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An abstract L^2 Fourier restriction theorem

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An L^2 Fourier restriction argument of Bak and Seeger is abstracted to the setting of locally compact abelian groups. This is used to prove new restriction estimates for varieties lying in modules over local fields or rings of integers $\mathbb{Z}/N\mathbb{Z}$.

1. Introduction

Let $n \geq 2$ and $\Sigma \subseteq \mathbb{R}^n$ be a hypersurface and μ a smooth, compactly supported density on Σ. Suppose that the Gaussian curvature of Σ does not vanish on the support of μ. The classical Stein–Tomas Fourier restriction theorem [43, 46] then asserts that the *a priori* estimate[1]

$$\|\hat{f}|_\Sigma\|_{L^2(\Sigma,\mu)} \lesssim_\mu \|f\|_{L^r(\mathbb{R}^n)}$$

is valid for all $1 \leq r \leq 2(n+1)/(n+3)$. A large number of variants and generalisations of this important inequality have appeared in the literature. For instance, one may relax the curvature condition on the hypersurface and prove estimates for a restricted range of r, or investigate measures supported on surfaces of larger co-dimension [6, 9, 14, 22, 23, 35]. The underlying surface can be removed entirely by working with abstract measures satisfying certain dimensional and Fourier-dimensional hypotheses [1, 36, 37]. Restriction theory can also be formulated in alternative algebraic settings, and in particular for varieties lying in vector spaces over finite fields [24–26, 28, 31, 33, 38].

The purpose of this brief note is to formulate an abstract L^2 Fourier restriction theorem over a certain class of locally compact abelian (LCA) groups (for the reader's convenience, a review of the basic elements of Fourier analysis on groups is appended). This result provides a unified approach to many of the generalisations of the Stein–Tomas theorem mentioned above

[1]If L is a list of objects and $X, Y \geq 0$, then the notation $X \lesssim_L Y$ or $Y \gtrsim_L X$ signifies $X \leq C_L Y$ where C_L is a constant which depends only on the objects featured in L.

(although it should be noted that it certainly fails to recover the deeper and more intricate results in the field such as those of [23] or [33]). Moreover, this abstract formulation allows one to easily develop L^2 Fourier restriction theory in new settings such as modules over local fields, their associated quotient rings or rings of integers modulo N.

Let G be a LCA group with Haar measure m and suppose G is equipped with a one parameter family of translation-invariant balls $\{B_\rho^G(x) : x \in G; \rho > 0\}$. The term 'balls' is used loosely here: the $B_\rho^G(x)$ are simply open sets and need not arise from a metric. They are required, however, to satisfy the following axioms:

i) Nesting: $B_\rho^G(0) \subseteq B_{\rho'}^G(0)$ for all $0 < \rho \leq \rho'$;

ii) Symmetry: $B_\rho^G(0) = -B_\rho^G(0)$ for all $0 < \rho$;

iii) Covering: $\bigcup_{\rho>0} B_\rho^G(0) = G$;

iv) Translation invariance: $B_\rho^G(x) = x + B_\rho^G(0)$ for all $x \in G$ and $0 < \rho$.

In addition, it is assumed that the balls satisfy the regularity condition

$$\text{(R)} \qquad m(B_\rho^G(0)) \leq C_1 \rho^n \text{ for all } 0 < \rho.$$

Let $\widehat{G}$ denote the Pontryagin dual group and $\hat{m}$ its Haar measure, which is normalised so that the inversion formula (and hence Plancherel's theorem) hold. Suppose $\widehat{G}$ is also equipped with a family of translation-invariant balls $\{B_\rho^{\widehat{G}}(\xi) : \xi \in \widehat{G}; \rho > 0\}$ (that is, a system of open sets satisfying i)–iv) above) and, further, that there is a system of real-valued Borel functions $\{\varphi_\rho\}_{\rho>0}$ on G such that $\varphi_\rho = 1$ on $B_\rho^G(0)$, $\mathrm{supp}(\varphi_\rho) \subset B_{2\rho}^G(0)$, $\|\varphi_\rho\|_{L^\infty(G,m)} \leq 1$ and

$$\text{(F)} \qquad |\hat{\varphi}_\rho(\xi)| \leq C_2 s^{-n} \text{ whenever } -\xi \notin B_s^{\widehat{G}}(0) \text{ and } s \geq 1/\rho.$$

Here $\hat{\varphi}_\rho$ denotes the Fourier transform of φ_ρ, given by

$$\hat{\varphi}_\rho(\xi) := \int_G \varphi_\rho(x)\xi(-x)\,\mathrm{d}m(x) \qquad \text{for all characters } \xi \in \widehat{G}.$$

In addition to this pointwise estimate for the $\hat{\varphi}_\rho$, it is also convenient to assume uniform $L^1(\widehat{G})$-boundedness; explicitly,

$$\text{(F}'\text{)} \qquad \int_{\widehat{G}} |\hat{\varphi}_\rho(\xi)|\,\mathrm{d}\hat{m}(\xi) \leq C_3.$$

The $\{\varphi_\rho\}_{\rho>0}$ can be used to construct a system of operators which can be thought of as 'smooth Littlewood–Paley projections'. As such, when all of the above criteria are satisfied, the ensemble $(G, \{B_\rho^G\}, \{B_\rho^{\widehat{G}}\}, \{\varphi_\rho\})$ is referred to as a *Littlewood–Paley system.*

Example 1.1. The prototypical example is, of course, given by $G = \mathbb{R}^n$ (so that $\widehat{G} \cong \mathbb{R}^n$) and taking the system of balls and dual balls to be simply those induced by the Euclidean metric. Here the Haar measure is Lebesgue measure and condition (R) is immediately satisfied with $C_1 \lesssim_n 1$. A system of projections is given by taking a radially decreasing Schwartz function φ satisfying $\varphi(x) = 1$ for $x \in B$ and $\mathrm{supp}(\varphi) \subset 2B$, where $B \subseteq \mathbb{R}^n$ is the unit ball, and defining $\varphi_\rho(x) := \varphi(\rho^{-1}x)$ for all $x \in \mathbb{R}^n$; the conditions (F) and (F$'$) are then readily verified with $C_2, C_3 \lesssim_n 1$. The φ_ρ define a system of Littlewood–Paley projections in the classical sense and applying the forthcoming analysis to this example recovers the results (and methods) of [1]. Note that one cannot take φ_r to be the sharp cutoff function $\chi_{B_\rho^G(0)}$: indeed, the lack of regularity of $\chi_{B_\rho^G(0)}$ leads to poor Fourier decay estimates. For comparison, in discrete and non-archimedean settings, as considered below, characteristic functions of balls are smooth (in the sense that they admit favourable Fourier decay-type estimates) and in these cases one may take $\varphi_\rho := \chi_{B_\rho^G(0)}$.

Example 1.2. If $G = \mathbb{F}_q^n$ is a vector space over a finite field (so that $\widehat{G} \cong \mathbb{F}_q^n$), then one may define

$$B_\rho^G(x) := \begin{cases} \emptyset & \text{if } 0 < \rho < 1 \\ \{x\} & \text{if } 1 \leq \rho < q \\ \mathbb{F}_q^n & \text{if } q \leq \rho < \infty; \end{cases} \qquad B_\rho^{\widehat{G}}(\xi) := \begin{cases} \emptyset & \text{if } 0 < \rho < 1/q \\ \{\xi\} & \text{if } 1/q \leq \rho < 1 \\ \mathbb{F}_q^n & \text{if } 1 \leq \rho < \infty. \end{cases}$$

Here the Haar measure is counting measure on G and condition (R) is immediately satisfied with $C_1 = 1$. A system of projections is given by $\varphi_\rho := \chi_{B_\rho^G(0)}$ for all $\rho > 0$. The conditions (F) and (F$'$) can be easily verified with $C_2 = C_3 = 1$, noting that here the Haar measure on $\widehat{G}$ is normalised to have mass 1.

Further examples are discussed in §2. From Example 1.2 above one observes that, in general, it is important that the families of balls $\{B_\rho^G\}, \{B_\rho^{\widehat{G}}\}$ do not necessarily arise from a metric, or even a pseudo-metric. This will also be the case in the basic application where the underlying LCA group G arises from a ring of integers modulo N.

The main result of the article is an abstract L^2 restriction theorem for Littlewood–Paley systems. In particular, restriction with respect to some finite (positive) measure μ on $\widehat{G}$ is investigated. Analogously to the results in the Euclidean setting [1, 36, 37], one assumes that the measure μ satisfies both a dimensional (or regularity) and Fourier-dimensional hypothesis; in particular, for some $0 < b \leq a < n$ assume the following hold:

$$\text{(R}\mu\text{)} \qquad \mu(B_\rho^{\widehat{G}}(\xi)) \leq A\rho^a \text{ for all } \xi \in \widehat{G}, \quad \text{and}$$

$$\text{(F}\mu\text{)} \qquad |\check{\mu}(x)| \leq B\rho^{-b/2} \text{ for all } x \notin B_\rho^G(0).$$

With the various definitions now in place, the main theorem is as follows.

Theorem 1.3. *Let* $(G, \{B_\rho^G\}, \{B_\rho^{\widehat{G}}\}, \{\varphi_\rho\})$ *be a Littlewood–Paley system,* $0 < b \leq a < n$ *and suppose* μ *is a finite measure on* $\widehat{G}$ *satisfying* (Rμ) *and* (Fμ). *Then*

$$\|\hat{f}\|_{L^2(\mu)} \leq C_r \|f\|_{L^r(G)} \tag{1.1}$$

holds for all $1 \leq r \leq r_0$ *where*

$$r_0 := \frac{4(n-a)+2b}{4(n-a)+b}. \tag{1.2}$$

Furthermore, the constant C_r *in* (1.1) *depends only on* $r, n, C_1, C_2, C_3, A, B, a$ *and* b.

Remark 1.4. 1) The proof will in fact show that the Fourier transform satisfies a stronger $L^{r_0,2}(G) - L^2(\mu)$ inequality.

2) One may extract an explicit value for the constant appearing in the statement of Theorem 1.3 from the proof presented below. In particular, for $r = r_0$ one may take $C = \bar{C}^{1/2}$ where $\bar{C}$ is a constant of the form

$$\bar{C} = C_{n,a,b}(C_1 + C_2)^{1-\theta} C_3^{(1-\theta)/(2-\theta)} A^{1-\theta} B^\theta \tag{1.3}$$

for the exponent θ given by

$$\theta := \frac{2(n-a)}{2(n-a)+b}. \tag{1.4}$$

In view of applications it is useful to track (at least roughly) the dependence on the constants. This is particularly relevant when considering

Fourier restriction in discrete settings such as finite fields, or rings of integers $\mathbb{Z}/N\mathbb{Z}$. In these cases one wishes to prove estimates that are 'essentially' independent of the cardinality of the underlying field or ring: see §2.

The theorem is proved by adapting the arguments of [1] so as to be applicable in the abstract setting of Littlewood–Paley systems.

This article is structured as follows: in §2 some examples of groups and measures satisfying the hypotheses of the theorem are discussed; the proof of Theorem 1.3 is then given in §3. The note proper concludes with a discussion of some related estimates for the convolution operators $f \mapsto f * \mu$ in §4. Reviews of the basic theory of Fourier analysis on groups and of the p-adic numbers are appended for the reader's convenience.

2. Examples

In this section examples of Littlewood–Paley systems are discussed, together with some prototypical measures μ that satisfy (Rμ) and (Fμ) with favourable values of a, b, A and B. For simplicity, the discussion is restricted to measures supported on smooth surfaces or algebraic varieties.

Euclidean spaces

Theorem 1.3 generalises the existing abstract restriction theory of Mockenhaupt [37], Mitsis [36] and Bak and Seeger [1].

Vector spaces over finite fields

Theorem 1.3 also generalises the basic finite field version of the Stein–Tomas theorem due to Mockenhaupt and Tao [38]. Here it is important to observe that the constants C_1, C_2, C_3 can all be chosen independently of the cardinality of the underlying finite field.

It is remarked that in many respects the finite field setting behaves in a somewhat different manner from the other examples featured in this section. Some of the important and interesting differences are highlighted at the end of this section.

Vector spaces over $\mathbb{Q}_p$

Let p be a fixed odd prime and consider the field of p-adic numbers $\mathbb{Q}_p$ with p-adic absolute value $|\cdot|_p$. For the reader's convenience, a brief review of the p-adic analysis is included in Appendix B. The vector space $G := \mathbb{Q}_p^n$ is self-dual as a LCA group and both G and $\widehat{G}$ are endowed with the family of (clopen) balls $B_\rho(x) := \{x \in \mathbb{Q}_p^n : \|x\| \leq \rho\}$ induced by the ℓ^∞-norm

$$\|x\| := \max_{1\leq j\leq n} |x_j|_p \quad \text{for all } x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n.$$

The Haar measures on both $\mathbb{Q}_p^n$ and the dual group are normalised so that the unit ball $\mathbb{Z}_p^n := \{x \in \mathbb{Q}_p^n : \|x\| \leq 1\}$ has measure 1. The regularity property (R) then holds with $C_1 = 1$.

Fix an additive character $e\colon \mathbb{Q}_p \to \mathbb{T}$ such that e restricts to the constant function 1 on $\mathbb{Z}_p$ and to a non-principal character on $p^{-1}\mathbb{Z}_p$. Then for any integrable $f\colon \mathbb{Q}_p^n \to \mathbb{C}$ the Fourier transform $\hat{f}$ is given by

$$\hat{f}(\xi) := \int_{\mathbb{Q}_p^n} f(x)e(-x\cdot\xi)\,\mathrm{d}m(x) \qquad \text{for all } \xi \in \mathbb{Q}_p^n;$$

here $x \cdot \xi := x_1\xi_1 + \cdots + x_n\xi_n$ for $x = (x_1, \dots, x_n)$, $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{Q}_p^n$. In particular, defining $\varphi_\rho := \chi_{B_\rho(0)}$ for $\rho > 0$ one may easily verify that $\hat{\varphi}_\rho(\xi) = p^{-n\nu}\chi_{B_{p^\nu}(0)}(\xi)$ where $\nu \in \mathbb{Z}$ is the smallest integer such that $p^{-\nu} \leq \rho$. The conditions (F) and (F′) immediately follow with $C_2 = C_3 = 1$. It is remarked that, by the non-archimedean nature of the absolute value, the φ_ρ are smooth functions on $\mathbb{Q}_p^n$ and, moreover, are natural p-adic analogues of Schwartz functions (in particular, they belong to the *Schwartz–Bruhat* class of functions on $\mathbb{Q}_p^n$: see [2, 39, 45]).

Let $h\colon \mathbb{Z}_p^{n-1} \to \mathbb{Z}_p$ be the mapping given by $h(\omega) := \omega_1^2 + \cdots + \omega_{n-1}^2$ and consider the paraboloid

$$\Sigma := \{(\omega, h(\omega)) : \omega \in \mathbb{Z}_p^{n-1}\} \subseteq \mathbb{Q}_p^n.$$

Take μ to be the measure on Σ given by the push-forward of the Haar measure on $\mathbb{Z}_p^{n-1}$ under the graphing function $\omega \mapsto (\omega, h(\omega))$. The condition (R$\mu$) is readily verified for μ with $A = 1$ and $a = n - 1$. On the other hand, the inverse Fourier transform of the measure is given by $\check{\mu}(x) = \prod_{j=1}^{n-1} G(x_j, x_n)$ where

$$G(a,b) := \int_{\mathbb{Z}_p} e(at + bt^2)\,\mathrm{d}m(t) \qquad \text{for } a, b \in \mathbb{Q}_p.$$

The integral $G(a,b)$ can be written in terms of classical Gauss sums and thereby evaluated or, alternatively, one may analyse $G(a,b)$ directly using the basic algebraic properties of the character e. In either case, it is not difficult to deduce that

$$|G(a,b)| = \begin{cases} |b|_p^{-1/2} & \text{if } |a|_p \leq |b|_p \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

for all $a, b \in \mathbb{Q}_p$ with $\max\{|a|_p, |b|_p\} > 1$. Indeed, adopting the latter of the two approaches described above, given $a, b \in \mathbb{Q}_p$ with $\max\{|a|_p, |b|_p\} > 1$ it follows that

$$\begin{aligned} |G(a,b)|^2 &= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} e(a(t-s) + b(t^2 - s^2))\,\mathrm{d}m(s)\mathrm{d}m(t) \\ &= \int_{\mathbb{Z}_p} e(at + bt^2)\Big(\int_{\mathbb{Z}_p} e(2bts)\,\mathrm{d}m(s)\Big)\mathrm{d}m(t). \end{aligned}$$

Note that the latter identity follows by the change of the t variable $t \mapsto t + s$; this is valid owing to the translation-invariance of the Haar measure.

The proof of (2.1) now relies on repeated application of the orthogonality relations for the characters on $\mathbb{Z}_p$ (see (A.1)). Since p is odd, $s \mapsto e(2bts)$ is a non-principal character on $\mathbb{Z}_p$ if and only if $|t|_p > |b|_p^{-1}$ and therefore

$$|G(a,b)|^2 = \int\limits_{\{t\in\mathbb{Z}_p : |t|_p \leq |b|_p^{-1}\}} e(at + bt^2)\,\mathrm{d}m(t) = \int\limits_{\{t\in\mathbb{Z}_p : |t|_p \leq |b|_p^{-1}\}} e(at)\,\mathrm{d}m(t).$$

If $|b|_p \leq 1$, then the right-hand integral is taken over the whole of $\mathbb{Z}_p$. Since, by hypothesis, $|a|_p > 1$ in this case, the character $e(a\,\cdot)$ is non-principal and therefore the Gauss sum vanishes. On the other hand, if $|b|_p > 1$, then $b^{-1} \in \mathbb{Z}_p$ and, by applying the p-adic change of variables[2] $t \mapsto b^{-1}t$, it follows that

$$|G(a,b)|^2 = |b|_p^{-1}\int_{\mathbb{Z}_p} e(ab^{-1}t)\,\mathrm{d}m(t).$$

The desired identity (2.1) now immediately follows from the orthogonality relations.

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[2]The change of variables formula for dilation mappings, as used here, can be easily deduced from the basic properties of the Haar measure on $\mathbb{Z}_p$. A systematic approach to p-adic change of variables can be found in, for instance, [21, §7.4], [42, A.7] or [18, pp. 97-98].

From these observations it follows that

$$|\check{\mu}(x)| \le \|x\|^{-(n-1)/2} \qquad \text{for all } x \in \mathbb{Q}_p^n$$

and therefore (Fμ) holds with $B = 1$ and $b = n-1$.

Combining these observations with Theorem 1.3 produces a p-adic variant of the classical Stein–Tomas theorem, which shares the

$$r_0 = 2(n+1)/(n+3)$$

numerology of the Euclidean case.

Modules over rings of integers $\mathbb{Z}/p^\alpha\mathbb{Z}$

Let $N \in \mathbb{N}$ and consider the module $G := [\mathbb{Z}/N\mathbb{Z}]^n$. For $k \in \mathbb{N}$ define a function $\|\cdot\| \colon [\mathbb{Z}/N\mathbb{Z}]^k \to \mathbb{N}$ by setting

$$\|\vec{x}\| := \frac{N}{\gcd(x_1, \dots, x_k, N)} \qquad \text{for all } \vec{x} = (x_1, \dots, x_k) \in [\mathbb{Z}/N\mathbb{Z}]^k.$$

The image of this 'norm' is thought of as a set of available scales in the module. It is natural to compare the scales under the division ordering $\preceq$, defined by $a \preceq b$ for $a, b \in \mathbb{N}$ if and only if $a \mid b$. One may isolate the elements lying at a given scale by defining

$$\mathcal{B}_d := \{\vec{x} \in [\mathbb{Z}/N\mathbb{Z}]^n : \|\vec{x}\| \preceq d\} \quad \text{for all divisors } d \mid N.$$

Fix an odd prime p and now specialise to the case $N = p^\alpha$ for some $\alpha \in \mathbb{N}$. In this situation the set of available scales is given by $\{1, p, \dots, p^\alpha\}$, which is totally ordered under $\preceq$. Define a collection of balls on $G := [\mathbb{Z}/p^\alpha\mathbb{Z}]^n$ by $B_\rho^G(\vec{x}\,) := \vec{x} + \mathcal{B}_{p^\nu}$ where $0 \le \nu \le \alpha$ is the largest value for which $p^\nu \le \rho$ (if $0 < \rho < 1$, then $B_\rho^G(\vec{x}\,) := \emptyset$). These balls do not arise from a metric, but nevertheless the satisfy the crucial properties i)–iv) listed in the introduction. Furthermore, the Haar measure on $[\mathbb{Z}/p^\alpha\mathbb{Z}]^n$ is simply counting measure and the regularity property (R) therefore holds with $C_1 = 1$. A system of dual balls on $\widehat{G}$ is given by $B_\rho^{\widehat{G}}(\vec{\xi}\,) := \vec{\xi} + \mathcal{B}_{p^{\alpha-\nu}}$ where now $0 \le \nu \le \alpha$ is the smallest value for which $p^\nu \ge 1/\rho$ (if $0 < \rho < p^{-\alpha}$, then $B_\rho^{\widehat{G}}(\vec{\xi}\,) := \emptyset$). Taking $\varphi_\rho := \chi_{B_\rho^G(\vec{0})}$ the properties (F) and (F$'$) are both easily seen to hold with $C_2 = C_3 = 1$.

Let $h\colon [\mathbb{Z}/p^\alpha\mathbb{Z}]^{n-1} \to \mathbb{Z}/p^\alpha\mathbb{Z}$ be the mapping given by $h(\vec{\omega}) := \omega_1^2 + \cdots + \omega_{n-1}^2$ and consider the paraboloid

$$\Sigma := \{(\vec{\omega}, h(\vec{\omega})) : \vec{\omega} \in [\mathbb{Z}/p^\alpha\mathbb{Z}]^{n-1}\} \subseteq [\mathbb{Z}/p^\alpha\mathbb{Z}]^n.$$

If μ denotes the normalised counting measure on Σ, then it is immediate that μ satisfies (Rμ) with $A = 1$ and $a = n-1$. The Fourier transform

$$\check{\mu}(\vec{x}\,) = \frac{1}{p^{(n-1)\alpha}} \sum_{\omega \in [\mathbb{Z}/p^\alpha\mathbb{Z}]^{n-1}} e^{2\pi i (x'\cdot\vec{\omega} + x_n h(\vec{\omega}))/p^\alpha} \quad \text{for } \vec{x} = (x', x_n) \in [\mathbb{Z}/p^\alpha\mathbb{Z}]^n$$

can be evaluated via the classical formulae for Gauss sums. In particular, it is not difficult to show that

$$|\check{\mu}(\vec{x}\,)| \le \|\vec{x}\,\|^{-(n-1)/2},$$

which implies that (Fμ) holds with $B = 1$ and $b = n-1$. See [20] for further details.

Applying Theorem 1.3, one deduces that the inequality

$$\left(\frac{1}{\#\Sigma} \sum_{\vec{\xi} \in \Sigma} |\hat{F}(\vec{\xi}\,)|^2\right)^{1/2} \lesssim_n \|F\|_{\ell^r([\mathbb{Z}/p^\alpha\mathbb{Z}]^n)}$$

holds for all $1 \le r \le 2(n+1)/(n+3)$. The important observation here is that the implied constant in this estimate is independent of both p and α (and therefore the cardinality of the underlying ring).[3] Thus, this result is a $[\mathbb{Z}/p^\alpha\mathbb{Z}]^n$-analogue of certain finite field restriction estimates of Mockenhaupt and Tao [38].

The analysis of this discrete example has many similarities with the continuous p-adic example. In fact, restriction theory over $\mathbb{Q}_p$ is equivalent to restriction theory over $[\mathbb{Z}/p^\alpha\mathbb{Z}]^n$ in a precise sense. In particular, there

[3] Indeed, the inequality

$$\left(\frac{1}{\#\Sigma} \sum_{\xi \in \Sigma} |\hat{F}(\xi)|^s\right)^{1/s} \lesssim_{n,p,\alpha} \|F\|_{\ell^r([\mathbb{Z}/p^\alpha\mathbb{Z}]^n)}$$

trivially holds for all Lebesgue exponents $1 \le r, s \le \infty$ (with a constant which now depends on p and α) as a consequence of the Riemann–Lebesgue lemma and the equivalence of norms on finite-dimensional vector spaces.

is a ‘correspondence principle’, which is a manifestation of the uncertainty principle, that allows one to ‘lift’ restriction problems over the discrete rings $\mathbb{Z}/p^\alpha\mathbb{Z}$ to the continuous setting of $\mathbb{Q}_p$. This lifting procedure is discussed in detail in [18] and [20].

Vector spaces over local fields and modules over their quotient rings

The two previous examples can be generalised to the setting of non-archimedean local fields. Let K be a field with a discrete non-archimedean absolute value $|\cdot|_K$, suppose $\pi \in K$ is a choice of uniformiser and let $\mathfrak{o} := \{x \in K : |x|_K \leq 1\}$ denote the ring of integers of K. Assume that the residue class field $\mathfrak{o}/\pi\mathfrak{o}$ is finite. For the details of the relevant definitions see, for instance, [29], [30] or [45]. Generalising the p-adic example, any finite-dimensional vector space K^n can be endowed with a natural Littlewood–Paley system by taking the balls to be those induced by the ℓ^∞-norm on K^n and the projections φ_ρ to be characteristic functions of balls. It is remarked that, by the non-archimedean nature of the absolute value, these φ_ρ are in fact smooth functions. Similarly, generalising the $\mathbb{Z}/p^\alpha\mathbb{Z}$ example, for each $\alpha \in \mathbb{N}$ the module $[\mathfrak{o}/\pi^\alpha\mathfrak{o}]^n$ can also be endowed with a natural Littlewood–Paley system. The restriction theories over the vector space K^n and over the modules $[\mathfrak{o}/\pi^\alpha\mathfrak{o}]^n$ are in some sense equivalent via a correspondence principle which extends that described above. The details may be found in [18].

It is well-known that any field K satisfying the above properties is isomorphic to either a finite extension of $\mathbb{Q}_p$ for some prime p or the field $\mathbb{F}_q((X))$ of formal Laurent series over a finite field $\mathbb{F}_q$. The local fields $\mathbb{F}_q((X))$ are particularly well-behaved spaces which act as simplified models of Euclidean space. For instance, Fourier analysis over $\mathbb{F}_2((X))$ corresponds to the study of Fourier-Walsh series, which has played a prominent rôle as a model for problems related to Carleson’s theorem and time-frequency analysis [7, 8]. Recently there has been increased interest in local field variants of other problems in Euclidean harmonic analysis and geometric measure theory, focusing on the Kakeya conjecture [5, 10, 12, 13, 20]. This has stemmed from Dvir’s solution [11] to Wolff’s finite field Kakeya conjecture, which has led to progress on the original Euclidean problem [4, 16, 17]. It is natural to also consider local field analogues of the restriction problem; this topic is investigated further in [18, 20].

Modules over rings of integers $\mathbb{Z}/N\mathbb{Z}$

Let $N \in \mathbb{N}$ and consider the ring of integers $\mathbb{Z}/N\mathbb{Z}$. If N is not a power of a fixed prime, but has multiple distinct prime factors, then the set of available scales for $\mathbb{Z}/N\mathbb{Z}$ is only partially ordered under $\preceq$. This introduces additional difficulties when one attempts to generalise the constructions described in the $\mathbb{Z}/p^\alpha\mathbb{Z}$ case. In particular, in order to ensure the nesting property, the balls $B^G_\rho(\vec{x}\,)$ and $B^{\widehat{G}}_\rho(\vec{\xi}\,)$ in $[\mathbb{Z}/N\mathbb{Z}]^n$ are now defined by

$$B^G_\rho(\vec{x}\,) := \vec{x} + \bigcup_{d|N:d\leq\rho} \mathcal{B}_d \quad \text{and} \quad B^{\widehat{G}}_\rho(\vec{\xi}\,) := \vec{\xi} + \bigcup_{d|N:d\geq 1/\rho} \mathcal{B}_{N/d}.$$

The verification of the properties (R), (F) and (F′) for these balls is more involved and an ε-loss in N must, in general, be included in the constants. The details are discussed in [20], where a theory of Fourier restriction over such rings of integers is systematically developed. The partially ordered scale structure on $\mathbb{Z}/N\mathbb{Z}$ tends to make the analysis more involved in this setting than over $\mathbb{R}^n$ (where the scales are, of course, totally ordered), and typically the arguments require additional number-theoretic input [19, 20].

Comparison between the different settings

To conclude this section some differences are highlighted between the various different settings for the restriction problem featured above. The principal observation is that, except for certain endpoint questions, the numerology associated to the restriction theory in all the featured settings tends to be the same (that is, it mirrors precisely the euclidean theory) *except for the finite field case*. As one example of this, as indicated above, Theorem 1.3 implies an L^2 restriction theorem for the paraboloid in each setting (with a suitable dependence on the constant: for instance, when working over $\mathbb{F}_q$ or $\mathbb{Z}/p^\alpha\mathbb{Z}$ one wishes the constant to be independent of the cardinality of the underlying group). In each case one obtains the same range of exponents $1 \leq r \leq 2(n+1)/(n+3)$. Variants of the well-known Knapp example show that this range is sharp for the L^2-restriction problem for the paraboloid over $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{Z}/p^\alpha\mathbb{Z}$ and $\mathbb{Z}/N\mathbb{Z}$ (there are some slight technicalities in formulating the problem over $\mathbb{Z}/N\mathbb{Z}$: see [20] for details). However, in the finite field setting no suitable analogue of the Knapp example is available and, indeed, the range $1 \leq r \leq 2(n+1)/(n+3)$ is no longer sharp. This was already observed in the foundational work of Mockenhaupt and Tao [38] where it was shown that the endpoint estimate for the parabola in $\mathbb{F}_q^2$ is $4/3$ rather than the exponent 6/5 given by Theorem 1.3. The conjectured sharp range of exponents for the finite field problem in general dimensions is a little complicated, depending on the congruence class of n modulo 4 and arithmetic properties of the underlying field (these conditions determine the existence of certain affine subspaces in the paraboloid): see [27] for details. Numerous authors have made progress in this direction [24–28, 31–33, 41]; attention is drawn to the recent results of Iosevich, Koh and Lewko [27] and Rudnev and Shkredov [41] which have established sharp L^2-restriction estimates for the finite field paraboloid in even dimensions.

In addition to the differences in numerology in the finite field problem, it is also pertinent to note that the techniques involved over $\mathbb{F}_q$ are often distinct from those used in the other settings. For instance, the study of L^2-restriction over finite fields has made heavy use of tools from incidence geometry (see, for instance, [41]). Tools from number theory, and in particular exponential sum estimates, also play a significant rôle in the finite field restriction problem: for instance, as observed in [26], decay estimates for the Fourier transform of the normalised counting measure on the sphere

$$\{\omega \in \mathbb{F}_q^n : \omega_1^2 + \cdots + \omega_n^2 = 1\}$$

can be obtained through appeal to the classical exponential sum estimates of A. Weil [47]. In the euclidean case, analogous decay estimates follow from much more elementary stationary phase estimates. Number-theoretic considerations along these lines also appear to be a significant feature of the $\mathbb{Z}/N\mathbb{Z}$ formulation: see [19, 20].

The differences between the finite field setting and other formulations of the Fourier restriction problem can largely be accounted for by the lack of available scales in finite fields. This perspective is pursued in the companion paper [20] where the finite field, real and $\mathbb{Z}/N\mathbb{Z}$ settings are contrasted in much more detail.

3. Proof of Theorem 1.3

The main ingredient is a Lorentz-space convolution inequality (for the definitions of the Lorentz spaces and the associated norms $L^{q,s}(G)$ see, for instance, [44, Chapter V]).

Proposition 3.1. *For G and μ as in the statement of Theorem 1.3 define $Tf := f * \check{\mu}$. If*

$$\sigma := \frac{2(n-a+b)(2(n-a)+b)}{2(n-a+b)(2(n-a)+b)-b^2}, \qquad \tau := \frac{2(n-a+b)}{b},$$

then whenever $\sigma < p < \tau'$ and q satisfies $1/p - 1/q = 2(n-a)/(2(n-a)+b)$, the estimate

$$\|Tf\|_{L^{q,s}(G)} \lesssim_{p,s} \bar{C}\|f\|_{L^{p,s}(G)}$$

holds for any $0 < s \le \infty$. Here $\bar{C}$ is the expression appearing in (1.3).

Theorem 1.3 is an immediate consequence of this estimate.

Proof (of Theorem 1.3). Note that $(p,q) := (r_0, r_0')$ satisfies the hypotheses of Proposition 3.1. By the Lorentz space version of Hölder's inequality together with a duality argument,

$$\begin{aligned}\int_{\widehat{G}} |\hat{f}(\xi)|^2 \,\mathrm{d}\mu(\xi) &= \int_G f(x)\overline{Tf(x)}\,\mathrm{d}m(x) \\ &\le \|f\|_{L^{r_0,2}(G)}\|Tf\|_{L^{r_0',2}(G)} \lesssim_{p,s} \bar{C}\|f\|^2_{L^{r_0,2}(G)}.\end{aligned}$$

Interpolating against the trivial $L^1(G) - L^\infty(\mu)$ inequality (using Marcinkiewicz interpolation: see, for instance, [44, Chapter V]) concludes the proof. □

Turning to the proof of Proposition 3.1, the first step is, in fact, to prove the restricted weak-type version of the estimate (1.1) for $r = r_0$. This is achieved via (an abstraction of) an L^2 restriction argument due to A. Carbery. The weak version of the Stein–Tomas theorem can then be applied to bound the convolution operator.

Lemma 3.2. *Under the hypotheses of Theorem 1.3,*[4] *the restricted weak-type estimate*

$$\|\hat{\chi}_E\|_{L^2(\mu)} \lesssim_{n,a} (C_1 + C_2)^{(1-\theta)/2} A^{(1-\theta)/2} B^{\theta/2} \|\chi_E\|_{L^{r_0}(G)}$$

holds for all Borel sets $E \subset G$.

[4] In fact, the hypotheses can be slightly weakened: here the symmetry property ii) of the balls and L^1 estimate (F′) are not required.

Proof. Decompose the measure μ by writing $\mu = \mu_1 + \mu_2$ where

$$\check{\mu}_1 := \varphi_\rho \cdot \check{\mu} \quad \text{and} \quad \check{\mu}_2 := (1-\varphi_\rho)\cdot \check{\mu} \tag{3.1}$$

for fixed value of $\rho > 0$ chosen so as to satisfy the later requirements of the proof. Thus, $T = T_1 + T_2$ where $T_j f := f * \check{\mu}_j$ for $j = 1, 2$.

Fixing a Borel set $E \subseteq \widehat{G}$ observe, by duality and Hölder's inequality, that

$$\int_{\widehat{G}} |\hat{\chi}_E(\xi)|^2 \,\mathrm{d}\mu(\xi) \le \|T_1\chi_E\|_{L^2(G)} m(E)^{1/2} + \|T_2\chi_E\|_{L^\infty(G)} m(E). \tag{3.2}$$

Since $\mu_1 = \hat{\varphi}_\rho * \mu$, it follows that

$$\begin{aligned}\mu_1(\xi) &= \int_{B^{\widehat{G}}_{1/\rho}(\xi)} \hat{\varphi}_\rho(\xi-\eta)\,\mathrm{d}\mu(\eta) + \sum_{k=1}^\infty \int_{B^{\widehat{G}}_{2^k/\rho}(\xi)\setminus B^{\widehat{G}}_{2^{k-1}/\rho}(\xi)} \hat{\varphi}_\rho(\xi-\eta)\,\mathrm{d}\mu(\eta)\\ &=: \mathrm{I} + \mathrm{II}\end{aligned}$$

Applying the Riemann–Lebesgue estimate $\|\hat{\varphi}_\rho\|_{L^\infty(\widehat{G})} \le m(B^G_{2\rho}(0))$ together with the hypotheses (R) and (Rμ), one deduces that

$$|\mathrm{I}| \le m\big(B^G_{2\rho}(0)\big)\mu\big(B^{\widehat{G}}_{1/\rho}(\xi)\big) \le 2^n C_1 A\rho^{n-a}.$$

Furthermore, for any $k \in \mathbb{N}$ the condition (F) implies that

$$|\hat{\varphi}_\rho(\xi-\eta)| \le C_2 2^{-(k-1)n}\rho^n \qquad \text{for all } \eta \notin B^{\widehat{G}}_{2^{k-1}/\rho}(\xi)$$

and so

$$|\mathrm{II}| \le C_2\left(\sum_{k=1}^\infty 2^{-(k-1)n}\mu\big(B^{\widehat{G}}_{2^k/\rho}(\xi)\big)\right)\rho^n \le 2^n C_2 \left(\sum_{k=1}^\infty 2^{-(n-a)k}\right) A\rho^{n-a}.$$

Combining these observations,

$$\|\mu_1\|_{L^\infty(\widehat{G})} \le 2^n\big(C_1 + (2^{n-a}-1)^{-1}C_2\big)A\rho^{n-a} \lesssim_{n,a} (C_1+C_2)A\rho^{n-a} \tag{3.3}$$

and so

$$\|T_1\chi_E\|_{L^2(G)} = \|\mu_1\hat{\chi}_E\|_{L^2(\widehat{G})} \lesssim_{n,a} (C_1+C_2)Am(E)^{1/2}\rho^{n-a}. \tag{3.4}$$

On the other hand, since $\mathrm{supp}(1-\varphi_\rho) \subseteq G \setminus B^G_\rho(0)$, it follows from (F$\mu$) that

$$\|\check{\mu}_2\|_{L^\infty(G)} \le 2B\rho^{-b/2} \tag{3.5}$$

and hence

$$\|T_2\chi_E\|_{L^\infty(G)} \leq \|\check{\mu}_2\|_{L^\infty(G)}\|\chi_E\|_{L^1(\widehat{G})} \lesssim Bm(E)\rho^{-b/2}. \tag{3.6}$$

Combining (3.2), (3.4) and (3.6) one concludes that

$$\|\hat{\chi}_E\|^2_{L^2(\mu)} \lesssim_{n,a} (C_1 + C_2)Am(E)\rho^{n-a} + Bm(E)^2\rho^{-b/2}$$

Thus, choosing ρ so that $\rho^{n-a+b/2} \sim_{n,a} (C_1 + C_2)^{-1}A^{-1}Bm(E)$ and recalling the definition (1.4), the desired inequality follows. □

Proof (of Proposition 3.1). Since T is essentially self-adjoint[5], it suffices to show that T is of restricted weak-type (σ, τ). Indeed, it then follows that T is also of restricted weak-type (τ', σ') and the desired result is then deduced by interpolating between these estimates (see, for instance, [44, Chapter V, Theorem 3.15] for a statement of the relevant interpolation theorem).

Decompose $T = T_1 + T_2$ as above; although the same notation is used, it is understood that this decomposition is made with respect to a new value of ρ, chosen so as to satisfy the later requirements of the proof. Applying (3.3), one observes that

$$\begin{aligned}
\|T_1\chi_E\|^2_{L^2(G)} &= \int_{\widehat{G}} |\hat{\chi}_E(\xi)|^2|\mu_1(\xi)|^2\,\mathrm{d}\hat{m}(\xi) \\
&\lesssim_{n,a} (C_1 + C_2)A\rho^{n-a}\int_{\widehat{G}} |\hat{\chi}_E(\xi)|^2|\mu_1(\xi)|\,\mathrm{d}\hat{m}(\xi) \\
&\leq (C_1 + C_2)A\rho^{n-a}\int_{\widehat{G}}\int_{\widehat{G}} |\hat{\chi}_E(\xi+\eta)|^2\mathrm{d}\mu(\eta)|\hat{\varphi}_\rho(\xi)|\,\mathrm{d}\hat{m}(\xi) \\
&\lesssim_{n,a,b} (C_1 + C_2)^{2-\theta}C_3A^{2-\theta}B^\theta m(E)^{2/r_0}\rho^{n-a}, \qquad (3.7)
\end{aligned}$$

where the final inequality is due to Lemma 3.2 and (F′). If $F \subseteq G$ is any Borel set, then

$$\begin{aligned}
\langle T\chi_E\,,\,\chi_F\rangle &= \langle T_1\chi_E\,,\,\chi_F\rangle + \langle T_2\chi_E\,,\,\chi_F\rangle \\
&\leq \|T_1\chi_E\|_{L^2(G)}m(F)^{1/2} + \|T_2\chi_E\|_{L^\infty(G)}m(F).
\end{aligned}$$

[5] In particular, $T^*g = g * \check{\tilde{\mu}}$ where $\tilde{\mu}$ is the measure on $\widehat{G}$ given by $\tilde{\mu}(E) := \mu(-E)$ for all Borel sets $E \subseteq \widehat{G}$. Note (R$\tilde{\mu}$) and (F$\tilde{\mu}$) hold if and only if (Rμ) and (Fμ) hold with the same constants A and B and so the subsequent arguments apply equally to T and T^*.

Thus, as a consequence of (3.6) and (3.7), the right-hand side of the above expression is dominated by

$$(C_1 + C_2)^{1-\theta/2} C_3^{1/2} A^{1-\theta/2} B^{\theta/2} m(E)^{1/r_0} m(F)^{1/2} \rho^{(n-a)/2} \\ + Bm(E)m(F)\rho^{-b/2}.$$

Choosing

$$\rho^{(n-a+b)/2} \sim_{n,a,b} ((C_1 + C_2)AB^{-1})^{-(1-\theta/2)} C_3^{-1/2} m(E)^{1/r_0'} m(F)^{1/2}$$

yields the estimate

$$\langle T\chi_E \,,\, \chi_F\rangle \leq \bar{C} m(E)^{1/\sigma} m(F)^{1/\tau'}$$

where σ, τ and $\bar{C}$ are as in the statement of the proposition. In particular, T is of restricted weak-type (σ, τ), as required. □

4. Some remarks on convolution operators

Recall that Theorem 1.3 was a direct consequence of an estimate for the convolution operator $f \mapsto \check{\mu} * f$. Using the method of proof of Lemma 3.2, one may also obtain $L^q(\widehat{G}) - L^r(\widehat{G})$ estimates for the related convolution operator $f \mapsto \mu * f$.

Lemma 4.1. *Let $(G, \{B_r^G\}, \{B_r^{\widehat{G}}\}, \{\varphi_r\})$ be a Littlewood–Paley system, $0 < b \leq a < n$ and suppose μ is a probability measure on $\widehat{G}$ satisfying* (Rμ) *and* (Fμ). *If $\mathcal{T}$ denotes the closed triangle with vertices $\{(0,0), (1,1), (1/r_0, 1/s_0)\}$ where*

$$r_0 := \frac{2(n-a)+b}{n-a+b} \qquad s_0 := \frac{2(n-a)+b}{n-a},$$

then

$$\|f * \mu\|_{L^s(\widehat{G})} \leq C\|f\|_{L^r(\widehat{G})} \tag{4.1}$$

holds whenever $(1/r, 1/s) \in \mathcal{T} \setminus \{(1/r_0, 1/s_0)\}$. Furthermore, the constant C depends only on n, C_1, C_2, A, B, a and b.

This is a partial extension of a classical generalised Radon transform estimate due to Littman [34]. The latter treats the case where μ is a smooth, compactly support density supported on a hypersurface in $\mathbb{R}^n$, under the assumption that the hypersurface has non-vanishing Gaussian curvature on the support of μ. In this case Littman [34] establishes (4.1) for the sharp range $(1/r, 1/s) \in \mathcal{T}$, including the $(1/r_0, 1/s_0)$ endpoint. Lemma 4.1 is known to hold (together with the endpoint estimate) in Euclidean space for general measures satisfying (Rμ) and (Fμ), although as far as the authors are aware this has not appeared in print (see, however, [15]). The finite field case has also been studied [3].

Proof (of Lemma 4.1). Since μ is a probability measure it follows that the convolution operator is bounded on $L^r(\widehat{G})$ for all $1 \leq r \leq \infty$. It therefore suffices to prove that $f \mapsto f * \mu$ satisfies a restricted weak-type (r_0, s_0) inequality.

Decompose μ by writing $\mu = \mu_1 + \mu_2$ where the μ_j are as defined in (3.1). Once again, $\rho > 0$ is a fixed value chosen so as to satisfy the later requirements of the proof.

For Borel sets $E, F \subset G$ it follows that

$$\begin{aligned}\langle \mu * \chi_E \,,\, \chi_F \rangle &\leq \|\mu_1 * \chi_E\|_{L^\infty(\widehat{G})} \hat{m}(F) + \|\mu_2 * \chi_E\|_{L^2(\widehat{G})} \hat{m}(F)^{1/2} \\ &\leq \|\mu_1\|_{L^\infty(\widehat{G})} \hat{m}(E)\hat{m}(F) + \|\check{\mu}_2\|_{L^\infty(G)} \hat{m}(E)^{1/2} \hat{m}(F)^{1/2}.\end{aligned}$$

The proof of Lemma 3.2, and in particular (3.3) and (3.5), now imply that

$$\langle \mu * \chi_E \,,\, \chi_F \rangle \lesssim_{n,a} (C_1 + C_2) A \rho^{n-a} m(E) m(F) + B \rho^{-b/2} m(E)^{1/2} m(F)^{1/2}.$$

Thus, choosing ρ so that $\rho^{n-a+b/2} \sim_{n,a} (C_1 + C_2)^{-1} A^{-1} B m(E)^{-1/2} m(F)^{-1/2}$, one concludes that

$$\langle \mu * \chi_E \,,\, \chi_F \rangle \lesssim_{n,a,b} (C_1 + C_2)^{1-\theta} A^{1-\theta} B^{\theta} m(E)^{r_0} m(F)^{s_0'}$$

for θ as defined in (1.4), as required. □

As a final remark, when $G = \mathbb{R}^n$ the strong-type (r_0, s_0) estimate can be obtained for $f \mapsto f * \mu$ by augmenting the above argument with standard inequalities for the Littlewood–Paley square function. It would be interesting to understand whether the endpoint estimate holds in general, given that the spaces in question do not fall under any existent Calderón–Zygmund theory.

Appendix A. Basic elements of Fourier analysis on Groups

For the reader's convenience, here the basic elements of Fourier analysis on locally compact abelian groups are reviewed. There are many classical treaties on this subject which may be consulted for further information and for proofs of the numerous assertions made below: see, for instance, [40].

A locally compact abelian group (LCA) is a locally compact Hausdorff topological space which also has the structure of an abelian group and has the property that $(x, y) \mapsto x - y$ is a continuous map from the product space $G \times G$ onto G. Any LCA group G admits a non-negative regular Borel measure m which is non-zero and translation-invariant in the sense that

$$m(E + x) = m(E) \qquad \text{for all } x \in G \text{ and } E \subseteq G \text{ Borel.}$$

Such a measure m is called a *Haar* measure on G and is unique up to multiplication by a positive scalar. There is a natural choice of normalisation for the Haar measure (which is dictated by the *inversion formula*, as discussed below); henceforth it is assumed that m is *the* Haar measure on G given by this choice normalisation.

Using the Haar measure one may define the Lebesgue spaces $L^p(G)$ on G. Furthermore, the translation-invariant property of m gives rise to a convolution operation between Borel functions. In particular, given Borel functions f, g on G define their convolution to be the function

$$f * g(x) = \int_G f(y)g(x - y)\,\mathrm{d}y,$$

provided that for at least almost every $x \in G$ the function $y \mapsto f(y)g(x - y)$ is indeed integrable. The classical Young inequality for convolution extends to this setting; in particular, $f * g$ is well-defined as an $L^r(G)$ function whenever $f \in L^p(G)$ and $g \in L^q(G)$ and $1/p + 1/q = 1/r + 1$. Moreover, in this case one has the inequality

$$\|f * g\|_{L^r(G)} \leq \|f\|_{L^p(G)}\|g\|_{L^q(G)}.$$

The convolution operator can also be defined between regular Borel measures μ on G with finite total variation $\|\mu\|$. If $M(G)$ denotes the space of all such measures and $\mu, \lambda \in M(G)$, then one may show that the measure $\mu * \lambda \in M(G)$ where

$$\mu * \lambda(E) := \int_G \int_G \chi_E(x + y)\mathrm{d}\mu(x)\mathrm{d}\mu(y) \qquad \text{for all } E \subseteq G \text{ Borel}$$

and $\|\mu * \lambda\| \leq \|\mu\|\|\lambda\|$.

A *character* of G is a continuous group homomorphism from G to the circle $\{z \in \mathbb{C} : |z| = 1\}$. The set of all characters forms a LCA group under pointwise multiplication and the compact-open topology. This group is called the *Pontryagin dual group* (or simply the *dual group*) of G and is denoted by $\widehat{G}$. It is remarked that any $x \in G$ defines a character on the dual group $\widehat{G}$ via evaluation: $\xi \mapsto \xi(x)$. The Pontryagin duality theorem states that all characters of $\widehat{G}$ arise in this manner and, moreover, this identification between elements of G and characters of $\widehat{G}$ forms an isomorphism and a homeomorphism between G and its double dual.

A character $\xi \in \widehat{G}$ is *non-principal* if it is not the identity element in $\widehat{G}$: that is, if there exists some $x \in G$ such that $\xi(x) \neq 1$ (accordingly, the identity element is referred to as the *principal character*). If the Haar measure m is finite, so that $\xi \in L^1(G)$ for all $\xi \in \widehat{G}$, then

$$\int_G \xi(x)\,\mathrm{d}m(x) = \begin{cases} m(G) & \text{if } \xi \text{ is principal} \\ 0 & \text{if } \xi \text{ is non-principal.} \end{cases} \tag{A.1}$$

For $f \in L^1(G)$ the *Fourier transform* of f is the function $\hat{f} \in L^\infty(\widehat{G})$ given by

$$\hat{f}(\xi) = \int_G f(x)\overline{\xi(x)}\,\mathrm{d}m(x) \qquad \text{for all } \xi \in \widehat{G}.$$

More generally, if $\mu \in M(G)$, then the Fourier transform of μ is the function $\hat{\mu} \in L^\infty(\widehat{G})$ given by

$$\hat{\mu}(\xi) = \int_G \overline{\xi(x)}\,\mathrm{d}\mu(x) \qquad \text{for all } \xi \in \widehat{G}.$$

A basic and useful property of the Fourier transform is that given $f, g \in L^1(G)$ or $\mu, \lambda \in M(G)$ one has

$$\widehat{f * g} = \hat{f} \cdot \hat{g} \quad \text{and} \quad \widehat{\mu * \lambda} = \hat{\lambda} \cdot \hat{\mu}.$$

It is remarked that if μ and λ are absolutely continuous with respect to m with Radon–Nikodym derivatives f and g, respectively, then the two definitions of the Fourier transform and the two convolution identities coincide (that is, $\hat{\mu} = \hat{f}$ and $\lambda * \mu$ has Radon–Nikodym derivative $f * g$ with respect to m).

If $f \in L^1(G)$ and $\hat{f} \in L^1(\widehat{G})$, then the *Fourier inversion formula*

$$f(x) = \int_{\widehat{G}} \hat{f}(\xi)\xi(x)\,\mathrm{d}m(x) \qquad \text{for all } x \in G \tag{A.2}$$

holds provided the Haar measure is correctly normalised (indeed, (A.2) dictates the choice of normalisation). In view of this formula, if $f \in L^1(\widehat{G})$, then the *inverse Fourier transform* of f is the function $\check{f} \in L^\infty(G)$ given by

$$\check{f}(x) = \int_{\widehat{G}} f(x)\xi(x)\,\mathrm{d}\check{m}(\xi) \qquad \text{for all } x \in G,$$

where $\check{m}$ denotes the Haar measure on $\widehat{G}$. More generally, if $\mu \in M(\widehat{G})$, then the inverse Fourier transform of μ is the function $\check{\mu} \in L^\infty(G)$ given by

$$\check{\mu}(x) = \int_{\widehat{G}} \xi(x)\,\mathrm{d}\mu(\xi) \qquad \text{for all } x \in G.$$

In light of the Pontryagin duality theorem, one may write $\check{f}(x) = \hat{f}(-x)$ and $\check{\mu}(x) = \hat{\mu}(-x)$, where here $-x$ is thought of as a character on $\widehat{\widehat{G}}$.

For the purposes of this article, a key tool is the Plancherel theorem which states that the Fourier transform restricted to $L^2(G) \cap L^1(G)$ is an isometry with respect to the L^2-norm. Thus, the Fourier transform operator can be uniquely extended to a mapping on the whole of $L^2(G)$ and the Plancherel identity

$$\|\hat{f}\|_{L^2(\widehat{G})} = \|f\|_{L^2(G)} \qquad \text{for all } f \in L^2(G)$$

holds.

Appendix B. Basic elements of *p*-adic analysis

In this section some basic facts regarding analysis over $\mathbb{Q}_p$ are reviewed. Fixing a prime p, recall that the p-adic absolute value

$$|\cdot|_p \colon \mathbb{Z} \to \{0, p^{-1}, p^{-2}, \dots\}$$

is defined by

$$|x|_p := \begin{cases} p^{-k} & \text{if } x \neq 0 \text{ and } p^k \,\|x \text{ for } k \in \mathbb{N}_0 \\ 0 & \text{otherwise,} \end{cases}$$

where the notation $p^k \,\|\theta$ is used to denote that p^k divides θ (that is, $p^k \mid \theta$) and no larger power of p divides θ. The function $|\cdot|_p$ uniquely extends to a non-archimedean absolute value on the rationals $\mathbb{Q}$.[6] The field of p-adic

[6]That is, $|\cdot|_p \colon \mathbb{Q} \to [0,\infty)$ satisfies the following properties:

i) (Positive definite) $|x|_p \geq 0$ for all $x \in \mathbb{Q}$ and $|x|_p = 0$ if and only if $x = 0$;

numbers $\mathbb{Q}_p$ is defined to be the metric completion of $\mathbb{Q}$ under the metric induced by $|\cdot|_p$. One may verify that $\mathbb{Q}_p$ indeed has a natural field structure and contains $\mathbb{Q}$ as a subfield.

Any element $x \in \mathbb{Q}_p \setminus \{0\}$ admits a unique p-adic series expansion

$$x = \sum_{j=J}^{\infty} x_j p^j \tag{B.1}$$

where $J \in \mathbb{Z}$, $x_j \in \{0, 1, \ldots, p-1\}$ for all $j \in \mathbb{Z}$ with $x_J \neq 0$ (and $x_j := 0$ for $j < J$). The sum is understood as the limit of a sequence of rationals, where the convergence is with respect to the p-adic absolute value. In this case, $|x|_p = p^{-J}$. The ring of p-adic numbers $\mathbb{Z}_p$ is defined to be the set comprised of 0 together with all the elements $x \in \mathbb{Q}_p \setminus \{0\}$ for which $J \geq 0$ in the expansion (B.1). Thus, $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$, and this clearly forms a subring of $\mathbb{Q}_p$ by the multiplicative property of the absolute value.

The field $\mathbb{Q}_p$ is a locally compact abelian group under the addition operation and the Haar measure is, as usual, denoted by m; this measure is normalised so that $m(\mathbb{Z}_p) = 1$ (this is consistent with the choice of normalisation described in Appendix A: it ensures that the inversion formula and Plancherel's theorem hold). For any $\rho > 0$ and $x \in \mathbb{Q}_p$ the clopen ball $B_\rho(x)$ is defined as in §2 by

$$B_\rho(x) := \{y \in \mathbb{Q}_p : |x - y|_p \leq \rho\}.$$

For each $\alpha \in \mathbb{Z}$ the ball $B_{p^\alpha}(0) = p^{-\alpha}\mathbb{Z}_p$ is an additive subgroup of $\mathbb{Q}_p$, and all other balls of radius p^α arise as cosets of $B_{p^\alpha}(0)$. It immediately follows from the translation invariance property of the Haar measure (together with the choice of normalisation) that $m\big(B_{p^\alpha}(x)\big) = p^\alpha$ for all $\alpha \in \mathbb{Z}$ and $x \in \mathbb{Q}_p$.

The p-adic field $\mathbb{Q}_p$ is self-dual in the Pontryagin sense (and, consequently, so too are the vector spaces $\mathbb{Q}_p^n$). In particular, if one fixes an additive character $e\colon \mathbb{Q}_p \to \mathbb{T}$ such that e restricts to the constant function 1 on $\mathbb{Z}_p$ and to a non-principal character on $p^{-1}\mathbb{Z}_p$, then every character of $\mathbb{Q}_p^n$ can be realised as a map $e_\xi(x)$ for some $\xi \in \mathbb{Q}_p^n$ where $e_\xi(x) := e(x \cdot \xi)$ for all $x \in \mathbb{Q}_p^n$. Moreover, the map $\xi \mapsto e_\xi$ is an isomorphism and a homeomorphism between $\mathbb{Q}_p^n$ and $\widehat{\mathbb{Q}}_p^n$ and therefore the two groups are henceforth tacitly identified. As noted in §2, for any integrable $f\colon \mathbb{Q}_p^n \to \mathbb{C}$ the Fourier

ii) (Multiplicative) $|xy|_p = |x|_p|y|_p$ for all $x, y \in \mathbb{Q}$;
iii) (Strong triangle inequality) $|x + y|_p \leq \max\{|x|_p, |y|_p\}$ for all $x, y \in \mathbb{Q}$.

transform $\hat{f}$ can therefore be defined by

$$\hat{f}(\xi) := \int_{\mathbb{Q}_p^n} f(x)e(-x \cdot \xi)\, \mathrm{d}m(x) \qquad \text{for all } \xi \in \mathbb{Q}_p^n.$$

It is instructive to consider an explicit choice of character e. Define the fractional part function $\{\,\cdot\,\}_p \colon \mathbb{Q}_p \to \mathbb{Q}$ as follows: given $x \in \mathbb{Q}_p$ with p-adic expansion $\sum_{j=J}^{\infty} x_j p^j$, let $\{x\}_p := \sum_{j=J}^{-1} x_j p^j$. Observe that $\{x\}_p = 0$ if and only if $x \in \mathbb{Z}_p$. Defining $e \colon \mathbb{Q}_p \to \mathbb{T}$ by

$$e(x) := e^{2\pi i \{x\}_p} \quad \text{for all } x \in \mathbb{Q}_p,$$

it is easy to check that this function has the desired properties.

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