_g : G \to G$ denotes right multiplication by a fixed $g \in G$ and $d_e R_g : \mathfrak{g} \to T_g G$ is the differential of R_g at $e \in G$. Note that (2) is simply the result of composing the right trivialization $G \times \mathfrak{g}^* \to T^*G$ with the isomorphism $G \times \mathfrak{g} \to G \times \mathfrak{g}^*$, $(g,x) \mapsto (g, x^\vee)$.

Let $\omega := \Psi^*(\omega_{T^*G})$ denote the induced holomorphic symplectic form on $G \times \mathfrak{g}$. Given $(g,x) \in G \times \mathfrak{g}$, one can verify that ω restricts to the following bilinear form on $T_{(g,x)}(G \times \mathfrak{g}) = T_g G \oplus \mathfrak{g}$:

$$(3) \quad \omega_{(g,x)}\big((d_e R_g(y_1), z_1), (d_e R_g(y_2), z_2)\big) = \langle y_1, z_2 \rangle - \langle y_2, z_1 \rangle - \langle x, [y_1, y_2] \rangle,$$

where $y_1, y_2, z_1, z_2 \in \mathfrak{g}$ (cf. [23, Sect. 5, Eqn. (14R)]).

Now let (ξ, h, η) be an $\mathfrak{sl}_2(\mathbb{C})$-triple with associated Slodowy slice $S(\xi, h, \eta) \subseteq \mathfrak{g}$. One has an inclusion of varieties $G \times S(\xi, h, \eta) \subseteq G \times \mathfrak{g}$, by virtue of which the former carries some interesting geometric structures. Indeed, the following is one of Bielawski's results (see [4]), phrased in terms of $G \times \mathfrak{g}$ instead of T^*G

Theorem 4 (Bielawski). *If (ξ, h, η) is an $\mathfrak{sl}_2(\mathbb{C})$-triple, then $G \times S(\xi, h, \eta)$ has a canonical hyperkähler structure whose underlying holomorphic symplectic form is obtained by restricting that of $G \times \mathfrak{g}$ to $G \times S(\xi, h, \eta)$.*

Now let G act on $G \times S(\xi, h, \eta)$ according to

$$(4) \qquad h \cdot (g,x) := (gh^{-1}, x), \quad g, h \in G, \ x \in S(\xi, h, \eta).$$

This action enjoys a few properties that we record here for future reference.

Proposition 5. *The action (4) is Hamiltonian with respect to the holomorphic symplectic form on $G \times S(\xi, h, \eta)$, and*

$$(5) \qquad \mu : G \times S(\xi, h, \eta) \to \mathfrak{g}^*$$
$$(g,x) \mapsto -\mathrm{Ad}^*_{g^{-1}}(x^\vee)$$

is a moment map.

Proof. Consider the following extension of (4) to an action on $G \times \mathfrak{g}$:

$$(6) \qquad h \cdot (g, x) := (gh^{-1}, x), \quad g, h \in G, \ x \in \mathfrak{g}.$$

We will show (6) to be a Hamiltonian action with respect to the symplectic form (3), and that

$$\begin{aligned} \tilde{\mu} \ : \ G \times \mathfrak{g} &\ \to\ \mathfrak{g}^* \\ (g, x) &\ \mapsto\ -\mathrm{Ad}^*_{g^{-1}}(x^\vee) \end{aligned}$$

is a moment map. Since $G \times S(\xi, h, \eta)$ is a G-invariant symplectic subvariety of $G \times \mathfrak{g}$, it will follow that (4) is a Hamiltonian action with moment map $\tilde{\mu}|_{G \times S(\xi, h, \eta)} = \mu$.

Now consider the following action of G on T^*G:

$$(7) \qquad h \cdot (g, \alpha) := (gh^{-1}, \alpha \circ (d_g R_{h^{-1}})^{-1}), \quad g, h \in G, \ \alpha \in T^*_g G.$$

This is the action on T^*G naturally induced by an action of G on itself, namely

$$(8) \qquad h \cdot g := gh^{-1}, \quad g, h \in G.$$

As such, (7) is necessarily a Hamiltonian action with the following moment map:

$$\Theta : T^*G \to \mathfrak{g}^*, \quad \Theta(g, \alpha)(y) = \alpha(X_y(g)), \quad g \in G, \ \alpha \in T^*_g G, \ y \in \mathfrak{g}$$

(see [26, Example 4.5.4]), where X_y is the fundamental vector field on G for the action (8) and the point $y \in \mathfrak{g}$. One can verify that $X_y(g) = -d_e L_g(y) \in T_g G$ for all $y \in \mathfrak{g}$ and $g \in G$.

Recall the isomorphism $\Psi : G \times \mathfrak{g} \to T^*G$ from (2). It is not difficult to verify that Ψ is G-equivariant with respect to the actions (6) and (7). Moreover, as $\omega = \Psi^*(\omega_{T^*G})$, it follows that Ψ is a G-equivariant symplectomorphism. Since (7) is a Hamiltonian action with moment map Θ, this implies that (6) is also a Hamiltonian action with moment map $\Theta \circ \Psi$. It then suffices to prove that $\tilde{\mu} = \Theta \circ \Psi$.

Given $g \in G$ and $x, y \in \mathfrak{g}$, we have

$$
\begin{aligned}
(\Theta \circ \Psi)(g, x)(y) &= \Theta(g, x^\vee \circ (d_e R_g)^{-1})(y) \\
&= (x^\vee \circ (d_e R_g)^{-1})(X_y(g)) \\
&= \langle x, (d_e R_g)^{-1}(X_y(g)) \rangle \\
&= \langle x, -(d_e R_g)^{-1}(d_e L_g(y)) \rangle \\
&= \langle x, -\operatorname{Ad}_g(y) \rangle \\
&= \langle -\operatorname{Ad}_{g^{-1}}(x), y \rangle \\
&= (-\operatorname{Ad}_{g^{-1}}(x))^\vee(y).
\end{aligned}
$$

It follows that $(\Theta \circ \Psi)(g, x) = (-\operatorname{Ad}_{g^{-1}}(x))^\vee$, and in turn the right-hand side becomes

$$
-\operatorname{Ad}^*_{g^{-1}}(x^\vee) = \tilde{\mu}(g, x)
$$

when one recalls that (1) intertwines the adjoint and coadjoint representations. $\qquad \square$

3. A canonical abstract integrable system

3.1. The map $G \times S_{\mathrm{reg}} \to \mathfrak{g}$ and its properties

For the duration of this article, (ξ, h, η) will be a fixed regular $\mathfrak{sl}_2(\mathbb{C})$-triple and $S_{\mathrm{reg}} := S(\xi, h, \eta)$ shall denote its associated Slodowy slice. One may then consider the holomorphic map

$$
\begin{aligned}
(9) \qquad\qquad \Phi : G \times S_{\mathrm{reg}} &\to \mathfrak{g} \\
(g, x) &\mapsto -\operatorname{Ad}_{g^{-1}}(x).
\end{aligned}
$$

A preliminary observation is that Φ is G-equivariant for the adjoint action on $\mathfrak{g}$ and the following G-action on $G \times S_{\mathrm{reg}}$:

$$
(10) \qquad\qquad h \cdot (g, x) := (gh^{-1}, x), \quad g, h \in G, \ x \in S_{\mathrm{reg}}.
$$

However, one can say considerably more about Φ.

Proposition 6. *The map Φ is a holomorphic submersion and its image is $\mathfrak{g}_{\mathrm{reg}}$.*

Proof. Theorem 3 gives the inclusion $S_{\mathrm{reg}} \subseteq \mathfrak{g}_{\mathrm{reg}}$, and the latter set is invariant under both the adjoint G-action and multiplication by -1. From this

last sentence, it follows that $-\operatorname{Ad}_{g^{-1}}(x) \in \mathfrak{g}_{\mathrm{reg}}$ for all $g \in G$ and $x \in S_{\mathrm{reg}}$, i.e. $\Phi(G \times S_{\mathrm{reg}}) \subseteq \mathfrak{g}_{\mathrm{reg}}$. For the opposite inclusion, suppose that $x \in \mathfrak{g}_{\mathrm{reg}}$. It follows that $-x$ belongs to a regular adjoint orbit, which by Theorem 3 must intersect S_{reg} at a point y. Note that $-x = \operatorname{Ad}_{g^{-1}}(y)$ for some $g \in G$, so that $\Phi(g, y) = x$. We conclude that $\mathfrak{g}_{\mathrm{reg}} \subseteq \Phi(G \times S_{\mathrm{reg}})$, completing our proof that $\mathfrak{g}_{\mathrm{reg}}$ is the image of Φ.

To show that Φ is submersive is to show that the differential of Φ at (g, x), $d_{(g,x)}\Phi$, is a surjective map of tangent spaces for all $(g, x) \in G \times S_{\mathrm{reg}}$. However, since Φ is G-equivariant in the sense discussed before this proposition, it will suffice prove that

$$d_{(e,x)}\Phi : T_{(e,x)}(G \times S_{\mathrm{reg}}) \to T_{-x}(\mathfrak{g}) = \mathfrak{g}$$

is surjective for all $x \in S_{\mathrm{reg}}$. To this end, note that $T_{(e,x)}(G \times S_{\mathrm{reg}})$ is canonically identified with $\mathfrak{g} \oplus Z_{\mathfrak{g}}(\eta)$. Also, given $(y, z) \in \mathfrak{g} \oplus Z_{\mathfrak{g}}(\eta)$, observe that $t \mapsto (\exp(ty), x + tz)$ is a curve in $G \times S_{\mathrm{reg}}$ having tangent vector (y, z) at $t = 0$. Using the previous two statements, we may present $d_{(e,x)}\Phi$ as a map

$$d_{(e,x)}\Phi : \mathfrak{g} \oplus Z_{\mathfrak{g}}(\eta) \to \mathfrak{g}$$

whose value at the tangent vector $(y, z) \in \mathfrak{g} \oplus Z_{\mathfrak{g}}(\eta)$ is calculated as follows:

$$\begin{aligned}
(d_{(e,x)}\Phi)(y, z) &= \frac{d}{dt}\bigg|_{t=0} \left(\Phi(\exp(ty), x + tz) \right) \\
&= \frac{d}{dt}\bigg|_{t=0} \left(-\operatorname{Ad}_{\exp(-ty)}(x + tz) \right) \\
&= \frac{d}{dt}\bigg|_{t=0} \left(-\exp(\operatorname{ad}_{-ty})(x + tz) \right) \\
&= [y, x] - z.
\end{aligned}$$

Noting that $T_x\mathcal{O}(x) = \{[y, x] : y \in \mathfrak{g}\}$, this calculation shows the image of $d_{(e,x)}\Phi$ to be precisely $T_x\mathcal{O}(x) + Z_{\mathfrak{g}}(\eta)$. Since $\mathcal{O}(x)$ and S_{reg} are transverse (by Theorem 3), this image is necessarily all of $\mathfrak{g}$. We conclude that $d_{(e,x)}\Phi$ is surjective for all $x \in S_{\mathrm{reg}}$, as required. $\qquad\square$

For future reference, we record the following immediate consequence of Proposition 6.

Corollary 7. *If $x \in \mathfrak{g}_{\mathrm{reg}}$, then $\Phi^{-1}(x)$ and its connected components are complex submanifolds of $G \times S_{\mathrm{reg}}$ having dimension $\mathrm{rk}(G)$.*

Proof. Proposition 6 implies that $\Phi^{-1}(x)$ and its connected components are complex submanifolds of dimension $\dim(G)+\dim(S_{\mathrm{reg}})-\dim(\mathfrak{g})=\dim(S_{\mathrm{reg}})$. Furthermore,

$$(11) \qquad \dim(S_{\mathrm{reg}}) = \dim(S(\xi,h,\eta)) = \dim(Z_{\mathfrak{g}}(\eta)) = \mathrm{rk}(G),$$

with the third equality in 11 following from the fact that η is regular. This completes the proof. $\qquad\square$

It turns out that Φ enjoys some additional structure. To describe it, recall from Section 2.1 that $\mathfrak{g}$ is canonically a holomorphic Poisson variety. At the same time, $G \times S_{\mathrm{reg}}$ is Poisson by virtue of inheriting a holomorphic symplectic form from $G \times \mathfrak{g}$ (see Theorem 4). These considerations give context for the following result.

Proposition 8. *The map Φ is a morphism of Poisson varieties.*

Proof. Since $\mathfrak{g}$ inherits its Poisson structure from $\mathfrak{g}^*$ and the isomorphism (1), it suffices to show that the composition of Φ with (1) is Poisson. However, this composite map coincides with the moment map μ from (5), and (equivariant) moment maps for Hamiltonian G-actions are necessarily Poisson (see [8, Lem. 1.4.2(ii)]). This completes the proof. $\qquad\square$

3.2. Structure of the fibres

Let us take a moment to examine the nonempty fibres of Φ, which by Proposition 6 are precisely those fibres of the form $\Phi^{-1}(x)$, $x \in \mathfrak{g}_{\mathrm{reg}}$. For each such element x, Theorem 3 implies that S_{reg} intersects $\mathcal{O}(-x)$ in a single point, $\tilde{x} \in \mathcal{O}(-x) \cap S_{\mathrm{reg}}$. Since $\tilde{x}$ belongs to the orbit of $-x$, there exists a (non-unique) $g \in G$ such that $\mathrm{Ad}_{g^{-1}}(\tilde{x}) = -x$. In what follows, we show $\Phi^{-1}(x)$ to be $R_g(Z_G(\tilde{x})) \times \{\tilde{x}\} \subseteq G \times S_{\mathrm{reg}}$.

Proposition 9. *If $x \in \mathfrak{g}_{\mathrm{reg}}$, then*

$$(12) \qquad\qquad \Phi^{-1}(x) = R_g(Z_G(\tilde{x})) \times \{\tilde{x}\},$$

where $\tilde{x}$ is the unique element in $\mathcal{O}(-x) \cap S_{\mathrm{reg}}$ and $g \in G$ is any element satisfying $\mathrm{Ad}_{g^{-1}}(\tilde{x}) = -x$.

Proof. To see the inclusion "$\supseteq$", one directly verifies $\Phi(hg, \tilde{x}) = x$ for all $h \in Z_G(\tilde{x})$. This is a straightforward exercise. As for the other inclusion,

suppose $(h, y) \in G \times S_{\mathrm{reg}}$ satisfies $\Phi(h, y) = x$, i.e. $- \mathrm{Ad}_{h^{-1}}(y) = x$. It follows that $y = \mathrm{Ad}_h(-x)$ belongs to $\mathcal{O}(-x)$, which together with the fact that $y \in S_{\mathrm{reg}}$ implies $y = \tilde{x}$. Furthermore,

$$\mathrm{Ad}_{h^{-1}}(\tilde{x}) = \mathrm{Ad}_{h^{-1}}(y) = -x = \mathrm{Ad}_{g^{-1}}(\tilde{x}),$$

which one can manipulate to show $\mathrm{Ad}_{hg^{-1}}(\tilde{x}) = \tilde{x}$. We conclude that $hg^{-1} \in Z_G(\tilde{x})$, so that $h \in R_g(Z_G(\tilde{x}))$. Hence $(h, y) \in R_g(Z_G(\tilde{x})) \times \{\tilde{x}\}$, as desired. $\square$

Proposition 10. *If $x \in \mathfrak{g}_{\mathrm{reg}}$, then $\Phi^{-1}(x)$ is an isotropic subvariety of $G \times S_{\mathrm{reg}}$.*

Proof. Let ω denote the holomorphic symplectic form on $G \times \mathfrak{g}$, as described in 2.3. Since the holomorphic symplectic form on $G \times S_{\mathrm{reg}}$ is obtained by restricting ω (see Theorem 4), proving the proposition amounts to showing that ω restricts to the zero-form on tangent spaces of $\Phi^{-1}(x)$. To identify these tangent spaces, let $\tilde{x} \in \mathcal{O}(-x) \cap S_{\mathrm{reg}}$ and $g \in G$ be as in the statement of Proposition 9. The proposition implies that each point in $\Phi^{-1}(x)$ has the form $(hg, \tilde{x})$ for $h \in Z_G(\tilde{x})$, and that the tangent space of $\Phi^{-1}(x)$ at $(hg, \tilde{x})$ is the following subspace of $T_{(hg, \tilde{x})}(G \times \mathfrak{g}) = T_{hg}G \oplus \mathfrak{g}$:

$$(13) \qquad T_{(hg, \tilde{x})}(\Phi^{-1}(x)) = T_{hg}(R_g(Z_G(\tilde{x}))) \oplus \{0\} \subseteq T_{hg}G \oplus \mathfrak{g}.$$

Now note that $T_{hg}(R_g(Z_G(\tilde{x})))$ is the right g-translate of the tangent space to $Z_G(\tilde{x})$ at h, i.e. $T_{hg}(R_g(Z_G(\tilde{x}))) = d_h R_g(T_h Z_G(\tilde{x}))$. At the same time, $T_h Z_G(\tilde{x})$ is the right h-translate of the tangent space to $Z_G(\tilde{x})$ at e, meaning $T_h Z_G(x) = d_e R_h(Z_{\mathfrak{g}}(\tilde{x}))$. It follows that

$$T_{hg}(R_g(Z_G(\tilde{x}))) = d_h R_g(T_h Z_G(\tilde{x})) = d_h R_g(d_e R_h(Z_{\mathfrak{g}}(\tilde{x}))) = d_e R_{hg}(Z_{\mathfrak{g}}(\tilde{x})).$$

In particular, one may rewrite (13) as the statement

$$T_{(hg, \tilde{x})}(\Phi^{-1}(x)) = d_e R_{hg}(Z_{\mathfrak{g}}(\tilde{x})) \oplus \{0\} \subseteq T_{hg}G \oplus \mathfrak{g}.$$

We are therefore reduced to verifying that

$$\omega_{(hg, \tilde{x})}\big((d_e R_{hg}(y_1), 0), (d_e R_{hg}(y_2), 0)\big) = 0$$

for all $y_1, y_2 \in Z_{\mathfrak{g}}(\tilde{x})$. To this end, (3) implies that

$$\omega_{(hg, \tilde{x})}\big((d_e R_{hg}(y_1), 0), (d_e R_{hg}(y_2), 0)\big) = -\langle \tilde{x}, [y_1, y_2] \rangle.$$

Using the Killing form's ad-invariance property, the right-hand-side becomes $-\langle[\tilde{x}, y_1], y_2\rangle$. This is necessarily zero, as $y_1 \in Z_{\mathfrak{g}}(\tilde{x})$. $\qquad\square$

We conclude this section by characterizing various fibres of Φ up to variety isomorphism. While not strictly essential to proving our main results, these characterizations are in keeping with the long-standing interest in understanding generic and non-generic fibres of integrable systems. We begin with the following proposition.

Proposition 11. *If $x \in \mathfrak{g}_{\mathrm{reg}}$, then $\Phi^{-1}(x)$ and $Z_G(x)$ are isomorphic as varieties.*

Proof. Let $\tilde{x}$ be as introduced in Proposition 9, so that (12) implies $\Phi^{-1}(x)$ and $Z_G(\tilde{x})$ are isomorphic as varieties. It then remains to prove that $Z_G(\tilde{x})$ and $Z_G(x)$ are isomorphic. Now, since $\tilde{x} \in \mathcal{O}(-x)$, one sees that $Z_G(\tilde{x})$ and $Z_G(-x)$ are conjugate in G. The latter stabilizer coincides with $Z_G(x)$, so that $Z_G(\tilde{x})$ and $Z_G(x)$ are conjugate in G. In particular, $Z_G(\tilde{x})$ and $Z_G(x)$ are isomorphic as varieties (in fact, as algebraic groups). $\qquad\square$

Now note that $Z_G(x)$ is a maximal torus of G whenever $x \in \mathfrak{g}_{\mathrm{reg}} \cap \mathfrak{g}_{\mathrm{ss}}$ (cf. [9, Lem. 2.1.9, Thm. 2.3.3]). Proposition 11 then implies that fibres of Φ over $\mathfrak{g}_{\mathrm{reg}} \cap \mathfrak{g}_{\mathrm{ss}}$ are isomorphic to maximal tori of G, or equivalently to $(\mathbb{C}^*)^{\mathrm{rk}(G)}$. Since $\mathfrak{g}_{\mathrm{reg}} \cap \mathfrak{g}_{\mathrm{ss}}$ is open and dense in $\mathfrak{g}_{\mathrm{reg}}$, it follows that generic fibres of Φ are isomorphic to $(\mathbb{C}^*)^{\mathrm{rk}(G)}$. This is not true of all fibres, however. To see this, suppose now that $x \in \mathcal{O}_{\mathrm{reg}}$. It is known that $Z_G(x)$ decomposes as an internal direct product $Z(G) \times U_x$, where $Z(G)$ is the centre of G and U_x is a connected closed unipotent subgroup of $Z_G(x)$ (see [30, Thm. 5.9(b)]). The centre is finite by virtue of our having taken G to be semisimple, so that $\dim(U_x) = \dim(Z_G(x)) = \mathrm{rk}(G)$. Also, as a connected unipotent group, U_x is necessarily isomorphic to its Lie algebra (see [16, Chapt. VIII, Thm. 1.1]). In particular, $U_x \cong \mathbb{C}^{\mathrm{rk}(G)}$ as varieties and it follows that $\Phi^{-1}(x) \cong Z_G(x) \cong Z(G) \times \mathbb{C}^{\mathrm{rk}(G)}$ has $|Z(G)|$ connected components, each isomorphic to $\mathbb{C}^{\mathrm{rk}(G)}$.

3.3. The abstract integrable system

While [11] discusses abstract integrable systems in considerable generality, we shall focus on systems arising in a specific way. To this end, we will need to review a few definitions involving a holomorphic symplectic manifold X and a holomorphic foliation $\mathcal{F}$ of X. Firstly, $\mathcal{F}$ is called an *isotropic foliation* if its leaves are isotropic submanifolds of X. Secondly, one calls $\mathcal{F}$ *Poisson*

complete if the Poisson bracket of two locally defined first integrals of $\mathcal{F}$ is always itself a first integral. We may now state a holomorphic counterpart of Proposition 2.18(2) from [11].

Proposition 12. *Let X be a holomorphic symplectic manifold with a holomorphic foliation $\mathcal{F}$. If $\mathcal{F}$ is isotropic and Poisson complete, then $(X, \mathcal{F})$ is an abstract integrable system.*

Remark. A true holomorphic counterpart of [11, Prop. 2.18(2)], as stated, would be slightly more general than what appears above. It would relax the requirement that X be holomorphic symplectic, instead taking X to be a holomorphic Poisson manifold having a *regular* Poisson structure. For further details, we refer the reader to [11].

Proposition 12 will be our main technical tool for realizing an abstract integrable system on $G \times S_{\text{reg}}$, which we now discuss. Indeed, recall from Proposition 6 that $\Phi : G \times S_{\text{reg}} \to \mathfrak{g}$ is a holomorphic submersion. It follows that the connected components of Φ's fibres are the leaves of a holomorphic foliation $\mathcal{F}_{\text{reg}}$ of $G \times S_{\text{reg}}$.

Theorem 13. *With $\mathcal{F}_{\text{reg}}$ as defined above, $(G \times S_{\text{reg}}, \mathcal{F}_{\text{reg}})$ is an abstract integrable system of rank equal to $\text{rk}(G)$.*

Proof. Corollary 7 implies that $\mathcal{F}_{\text{reg}}$ is a $\text{rk}(G)$-dimensional foliation. It then just remains to prove that $(G \times S_{\text{reg}}, \mathcal{F}_{\text{reg}})$ is an abstract integrable system, which we will accomplish by showing the hypotheses of Proposition 12 to be satisfied. To begin, Proposition 10 implies that $\mathcal{F}_{\text{reg}}$ is an isotropic foliation. Also, as Φ is a Poisson submersion (see Propositions 6 and 8), Example 2.14 from [11] explains that $\mathcal{F}_{\text{reg}}$ is necessarily Poisson complete. This concludes the proof.   □

3.4. Moment maps and abstract integrable systems

We now discuss a generalization of Theorem 13. To this end, recall that $\Phi : G \times S_{\text{reg}} \to \mathfrak{g}$ induces the abstract integrable system $(G \times S_{\text{reg}}, \mathcal{F}_{\text{reg}})$ as follows: the leaves of $\mathcal{F}_{\text{reg}}$ are the connected components of Φ's fibres. We also know Φ to be a ($\mathfrak{g}$-valued) moment map (by Proposition 5), and it is natural to imagine that there are some general conditions under which a moment map will, analogously to Φ, induce an abstract integrable system. This is indeed the case, as we shall establish. We will work in the holomorphic category for the sake of consistency with the rest of the paper, and the reader

should interpret all relevant notions accordingly (ex. manifolds as complex manifolds, maps as holomorphic maps, etc.). Nevertheless, many parts of our discussion will also hold in the smooth category.

Let all notation be as established in Section 2, and let X be a holomorphic symplectic manifold. Suppose that X carries a Hamiltonian action of G. Using the isomorphism (1) to identify $\mathfrak{g}^*$ with $\mathfrak{g}$, we will present the moment map as $\mu : X \to \mathfrak{g}$. Also, given $x \in X$, let $Z_G(x) \subseteq G$ and $Z_{\mathfrak{g}}(x) \subseteq \mathfrak{g}$ denote the G-stabilizer of x and its Lie algebra, respectively. We shall assume that the G-action on X is *locally free*, meaning that $Z_{\mathfrak{g}}(x) = \{0\}$ for all $x \in X$. This is equivalent to μ being a submersion (see [2, Prop. III.2.3]), and we may define $\mathcal{F}_{\mu}$ to be the holomorphic foliation of X whose leaves are the connected components of μ's fibres. With this in mind, our generalization of Theorem 13 will take the following form: finding conditions on X and μ under which the proof of Theorem 13 will show $(X, \mathcal{F}_{\mu})$ to be an abstract integrable system after we replace $G \times S_{\mathrm{reg}}$, Φ, and $\mathcal{F}_{\mathrm{reg}}$ with X, μ, and $\mathcal{F}_{\mu}$, respectively. Referring to the proof of Theorem 13, one readily sees that there is only one possible issue — whether $\mathcal{F}_{\mu}$ is an isotropic foliation, or equivalently, all fibres of μ are isotropic in X.

Theorem 14. *Let X be a holomorphic symplectic manifold on which G acts locally freely and in a Hamiltonian fashion with moment map $\mu : X \to \mathfrak{g}$. Then, $\mathcal{F}_{\mu}$ is an isotropic foliation of X if and only if $\mu(X) \subseteq \mathfrak{g}_{\mathrm{reg}}$ and $\dim(X) = \dim(G) + \mathrm{rk}(G)$. In this case, $(X, \mathcal{F}_{\mu})$ is an abstract integrable system of rank equal to $\mathrm{rk}(G)$.*

Proof. Let ω denote the holomorphic symplectic form on X and $\omega|_{\mu^{-1}(\mu(x))}$ its pullback to the level set $\mu^{-1}(\mu(x))$, $x \in X$. It follows that $\mathcal{F}_{\mu}$ is an isotropic foliation if and only if

(14) $$\omega|_{\mu^{-1}(\mu(x))} = 0 \text{ for all } x \in X.$$

Now let β_x denote the bilinear form on $T_x(\mu^{-1}(\mu(x)))$ obtained by evaluating $\omega|_{\mu^{-1}(\mu(x))}$ at x, noting that (14) holds if and only if

(15) $$\beta_x = 0 \text{ for all } x \in X.$$

The kernel of β_x is the tangent space to the $Z_G(\mu(x))$-orbit of x, i.e. $T_x(Z_G(\mu(x)) \cdot x)$ (see [2, Lemma III.2.11]), so that (15) holds if and only

if

$$T_x(Z_G(\mu(x)) \cdot x) = T_x(\mu^{-1}(\mu(x))) \text{ for all } x \in X. \tag{16}$$

As $T_x(Z_G(\mu(x)) \cdot x)$ is a subspace of $T_x(\mu^{-1}(\mu(x)))$, (16) is true if and only if

$$\dim(T_x(Z_G(\mu(x)) \cdot x)) = \dim(T_x(\mu^{-1}(\mu(x)))) \text{ for all } x \in X. \tag{17}$$

In the interest of modifying (17), we make two observations. Firstly, μ being a submersion implies

$$\dim(T_x(\mu^{-1}(\mu(x)))) = \dim(X) - \dim(\mathfrak{g}) = \dim(X) - \dim(G).$$

Secondly, since the G-action is locally free, we must have

$$\dim(T_x(Z_G(\mu(x)) \cdot x)) = \dim(Z_G(\mu(x))).$$

It follows that (17) holds if and only if

$$\dim(Z_G(\mu(x))) = \dim(X) - \dim(G) \text{ for all } x \in X. \tag{18}$$

By virtue of the discussion above, we are reduced to showing that (18) holds if and only if $\mu(X) \subseteq \mathfrak{g}_{\mathrm{reg}}$ and $\dim(X) = \dim(G) + \mathrm{rk}(G)$. To this end, assume that (18) is satisfied. Since μ is a submersion, its image $\mu(X)$ is necessarily open in $\mathfrak{g}$. The set of regular elements is dense in $\mathfrak{g}$ (as discussed in Section 2.1) and must therefore intersect $\mu(X)$, i.e. $\mu(y) \in \mathfrak{g}_{\mathrm{reg}}$ for some $y \in X$. Note that $\dim(Z_G(\mu(y))) = \mathrm{rk}(G)$, which together with (18) gives $\dim(X) = \dim(G) + \mathrm{rk}(G)$. Moreover, (18) now reads as

$$\dim(Z_G(\mu(x))) = \mathrm{rk}(G) \text{ for all } x \in X.$$

This is the statement that $\mu(x) \in \mathfrak{g}_{\mathrm{reg}}$ for all $x \in X$, or equivalently $\mu(X) \subseteq \mathfrak{g}_{\mathrm{reg}}$.

Conversely, assume that $\mu(X) \subseteq \mathfrak{g}_{\mathrm{reg}}$ and $\dim(X) = \dim(G) + \mathrm{rk}(G)$. It is then immediate that both sides of (18) coincide with $\mathrm{rk}(G)$ for all $x \in X$, so that (18) holds. This completes the proof. $\square$

4. Some integrable systems

While the abstract integrable system $(G \times S_{\mathrm{reg}}, \mathcal{F}_{\mathrm{reg}})$ has the virtue of being completely canonical, it lacks the explicit Hamiltonian functions of a

traditional integrable system. Nevertheless, it is possible to construct integrable systems on $G \times S_{\mathrm{reg}}$. We shall illustrate this in two ways, devoting Section 4.1 to the first and Section 4.2 to the second.

4.1. An integrable system of rank equal to rk(G)

Consider the algebra $\mathbb{C}[\mathfrak{g}] := \mathrm{Sym}(\mathfrak{g}^*)$ of polynomial functions on the variety $\mathfrak{g}$. The Poisson structure on $\mathfrak{g}$ (discussed in Section 2.1) gives $\mathbb{C}[\mathfrak{g}]$ the structure of a Poisson algebra. Also, the adjoint action induces a representation of G on $\mathbb{C}[\mathfrak{g}]$, and one can form the subalgebra $\mathbb{C}[\mathfrak{g}]^G \subseteq \mathbb{C}[\mathfrak{g}]$ of G-invariant polynomials. Each of these invariant polynomials Poisson-commutes with every polynomial on $\mathfrak{g}$, i.e.

(19) $\{f, h\} = 0$ for all $f \in \mathbb{C}[\mathfrak{g}]^G, \ h \in \mathbb{C}[\mathfrak{g}]$.

Also, it is a celebrated fact that $\mathbb{C}[\mathfrak{g}]^G$ is generated by rk(G) algebraically independent homogeneous generators. Let $f_1, f_2, \ldots, f_{\mathrm{rk}(G)} \in \mathbb{C}[\mathfrak{g}]^G$ be a choice of such generators, fixed for the rest of this paper, and consider the map

$$
\begin{aligned}
F \ : \ \mathfrak{g} \ &\to \ \mathbb{C}^{\mathrm{rk}(G)} \\
x \ &\mapsto \ (f_1(x), f_2(x), \ldots, f_{\mathrm{rk}(G)}(x)).
\end{aligned}
$$

It is known that $\mathfrak{g}_{\mathrm{reg}}$ is the locus on which $df_1, df_2, \ldots, df_{\mathrm{rk}(G)}$ are linearly independent, or equivalently

(20) $\mathfrak{g}_{\mathrm{reg}} = \{x \in \mathfrak{g} : \mathrm{rank}(d_x F) = \mathrm{rk}(G)\}$

(see [20, Thm. 9]).

Now choose a basis $\{\theta_1, \theta_2, \ldots, \theta_{\dim(G)}\}$ of $\mathfrak{g}^*$, viewed as a system of global holomorphic coordinates on $\mathfrak{g}$. For each $x \in \mathfrak{g}$, let $[d_x F]$ denote the rk(G) $\times$ dim(G) Jacobian matrix representative of $d_x F$, i.e.

(21) $[d_x F]_{ij} := \dfrac{\partial f_i}{\partial \theta_j}(x), \quad i = 1, \ldots, \mathrm{rk}(G), \ j = 1, \ldots, \dim(G).$

Choosing a point $y \in \mathfrak{g}_{\mathrm{reg}}$, we may use (20) and conclude that $[d_y F]$ has rank equal to rk(G). It follows that

(22) $\det(M(y)) \neq 0$

some rk(G) $\times$ rk(G) submatrix $M(y)$ of $[d_y F]$. Reordering our basis vectors if necessary, we may assume that this submatrix consists of the first rk(G)

columns. Now consider the holomorphic functions $\Phi_1, \Phi_2, \ldots, \Phi_{\dim(G)} : G \times S_{\text{reg}} \to \mathbb{C}$ defined as follows:

$$(23) \qquad \Phi_i = \begin{cases} f_i \circ \Phi & \text{if } i = 1, \ldots, \text{rk}(G) \\ \theta_i \circ \Phi & \text{if } i = \text{rk}(G) + 1, \ldots, \dim(G). \end{cases}$$

Theorem 15. *The functions Φ_i form an integrable system on $G \times S_{\text{reg}}$, and the rank of this system is $\text{rk}(G)$.*

Proof. Using (19), one sees that

$$\{f_i, h\} = 0 \text{ for all } i \in \{1, \ldots, \text{rk}(G)\} \text{ and } h \in \mathbb{C}[\mathfrak{g}].$$

Since Φ is a Poisson morphism (by Proposition 8), it follows that

$$\{f_i \circ \Phi, h \circ \Phi\} = 0 \text{ for all } i \in \{1, \ldots, \text{rk}(G)\} \text{ and } h \in \mathbb{C}[\mathfrak{g}].$$

Replacing $f_i \circ \Phi$ with Φ_i and choosing h appropriately, we obtain

$$\{\Phi_i, \Phi_j\} = 0 \text{ for all } i \in \{1, \ldots, \text{rk}(G)\}, \ j \in \{1, \ldots, \dim(G)\}.$$

It remains only to prove that $d\Phi_1, d\Phi_2, \ldots, d\Phi_{\dim(G)}$ are linearly independent on an open dense subset of $G \times S_{\text{reg}}$. To this end, consider the holomorphic map

$$\begin{aligned} H : \mathfrak{g} &\to \mathbb{C}^{\dim(G)} \\ x &\mapsto (f_1(x), \ldots, f_{\text{rk}(G)}(x), \theta_{\text{rk}(G)+1}(x), \ldots, \theta_{\dim(G)}(x)), \end{aligned}$$

and note that

$$(H \circ \Phi)(g, x) = (\Phi_1(g, x), \Phi_2(g, x), \ldots, \Phi_{\dim(G)}(g, x)), \quad (g, x) \in G \times S_{\text{reg}}.$$

It follows that the linear independence of $d\Phi_1, d\Phi_2, \ldots, d\Phi_{\dim(G)}$ at a point is equivalent to $d(H \circ \Phi)$ having full rank at the same point. Also, as Φ is a submersion (by Proposition 6), $d(H \circ \Phi)$ has full rank at $(g, x) \in G \times S_{\text{reg}}$ if and only if dH has full rank at $\Phi(g, x)$. By virtue of these last two sentences, it will suffice to prove that the open set

$$(24) \qquad U := \Phi^{-1}\big(\{x \in \mathfrak{g} : \text{rank}(d_x H) = \dim(G)\}\big) \subseteq G \times S_{\text{reg}}$$

is dense. Accordingly, recall the Jacobian matrix construction (21). One analogously has a Jacobian matrix representative $[d_x H]$ for each linear map

$d_x H$, $x \in \mathfrak{g}$. It is not difficult to see that $[d_x H]$ is block upper-triangular with two diagonal blocks, one consisting of the first $\mathrm{rk}(G)$ columns of $[d_x F]$ and the other an identity matrix. The former block, to be denoted $M(x)$, must therefore have determinant equal to that of $[d_x H]$. We shall let $\rho(x)$ denote this common determinant, i.e.

$$\rho(x) := \det(M(x)) = \det([d_x H]), \quad x \in \mathfrak{g}.$$

Now note that $\mathrm{rank}(d_x H) = \dim(G)$ if and only if $\rho(x) \neq 0$, so that (24) becomes

(25) $$U = \Phi^{-1}(\mathfrak{g} \setminus \rho^{-1}(0)) = (G \times S_{\mathrm{reg}}) \setminus (\rho \circ \Phi)^{-1}(0).$$

Moreover, (22) implies that $\rho(y) \neq 0$ for some $y \in \mathfrak{g}_{\mathrm{reg}}$. Since $\mathfrak{g}_{\mathrm{reg}}$ is the image of Φ (by Proposition 6), we can write $y = \Phi(g, x)$ for some $(g, x) \in G \times S_{\mathrm{reg}}$. Note that the condition $\rho(y) \neq 0$ then becomes $(\rho \circ \Phi)(g, x) \neq 0$, meaning that $\rho \circ \Phi$ is not identically zero. As a holomorphic function with this property, the complement of its vanishing locus is necessarily dense in $G \times S_{\mathrm{reg}}$. This complement is precisely U by (25), completing the proof. $\square$

Remark. Recall the passage from integrable systems to abstract integrable systems, as explained in the text between Definitions 1 and 2. Conversely, one might begin with an abstract integrable system and hope to find an integrable system from which it arises. Theorem 15 is a successful instance of this, in which one starts with the abstract integrable system from Theorem 13.

4.2. A family of completely integrable systems

While Theorem 15 provides an integrable system, the system itself is not completely integrable. Indeed, the dimension of $G \times S_{\mathrm{reg}}$ (equal to $\dim(G) + \mathrm{rk}(G)$) is more than twice the rank of this integrable system (equal to $\mathrm{rk}(G)$, by Theorem 15). In what follows, however, we will show that $G \times S_{\mathrm{reg}}$ carries a family of completely integrable systems parametrized by the regular semisimple elements $\mathfrak{g}_{\mathrm{reg}} \cap \mathfrak{g}_{\mathrm{ss}}$. Our arguments will make extensive use of results on maximal Poisson-commutative subalgebras of polynomial algebras, developed by Mishchenko and Fomenko in [24] and summarized by Rybnikov in [28]. In more detail, we may associate to each $\beta \in \mathfrak{g}_{\mathrm{reg}} \cap \mathfrak{g}_{\mathrm{ss}}$ the following

family of polynomials:

$$(26) \qquad f_{ij}^{\beta} := (\partial_\beta)^j (f_i) \in \mathbb{C}[\mathfrak{g}], \quad i = 1, \dots, \mathrm{rk}(G), \quad j = 0, \dots, \deg(f_i),$$

where ∂_β is the operator for taking a directional derivative in the direction β. The following is one of Mishchenko and Fomenko's results, as presented in Section 2 of [28].

Theorem 16 (Mishchenko-Fomenko). *If $\beta \in \mathfrak{g}_{\mathrm{reg}} \cap \mathfrak{g}_{\mathrm{ss}}$, then (26) is a list of $\frac{1}{2}(\dim(G) + \mathrm{rk}(G))$ algebraically independent polynomials on $\mathfrak{g}$, and these polynomials generate a maximal Poisson-commutative subalgebra of $\mathbb{C}[\mathfrak{g}]$.*

Now fix $\beta \in \mathfrak{g}_{\mathrm{reg}} \cap \mathfrak{g}_{\mathrm{ss}}$ and consider the holomorphic functions on $G \times S_{\mathrm{reg}}$ obtained by pulling back those in (26) along Φ, i.e.

$$(27) \qquad \Phi_{ij}^{\beta} := f_{ij}^{\beta} \circ \Phi, \quad i = 1, \dots, \mathrm{rk}(G), \quad j = 0, \dots, \deg(f_i).$$

Theorem 17. *If $\beta \in \mathfrak{g}_{\mathrm{reg}} \cap \mathfrak{g}_{\mathrm{ss}}$, then the functions in (27) form a completely integrable system on $G \times S_{\mathrm{reg}}$.*

Proof. We begin by noting that (27) is a list of $\frac{1}{2}(\dim(G) + \mathrm{rk}(G))$ functions (see Theorem 16). This number is precisely half the dimension of $G \times S_{\mathrm{reg}}$, so we need only prove that the Φ_{ij}^{β} Poisson commute in pairs and have linearly independent differentials on an open dense subset of $G \times S_{\mathrm{reg}}$. To establish the first of these properties, recall from Proposition 8 that Φ is a Poisson morphism. In particular, if two functions on $\mathfrak{g}$ Poisson commute, their respective pullbacks under Φ must also Poisson commute. Since the f_{ij}^{β} Poisson commute in pairs (by Theorem 16), it follows that the same must be true of the pullbacks $\Phi^*(f_{ij}^{\beta}) = \Phi_{ij}^{\beta}$.

To see that the Φ_{ij}^{β} have linearly independent differentials on an open dense subset of $G \times S_{\mathrm{reg}}$, set $p := \frac{1}{2}(\dim(G) + \mathrm{rk}(G))$, re-index the f_{ij}^{β} as $f_1^{\beta}, f_2^{\beta}, \dots, f_p^{\beta}$, and consider the holomorphic map

$$\begin{aligned} H \;:\; \mathfrak{g} &\;\to\; \mathbb{C}^p \\ x &\;\mapsto\; (f_1^{\beta}(x), f_2^{\beta}(x), \dots, f_p^{\beta}(x)). \end{aligned}$$

By arguments analogous to those appearing in the proof of Theorem 15, we need only show that the open subset

$$U := \Phi^{-1}\big(\{x \in \mathfrak{g} : \mathrm{rank}(d_x H) = p\}\big) \subseteq G \times S_{\mathrm{reg}}$$

is dense. To this end, recall the global holomorphic coordinates on $\mathfrak{g}$ fixed in Section 4.1, and let $[d_x H]$ denote the resulting Jacobian matrix representative of $d_x H$. Using the fact that the f_{ij}^{β} are algebraically independent (see Theorem 16), it is straightforward to find a $p \times p$ submatrix $M(x)$ of $[d_x H]$ whose determinant is not identically zero as a function of x (see [17, Section 9], for instance). In other words, the function

$$\rho : \mathfrak{g} \rightarrow \mathbb{C}$$
$$x \mapsto \det(M(x))$$

is not identically zero. Now note that $\mathrm{rank}(d_x H) = p$ whenever $\rho(x) \neq 0$, or equivalently

$$\mathfrak{g} \setminus \rho^{-1}(0) \subseteq \{x \in \mathfrak{g} : \mathrm{rank}(d_x H) = p\}.$$

Taking preimages under Φ, we obtain

$$(G \times S_{\mathrm{reg}}) \setminus (\rho \circ \Phi)^{-1}(0) \subseteq U.$$

We are therefore reduced to showing that $(G \times S_{\mathrm{reg}}) \setminus (\rho \circ \Phi)^{-1}(0)$ is dense in $G \times S_{\mathrm{reg}}$, for which our argument is similar to one given in the proof of Theorem 15. Specifically, as ρ is not identically zero, the complement of its vanishing locus is nonempty and must intersect the dense subset $\mathfrak{g}_{\mathrm{reg}} \subseteq \mathfrak{g}$. Since $\mathfrak{g}_{\mathrm{reg}}$ is the image of Φ (by Proposition 6), it follows that $\rho \circ \Phi$ is not identically zero. As argued in the proof of Theorem 15, this shows that $(G \times S_{\mathrm{reg}}) \setminus (\rho \circ \Phi)^{-1}(0)$ is dense in $G \times S_{\mathrm{reg}}$. $\square$

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