

Irreducible components of extended eigenvarieties and interpolating Langlands functoriality

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We study the basic geometry of a class of analytic adic spaces that arise in the study of the extended (or adic) eigenvarieties constructed by Andreatta–Iovita–Pilloni, Gulotta and the authors. We apply this to prove a general interpolation theorem for Langlands functoriality, which works for extended eigenvarieties and improves upon existing results in characteristic 0. As an application, we show that the characteristic p locus of the extended eigenvariety for GL_2/F , where F/\mathbb{Q} is a cyclic extension, contains non-ordinary components of dimension at least $[F : \mathbb{Q}]$.

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1. Introduction

1.1. Previous work

In a previous work [22], we have described a general construction of ‘extended eigenvarieties’: analytic adic spaces (of mixed characteristic), which contain the rigid analytic eigenvarieties constructed by Hansen [16] as an open subspace. This follows work of Andreatta, Iovita and Pilloni [2], who gave a (different) construction of an extended eigencurve, and there is also

independent work of Gulotta [14] which constructs extended versions of Urban’s eigenvarieties [28]. These newly constructed objects appear to be natural spaces in which to consider families of finite slope automorphic representations. Moreover, they provide a new perspective on the geometry of rigid analytic eigenvarieties. We refer to [2, 22] for more remarks and questions related to these extended eigenvarieties.

One basic question about extended eigenvarieties is: do they contain rigid analytic eigenvarieties as a *proper subset*? Results of Bergdall–Pollack [6] and Liu–Wan–Xiao [23] show that the extended eigencurve (and an analogue constructed using definite quaternion algebras over \mathbb{Q}) does indeed contain the Coleman–Mazur eigencurve as a proper subset. More precisely, they show that the extended eigencurve contains infinitely many non-ordinary points in characteristic p . One motivation for this article is to bootstrap this result, using a new result on p -adic interpolation of Langlands functoriality, to show that other extended eigenvarieties contain their rigid analytic counterpart as a proper subset, and give lower bounds on the dimension of the characteristic p locus.

1.2. Contents of this article

In this article, we begin by studying the types of analytic adic spaces that appear in the construction of extended eigenvarieties. We tentatively call them *pseudorigid spaces*, since they generalise rigid spaces (over complete discretely valued fields) and have the same key features. Our main focus is to establish some basic results about the geometry of pseudorigid spaces, including the existence of a Zariski topology, normalizations, irreducible components and a well behaved dimension theory. We refer to [11] for these notions in the setting of rigid analytic spaces, and our exposition is heavily influenced by this reference.

We then define a very general and elementary version of an abstract ‘eigenvariety datum’, which is more flexible than existing notions in the literature. Using it we prove an interpolation theorem (Thm. 3.2.1) which has the following somewhat imprecise form:

Theorem. *Suppose we have two extended eigenvarieties \mathcal{X}, \mathcal{Y} , a homomorphism between appropriate abstract Hecke algebras $\sigma : \mathbb{T}_{\mathcal{Y}} \rightarrow \mathbb{T}_{\mathcal{X}}$ and a collection of ‘classical points’ \mathcal{X}^{cl} whose associated systems of Hecke eigenvalues, when composed with σ , also appear in \mathcal{Y} . Then the associated map $\mathcal{X}^{\text{cl}} \rightarrow \mathcal{Y}$ interpolates into a canonical morphism from the Zariski closure of \mathcal{X}^{cl} in \mathcal{X} to \mathcal{Y} .*

This result gives a substantial and essentially optimal generalisation of previous interpolation theorems which have been used to study p -adic Langlands functoriality [10, Prop. 3.5], [5, Prop. 7.2.8], [16, Thm. 5.1.6] (see Remark 3.2.2 for more precise comments).

Somewhat surprisingly, both the statement and the proof of our theorem seem to be significantly simpler than existing results, and it seems to be very well-adapted to interpolating known cases of Langlands functoriality¹. In particular, our methods are rather different from those of Chenevier, Bellaïche–Chenevier and of Hansen, and ultimately rely on a simple trick that allows us to consider two eigenvarieties as Zariski closed subspaces inside a common, bigger, eigenvariety. After proving our interpolation theorem, we then briefly review our construction of extended eigenvarieties from [22]. We address, at the suggestion of a referee, a question not touched upon in [22] concerning the dependence of our construction on the choice of controlling operator, and we also generalise a result giving a lower bound for the dimensions of irreducible components of eigenvarieties [16, Prop. B.1] to extended eigenvarieties (Prop. 3.3.2).

Finally, as an example application, we apply the interpolation theorem to establish the existence of a base change map between the extended eigencurve and the extended eigenvarieties for GL_2 over a cyclic extension F/\mathbb{Q} (Thm. 4.3.1). Combining this with our result on the dimensions of irreducible components shows that the characteristic p loci of these extended eigenvarieties contain (non-ordinary) components of dimension at least $[F : \mathbb{Q}]$ (Cor. 4.3.2). In particular, they strictly contain their rigid analytic loci (where p is invertible). We also outline how the same strategy would allow one to give generalisations of this result (Rem 4.3.3).

To conclude this introduction, we remark that recent work of Lourenço [24] studies the geometry of pseudorigid spaces further, in particular extending results of Bartenwerfer and Lütkebohmert on extending (bounded) functions on normal rigid spaces to normal pseudorigid spaces.

¹More precisely (but still somewhat imprecisely), Langlands functoriality implies transfer of systems of Hecke eigenvalues, and our theorem and its predecessors allow for interpolation of the transfer of systems of Hecke eigenvalues. In general this information is coarser than the transfer of L-packets.

2. Irreducible components and normalization

2.1. The Zariski topology for locally Noetherian adic spaces

For basics on adic spaces we refer to [18–20]. Following now common terminology, we will say that an adic space X is *locally Noetherian* if it is locally of the form $\mathrm{Spa}(R, R^+)$, where R has a Noetherian ring of definition over which R is finitely generated, or locally of the form $\mathrm{Spa}(S, S^+)$, where S is strongly Noetherian. To make things slightly easier, we will say that an affinoid ring (A, A^+) is Noetherian if A is strongly Noetherian, or admits a Noetherian ring of definition over which it is finitely generated. All affinoid rings in this paper will be assumed complete unless otherwise stated.

Let us start with a general observation. If (R, R^+) is any affinoid ring, then there is a continuous map $\mathrm{Spa}(R, R^+) \rightarrow \mathrm{Spec}(R)$ sending a valuation v to its kernel (or support) $\mathrm{Ker} v$. This map is functorial in (R, R^+) . One may define the *Zariski topology* on $\mathrm{Spa}(R, R^+)$ to be the topology whose open sets are the preimages of the open sets in $\mathrm{Spec}(R)$. It is not obvious to us how one would extend this construction to arbitrary adic spaces, since one cannot simply ‘glue’ these topologies (if $\mathrm{Spa}(S, S^+) \subseteq \mathrm{Spa}(R, R^+)$ is a rational subset, then the inclusion is continuous for the Zariski topology, but there is no reason that it should be an open embedding). Our goal in this section is to give a definition of the Zariski topology for arbitrary *locally Noetherian* adic spaces which recovers the definition above in the affinoid case, and show that this gives a natural notion of irreducibility and irreducible components.

Let X be a locally Noetherian adic space. By [19, Theorem 2.5], X has a good theory of coherent \mathcal{O}_X -modules. If $X = \mathrm{Spa}(R, R^+)$ is affinoid with (R, R^+) Noetherian, then there is an equivalence of abelian categories between finitely generated R -modules and coherent \mathcal{O}_X -modules, given by sending a coherent \mathcal{O}_X -module \mathcal{F} to its global sections $\mathcal{F}(X)$, and sending a finitely generated R -module M to the sheaf \widetilde{M} defined by

$$\widetilde{M}(U) = M \otimes_R \mathcal{O}_X(U)$$

whenever $U \subseteq X$ is a rational subset. We say that a coherent \mathcal{O}_X -module \mathcal{I} is a *coherent \mathcal{O}_X -ideal* (or *coherent ideal* for short, if no confusion seems likely to arise) if it is a sub- \mathcal{O}_X -module of \mathcal{O}_X . By the construction in [20, (1.4.1)], any coherent ideal \mathcal{I} gives rise to a *closed adic subset*

$$V(\mathcal{I}) = \{x \in X : \mathcal{I}_x \neq \mathcal{O}_{X,x}\}$$

of X (in particular, this is a closed subset of X). Locally, if $X = \text{Spa}(R, R^+)$ with (R, R^+) Noetherian and $\mathcal{I} = \tilde{I}$, we have $V(\mathcal{I}) = \text{Spa}(R/I, (R/I)^+)$, where $(R/I)^+$ is defined to be the integral closure of $R^+/(I \cap R^+)$ in R/I .

Let us return to the case of general locally Noetherian X . We wish to check that the closed adic subsets of X form the closed subsets of a topology. To do this, let us define some operations on coherent ideals. If $(\mathcal{I}_j)_{j \in J}$ is a collection of coherent ideals, then we define $\sum_{j \in J} \mathcal{I}_j$ to be the sheaf associated with the presheaf

$$U \mapsto \sum_{j \in J} \mathcal{I}_j(U) \subseteq \mathcal{O}_X(U).$$

It is a subsheaf of \mathcal{O}_X by construction. If \mathcal{I}_1 and \mathcal{I}_2 are two coherent ideals, we define their intersection $\mathcal{I}_1 \cap \mathcal{I}_2$ to be the sheaf $U \mapsto \mathcal{I}_1(U) \cap \mathcal{I}_2(U)$. Note that these constructions commute with restriction to open subsets. We record the following elementary lemma.

Lemma 2.1.1. *Let (R, R^+) be Noetherian. If $(I_j)_{j \in J}$ is a collection of ideals of R and $I = \sum_{j \in J} I_j$, then $\tilde{I} = \sum_{j \in J} \tilde{I}_j$. If I_1 and I_2 are two ideals of R , then $\widetilde{I_1 \cap I_2} = \tilde{I}_1 \cap \tilde{I}_2$.*

Proof. The second statement is straightforward to deduce from the definitions, so we content ourselves with proving the first statement. By the definitions, \tilde{I} is the sheafification of a presheaf which on rational subsets U is given by

$$U \mapsto \sum_{j \in J} (I_j \otimes_R \mathcal{O}_X(U)).$$

By flatness of $R \rightarrow \mathcal{O}_X(U)$, we have $I_j \otimes_R \mathcal{O}_X(U) = I_j \mathcal{O}_X(U)$ for all $j \in J$, so it suffices to prove that $\sum_{j \in J} I_j \mathcal{O}_X(U) = I \mathcal{O}_X(U)$ (since the latter is equal to $I \otimes_R \mathcal{O}_X(U)$, again by flatness). By definition I is the image of the the natural map $\bigoplus_{j \in J} I_j \rightarrow R$, and $\sum_{j \in J} I_j \mathcal{O}_X(U)$ is the image of the same map after applying $- \otimes_R \mathcal{O}_X(U)$. The statement now follows from flatness of $R \rightarrow \mathcal{O}_X(U)$. \square

Let us now prove that when $X = \text{Spa}(R, R^+)$ with (R, R^+) Noetherian, the Zariski closed subsets of $\text{Spa}(R, R^+)$ (according to our general definition) are exactly the closed adic subsets.

Proposition 2.1.2. *Let (R, R^+) be Noetherian and put $X = \text{Spa}(R, R^+)$. Let I be an ideal of R . Then $V(\tilde{I})$ is the preimage of the closed subset of $\text{Spec}(R)$ corresponding to I .*

Proof. Since all ideals of R are closed and R/I carries the quotient topology coming from R , this is essentially trivial; any $v \in X$ belongs to

$$\mathrm{Spa}(R/I, (R/I)^+)$$

if and only if $I \subseteq \mathrm{Ker} v$. □

Corollary 2.1.3. *Let X be a locally Noetherian adic space.*

- 1) $V(0) = X$, and $V(\mathcal{O}_X) = \emptyset$.
- 2) If $(\mathcal{I}_j)_{j \in J}$ is a collection of coherent ideals, then $\mathcal{I} = \sum_{j \in J} \mathcal{I}_j$ is a coherent ideal, and $V(\mathcal{I}) = \bigcap_{j \in J} V(\mathcal{I}_j)$.
- 3) If \mathcal{I}_1 and \mathcal{I}_2 are two coherent ideals, then $\mathcal{I}_1 \cap \mathcal{I}_2$ is a coherent ideal and $V(\mathcal{I}_1 \cap \mathcal{I}_2) = V(\mathcal{I}_1) \cup V(\mathcal{I}_2)$.

Proof. It is enough to prove these statements locally, so we assume that $X = \mathrm{Spa}(R, R^+)$ with (R, R^+) Noetherian. The Corollary then follows directly from Lemma 2.1.1 and Proposition 2.1.2. □

It now makes sense to make the following definition.

Definition 2.1.4. Let X be a locally Noetherian adic space. We define the *Zariski topology* of X to be the topology on X whose closed sets are the closed adic subsets of X .

In light of this definition we will refer to closed adic subsets of X as *Zariski closed subsets of X* . Next, we will discuss more general coherent sheaves. Let X be a locally Noetherian adic space and let \mathcal{F} be a coherent \mathcal{O}_X -module. We define its support $\mathrm{Supp}(\mathcal{F})$ by $\{x \in X \mid \mathcal{F}_x \neq 0\}$. Note that $V(\mathcal{I}) = \mathrm{Supp}(\mathcal{O}_X/\mathcal{I})$.

We now show that $\mathrm{Supp}(\mathcal{F})$ is closed (in the usual topology). It is enough to verify this locally so assume $X = \mathrm{Spa}(R, R^+)$ with (R, R^+) Noetherian and $\mathcal{F} = \widetilde{M}$. Suppose $x \notin \mathrm{Supp}(\mathcal{F})$. We have $\widetilde{M}_x = \varinjlim_{U \ni x} M \otimes_R \mathcal{O}_X(U) = 0$ where U is rational. If m_1, \dots, m_r are generators of \widetilde{M} , we see that there is a U such that the m_i vanish in $M \otimes_R \mathcal{O}_X(U)$. It follows that $\mathcal{F}(U) = M \otimes_R \mathcal{O}_X(U) = 0$, and we deduce that $\mathrm{Supp}(\mathcal{F})$ is closed. Our next goal is to show that it is Zariski closed. We define the *annihilator* $\mathrm{Ann}(\mathcal{F})$ of \mathcal{F} by

$$(\mathrm{Ann}(\mathcal{F}))(U) = \{f \in \mathcal{O}_X(U) \mid f\mathcal{F}_x = 0 \ \forall x \in U\}.$$

This is an \mathcal{O}_X -subsheaf of \mathcal{O}_X . Since \mathcal{F} is coherent, we have that $\mathrm{Ann}(\mathcal{F})_x = \mathrm{Ann}(\mathcal{F}_x)$, where the right hand side is the annihilator of the $\mathcal{O}_{X,x}$ -module \mathcal{F}_x .

Lemma 2.1.5. *Let (R, R^+) be Noetherian and let M be a finitely generated R -module. Then $\text{Ann}(\widetilde{M}) = \widetilde{\text{Ann}(M)}$.*

Proof. Recall that the formation of annihilators of finitely generated modules commute with flat base change [26, Tag 07T8]. Let U be a rational subset of $X = \text{Spa}(R, R^+)$. First note that $\text{Ann}(M)\mathcal{O}_X(U) = \text{Ann}(M \otimes_R \mathcal{O}_X(U)) \subseteq \text{Ann}(\widetilde{M})(U)$, where we have used the above fact for the first equality. We need to prove the opposite inclusion. Let $f \in \text{Ann}(\widetilde{M})(U)$ and let $x \in U$. f kills $\widetilde{M}_x = \varinjlim_{x \in V \subseteq U} M \otimes_R \mathcal{O}_X(V)$, and hence (by finite generation) has to kill $M \otimes_R \mathcal{O}_X(U_x)$ for some rational subset $U_x \subseteq U$ containing x . It follows that we may find an open cover $U = \bigcup_{x \in U} U_x$ of rational subsets such that $f \in \text{Ann}(M \otimes_R \mathcal{O}_X(U_x)) = \text{Ann}(M)\mathcal{O}_X(U_x)$. This implies that $f \in \text{Ann}(M)\mathcal{O}_X(U)$, as desired. \square

Corollary 2.1.6. *Let X be a locally Noetherian adic space and \mathcal{F} a coherent \mathcal{O}_X -module. Then $\text{Ann}(\mathcal{F})$ is coherent \mathcal{O}_X -ideal and*

$$\text{Supp}(\mathcal{F}) = V(\text{Ann}(\mathcal{F})),$$

so $\text{Supp}(\mathcal{F})$ is Zariski closed. If $f : X' \rightarrow X$ is a morphism of locally Noetherian adic spaces and $Z \subseteq X$ is Zariski closed, then $f^{-1}(Z) \subseteq X'$ is Zariski closed. If f is finite and $Z' \subseteq X'$ is Zariski closed, then $f(Z') \subseteq X$ is Zariski closed.

Proof. $\text{Ann}(\mathcal{F})$ is a coherent ideal by Lemma 2.1.5. To see that $\text{Supp}(\mathcal{F}) = V(\text{Ann}(\mathcal{F}))$, note that, for any $x \in X$, $\text{Ann}(\mathcal{F})_x = \text{Ann}(\mathcal{F}_x)$ and that $\text{Ann}(\mathcal{F}_x) = \mathcal{O}_{X,x}$ if and only if $\mathcal{F}_x = 0$. For the second statement, choose a coherent \mathcal{O}_X -ideal \mathcal{I} such that $Z = V(\mathcal{I})$. Then $f^{-1}(Z) = \text{Supp}(f^*(\mathcal{O}_X/\mathcal{I}))$ is Zariski closed by the first statement of the corollary.

It remains to prove the final statement. Replacing X' by Z' (choosing some coherent $\mathcal{O}_{X'}$ -ideal \mathcal{J} with $Z' = V(\mathcal{J})$) we may assume that $X' = Z'$. We wish to prove that $f(X') = \text{Supp}(f_*\mathcal{O}_{X'})$. Since f is finite, $f(X')$ is closed, so we must have $\text{Supp}(f_*\mathcal{O}_{X'}) \subseteq f(X')$. If the inclusion is proper, we can find an open affinoid set $U \subseteq X$ which intersects $f(X')$ but is disjoint from $\text{Supp}(f_*\mathcal{O}_{X'})$, since $\text{Supp}(f_*\mathcal{O}_{X'})$ is closed. But then we must have $(f_*\mathcal{O}_{X'})|_U = 0$, but also $(f_*\mathcal{O}_{X'})(U) = \mathcal{O}_{X'}(f^{-1}(U)) \neq 0$ since $f^{-1}(U) \neq \emptyset$, a contradiction. This finishes the proof. \square

In particular, if $f : X \rightarrow Y$ is a morphism of locally Noetherian adic spaces, then it is continuous with respect to the Zariski topologies on X and

Y . Let us record a few basic general topology facts about the Zariski topology. Recall that a topological space Z is said to be *Noetherian* if it satisfies the descending chain condition for closed subsets. We say that a topological space is *locally Noetherian* if it has an open cover by Noetherian spaces. Note that if a topological space is quasicompact and locally Noetherian, then it is Noetherian.

Lemma 2.1.7. *Let X be a locally Noetherian adic space.*

- 1) *If $X = \mathrm{Spa}(R, R^+)$ is affinoid with (R, R^+) Noetherian, then X is Noetherian for the Zariski topology.*
- 2) *X is locally Noetherian for the Zariski topology, and hence if X is quasicompact then X is Noetherian for the Zariski topology.*

Proof. For (1), note that if (R, R^+) is Noetherian then R is a Noetherian ring, so the statement follows from Proposition 2.1.2 and the fact that $\mathrm{Spec}(R)$ is Noetherian. For (2), note that if $U = \mathrm{Spa}(S, S^+) \subseteq X$ is an open affinoid with (S, S^+) Noetherian, then the subspace topology on U coming from the Zariski topology on X is coarser than the Zariski topology on U . It follows from (1) that U is Noetherian with respect to the subspace topology, which proves that X is locally Noetherian. \square

Let us start to discuss irreducible components, by making the following standard definition:

Definition 2.1.8. Let X be a locally Noetherian adic space. A Zariski closed subset Z is said to be *irreducible* if, for any Zariski closed subsets Z_1, Z_2 of X , $Z \subseteq Z_1 \cup Z_2$ implies $Z \subseteq Z_1$ or $Z \subseteq Z_2$. Z is said to be an *irreducible component* of X if it is irreducible and not properly contained in any other irreducible Zariski closed set.

There are a number of things we would expect from a satisfactory theory of irreducible components. In particular, we would like to be able to write X as the union of its irreducible components (and that the union of any proper subset of irreducible components is not the whole of X), and we would like to be able to give any Zariski closed subset a *canonical* structure of a reduced locally Noetherian adic space. For now, we will not say so much about the first question, except to note that if X is quasicompact, then the Zariski topology is Noetherian by above and we have a satisfactory notion of irreducible components. Our intended applications, however, are to eigenvarieties, which are not quasicompact.

2.2. Pseudorigid spaces

Let K be a complete discretely valued field with ring of integers \mathcal{O}_K , a uniformizer π_K and residue field k . We start by giving a name to the family of Tate rings over \mathcal{O}_K that we will be working with in this paper. Whenever we drop R^+ from the notation, it is assumed that $R^+ = R^\circ$. Accordingly, we will often conflate an f-adic ring R with the affinoid ring (R, R°) .

Definition 2.2.1. Let R be a Tate \mathcal{O}_K -algebra. We say that R is a *Tate ring formally of finite type over \mathcal{O}_K* ² if R has a ring of definition R_0 which is formally of finite type over \mathcal{O}_K in the usual sense (i.e. has a radical ideal of definition I such that R_0/I is a finitely generated k -algebra).

Remark 2.2.2. The topology on a ring of definition R_0 for a Tate \mathcal{O}_K -algebra R is the (ϖ) -adic topology for some topologically nilpotent unit $\varpi \in R$. We obtain a map $\mathcal{O}_K[[X]] \rightarrow R_0$ defined by $X \mapsto \varpi$ and R_0 is formally of finite type over \mathcal{O}_K if and only if it is topologically of finite type over $\mathcal{O}_K[[X]]$, i.e. if and only if it is (topologically) isomorphic to a quotient of $\mathcal{O}_K[[X]]\langle Y_1, \dots, Y_n \rangle$ for some n .

We note that any ideal I of a Tate ring R formally of finite type over \mathcal{O}_K is closed, and that if S is an f-adic ring which is topologically of finite type over R then S is a Tate ring formally of finite type over \mathcal{O}_K as well. In particular, all quotients and all rational localizations of R are Tate rings formally of finite type over \mathcal{O}_K . Let us recall a few results about Tate rings formally of finite type over \mathcal{O}_K from [22, Appendix A].

Proposition 2.2.3. *Let R and S be Tate rings formally of finite type over \mathcal{O}_K , and let $f : R \rightarrow S$ be a continuous homomorphism which induces an open immersion $\mathrm{Spa}(S, S^+) \rightarrow \mathrm{Spa}(R)$ for some ring S^+ of integral elements in S . Then the following hold:*

- 1) R is Jacobson and excellent.
- 2) $S^+ = S^\circ$.
- 3) If \mathfrak{m} is a maximal ideal of S , then $f^{-1}(\mathfrak{m})$ is a maximal ideal of R , and the natural map $R_{f^{-1}(\mathfrak{m})} \rightarrow S_{\mathfrak{m}}$ induces an isomorphism on completions.

²In [24], these are called pseudoaffinoid \mathcal{O}_K -algebras.

- 4) If $R \rightarrow T$ is finite and $(R, R^\circ) \rightarrow (T, T^+)$ is the corresponding finite morphism of affinoid rings, then $T^+ = T^\circ$.
- 5) If R is reduced, then S is reduced.

Proof. This consists of collecting well known results together with results from [22] (which in turn are deduced mostly from well known results and results from [1]). We start with part (1). R is Jacobson by [22, Lemma A.1]. If R_0 is a ring of definition of R which is formally of finite type over \mathcal{O}_K , then R_0 is excellent by [29, Proposition 7] and [30, Theorem 9], and hence R is excellent since it is finitely generated over R_0 . Part (2) follows from [22, Theorem A.7] in the reduced case, but the argument to show that $S^+ = S^\circ$ works without the reducedness assumption. Part (3) follows from [22, Proposition A.15(2)]. Part (4) follows from [22, Lemma A.3]. Part (5) follows from parts (1) and (2) together with standard properties reducedness and excellence, cf. the first paragraph of [11, §1.2], or alternatively from [22, Theorem A.7]. \square

Next, we make a global definition.

Definition 2.2.4. Let X be an adic space over \mathcal{O}_K . We say that X is a *pseudorigid space over \mathcal{O}_K* if it is locally of the form $\mathrm{Spa}(R)$, for R a Tate ring formally of finite type over \mathcal{O}_K .

We remark that any rigid space over K is a pseudorigid space over \mathcal{O}_K , where by rigid space over K we mean an adic space X over K which is locally of the form $\mathrm{Spa}(R)$, for R topologically of finite type over K . We will say that X is an *affinoid pseudorigid space over \mathcal{O}_K* if it is equal to $\mathrm{Spa}(R)$ for some R which is a Tate ring formally of finite type over \mathcal{O}_K . Our next goal is to single out the points on a pseudorigid space that locally come from maximal ideals.

Lemma 2.2.5. *Let R be a Tate ring formally of finite type over \mathcal{O}_K , and let $\mathfrak{m} \subseteq R$ be a maximal ideal. Then there exists a unique $v \in \mathrm{Spa}(R)$ with $\mathrm{Ker} v = \mathfrak{m}$. This gives a canonical embedding of the spectrum $\mathrm{Max}(R)$ of maximal ideals into $\mathrm{Spa}(R)$, compatible with the morphism $\mathrm{Spa}(R) \rightarrow \mathrm{Spec}(R)$.*

Proof. The locus of $v \in \mathrm{Spa}(R)$ with $\mathrm{Ker} v = \mathfrak{m}$ is equal to $\mathrm{Spa}(R/\mathfrak{m})$ (here we use Prop. 2.2.3(4)), so we have to prove that this is a singleton. Let R_0 be a ring of definition formally of finite type over \mathcal{O}_K , and let $\varpi \in R$ be a topologically nilpotent unit, which we assume lies in R_0 . Let $\mathfrak{p} = R_0 \cap \mathfrak{m}$.

The quotient ring R/\mathfrak{m} carries the topology where R_0/\mathfrak{p} is open and carries the $\overline{\varpi}$ -adic topology, where $\overline{\varpi}$ is the reduction of ϖ . Since \mathfrak{p} is closed point in $\mathrm{Spec}(R_0) \setminus \{\varpi = 0\} = \mathrm{Spec}(R)$, R_0/\mathfrak{p} is a 1-valuation order. Here we refer to [1, Definition 1.11.1] or [22, Definition A.11] for the definition of a 1-valuation order (at least in the Noetherian setting), and [1, Proposition 1.11.8] for the implication (which is an equivalence). As a result, $R/\mathfrak{m} = (R_0/\mathfrak{p})[1/\overline{\varpi}]$ is a complete discretely valued field, with valuation ring the integral closure of R_0/\mathfrak{p} , and R_0/\mathfrak{p} is open in the valuation topology. It follows that the valuation topology is equal to the quotient topology coming from R , and hence that $\mathrm{Spa} R/\mathfrak{m}$ is a singleton, as desired. This proves the first statement, and the remaining statements are immediate. \square

Proposition 2.2.6. *Let R and S be Tate rings formally of finite type over \mathcal{O}_K , and let $f : R \rightarrow S$ be a continuous \mathcal{O}_K -homomorphism. Then f is topologically of finite type, and preimages of maximal ideals are maximal ideals. Hence the induced map $\phi : \mathrm{Spa}(S) \rightarrow \mathrm{Spa}(R)$ maps $\mathrm{Max}(S)$ into $\mathrm{Max}(R)$. Moreover, $\mathrm{Max}(R)$ is dense in $\mathrm{Spa}(R)$.*

Proof. The first part is [22, Corollary A.14] (the statement is only for $K = \mathbb{Q}_p$, but the proof works the same). The second part follows immediately. For the third part, note that if $U \subseteq \mathrm{Spa}(R)$ is a rational subset, then the first part implies that $\mathrm{Max}(\mathcal{O}_X(U)) = U \cap \mathrm{Max}(R)$. It follows that $U \cap \mathrm{Max}(R)$ is non-empty if U is, and hence that $\mathrm{Max}(R)$ is dense. \square

Armed with this we may generalize the notion of ‘classical points’ on a rigid space to pseudorigid spaces.

Definition 2.2.7. Let X be a pseudorigid space. We say that a point $x \in X$ is *maximal* if there exists an open affinoid pseudorigid neighbourhood $U = \mathrm{Spa}(R)$ of x such that $x \in \mathrm{Max}(R) \subseteq U$. We denote the set of maximal points by $\mathrm{Max}(X)$.

If $x \in \mathrm{Max}(X)$ and $V = \mathrm{Spa}(S) \subseteq X$ is an open affinoid pseudorigid neighbourhood, then by Proposition 2.2.6 $x \in \mathrm{Max}(S)$. Moreover, it also follows that $\mathrm{Max}(X)$ is dense, that $\mathrm{Max}(X) = \mathrm{Max}(R)$ if $X = \mathrm{Spa}(R)$ is affinoid pseudorigid, and that any morphism $f : X \rightarrow Y$ of pseudorigid spaces maps $\mathrm{Max}(X)$ into $\mathrm{Max}(Y)$.

Let us now discuss the Zariski topology for pseudorigid spaces. To do this, we need the radical of a coherent ideal, following [7, (9.5.1)]. Let X be a pseudorigid space and let \mathcal{I} be a coherent \mathcal{O}_X -ideal. We define the radical

$\sqrt{\mathcal{I}}$ of \mathcal{I} to be the sheaf on X associated with the presheaf

$$U \mapsto \sqrt{\mathcal{I}(U)},$$

where, if A is any ring and $J \subseteq A$ is an ideal, \sqrt{J} denotes the radical of J . This construction commutes with restriction to open subsets, and defines an \mathcal{O}_X -ideal, which we would like to be coherent. One checks, using that the formation of the (usual) radical commutes with filtered direct limits, that [7, §9.5.1 Proposition 1] goes through in our setting, with the same proof. In particular, if $\sqrt{\mathcal{I}}$ is coherent, then $V(\mathcal{I}) = V(\sqrt{\mathcal{I}})$ and $\sqrt{\mathcal{I}}|_U$ is the sheaf attached to the ideal $\sqrt{\mathcal{I}(U)}$ for any affinoid open $U = \text{Spa}(S, S^+)$ with (S, S^+) Noetherian. So, to prove that $\sqrt{\mathcal{I}}$ is coherent, we may reduce to the affinoid case.

Proposition 2.2.8. *Let $X = \text{Spa}(R)$ be an affinoid pseudorigid space. Let $I \subseteq R$ be an ideal and put $\mathcal{I} = \tilde{I}$. Then $\sqrt{\mathcal{I}}$ is a coherent ideal, associated with \sqrt{I} .*

Proof. The argument in the proof of [7, §9.5.1 Proposition 2] goes through, if we can verify that, for any rational subset $U \subseteq X$, $\sqrt{I}\mathcal{O}_X(U)$ is a radical ideal in $\mathcal{O}_X(U)$. But $\mathcal{O}_X(U)/\sqrt{I}\mathcal{O}_X(U)$ is a rational localization of the reduced Tate ring R/\sqrt{I} , so it is reduced by Proposition 2.2.3(5), which is what we wanted to prove. \square

We also have the following familiar property of the radical.

Proposition 2.2.9. *Let X be a pseudorigid space over \mathcal{O}_K and let \mathcal{I}, \mathcal{J} be two coherent \mathcal{O}_X -ideals with $V(\mathcal{I}) = V(\mathcal{J})$. Then $\sqrt{\mathcal{I}} = \sqrt{\mathcal{J}}$.*

Proof. It suffices to prove this locally, so we may reduce to the case $X = \text{Spa}(R)$ and $\mathcal{I} = \tilde{I}$, $\mathcal{J} = \tilde{J}$ with I, J ideals of R . If $\text{Spa}(R/I) = V(\mathcal{I}) = V(\mathcal{J}) = \text{Spa}(R/J)$ as subsets of $\text{Spa}(R)$, it follows that $\text{Max}(R/I) = \text{Max}(R/J)$ as subsets of $\text{Spec}(R)$. Since R/I and R/J are Jacobson (by Proposition 2.2.3(1)), $\text{Max}(R/I)$ is dense in $\text{Spec}(R/I)$ and similarly for R/J , so we deduce that $\text{Spec}(R/I) = \text{Spec}(R/J)$ and hence that $\sqrt{I} = \sqrt{J}$, which finishes the proof. \square

We formulate the upshot of the previous two propositions in the following Corollary.

Corollary 2.2.10. *Let X be a pseudorigid space over \mathcal{O}_K . Then any Zariski closed subset $Z = V(\mathcal{I})$ of X has a canonical structure of a reduced locally*

Noetherian adic space, with structure sheaf $\mathcal{O}_X/\sqrt{\mathcal{I}}$. We call this the reduced structure on Z .

Proof. $\sqrt{\mathcal{I}}$ is coherent and $Z = V(\sqrt{\mathcal{I}})$ by Proposition 2.2.8 and the discussion preceding it, and $\sqrt{\mathcal{I}}$ only depends on Z (and not on the choice of \mathcal{I}) by Proposition 2.2.9. \square

2.3. Normalizations of pseudorigid spaces

The goal of this subsection is to construct a theory of normalizations for pseudorigid spaces over \mathcal{O}_K .

Definition 2.3.1. Let X be a pseudorigid space over \mathcal{O}_K . We say that X is *normal* if X is locally of the form $\mathrm{Spa}(R)$, where R is a normal Tate ring formally of finite type over \mathcal{O}_K .

This definition is well behaved, in the following sense.

Lemma 2.3.2. *Let $X = \mathrm{Spa}(R)$ be an affinoid pseudorigid space over \mathcal{O}_K .*

- 1) *If R is normal and $U \subseteq X$ is a rational subset, then $\mathcal{O}_X(U)$ is normal.*
- 2) *If X is normal, then R is normal.*

Proof. This follows from Proposition 2.2.3 by standard arguments. We start with (1). Put $S = \mathcal{O}_X(U)$ and let \mathfrak{n} be any maximal ideal of S . It suffices to prove that $S_{\mathfrak{n}}$ is normal. Let \mathfrak{m} be the preimage of \mathfrak{n} in R . This is a maximal ideal and the map $R_{\mathfrak{m}} \rightarrow S_{\mathfrak{n}}$ induces an isomorphism on completions. Since R is normal and excellent, so is $R_{\mathfrak{m}}$ and hence its completion. By excellence of $S_{\mathfrak{n}}$, $S_{\mathfrak{n}}$ is therefore normal as well, which finishes the proof. The same argument, but reversed, proves (2). \square

We will need the following lemma, which is a partial generalization of [11, Lemma 2.1.4].

Lemma 2.3.3. *Let X be a normal connected pseudorigid space over \mathcal{O}_K and let $Z \subseteq X$ be a Zariski closed subset. If Z contains a nonempty open subset of X , then $Z = X$.*

Proof. Let us first assume that $X = \mathrm{Spa}(R)$ is affinoid pseudorigid, in which case R is normal by Lemma 2.3.2. Since X is connected, R is in fact a normal domain. Let $Z = V(\tilde{I})$ for some ideal I of R and assume that it contains

an open subset $U \subseteq X$, which we may assume to be a connected rational subset. Set $S = \mathcal{O}_X(U)$; S is a normal domain. Pick $f \in I$, we want to show that $f = 0$. Since f vanishes on Z , it must map to 0 in S , so it suffices to prove that the map $R \rightarrow S$ is injective. Pick any maximal ideal \mathfrak{n} of S and let \mathfrak{m} be its preimage in R . Composing $R \rightarrow S$ with the natural map $S \rightarrow \widehat{S}_{\mathfrak{n}}$, it suffices to prove that $R \rightarrow \widehat{R}_{\mathfrak{m}}$ is injective. This morphism factors as

$$R \rightarrow R_{\mathfrak{m}} \rightarrow \widehat{R}_{\mathfrak{m}} \rightarrow \widehat{S}_{\mathfrak{n}}$$

so it suffices to prove that these three maps are injective. The first is injective since R is a domain, the second is injective by Krull's intersection theorem, and the third is an isomorphism by Proposition 2.2.3(3). This finishes the proof in the affinoid case.

We now do the general case. Let \mathcal{I} be a coherent ideal such that $Z = V(\mathcal{I})$; we wish to show that $\mathcal{I} = 0$. Define Σ to be the set of open nonempty affinoid pseudorigid subspaces $V \subseteq X$ such that $\mathcal{I}|_V = 0$, and define Δ to be the set of open nonempty affinoid pseudorigid subspaces $W \subseteq X$ such that $\mathcal{I}|_W \neq 0$. Clearly $\Sigma \cap \Delta = \emptyset$, and we claim that if $V \in \Sigma$ and $W \in \Delta$ then $V \cap W = \emptyset$. Assume not, and let $G \subseteq V \cap W$ be nonempty open affinoid pseudorigid. Since $G \subseteq V$ we must have $\mathcal{I}|_G = 0$. But this means that $Z \cap W$, which is Zariski closed in W but not equal to W , contains a nonempty open subset $G \subseteq W$, which contradicts the Lemma in the affinoid case. It follows that $V \cap W = \emptyset$. Now set $U_{\Sigma} = \bigcup_{V \in \Sigma} V$ and $U_{\Delta} = \bigcup_{W \in \Delta} W$. These are disjoint open subsets and $X = U_{\Sigma} \cup U_{\Delta}$. By assumption we know that $U_{\Sigma} \neq \emptyset$, so by connectedness of X we must have $\Delta = \emptyset$, and hence $\mathcal{I} = 0$ and $Z = X$. \square

Note that Lemma 2.3.3 implies that a normal connected quasicompact pseudorigid space X is irreducible. Indeed, normality implies that it is reduced, and if we write X as union $X = Z_1 \cup Z_2$ of two Zariski closed subsets with $Z_1 \neq X$, then Z_2 contains the nonempty open subset $X \setminus Z_1$ and therefore has to equal X .

Let R be a Tate ring formally of finite type over \mathcal{O}_K . We denote by \widetilde{R} the *normalization* of R , i.e. the integral closure of R in its total ring of fractions. Since R is excellent, \widetilde{R} is finite over R and hence a Tate ring formally of finite type over \mathcal{O}_K . If $X = \text{Spa}(R)$, we put $\widetilde{X} = \text{Spa}(\widetilde{R})$ and call it, together with its canonical map $p : \widetilde{X} \rightarrow X$, the *normalization of X* . The morphism $p : \widetilde{X} \rightarrow X$ is finite and surjective (to see that it is surjective, use for example that p is closed since it is finite, and maps $\text{Max}(\widetilde{X})$ onto $\text{Max}(X)$). Next we show this construction glues, for which we need to verify that it commutes with rational localization.

Lemma 2.3.4. *Let R be a Tate ring formally of finite type and S a rational localization of R . Then there is a natural isomorphism $\tilde{R} \otimes_R S \cong \tilde{S}$, which is unique over S .*

Proof. Let I be the nilradical of R (which is also the kernel of $R \rightarrow \tilde{R}$). By the theory of the radical of coherent $\mathcal{O}_{\mathrm{Spa}(R)}$ -ideals developed above, IS is the nilradical of S . One then checks that $\tilde{R} \otimes_R S \cong \tilde{R} \otimes_{R/I} S/IS$ and that $\widetilde{S/IS} = \tilde{S}$, so we may reduce to the case when R , and hence S , is reduced.

It then suffices to show that $S \rightarrow \tilde{R} \otimes_R S$ is a normalization, since a normalization of S is unique up to unique isomorphism over S . To do this we will verify that conditions of [11, Theorem 1.2.2] are satisfied. R is Japanese since it is excellent, and S is flat over R since it is a rational localization of R . Note that $\tilde{R} \otimes_R S$ is normal by Lemma 2.3.2 since it is a rational localization of \tilde{R} . The last thing to verify is that if \mathfrak{p} is a minimal prime of R , then $S/\mathfrak{p}S$ is reduced, which follows from Proposition 2.2.3(5) since it is a rational localization of R/\mathfrak{p} . \square

We may then globalize the construction.

Definition 2.3.5. Let X be a pseudorigid space over \mathcal{O}_K . Then we may construct the *normalization* $\tilde{X} \rightarrow X$ by gluing together the normalizations of the open affinoid pseudorigid subspaces of X . The canonical map $p : \tilde{X} \rightarrow X$ is finite and surjective.

Using normalizations, we can deduce the following strengthening of Lemma 2.3.3.

Lemma 2.3.6. *Let X be an irreducible pseudorigid space over \mathcal{O}_K . Then \tilde{X} is connected. Let $Z \subseteq X$ be a Zariski closed subset. If Z contains a nonempty open subset of X , then $Z = X$.*

Proof. Let $p : \tilde{X} \rightarrow X$ be the normalization. Assume that $\tilde{X} = U \amalg V$ with U, V open. We claim that $p^{-1}(p(U)) - U$ is nowhere dense in \tilde{X} (i.e. contains no nonempty open subset). For now we assume the claim. Observe that U and V are Zariski open and closed. Since $X = p(U) \cup p(V)$ and X is irreducible we have $X = p(U)$ or $X = p(V)$. We may as well assume $X = p(U)$. Since $p^{-1}(p(U)) - U = \tilde{X} - U$ is nowhere dense, we deduce that V is empty. We now check the claim. It suffices to do this locally on X , so we assume that $X = \mathrm{Spa}(R)$ is an affinoid pseudorigid space (not necessarily irreducible). U is a union of connected (hence irreducible) components of $\tilde{X} = \mathrm{Spa}(\tilde{R})$. Taking intersections with $\mathrm{Max}(\tilde{R})$ reduces the claim to the

same statement for the map of Jacobson schemes $p : \text{Spec}(\tilde{R}) \rightarrow \text{Spec}(R)$, where the claim follows from the fact that $p^{-1}(p(U)) - U$ is a union of intersections of distinct irreducible components.

To show the second part, we consider $\tilde{U} \subseteq p^{-1}(Z) \subseteq \tilde{X}$. By the first part and Lemma 2.3.3, $p^{-1}(Z) = \tilde{X}$, and hence $Z = X$. \square

When X is quasicompact, the normalization relates to the irreducible components in the following way.

Proposition 2.3.7. *Let X be a quasicompact pseudorigid space over \mathcal{O}_K . Then the irreducible components of X are exactly the images of the connected components of \tilde{X} .*

Proof. Let $\tilde{X}_1, \dots, \tilde{X}_r$ denote the connected components of \tilde{X} and let $X_i = p(\tilde{X}_i)$. By Corollary 2.1.6 the X_i are Zariski closed, and they cover X since p is surjective. If $Z_1, Z_2 \subseteq X$ are Zariski closed subsets such that $X_i \subseteq Z_1 \cup Z_2$, then, considering the map $\tilde{X}_i \rightarrow X$, we may take their preimages to get Zariski closed subsets $Z'_1, Z'_2 \subseteq \tilde{X}_i$ with $X_i = Z'_1 \cup Z'_2$. It follows from Lemma 2.3.3 that $\tilde{X}_i = Z'_j$ for some $j \in \{1, 2\}$, and hence that $X_i \subseteq Z_j$, so X_i is irreducible. It remains to show that $X_i \subseteq X_k$ only if $i = k$. If $U \subseteq X$ is open quasicompact, then $\tilde{X}_i \cap \tilde{U}$ and $\tilde{X}_k \cap \tilde{U}$ are closed and open subsets of \tilde{U} , so one sees that it suffices to verify this statement locally on X . We may therefore reduce to the case of an affinoid pseudorigid space, where it follows from the analogous statement for spectra of maximal ideals. \square

For non-quasicompact spaces we do not have a decomposition into irreducible components a priori, but we will now show that Proposition 2.3.7 holds in this case as well. Recall that we have defined irreducible components as maximal irreducible Zariski closed sets (with respect to inclusion).

Proposition 2.3.8. *Let X be a pseudorigid space over \mathcal{O}_K with normalization $p : \tilde{X} \rightarrow X$. Let $(\tilde{X}_i)_{i \in I}$ be the set of connected components of \tilde{X} (here I is some index set) and let $X_i = p(\tilde{X}_i)$. Then the X_i are exactly the irreducible components of X , and their union is X .*

Proof. That the X_i are irreducible, distinct and cover X follows as in the proof of Proposition 2.3.7; it remains to show that they are maximal irreducible sets. Let $Z \subseteq X$ be a nonempty irreducible Zariski closed set. Let $U \subseteq X$ be a quasicompact open set with intersects Z . By quasicompactness of p we can find a finite set $S \subset I$ such that $\tilde{U} \subseteq \bigcup_{i \in S} \tilde{X}_i$, and hence $U \subseteq \bigcup_{i \in S} X_i$. Since $U \cap Z \subseteq Z$ is open and contained in the Zariski closed

subset $\bigcup_{i \in S} Z \cap X_i$ of the irreducible set Z , we must have $Z \subseteq X_i$ for some $i \in S$, as desired. \square

2.4. Dimension theory

If $X = \text{Spa}(A, A^+)$ is an affinoid adic space, then it is natural to define its dimension as the Krull dimension of the spectral space X . This definition globalizes in a natural way. However, for our purposes it will be more convenient to use a more algebraic definition for the dimension of a pseudorigid space. Before we define the dimension we make an ad hoc definition which is somewhat overdue.

Definition 2.4.1. Let X be a pseudorigid space over \mathcal{O}_K and let $x \in \text{Max}(X)$. Then we define the *completed local ring* $\widehat{\mathcal{O}}_{X,x}$ of X at x to be the ring $\widehat{A}_{\mathfrak{m}}$, where $U = \text{Spa}(A) \subseteq X$ is any open affinoid pseudorigid space containing x and \mathfrak{m} is the maximal ideal in A corresponding to x . Note that this is independent of the choice of U by Proposition 2.2.3(3), hence well defined.

In this section we will freely make use of the fact that a Noetherian local ring has the same Krull dimension as its completion.

Definition 2.4.2. Let X be a pseudorigid space over \mathcal{O}_K . We define the *dimension* of X to be

$$\dim X = \sup_{x \in \text{Max}(X)} \dim \widehat{\mathcal{O}}_{X,x}$$

(taken to be $+\infty$ if the supremum does not exist). We say that X is *equidimensional* if $\dim \widehat{\mathcal{O}}_{X,x}$ is independent of $x \in \text{Max}(X)$.

A few remarks are in order. First, we could also have defined $\dim X$ using open affinoid pseudorigid spaces; one has

$$\dim X = \sup_{U \subseteq X} \dim \mathcal{O}_X(U),$$

where U runs through the open affinoid pseudorigid subspaces of X . When $X = \text{Spa}(R)$ is an affinoid pseudorigid space, we have $\dim X = \dim R$. Second, we would expect this definition to agree with the Krull dimension of the locally spectral space X (for rigid analytic varieties this was proved by

Huber, cf. [20, Lemma 1.8.6, Proposition 1.8.11]), but we have not tried to prove this and we will not need it³.

The main result we will need on dimensions is that if X is irreducible, then X is equidimensional.

Lemma 2.4.3. *Let \mathfrak{m} be a maximal ideal of*

$$A = \mathcal{O}_K[[X_1, \dots, X_m]]\langle T_1, \dots, T_n \rangle.$$

Then $\dim \widehat{A}_{\mathfrak{m}} = m + n + 1$.

Proof. This is probably well known, but we sketch a proof for the convenience of the reader. Let $I \subseteq A$ be the ideal generated by π_K, X_1, \dots, X_m ; this is an ideal of definition of A , and A is complete with respect to I , so I is in the Jacobson radical of A (in fact it is the Jacobson radical). The maximal ideals of A are therefore in bijection with those of $A/I = k[T_1, \dots, T_n]$, hence parametrised by elements in \bar{k}^n (where \bar{k} is an algebraic closure of k). Making a finite unramified extension of K if necessary, we may assume that \mathfrak{m} is defined by a tuple in k^n , and applying a translation we may assume that this tuple is 0. In other words, $\mathfrak{m} = \langle \varpi, X_1, \dots, X_m, T_1, \dots, T_n \rangle$. Then $\widehat{A}_{\mathfrak{m}} \cong \mathcal{O}_K[[X_1, \dots, X_m, T_1, \dots, T_n]]$, which has dimension $m + n + 1$. \square

Corollary 2.4.4. *Let R be a Tate ring topologically of finite type over \mathcal{O}_K , and let $X = \text{Spa}(R)$. Assume that R is an integral domain. Then X is equidimensional.*

Proof. Let $R_0 \subseteq R$ be a ring of definition which is formally of finite type over \mathcal{O}_K , and choose a surjection $A = \mathcal{O}_K[[X]]\langle T_1, \dots, T_n \rangle \twoheadrightarrow R_0$. The kernel of this surjection is a prime ideal which we will call P . Let \mathfrak{m} be a maximal ideal in R and put $\mathfrak{p} = \mathfrak{m} \cap R_0$. Recall that R_0/\mathfrak{p} is local of dimension 1, and let $\mathfrak{q} \subseteq R_0$ denote the unique maximal ideal of R_0 above \mathfrak{p} . Let Q denote the preimage of \mathfrak{q} in A . We have

$$\dim R_{\mathfrak{m}} = \dim(R_0)_{\mathfrak{p}}.$$

Since R_0 is catenary, we have

$$\dim(R_0)_{\mathfrak{p}} = \dim(R_0)_{\mathfrak{q}} - 1.$$

³This has now been proved to be true by Lourenço, see [24, Corollary 4.13].

Since A is catenary we have

$$\dim(R_0)_{\mathfrak{q}} = \dim A_Q - \dim A_P = n + 2 - \dim A_P;$$

where we have used Lemma 2.4.3 in the last equality. Therefore $\dim R_{\mathfrak{m}} = n + 1 - \dim A_P$ is independent of \mathfrak{m} , as desired. \square

We can now globalize this.

Theorem 2.4.5. *Let X be an irreducible pseudorigid space over \mathcal{O}_K . Then X is equidimensional, and $\dim X = \dim \tilde{X}$.*

Proof. First assume that X is normal. Let $U = \text{Spa}(R) \subseteq X$ be an open connected affinoid pseudorigid space. Then R is a normal domain, and therefore U is equidimensional by Corollary 2.4.4. By connectedness of X it follows that X is equidimensional.

We now do the general case. By the above we know that \tilde{X} is equidimensional. Let $x \in \text{Max}(X)$. Choose an open affinoid pseudorigid space $U = \text{Spa}(R) \subseteq X$ containing x and let $\mathfrak{m} \subseteq R$ be the maximal ideal corresponding to x . We want to show that $\dim R_{\mathfrak{m}} = \dim \tilde{X}$. Let \tilde{R} be the normalization of R . $\tilde{U} = \text{Spa}(\tilde{R})$ is the preimage of U in \tilde{X} by construction. By the going-up theorem, we know that

$$\dim R_{\mathfrak{m}} = \max_{\tilde{\mathfrak{m}}} \dim \tilde{R}_{\tilde{\mathfrak{m}}};$$

where $\tilde{\mathfrak{m}}$ ranges over the maximal ideals of \tilde{R} lying over \mathfrak{m} . Since \tilde{X} , and therefore \tilde{U} , is equidimensional, the right hand side is equal to $\dim \tilde{X}$, as desired. \square

We end this subsection by recording the relation between the irreducible components passing through a maximal point $x \in X$ and the minimal primes of the completed local ring $\hat{\mathcal{O}}_{X,x}$.

Proposition 2.4.6. *Let X be a pseudorigid space over \mathcal{O}_K and let $x \in \text{Max}(X)$. There is canonical surjective map Ψ from the minimal primes \mathfrak{p} of $\hat{\mathcal{O}}_{X,x}$ to the irreducible components of X containing x , and $\dim \Psi(\mathfrak{p}) = \dim \hat{\mathcal{O}}_{X,x}/\mathfrak{p}$.*

Proof. If Y is a pseudorigid space, let $\text{Irr}(Y)$ denote the set of irreducible components of Y , and if $y \in Y$ is a maximal point, we let $\text{Irr}(Y, y) \subseteq \text{Irr}(Y)$ be the set of irreducible components containing y . Choose an open affinoid

pseudorigid $U = \text{Spa}(R) \subseteq \widetilde{X}$ containing x . Then the morphism $\widetilde{U} \rightarrow \widetilde{X}$ induces a map $\text{Irr}(U) = \text{Irr}(\widetilde{U}) \rightarrow \text{Irr}(\widetilde{X}) = \text{Irr}(X)$, which restricts to a surjection $\text{Irr}(U, x) \twoheadrightarrow \text{Irr}(X, x)$, and preserves dimensions.

By our definitions, $\text{Irr}(U, x)$ can be identified with $\text{Irr}(\text{Max}(R), \mathfrak{m})$ of irreducible components of $\text{Max}(R)$ containing the maximal ideal $\mathfrak{m} \subseteq R$ corresponding to x , and this identification preserves dimensions. $\text{Irr}(\text{Max}(R), \mathfrak{m})$ can in turn be identified with the minimal primes of $R_{\mathfrak{m}}$. Since $R_{\mathfrak{m}} \rightarrow \widehat{R}_{\mathfrak{m}}$ is faithfully flat, it follows from the going down property and the invariance of dimension on completion that there is a dimension-preserving surjection from the minimal primes of $\widehat{\mathcal{O}}_{X,x} = \widehat{R}_{\mathfrak{m}}$ to $\text{Irr}(\text{Max}(R), \mathfrak{m})$, given by pull-back along $R \rightarrow \widehat{R}_{\mathfrak{m}}$. Finally, one takes the composition with the map above to get a map to $\text{Irr}(X, x)$, and check that this is independent of the choice of U . \square

2.5. Factoriality of normalized weight space

The theory of Fredholm series and hypersurfaces is fundamental to the theory of eigenvarieties, which is our intended application. The study of irreducible components of Fredholm hypersurfaces provided motivation for developing the general theory of irreducible components of rigid spaces. It is based upon the notion of a rigid space X being *locally relatively factorial* (in m variables), which means that X has a cover by open affinoids $U_i = \text{Sp}(A_i)$ such that the relative Tate algebras $A_i\langle X_1, \dots, X_m \rangle$ are factorial (i.e. are unique factorization domains). When one moves away from eigenvarieties that are equidimensional over weight space, the role of Fredholm hypersurfaces becomes less important. For this reason, we have not pursued the generalization of the results of [11, §4] to the setting of pseudorigid spaces (we believe that this should be mostly straightforward, but we have not checked the details). One thing, however, that is perhaps not so clear, is that the weight spaces that occur for the extended eigenvarieties in [22] are relatively factorial. While we will not need this fact in this paper, it seems worth recording. Our proof is based upon the following criterion for factoriality. Recall that an integral domain R is *locally factorial* if the localizations $R_{\mathfrak{p}}$ are factorial for all prime ideals \mathfrak{p} of R (it suffices to check this for maximal ideals).

Lemma 2.5.1. *Let R be a locally factorial Noetherian ring. Suppose R is an integral domain and that there is an element x of the Jacobson radical of R such that R/xR is factorial. Then R is factorial.*

Proof. This is [25, Ch. 2, Lemma 2.2]; we sketch the proof for the convenience of the reader. It suffices to show that every height one prime ideal in R is principal. Let $I \subset R$ be a height one prime ideal. By the locally factorial assumption, I is locally principal and is therefore projective as an R -module. Note that since R/xR is a domain, xR is a prime ideal.

Suppose $x \in I$. Then $xR \subset I$ and since I has height one we have $xR = I$ and therefore I is principal. Now suppose $x \notin I$. Then $I \cap xR = xI$, so $I/xI = I/I \cap xR \cong (I + xR)/xR$. Since I is projective over R , $(I + xR)/xR$ is a projective ideal in R/xR , and by factoriality of R/xR it is therefore a principal ideal. By Nakayama’s lemma, I is principal. \square

The weight spaces that occur in [22] have the form $\text{Spa}(\mathbb{Z}_p[[S]])^{an}$, where $S \cong F \times \mathbb{Z}_p^n$ as p -adic analytic groups for some finite group F . It might happen that F has p -torsion, in which case $\text{Spa}(\mathbb{Z}_p[[S]])^{an}$ may fail to be a domain locally. One can take the normalization, in which case it will be locally of the form $\text{Spa}(\mathbb{Z}_p[[T_1, \dots, T_n]])^{an}$. We will sketch a proof that these spaces are locally relatively factorial. Let K be a discrete valued field.

Definition 2.5.2. Let $X = \text{Spa}(A)$ be an affinoid pseudorigid space over \mathcal{O}_K . We say that X is *relatively factorial* (in m variables) if $A\langle X_1, \dots, X_m \rangle$ is factorial. If X is a general pseudorigid space, we say that X is *locally relatively factorial* (in m variables) if there is an open cover of X by affinoid pseudorigid spaces $U_i = \text{Spa}(A_i)$ such that A_i is relatively factorial in m variables.

Now consider the pseudorigid space $\text{Spa}(\mathcal{O}_K[[T_1, \dots, T_n]])^{an}$. It has an open cover given by the affinoid pseudorigid space $U_0 = \{|T_1|, \dots, |T_n| \leq |p| \neq 0\}$ and

$$U_i = \{|p|, |T_1|, \dots, |T_n| \leq |T_i| \neq 0\},$$

for $i = 1, \dots, n$. U_0 is the adic spectrum of a Tate algebra in n variables over K , and hence relatively factorial (in any number of variables) by [7, 5.2.6/1].

Theorem 2.5.3. *Let $m \in \mathbb{Z}_{\geq 0}$ and let $1 \leq i \leq n$. Then U_i is relatively factorial in m variables.*

Proof. We will content ourselves with a sketch, since this result will not be used in the rest of the paper. By symmetry, it suffices to do the case $i = 1$. Set $U := U_1 = \text{Spa}(R)$ and $A = R\langle X_1, \dots, X_m \rangle$. A has a ring of definition $A_0 = R^\circ\langle X_1, \dots, X_m \rangle$ which carries the $T := T_1$ -adic topology; these

are Noetherian rings. We may define an increasing filtration on A by $F_j = T^{-j}A_0$, and one may compute the corresponding graded ring $gr(A)$; we have

$$gr(A) = k[u_0, u = u_1, u^{-1}, u_2, \dots, u_n, v_1, \dots, v_n]$$

where k is the residue field of K , u_0 is the symbol of p and has degree 1, u_i is the symbol of T_i and has degree 1, and v_j is the symbol of X_j and has degree 0. In particular, $gr(A)$ is a domain, so A and A_0 are domains. We have

$$gr(A_0) = k[u, u_0u^{-1}, u_2u^{-1}, \dots, u_nu^{-1}, v_1, \dots, v_m]$$

so

$$A_0/TA_0 = k[u_0u^{-1}, u_2u^{-1}, \dots, u_nu^{-1}, v_1, \dots, v_m].$$

We now wish to apply Lemma 2.5.1 to show that A_0 , and hence A , is factorial. We have verified that A_0 is a Noetherian domain and that A_0/TA_0 is factorial, and that T is in the Jacobson radical of A_0 (since A_0 is T -adically complete). It remains to show that A_0 is locally factorial. Let \mathfrak{m} be a maximal ideal of A_0 ; it contains T . By above the ring A_0/TA_0 is regular, so $(A_0/TA_0)_{\mathfrak{m}} = (A_0)_{\mathfrak{m}}/T(A_0)_{\mathfrak{m}}$ is regular. Since T is a non-zero-divisor, it follows that $(A_0)_{\mathfrak{m}}$ is a regular local ring [26, Tag 00NU] and hence factorial. \square

3. An interpolation theorem

In [10], Chenevier introduced an abstract interpolation theorem that allows one to show that a set-theoretic map between subsets of two eigenvarieties, in certain circumstances, extends to a rigid analytic morphism of *reduced* eigenvarieties. Chenevier's interpolation theorem was formulated in terms of the input datum of the general eigenvariety construction of Buzzard [8]. In [16], Hansen proved a generalization of Chenevier's theorem for the abstract eigenvariety construction considered in that paper, and gave some applications. In this section we prove a generalization of Hansen's theorem that we believe is close to optimal, and also applies to the extended eigenvarieties constructed in [22].

3.1. Eigenvariety data

We abstract the ingredients of the construction of extended eigenvarieties in [22, §4], generalizing [16, Definition 4.2.1]. We fix a complete discretely valued field K with ring of integers \mathcal{O}_K and a uniformizer π_K .

Definition 3.1.1. An *eigenvariety datum* is a tuple $\mathfrak{D} = (\mathcal{Z}, \mathcal{H}, \mathbb{T}, \psi)$ consisting of a pseudorigid space \mathcal{Z} over \mathcal{O}_K , \mathcal{H} a coherent $\mathcal{O}_{\mathcal{Z}}$ -module, \mathbb{T} a commutative \mathbb{Z}_p -algebra, and $\psi : \mathbb{T} \rightarrow \text{End}_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{H})$ a \mathbb{Z}_p -algebra homomorphism.

We remark that in practice, an eigenvariety datum as above is the penultimate step in the construction of an eigenvariety. In these situations, \mathcal{Z} is typically (isomorphic to) $\mathbb{A}_{\mathcal{W}}^n$ for some $n \geq 1$ (or $\mathbb{G}_{m, \mathcal{W}}^n$), where \mathcal{W} is the ‘weight space’. In fact, in most eigenvariety constructions one will have $n = 1$ and \mathcal{Z} can alternatively be taken to be a Fredholm hypersurface in $\mathbb{A}_{\mathcal{W}}^1$; this is where the use of the letter ‘Z’ comes from. Let us recall that a Fredholm series over \mathcal{W} is an entire power series $F(X) \in \mathcal{O}_{\mathbb{A}_{\mathcal{W}}^1}(\mathbb{A}_{\mathcal{W}}^1)$ with $F(0) = 1$, and a Fredholm hypersurface is a Zariski closed subspace of $\mathbb{A}_{\mathcal{W}}^1$ defined by $\{F = 0\}$, where F is a Fredholm series.

As in [16, Theorem 4.2.2], an eigenvariety datum \mathfrak{D} has an associated eigenvariety. Before giving the construction, we record the following commutative algebra result (cf. [27, Lemma 2.2(1)]).

Lemma 3.1.2. *Let A, B and T be rings with A and B Noetherian and let M be a finitely generated A -module. Assume that we have ring homomorphisms $f : A \rightarrow B$ and $\psi_A : T \rightarrow \text{End}_A(M)$ and let ψ_B be the composition of ψ_A with the natural map $\text{End}_A(M) \rightarrow \text{End}_B(M \otimes_A B)$ coming from f . Let T_A (resp. T_B) be the A -subalgebra (resp. B -subalgebra) generated by the image of ψ_A (resp. ψ_B). Then the natural B -linear map $T_A \otimes_A B \rightarrow T_B$ is a surjection with nilpotent kernel in general, and if f is flat then it is an isomorphism.*

Proof. We have B -linear maps

$$T \otimes_{\mathbb{Z}} B \rightarrow \text{End}_A(M) \otimes_A B \rightarrow \text{End}_B(M \otimes_A B)$$

induced by ψ_A and f respectively. The left map factors through $T_A \otimes_A B$ and T_B is the image of the composition of the two maps. This gives the natural map and shows its surjectivity since $T \otimes_{\mathbb{Z}} B \rightarrow T_A \otimes_A B$ is surjective. To prove that the kernel is nilpotent, it suffices to show that the support of the $T_A \otimes_A B$ -module $M \otimes_A B$ is all of $\text{Spec}(T_A \otimes_A B)$, since $M \otimes_A B$ is a faithful T_B -module by construction. Since $M \otimes_A B \cong M \otimes_{T_A} (T_A \otimes_A B)$, the support of $M \otimes_A B$ is the preimage of the support of M under the natural map $\text{Spec}(T_A \otimes_A B) \rightarrow \text{Spec}(T_A)$. But M is a faithful T_A -module by construction, so we get what we want.

Now assume that f is flat. Then the natural map $\text{End}_A(M) \otimes_A B \rightarrow \text{End}_B(M \otimes_A B)$ is an isomorphism and $T_A \otimes_A B$ is the image of $T \otimes_{\mathbb{Z}} B \rightarrow \text{End}_A(M) \otimes_A B$, which gives that $T_A \otimes_A B \rightarrow T_B$ is an isomorphism. \square

Let us now record the construction of the eigenvariety of an eigenvariety datum.

Proposition 3.1.3. *Given an eigenvariety datum $\mathfrak{D} = (\mathcal{Z}, \mathcal{H}, \mathbb{T}, \psi)$, there is a pseudorigid space $\mathcal{X} = \mathcal{X}(\mathfrak{D})$ over \mathcal{O}_K , together with a finite morphism $\pi : \mathcal{X} \rightarrow \mathcal{Z}$, a \mathbb{Z}_p -algebra homomorphism $\phi_{\mathcal{X}} : \mathbb{T} \rightarrow \mathcal{O}(\mathcal{X})$, and a faithful coherent $\mathcal{O}_{\mathcal{X}}$ -module \mathcal{H}^\dagger . There is a canonical isomorphism $\pi_* \mathcal{H}^\dagger \cong \mathcal{H}$, which is compatible with the actions of \mathbb{T} on both sides (via $\phi_{\mathcal{X}}$ and ψ , respectively).*

The space \mathcal{X} is characterized by the following local description: for $U \subset \mathcal{Z}$ a pseudorigid affinoid open we have $\mathcal{X}_U = \text{Spa}(\mathcal{T}_U)$ where \mathcal{T}_U is the $\mathcal{O}_{\mathcal{X}}(U)$ -subalgebra of $\text{End}_{\mathcal{O}_{\mathcal{X}}(U)}(\mathcal{H}(U))$ generated by the image of \mathbb{T} . Note that $\mathcal{H}(U)$ is canonically a \mathcal{T}_U -module, and this gives \mathcal{H}^\dagger over \mathcal{X}_U .

Proof. The proof is (essentially) the same as the proof of [16, Theorem 4.2.2], we sketch it for completeness. For $U \subseteq \mathcal{Z}$ affinoid open pseudorigid, \mathcal{T}_U is commutative and finite over $\mathcal{O}_{\mathcal{X}}(U)$, and hence is a Tate ring formally of finite type over \mathcal{O}_K . The space $\mathcal{X}_U = \text{Spa}(\mathcal{T}_U)$ carries a canonical finite map $\mathcal{X}_U \rightarrow U$ and $\mathcal{H}(U)$ is a finitely generated \mathcal{T}_U -module. By Lemma 3.1.2 and flatness of rational localization for affinoid pseudorigid spaces over \mathcal{O}_K , these constructions glue together and satisfy the assertions of the theorem. \square

From Lemma 3.1.2 we get the following compatibility of the eigenvariety construction with base change.

Proposition 3.1.4. *Let $\mathfrak{D} = (\mathcal{Z}, \mathcal{H}, \mathbb{T}, \psi)$ be an eigenvariety datum with eigenvariety \mathcal{X} . Let $f : \mathcal{Z}' \rightarrow \mathcal{Z}$ be a map of pseudorigid spaces over \mathcal{O}_K . Form the eigenvariety datum*

$$\mathfrak{D}' = (\mathcal{Z}', \mathcal{H}' = f^* \mathcal{H}, \mathbb{T}, \psi'),$$

where ψ' is the composition of ψ with the natural map

$$\text{End}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{H}) \rightarrow \text{End}_{\mathcal{O}_{\mathcal{X}'}}(\mathcal{H}').$$

Let \mathcal{X}' be the eigenvariety attached to \mathfrak{D}' . Then there is a natural map $\mathcal{X}' \rightarrow \mathcal{X}$ over \mathcal{Z} , and the induced map $\mathcal{X}' \rightarrow \mathcal{X} \times_{\mathcal{X}} \mathcal{Z}'$ induces an isomorphism $(\mathcal{X}')^{\text{red}} \rightarrow (\mathcal{X} \times_{\mathcal{X}} \mathcal{Z}')^{\text{red}}$. If f is flat, then $\mathcal{X}' \rightarrow \mathcal{X} \times_{\mathcal{X}} \mathcal{Z}'$ is an isomorphism.

Proof. By the local nature of the construction of eigenvarieties in Proposition 3.1.3, the assertion is local both on \mathcal{Z}' and \mathcal{Z} , so we may assume that they are both affinoid pseudorigid spaces over \mathcal{O}_K . Then the proposition follows directly from Lemma 3.1.2. \square

We single out a special case of Proposition 3.1.4 which characterises the points of eigenvarieties.

Corollary 3.1.5. *Let $\mathfrak{D} = (\mathcal{Z}, \mathcal{H}, \mathbb{T}, \psi)$ be an eigenvariety datum with eigenvariety $\pi : \mathcal{X} \rightarrow \mathcal{Z}$. Fix $z \in \text{Max}(\mathcal{Z})$. Then the set $\pi^{-1}(z) \subseteq \text{Max}(\mathcal{X})$ is in natural bijection with the systems of Hecke eigenvalues appearing in the fibre $\mathcal{H}(z)$, i.e. the maximal ideals lying above the kernel of the natural map $\mathbb{T} \otimes_{\mathbb{Z}_p} k(z) \rightarrow \text{End}_{k(z)}(\mathcal{H}(z))$.*

Proof. This follows from applying Proposition 3.1.4 with f the closed immersion $\mathcal{Z}' = z \hookrightarrow \mathcal{Z}$. \square

We finish with a simple reconstruction theorem for eigenvariety data.

Proposition 3.1.6. *Let $\pi : \mathcal{X} \rightarrow \mathcal{Z}$ be a finite morphism of pseudorigid spaces and let \mathbb{T} be a \mathbb{Z}_p -algebra. Assume that there is a ring homomorphism $\phi : \mathbb{T} \rightarrow \mathcal{O}(\mathcal{X})$ such that $\mathcal{O}_{\mathcal{X}}$ is generated by the image of ϕ over $\mathcal{O}_{\mathcal{Z}}$, and assume that \mathcal{H}^\dagger is a faithful coherent $\mathcal{O}_{\mathcal{X}}$ -module. We may form an eigenvariety datum*

$$\mathfrak{D} = (\mathcal{Z}, \mathcal{H}, \mathbb{T}, \psi),$$

where $\mathcal{H} = \pi_* \mathcal{H}^\dagger$ and ψ is the composition $\mathbb{T} \xrightarrow{\phi} \mathcal{O}(\mathcal{X}) \rightarrow \text{End}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{H})$. Then $\pi : \mathcal{X} \rightarrow \mathcal{Z}$ is the eigenvariety attached to \mathfrak{D} , and $\phi = \phi_{\mathcal{X}}$.

Proof. The assertion is local on \mathcal{Z} , so we may assume that $\mathcal{Z} = \text{Spa}(A)$, $\mathcal{X} = \text{Spa}(T)$, that \mathcal{H}^\dagger is the coherent sheaf attached to a finitely generated faithful T -module H , and that the map $\mathbb{T} \otimes_{\mathbb{Z}_p} A \rightarrow T$ induced by ϕ is surjective. The eigenvariety attached to \mathfrak{D} is then the adic spectrum of the image of the map $\mathbb{T} \otimes_{\mathbb{Z}_p} A \rightarrow \text{End}_A(H)$. This map factors as

$$\mathbb{T} \otimes_{\mathbb{Z}_p} A \rightarrow T \rightarrow \text{End}_T(H) \rightarrow \text{End}_A(H)$$

and the first map is surjective and the second and third are injective (using that H is a faithful T -module), so this image is (canonically isomorphic to) T and the identifications of π and $\phi_{\mathcal{X}}$ follow easily. \square

We remark that, as a result of this, any Zariski closed subset \mathcal{X}' of the eigenvariety \mathcal{X} of an eigenvariety datum is naturally the eigenvariety of an eigenvariety datum.

3.2. The interpolation theorem

We are now ready to prove our interpolation theorem (which could also be considered as a rigidity theorem). If \mathcal{U} is a pseudorigid space over \mathcal{O}_K and \mathcal{M} is a coherent $\mathcal{O}_{\mathcal{U}}$ -module, then we continue to write $\mathcal{M}(u)$ for the fibre of \mathcal{M} at any $u \in \text{Max}(\mathcal{U})$, which is a vector space over the residue field $k(u)$.

Theorem 3.2.1. *Let $\mathfrak{D}_i = (\mathcal{L}_i, \mathcal{H}_i, \mathbb{T}_i, \psi_i)$, for $i = 1, 2$, be eigenvariety data. Assume that we have the following data:*

- A morphism $j : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ of adic spaces;
- A \mathbb{Z}_p -algebra homomorphism $\sigma : \mathbb{T}_2 \rightarrow \mathbb{T}_1$;
- A subset $\mathcal{X}^{\text{cl}} \subseteq \text{Max}(\mathcal{X}_1)$ such that the \mathbb{T}_2 -eigensystem of x (by which we mean the \mathbb{T}_1 -eigensystem of x composed with σ) appears in $\mathcal{H}_2(j(\pi_1(x)))$ for all $x \in \mathcal{X}^{\text{cl}}$.

Let $\overline{\mathcal{X}}$ denote the Zariski closure of \mathcal{X}^{cl} in \mathcal{X}_1 , with its induced reduced structure. Then there is a canonical morphism

$$i : \overline{\mathcal{X}} \rightarrow \mathcal{X}_2$$

lying over $j : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ such that $\phi_{\overline{\mathcal{X}}} \circ \sigma = i^* \circ \phi_{\mathcal{X}_2}$. The morphism i inherits the following properties from j :

- If j is (partially) proper (resp. finite), then i is (partially) proper (resp. finite);
- If j is a closed immersion and σ is a surjection, then i is a closed immersion.

Proof. We start with a series of reduction steps. First, we form the eigenvariety datum

$$\mathfrak{D}_1^\sigma = (\mathcal{L}_1, \mathcal{H}_1, \mathbb{T}_2, \psi_1 \circ \sigma)$$

with corresponding eigenvariety $\pi_1^\sigma : \mathcal{X}_1^\sigma \rightarrow \mathcal{L}_1$. If $U \subseteq \mathcal{L}_1$ is an open affinoïd, we have a natural inclusion $\mathcal{O}_{\mathcal{X}_1^\sigma}((\pi_1^\sigma)^{-1}(U)) \subseteq \mathcal{O}_{\mathcal{X}_1}(\pi_1^{-1}(U))$ directly from the definitions, and they glue together to a finite dominant (in fact

surjective) map $f : \mathcal{X}_1 \rightarrow \mathcal{X}_1^\sigma$ over \mathcal{Z}_1 . Note that if σ is a surjection, then the inclusion in the previous sentence is an equality, and f is an equality. Form the analogue $\overline{\mathcal{X}}^\sigma$ of $\overline{\mathcal{X}}$ for \mathcal{X}_1^σ . One checks easily that $f(\overline{\mathcal{X}}) \subseteq \overline{\mathcal{X}}^\sigma$, so it suffices to prove the theorem after replacing \mathfrak{D}_1 by \mathfrak{D}_1^σ . In other words, we may assume that $\mathbb{T}_1 = \mathbb{T}_2 =: \mathbb{T}$.

Next, we pull back the eigenvariety datum \mathfrak{D}_2 along $j : \mathcal{Z}_1 \rightarrow \mathcal{Z}_2$ to reduce to the case $\mathcal{Z}_1 = \mathcal{Z}_2 =: \mathcal{Z}$, $j = id$, by Proposition 3.1.4. To check that this reduction works, note that if $x \in \mathcal{X}^{cl}$, then the \mathbb{T} -eigensystem of x appears in $\mathcal{H}_2(j(\pi_1(x)))$ if and only if it appears in $(j^* \mathcal{H}_2)(\pi_1(x))$, since $(j^* \mathcal{H}_2)(\pi_1(x)) = \mathcal{H}_2(j(\pi_1(x))) \otimes_{k(j(\pi_1(x)))} k(\pi_1(x))$.

Having done these reductions, we form the eigenvariety datum

$$\mathfrak{D}_3 = (\mathcal{Z}, \mathcal{H}_3 = \mathcal{H}_1 \oplus \mathcal{H}_2, \mathbb{T}, \psi_3 = (\psi_1, \psi_2)).$$

Let $\pi_3 : \mathcal{X}_3 \rightarrow \mathcal{Z}$ be the associated eigenvariety; it has both \mathcal{X}_1 and \mathcal{X}_2 appearing as Zariski closed subspaces in it since the coherent $\mathcal{O}_{\mathcal{Z}}$ -algebra $\pi_{3,*} \mathcal{O}_{\mathcal{X}_3}$ has both $\pi_{1,*} \mathcal{O}_{\mathcal{X}_1}$ and $\pi_{2,*} \mathcal{O}_{\mathcal{X}_2}$ naturally as quotients (this follows by examining the construction in Proposition 3.1.3). We need to show that $\mathcal{X}^{cl} \subseteq \mathcal{X}_2$; if so then $\overline{\mathcal{X}} \subseteq \mathcal{X}_2$ since \mathcal{X}_2 is Zariski closed in \mathcal{X}_3 . Let $x \in \mathcal{X}^{cl}$ and set $z = \pi_3(x) = \pi_1(x)$. By Corollary 3.1.5, we need to show that the \mathbb{T} -eigensystem of x appears in $\mathcal{H}_2(z)$. But this is exactly our assumption. This establishes the existence of i . For uniqueness, we can reduce to the case when \mathcal{Z}_1 and \mathcal{Z}_2 are both affinoid (since equality of morphisms can be checked locally on the source), and then the requirement $\phi_{\overline{\mathcal{X}}} \circ \sigma = i^* \circ \phi_{\mathcal{X}_2}$ uniquely determines i^* , and hence i , since the image of $\phi_{\overline{\mathcal{X}}}$ generates $\mathcal{O}(\overline{\mathcal{X}})$ over $\mathcal{O}(\mathcal{Z}_1)$.

It remains to prove that i inherits properties from j . If we summarize the construction, i is a composition of a finite map $\overline{\mathcal{X}} \rightarrow \overline{\mathcal{X}}^\sigma$, a closed immersion $\overline{\mathcal{X}}^\sigma \rightarrow (\mathcal{X}_2 \times_{\mathcal{Z}_2} \mathcal{Z}_1)^{red}$ and the canonical morphism $(\mathcal{X}_2 \times_{\mathcal{Z}_2} \mathcal{Z}_1)^{red} \rightarrow \mathcal{X}_2$ coming from the projection. If j is (partially) proper (resp. finite), then $(\mathcal{X}_2 \times_{\mathcal{Z}_2} \mathcal{Z}_1)^{red} \rightarrow \mathcal{X}_2$ is (partially) proper (resp. finite), and hence the composition is (partially) proper (resp. finite, since these properties are stable under base change and composition), proving the first assertion. If σ is surjective, then $\overline{\mathcal{X}} \rightarrow \overline{\mathcal{X}}^\sigma$ is an isomorphism (as noted above), so if j is a closed immersion, then $(\mathcal{X}_2 \times_{\mathcal{Z}_2} \mathcal{Z}_1)^{red} \rightarrow \mathcal{X}_2$ is a closed immersion and hence i is a closed immersion. \square

Remark 3.2.2. We should note that this theorem is not, strictly speaking, a generalization of [16, Theorem 5.1.6]. That theorem assumes a certain divisibility of determinants instead of the assumption on eigensystems in our

theorem. This divisibility is a weaker assumption; from such a result one typically deduces a result about eigensystems by a separation of eigenvalues argument. In practice (e.g. attempts to interpolate known cases of Langlands functoriality) this separation of eigenvalues has been done, so we think that our slightly stronger assumption is natural.

With this caveat, our theorem appears to be essentially optimal, and the (rather elementary) method of proof appears to be new. We note in particular that the use of the global geometry of the eigenvariety instead of a reduction to affinoids eliminates the need for \mathcal{X}^{cl} to be *very Zariski dense*. It should be noted, however, that the only technique currently known (to the authors) to control the Zariski closure of interesting sets \mathcal{X}^{cl} occurring in practice is to show that they are very Zariski dense inside a union of irreducible components of \mathcal{X} .

3.3. Extended eigenvarieties for overconvergent cohomology

In this section we briefly recall the extended eigenvarieties constructed in [22]. We refer to [22, §3.3, §4] for precise definitions and any undefined notation. Let F be a number field and let \mathbf{H}/F be a connected reductive group which is split at all places above p . We set $\mathbf{G} = \text{Res}_{\mathbb{Q}}^F \mathbf{H}$. Choosing split models $\mathbf{H}_{\mathcal{O}_{F_v}}$ of \mathbf{H} over \mathcal{O}_{F_v} for all $v|p$ and maximal tori and Borel subgroups $\mathbf{T}_v \subseteq \mathbf{B}_v \subseteq \mathbf{H}_{\mathcal{O}_{F_v}}$, we obtain a model $\mathbf{G}_{\mathbb{Z}_p} = \prod_{v|p} \text{Res}_{\mathbb{Z}_p}^{\mathcal{O}_{F_v}} \mathbf{H}_{\mathcal{O}_{F_v}}$ of \mathbf{G} over \mathbb{Z}_p , and closed subgroup schemes $\mathbf{T} = \prod_{v|p} \text{Res}_{\mathbb{Z}_p}^{\mathcal{O}_{F_v}} \mathbf{T}_v \subseteq \mathbf{B} = \prod_{v|p} \text{Res}_{\mathbb{Z}_p}^{\mathcal{O}_{F_v}} \mathbf{B}_v \mathbf{G}_{\mathbb{Z}_p}$. Set $T_0 = \mathbf{T}(\mathbb{Z}_p)$ and let I be the preimage of $\mathbf{B}(\mathbb{F}_p)$ under the map $\mathbf{G}_{\mathbb{Z}_p}(\mathbb{Z}_p) \rightarrow \mathbf{G}_{\mathbb{Z}_p}(\mathbb{F}_p)$. We define Σ to be the kernel of a choice of splitting of the inclusion $T_0 \subseteq \mathbf{T}(\mathbb{Q}_p)$. Inside Σ , we have a certain submonoid Σ^+ and a subset $\Sigma^{cpt} \subseteq \Sigma^+$. Fix compact open subgroups $K_\ell \subseteq \mathbf{G}(\mathbb{Q}_\ell)$ for all $\ell \neq p$ such that $K_\ell = \mathcal{G}(\mathbb{Z}_\ell)$ for all but finitely many ℓ , where \mathcal{G} is a reductive model of \mathbf{G} over $\mathbb{Z}[1/M]$ for some $M \in \mathbb{Z}_{\geq 1}$. Set $K^p = \prod_{\ell \neq p} K_\ell$ and $K = K^p I$. Let \mathbf{Z} denote the center of \mathbf{G} , put $Z(K) = \mathbf{Z}(\mathbb{Q}) \cap K$ and let $\overline{Z(K)} \subseteq T_0$ be the p -adic closure. Finally let $K_\infty \subseteq \mathbf{G}(\mathbb{R})$ be a maximal compact and connected subgroup and let $Z_\infty^\circ \subseteq Z_\infty = \mathbf{Z}(\mathbb{R})$ be the identity component.

If R is a Banach–Tate \mathbb{Z}_p -algebra [22, §3.1] and $\kappa : T_0/\overline{Z(K)} \rightarrow R^\times$ is a continuous character, then (under the assumption that the norm of R is adapted to κ [22, Def. 3.3.2]) we defined distribution modules \mathcal{D}_κ^r for $r < 1$ that are close enough to 1. \mathcal{D}_κ^r has actions of I and Σ^+ , and elements of Σ^{cpt} act as compact operators. \mathcal{D}_κ^r may be considered as a local system on

the locally symmetric space $X_K = \mathbf{G}(\mathbb{Q}) \backslash G(\mathbb{A})/KK_\infty Z_\infty^\circ$. A choice of triangulation of the Borel–Serre compactification of X_K and choices of homotopies between the corresponding simplicial chain complex and the singular chain complex gives a complex $C^\bullet(K, \mathcal{D}_\kappa^r)$ that computes the cohomology $H^*(X_K, \mathcal{D}_\kappa^r)$. Let \mathbb{T}_ℓ be the spherical Hecke algebra with respect to K_ℓ for any $\ell \neq p$ such that $K_\ell = \mathcal{G}(\mathbb{Z}_\ell)$ and set $\mathbb{T} = \bigotimes_\ell \mathbb{T}_\ell$. Choosing an element $t \in \Sigma^{cpt}$, one may define an eigenvariety datum

$$(\mathcal{Z}, \mathcal{H}, \mathbb{T}, \psi)$$

when R is a Tate ring formally of finite type over \mathbb{Z}_p , using the method of [22, §4]. Roughly speaking, the coherent sheaf \mathcal{H} is constructed out of the finite slope part of $H^*(X_K, \mathcal{D}_\kappa^r)$ with respect to the Hecke operator $U_t = [KtK]$. $\mathcal{Z} \subset \mathbb{A}_R^1$ is the Fredholm hypersurface for the Hecke operator U_t acting on the complex $C^\bullet(K, \mathcal{D}_\kappa^r)$ (this action depends on the choices of homotopies we made above, and in particular will only commute up to homotopy with $U_{t'}$ for a different choice of $t' \in \Sigma^{cpt}$). The homomorphism ψ comes from the action of \mathbb{T} on $H^*(X_K, \mathcal{D}_\kappa^r)$. The eigenvariety datum is independent of the choice of r and the choice of norm on R , and is compatible with open immersions $\mathrm{Spa}(S) \rightarrow \mathrm{Spa}(R)$. If we replace \mathcal{Z} by \mathbb{A}_R^1 , and \mathcal{H} by its pushforward under the closed immersion $\mathcal{Z} \subset \mathbb{A}_R^1$ we obtain an eigenvariety datum (with the same associated eigenvariety) which is moreover independent of our chosen triangulation of the Borel–Serre compactification of X_K and the choices of homotopies between the corresponding simplicial chain complex and the singular chain complex. However, the associated eigenvariety will in general depend on the choice of controlling operator U_t , since we have not incorporated any other Hecke operators at p into the eigenvariety datum. We refer to section 3.4 for more discussion of this issue.

Fix R and κ as above and put $\mathcal{W} = \mathrm{Spa}(R)$. Let $w \in \mathrm{Max}(\mathcal{W})$ with residue field $k(w)$ and write κ_w for the induced character $T_0/\overline{Z}(K) \rightarrow k(w)^\times$. Consider the eigenvariety datum $(\mathcal{Z}, \mathcal{H}, \mathbb{T}, \psi)$ and let $z = (w, \lambda) \in \mathrm{Max}(\mathcal{Z})$. The following proposition is a simple corollary of [22, Corollary 4.2.3], and will be used in the next section to interpolate some known cases of Langlands functoriality.

Proposition 3.3.1. *The systems of eigenvalues for \mathbb{T} occurring in the fibre $\mathcal{H}(z)$ are the same as the systems of eigenvalues of \mathbb{T} occurring in the generalized λ^{-1} -eigenspace of U_t on $H^*(K, \mathcal{D}_w^r)$ (for any allowed choice of r).*

Proof. Choosing a slope datum (U, h) for \mathcal{Z} with $z \in \mathcal{Z}_{U,h}$ (see [22, Def. 2.3.1] for the notion of a slope datum for a Fredholm hypersurface), then by construction $\mathcal{H}(\mathcal{Z}_{U,h}) = H^*(K, \mathcal{D}_U^r)_{\leq h}$ (and this is independent of r). Set

$$\begin{aligned} \mathbb{T}_{w,h} &= \text{Im} \left(\mathbb{T} \otimes_{\mathbb{Z}_p} k(w) \rightarrow \text{End}_{k(w)}(H^*(K, \mathcal{D}_w^r)_{\leq h}) \right); \\ \mathbb{T}_{U,h} &= \text{Im} \left(\mathbb{T} \otimes_{\mathbb{Z}_p} \mathcal{O}(U) \rightarrow \text{End}_{\mathcal{O}(U)}(H^*(K, \mathcal{D}_U^r)_{\leq h}) \right). \end{aligned}$$

By [22, Corollary 4.2.3] we have $(\mathbb{T}_{U,h} \otimes_{\mathcal{O}(U)} k(w))^{red} \cong \mathbb{T}_{w,h}^{red}$, so the systems of eigenvalues for \mathbb{T} occurring in the spaces

$$H^*(K, \mathcal{D}_w^r)_{\leq h} \quad \text{and} \quad H^*(K, \mathcal{D}_U^r)_{\leq h} \otimes_{\mathcal{O}(U)} k(w)$$

agree. Let $\mathfrak{m} \subseteq \mathcal{O}(\mathcal{Z}_{U,h})$ be the maximal ideal corresponding to z .

Now $\mathcal{H}(z) = H^*(K, \mathcal{D}_U^r)_{\leq h} \otimes_{\mathcal{O}(\mathcal{Z}_{U,h})} k(z)$ and the system of eigenvalues of \mathbb{T} occurring in $\mathcal{H}(z)$ are the maximal ideals of $\mathbb{T}_{U,h} \otimes_{\mathcal{O}(\mathcal{Z}_{U,h})} k(z)$, by Lemma 3.1.2. By the same lemma, these are the same as the systems of eigenvalues occurring in $\mathbb{T}_{U,h} \otimes_{\mathcal{O}(\mathcal{Z}_{U,h})} \mathcal{O}(\mathcal{Z}_{U,h})/\mathfrak{m}^n$, for any $n \geq 1$. By the above, these systems of eigenvalues are the same as those occurring in $H^*(K, \mathcal{D}_w^r)_{\leq h} \otimes_{\mathcal{O}(\mathcal{Z}_{U,h})} \mathcal{O}(\mathcal{Z}_{U,h})/\mathfrak{m}^n$, and for $n \gg 1$ this is the generalized λ^{-1} -eigenspace of U_t . \square

To finish this subsection, we note that the second author’s result [16, Proposition B.1] giving a lower bound for the dimensions of irreducible components of eigenvarieties (see also [28, Prop. 5.7.4]) holds for extended eigenvarieties. We retain the notation from above, and write $\pi : \mathcal{X} \rightarrow \mathcal{Z}$ for the eigenvariety attached to $(\mathcal{Z}, \mathcal{H}, \mathbb{T}, \psi)$.

Proposition 3.3.2. *Let $x \in \text{Max}(\mathcal{X})$ and put $(w, \lambda) = \pi(x)$. Let $h \in \mathbb{Q}_{\geq 0}$ be such that $H^*(K, \mathcal{D}_w^r)_{\leq h,x} \neq 0$. Set*

$$l(x) = \sup \{i \mid H^i(K, \mathcal{D}_w^r)_{\leq h,x} \neq 0\} - \inf \{i \mid H^i(K, \mathcal{D}_w^r)_{\leq h,x} \neq 0\}.$$

Note that $l(x)$ is independent of the choice of h . Assume that $\widehat{\mathcal{O}}_{\mathcal{W},w}$ is Cohen–Macaulay and, for simplicity, that \mathcal{W} is equidimensional. Then the dimension of any irreducible component containing x is greater than or equal to $\dim \mathcal{W} - l(x)$.

Proof. Choose a slope datum (U, h) such that $\pi(x) \in \mathcal{Z}_{U,h}$ and set $\mathcal{X}_{U,h} = \pi^{-1}(\mathcal{Z}_{U,h})$. By Proposition 2.4.6, it suffices to show that any minimal prime

of $\widehat{\mathcal{O}}_{\mathcal{X},x}$ has coheight $\geq \dim \mathcal{W} - l(x)$. From the construction of the eigenvariety datum, we get a complex $C^\bullet(K, \mathcal{D}_U^r)_{\leq h}$ of finite projective $\mathcal{O}(U)$ -modules that computes

$$\mathcal{H}(\mathcal{Z}_{U,h}) = H^*(K, \mathcal{D}_U^r)_{\leq h}.$$

Applying $- \otimes_{\mathcal{O}(\mathcal{X}_{U,h})} \widehat{\mathcal{O}}_{\mathcal{X},x}$, we get a complex C_x^\bullet of finite projective $\widehat{\mathcal{O}}_{\mathcal{W},w}$ -modules that computes $H_x = \mathcal{H}(\mathcal{Z}_{U,h}) \otimes_{\mathcal{O}(\mathcal{X}_{U,h})} \widehat{\mathcal{O}}_{\mathcal{X},x}$. Since H_x is a finite faithful $\widehat{\mathcal{O}}_{\mathcal{X},x}$ -module, it suffices to show that any minimal prime in the support of H_x in $\widehat{\mathcal{O}}_{\mathcal{W},w}$ has height $\leq l(x)$. This follows from [15, Theorem 2.1.1(1)], upon noting that $l(x)$ is equal to the amplitude of C_x^\bullet , in the notation of *loc. cit.* \square

3.4. Independence of the choice of controlling operator

The eigenvariety construction of the previous subsection involve making a choice of ‘controlling operator’ U_t , depending on a choice of $t \in \Sigma^{cpt}$. In this subsection, we describe a variant construction which incorporates all Atkin–Lehner Hecke operators at p into the eigenvariety construction. We then show that this construction gives an eigenvariety which is independent of the choice of $t \in \Sigma^{cpt}$.

We retain all the notations of the previous subsection⁴, and begin by defining some additional notation. We have a commutative subalgebra of the Iwahori–Hecke algebra at p

$$\mathcal{A}_p^+ \subset \mathbb{Z}_p[\mathbf{G}(\mathbb{Q}_p)//I]$$

generated by the characteristic functions $\mathbf{1}_{[sI]}$ for all $s \in \Sigma^+$ (see [5, Proposition 6.4.1]).

Now choosing an element $t \in \Sigma^{cpt}$, one may define an eigenvariety datum

$$(\mathcal{Z}^t, \mathcal{H}^t, \mathbb{T} \otimes_{\mathbb{Z}_p} \mathcal{A}_p^+, \psi^t)$$

and associated eigenvariety $\mathcal{X}^{\mathcal{A}_p^+, t}$ with $\phi^t : \mathbb{T} \otimes_{\mathbb{Z}_p} \mathcal{A}_p^+ \rightarrow \mathcal{O}(\mathcal{X}^{\mathcal{A}_p^+, t})$ as in the previous section (we now explicitly record the dependence on t in the notation), by incorporating the action of \mathcal{A}_p^+ on $H^*(X_K, \mathcal{D}_K^r)$.

⁴In particular, our weight space \mathcal{W} is affinoid, but everything discussed in this section glues to handle the general situation.

Lemma 3.4.1. *For all $s \in \Sigma^+$ the action of $[IsI]$ on \mathcal{H}^t is invertible. Equivalently, $\phi^t([IsI])$ is a unit in $\mathcal{O}(\mathcal{X}^{\mathcal{A}_p^+, t})$.*

Proof. This follows from the two following claims: the action of $U_t = [ItI]$ on \mathcal{H}^t is invertible and for any $s \in \Sigma^+$ there exists $k \geq 0$ and $s' \in \Sigma^+$ such that $ss' = t^k$. The first claim is an immediate consequence of the construction of \mathcal{H}^t using slope decompositions for U_t . For the second claim we use the notation in [22, §3.3]: there is an integer $r \geq 1$ such that $s^{-1}\overline{N}_r s \subset \overline{N}_1$, and for k sufficiently large we have $t^k \overline{N}_1 t^{-k} \subset \overline{N}_r$. This implies that $s^{-1}t^k \in \Sigma^+$. \square

Now we denote by $\widehat{\Sigma} = \text{Hom}(\Sigma, \mathbb{G}_{m, \mathcal{W}})$ the pseudorigid space over \mathcal{W} representing $S \mapsto \text{Hom}(\Sigma, S^\times)$ for affinoid pseudorigid R -algebras S . If we fix a \mathbb{Z} -basis for Σ we get an isomorphism $\widehat{\Sigma} \cong \mathbb{G}_{m, \mathcal{W}}^{\text{rk}_{\mathbb{Z}}(\Sigma)}$.

For each $s \in \Sigma$ we write $s = s'(s'')^{-1}$ for $s', s'' \in \Sigma^+$ and obtain an element

$$\phi([Is'I])\phi([Is''I])^{-1} \in \mathcal{O}(\mathcal{X}^{\mathcal{A}_p^+, t})^\times.$$

This defines a map $\pi^{\mathcal{A}_p^+, t} : \mathcal{X}^{\mathcal{A}_p^+, t} \rightarrow \widehat{\Sigma}$ which is finite because the finite map $\pi : \mathcal{X}^{\mathcal{A}_p^+, t} \rightarrow \mathcal{Z} \subset \mathbb{A}_{\mathcal{W}}^1$ is a composition of $\pi^{\mathcal{A}_p^+, t}$ and the separated map $\widehat{\Sigma} \rightarrow \mathbb{A}_{\mathcal{W}}^1$ given by evaluation at t^{-1} . Applying Proposition 3.1.6, we see that $\mathcal{X}^{\mathcal{A}_p^+, t}$ can be identified with the eigenvariety associated to the eigenvariety datum

$$(\widehat{\Sigma}, \pi_*^{\mathcal{A}_p^+, t} \mathcal{H}^{t, \dagger}, \mathbb{T} \otimes_{\mathbb{Z}_p} \mathcal{A}_p^+, \psi^{\mathcal{A}_p^+, t})$$

where $\psi^{\mathcal{A}_p^+, t}$ is the composition of ϕ^t with the map

$$\mathcal{O}(\mathcal{X}^{\mathcal{A}_p^+, t}) \rightarrow \text{End}_{\mathcal{O}_{\widehat{\Sigma}}}(\pi_*^{\mathcal{A}_p^+, t} \mathcal{H}^{t, \dagger}).$$

We can now show that the nilreduction of $\mathcal{X}^{\mathcal{A}_p^+, t}$ is independent of the choice of $t \in \Sigma^{\text{cpt}}$. In particular, $U_{t'}^{-1}$ induces a finite map $(\mathcal{X}^{\mathcal{A}_p^+, t})^{\text{red}} \rightarrow \mathbb{A}_{\mathcal{W}}^1$ and we will use this consequence to show that $\mathcal{X}^{\mathcal{A}_p^+, t}$ itself is independent of the choice of t .

Lemma 3.4.2. *Let $t, t' \in \Sigma^{\text{cpt}}$. There is a canonical $\widehat{\Sigma}$ -isomorphism*

$$(\mathcal{X}^{\mathcal{A}_p^+, t})^{\text{red}} \cong (\mathcal{X}^{\mathcal{A}_p^+, t'})^{\text{red}}$$

which is compatible with the maps $\phi^t : \mathbb{T} \otimes_{\mathbb{Z}_p} \mathcal{A}_p^+ \rightarrow \mathcal{O}(\mathcal{X}^{\mathcal{A}_p^+, t})^{\text{red}}$ and $\phi^{t'} : \mathbb{T} \otimes_{\mathbb{Z}_p} \mathcal{A}_p^+ \rightarrow \mathcal{O}(\mathcal{X}^{\mathcal{A}_p^+, t'})^{\text{red}}$.

Proof. We apply Theorem 3.2.1 to the two eigenvariety data

$$(\widehat{\Sigma}, \pi_*^{\mathcal{A}_p^+, t} \mathcal{H}^{t, \dagger}, \mathbb{T} \otimes_{\mathbb{Z}_p} \mathcal{A}_p^+, \psi^{\mathcal{A}_p^+, t}), \quad (\widehat{\Sigma}, \pi_*^{\mathcal{A}_p^+, t'} \mathcal{H}^{t', \dagger}, \mathbb{T} \otimes_{\mathbb{Z}_p} \mathcal{A}_p^+, \psi^{\mathcal{A}_p^+, t'}),$$

with j and σ (in the notation of Theorem 3.2.1) the identity maps and the subset \mathcal{X}^{cl} of ‘classical points’ equal to all the maximal points. A simple variant of Proposition 3.3.1 gives a canonical bijection between the maximal points of the two eigenvarieties being considered. \square

Finally, we show that the eigenvariety $\mathcal{X}^{\mathcal{A}_p^+, t}$ (not just the nilreduction) is independent of the choice of $t \in \Sigma^{cpt}$. We will need a basic lemma about finite maps of pseudorigid spaces:

Lemma 3.4.3. *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a map of pseudorigid spaces. Suppose that the induced map $f^{red} : \mathcal{X}^{red} \rightarrow \mathcal{Y}$ is finite. Then f is finite.*

Proof. Maps of pseudorigid spaces are necessarily locally of finite type (by Proposition 2.2.3(4) and Proposition 2.2.6). So we may apply [20, Prop. 1.5.5] which says that (for locally of finite type maps of analytic adic spaces) finite is equivalent to quasi-finite and proper. Since f and f^{red} are the same on underlying topological spaces, it is easy to see that the quasi-finiteness and properness of f^{red} implies these properties for f . \square

Proposition 3.4.4. *Let $t, t' \in \Sigma^{cpt}$. There is a canonical $\widehat{\Sigma}$ -isomorphism*

$$\mathcal{X}^{\mathcal{A}_p^+, t} \cong \mathcal{X}^{\mathcal{A}_p^+, t'}$$

which is compatible with the maps $\phi^t : \mathbb{T} \otimes_{\mathbb{Z}_p} \mathcal{A}_p^+ \rightarrow \mathcal{O}(\mathcal{X}^{\mathcal{A}_p^+, t})$ and $\phi^{t'} : \mathbb{T} \otimes_{\mathbb{Z}_p} \mathcal{A}_p^+ \rightarrow \mathcal{O}(\mathcal{X}^{\mathcal{A}_p^+, t'})$.

Proof. First we note that Lemma 3.4.1 implies that there is a map $\mathcal{X}^{\mathcal{A}_p^+, t} \xrightarrow{U_t^{-1}} \mathbb{A}_{\mathcal{W}}^1$. By Lemma 3.4.2, this induces a finite map $(\mathcal{X}^{\mathcal{A}_p^+, t})^{red} \rightarrow \mathbb{A}_{\mathcal{W}}^1$ which factors through the spectral variety $\mathcal{Z}^{t'}$. It follows from Lemma 3.4.3 that the map $U_{t'}^{-1}$ is finite and factors through a closed adic subspace $\widetilde{\mathcal{Z}}^{t'}$ of $\mathbb{A}_{\mathcal{W}}^1$ with the same underlying topological space as $\mathcal{Z}^{t'}$.

Now we consider the finite map $p : \mathcal{X}^{\mathcal{A}_p^+, t} \rightarrow \mathbb{A}_{\mathcal{W}}^2$ given by $(U_t^{-1}, U_{t'}^{-1})$, which factors through the fibre product $\mathcal{Z} := \mathcal{Z}^t \times_{\mathcal{W}} \widetilde{\mathcal{Z}}^{t'}$.

By Proposition 3.1.6, it now suffices to show that the coherent sheaf $p_* \mathcal{H}^{t, \dagger}$ has a description which is symmetric in t and t' . We can cover \mathcal{Z}^t by affinoid opens $\mathcal{Z}_{U, h}^t$, with (U, h) running over slope data (for the characteristic power series of U_t), and similarly for $\mathcal{Z}^{t'}$. Since $\mathcal{Z}^{t'} \rightarrow \widetilde{\mathcal{Z}}^{t'}$ is a

nilpotent thickening this gives a cover of $\widetilde{\mathcal{Z}}^{t'}$ by affinoid opens $\widetilde{\mathcal{Z}}_{V,k}^{t'}$. We have $\widetilde{\mathcal{Z}}_{V,k}^{t'} \cong \text{Spa}(\mathcal{O}(V)[X]/I_{V,k})$ with $I_{V,k}$ an ideal of $\mathcal{O}(V)[X]$ with $(Q'(X)^N) \subset I_{V,k} \subset (Q'(X))$ where $Q'(X)$ is the multiplicative polynomial in the slope factorisation of the characteristic power series of $U_{t'}$ corresponding to the slope datum (V, k) .

We obtain an open cover of \mathcal{Z} by the fibre products $\mathcal{Z}_{U,h}^t \times_{\mathcal{W}} \widetilde{\mathcal{Z}}_{V,k}^{t'}$ which are finite over $U \cap V \subset \mathcal{W}$. Since \mathcal{W} is quasi-separated, we can cover \mathcal{Z} by affinoid opens $\mathcal{Z}_{U,h,k} = \mathcal{Z}_{U,h}^t \times_{\mathcal{W}} \widetilde{\mathcal{Z}}_{U,k}^{t'}$, finite over affinoid opens U in \mathcal{W} such that (U, h) is a slope datum for U_t and (U, k) is a slope datum for $U_{t'}$.

We claim that the restriction of $p_*\mathcal{H}^{t,\dagger}$ to $\mathcal{Z}_{U,h,k}$ is the coherent sheaf associated to

$$H^*(X_K, \mathcal{D}_U^r)_{U_t \leq h} \cap H^*(X_K, \mathcal{D}_U^r)_{U_{t'} \leq k}$$

where the subscripts denote taking the slope decompositions for U_t and $U_{t'}$, which exist since (U, h) and (U, k) are slope data for the respective characteristic power series. In particular, we can take $\widetilde{\mathcal{Z}}^{t'} = \mathcal{Z}^{t'}$ and the symmetry between t and t' in our description of $p_*\mathcal{H}^{t,\dagger}$ implies the Proposition. So it remains to justify our claim.

By definition $\mathcal{H}^{t,\dagger}|_{\mathcal{Z}_{U,h}^t}$ is the coherent sheaf associated to

$$H^*(X_K, \mathcal{D}_U^r)_{U_t \leq h}.$$

Recall that we have multiplicative polynomials Q and Q' over $\mathcal{O}(U)$ such that $H^*(X_K, \mathcal{D}_U^r)_{U_t \leq h} = \ker(Q^*(U_t))$ and $Q^*(U_t)$ is an isomorphism on $H^*(X_K, \mathcal{D}_U^r)_{U_t > h}$, and similarly for the slope $\leq k$ decomposition with respect to $U_{t'}$. Since U_t and $U_{t'}$ commute (hence $Q^*(U_t)$ and $Q'^*(U_{t'})$ also commute) on $H^*(X_K, \mathcal{D}_U^r)$ the idempotent projectors

$$(Q^*(U_t)|_{\text{Im}Q^*(U_t)})^{-1} \circ Q^*(U_t), \quad (Q'^*(U_{t'})|_{\text{Im}Q'^*(U_{t'})})^{-1} \circ Q'^*(U_{t'})$$

giving their slope decompositions commute and we have a decomposition

$$\begin{aligned} H^*(X_K, \mathcal{D}_U^r)_{U_t \leq h} &= H^*(X_K, \mathcal{D}_U^r)_{U_t \leq h} \cap H^*(X_K, \mathcal{D}_U^r)_{U_{t'} \leq k} \\ &\quad \oplus H^*(X_K, \mathcal{D}_U^r)_{U_t \leq h} \cap H^*(X_K, \mathcal{D}_U^r)_{U_{t'} > k}. \end{aligned}$$

The restriction of $p_*\mathcal{H}^{t,\dagger}$ to $\mathcal{Z}_{U,h,k}$ corresponds to the module

$$H^*(X_K, \mathcal{D}_U^r)_{U_t \leq h} \otimes_{\mathcal{O}(U)[X]} \mathcal{O}(U)[X]/I_{U,k}$$

where X acts by $U_{t'}^{-1}$. Since $Q(U_{t'}^{-1})$ is an isomorphism on $H^*(X_K, \mathcal{D}_U^r)_{U_{t'} > k}$ and $Q(U_{t'}^{-1})$ is zero on $H^*(X_K, \mathcal{D}_U^r)_{U_{t'} \leq k}$, this tensor product can be identified with $H^*(X_K, \mathcal{D}_U^r)_{U_t \leq h} \cap H^*(X_K, \mathcal{D}_U^r)_{U_{t'} \leq k}$, as claimed. \square

4. Application

We give a sample application of our interpolation theorem Thm. 3.2.1, interpolating cyclic base change and using Prop. 3.3.2 to show that there exist large characteristic p loci in the extended eigenvarieties for GL_2 over number fields.

4.1. Cyclic base change

The following theorem is a consequence of [3], see [4, Lem. 5.1.1].

Theorem 4.1.1. *Suppose L/K is a cyclic extension of number fields. Let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_K)$, and suppose that $\pi \not\cong \pi \otimes (\chi \circ \mathrm{Art}_K \circ \det)$ for any character χ of $\mathrm{Gal}(L/K)$. Then there is a cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_L)$ such that for all places w of L lying over a place v of K we have $\mathrm{rec}(\Pi_w) \cong \mathrm{rec}(\pi_v)|_{W_{L_w}}$.*

4.2. Examples of eigenvariety data

4.2.1. The extended Coleman–Mazur eigencurve. With the notation of Section 3.3, we consider the case $\mathbf{G} = \mathrm{GL}_2/\mathbb{Q}$. We fix an integer $N \geq 5$ which is prime to p . We let \mathscr{W} denote the analytic adic weight space parametrising continuous characters $\kappa = (\kappa_1, \kappa_2)$ of

$$T_0 = \begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ 0 & \mathbb{Z}_p^\times \end{pmatrix}$$

(see [22, Defn. 4.1.1]) and let $\mathscr{W}_0 \subset \mathscr{W}$ denote the closed subspace where $\kappa_2 = 1$.

We let $K^p \subset \mathrm{GL}_2(\mathbb{A}_f^p)$ be the compact open subgroup given by

$$K^p = \{g \in \mathrm{GL}_2(\widehat{\mathbb{Z}}^p) \mid g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{N}\}$$

let $I \subset \mathrm{GL}_2(\mathbb{Q}_p)$ be the upper triangular Iwahori subgroup and set $K = K^p I \subset \mathrm{GL}_2(\mathbb{A}_f)$.

Let S be a finite set of primes, containing all the primes dividing pN , and let

$$\mathbb{T}_{\mathrm{GL}_2/\mathbb{Q}}^S = \otimes_{l \notin S} \mathbb{T}(\mathrm{GL}_2(\mathbb{Q}_l), \mathrm{GL}_2(\mathbb{Z}_l))$$

be the product of spherical Hecke algebras over \mathbb{Z}_p for places away from S .

Let $f \in \mathbb{Z}_{\geq 1}$. Using the construction recalled in Section 3.3, and gluing over a pseudorigid affinoid open cover of \mathcal{W}_0 , we obtain an eigenvariety datum

$$(\mathcal{Z}_{\mathrm{GL}_2/\mathbb{Q}}^f, \mathcal{H}_{\mathrm{GL}_2/\mathbb{Q}}, \mathbb{T}_{\mathrm{GL}_2/\mathbb{Q}}^S, \psi_{\mathrm{GL}_2/\mathbb{Q}})$$

and associated eigenvariety $\mathcal{E}^{S,f}$ whose maximal points correspond to systems of Hecke eigenvalues for \mathbb{T}^S and U_p^f , where $U_p = [K \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} K]$, with non-zero U_p^f -eigenvalue, appearing in the overconvergent cohomology spaces $H^1(K, \mathcal{D}_\kappa^r)$.

Remark 4.2.1. For any $f \geq 1$ there is a natural finite map $\mathcal{E}^{S,1} \rightarrow \mathcal{E}^{S,f}$.

We now define some notation: if \mathcal{X} is a pseudorigid space over \mathcal{O}_K we denote by $\mathcal{X}^{rig} \subset \mathcal{X}$ the Zariski open subspace of \mathcal{X} given by the locus where $\pi_K \neq 0$. We have $\mathcal{X}^{rig} = \mathcal{X} \times_{\mathrm{Spa}(\mathcal{O}_K, \mathcal{O}_K)} \mathrm{Spa}(K, \mathcal{O}_K)$ and \mathcal{X}^{rig} is (the adic space associated to) a rigid space over K .

Lemma 4.2.2. *The subspace $(\mathcal{E}^{S,f})^{rig}$ is Zariski dense in $\mathcal{E}^{S,f}$. In particular, any Zariski dense subset of $(\mathcal{E}^{S,f})^{rig}$ is Zariski dense in $\mathcal{E}^{S,f}$.*

Proof. Let (U, h) be a slope datum for $\mathcal{E}^{S,f}$. We recall (the discussion after [22, Lem. 6.1.3] gives more details) that $\mathcal{E}^{S,f}$ has an open cover by affinoid pseudorigid subspaces $\mathcal{E}_{U,h}^{S,f}$ where (U, h) is a slope datum, and $\mathcal{E}_{U,h}^{S,f}$ is finite over a Fredholm hypersurface $\mathcal{Z}_{\mathrm{GL}_2/\mathbb{Q}, U, h}$ for a Fredholm polynomial with coefficients in $\mathcal{O}(U)$. Now U^{rig} is Zariski dense in U , so it follows from [9, Lem. 6.2.8] (irreducible components of $\mathcal{E}_{U,h}^{S,f}$ have dimension 1, so they surject onto irreducible components of U and this lemma applies) that its inverse image $(\mathcal{E}_{U,h}^{S,f})^{rig}$ is Zariski dense in $\mathcal{E}_{U,h}^{S,f}$. Since the $\mathcal{E}_{U,h}^{S,f}$ cover $\mathcal{E}^{S,f}$ it follows that $(\mathcal{E}^{S,f})^{rig}$ is Zariski dense in $\mathcal{E}^{S,f}$. \square

We now denote by $\mathcal{E}_{cusp}^{S,f}$ the Zariski closure of the classical cuspidal points of $(\mathcal{E}^{S,f})^{rig}$, with its induced reduced structure. The preceding lemma implies that $\mathcal{E}_{cusp}^{S,f}$ is equal to the Zariski closure of $(\mathcal{E}^{S,f})_{cusp}^{rig}$, the Zariski closure in $(\mathcal{E}^{S,f})^{rig}$ of the classical cuspidal points, which is a union of irreducible components of $(\mathcal{E}^{S,f})^{rig}$. This furthermore implies that $\mathcal{E}_{cusp}^{S,f}$ is a union of irreducible components of $\mathcal{E}^{S,f}$. Finally, if F is a number field we denote by $\mathcal{E}_{cusp, F\text{-ncm}}^{S,f}$ the Zariski closure of the classical cuspidal points of $(\mathcal{E}^{S,f})^{rig}$ which do not have CM by an imaginary quadratic subfield of F . This is again a union of irreducible components of $\mathcal{E}^{S,f}$ and contains all the non-ordinary components.

4.2.2. Eigenvarieties for GL_2/F . We now allow F to be an arbitrary number field and, with the notation of Section 3.3, we consider the case $\mathbf{H} = \mathrm{GL}_2/F$. We let $K_F^p \subset \mathrm{GL}_2(\mathbb{A}_{F,f}^p)$ be the compact open subgroup given by

$$K_F^p = \{g \in \mathrm{GL}_2(\widehat{\mathcal{O}}_F^p) \mid g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{N}\}$$

let $I_F \subset \prod_{v|p} \mathrm{GL}_2(F_v)$ be the upper triangular Iwahori subgroup and set $K_F = K_F^p I_F \subset \mathrm{GL}_2(\mathbb{A}_{F,f})$. We let \mathscr{W}_F denote the analytic adic weight space parametrising continuous characters κ of

$$T_{F,0} = \begin{pmatrix} \prod_{v|p} \mathcal{O}_{F,v}^\times & 0 \\ 0 & \prod_{v|p} \mathcal{O}_{F,v}^\times \end{pmatrix}$$

which are trivial on (the closure of) the image of $Z(K) = F^\times \cap K$ in $T_{F,0}$.

The dimension of \mathscr{W}_F is equal to the difference between $2[F : \mathbb{Q}]$ and the \mathbb{Z}_p -rank of the closure of \mathcal{O}_F^\times in $\mathcal{O}_{F,p}^\times$. This latter rank is $r_1 + r_2 - 1 - \mathfrak{d}_{F,p}$, where $\mathfrak{d}_{F,p}$, by definition, measures the defect in Leopoldt’s conjecture for F and p and r_1, r_2 denote the number of real and complex places of F .

Again we have an eigenvariety datum

$$(\mathscr{L}_{\mathrm{GL}_2/F}, \mathscr{H}_{\mathrm{GL}_2/F}, \mathbb{T}_{\mathrm{GL}_2/F}^S, \psi_{\mathrm{GL}_2/F})$$

where

$$\mathbb{T}_{\mathrm{GL}_2/F}^S = \otimes_{v \notin S} \mathbb{T}(\mathrm{GL}_2(F_v), \mathrm{GL}_2(\mathcal{O}_{F_v}))$$

is a tensor product over finite places v which do not divide any of the primes in S . We denote the associated eigenvariety by \mathscr{X}_F^S whose points correspond to systems of Hecke eigenvalues for $\mathbb{T}_{\mathrm{GL}_2/F}^S$ and $U_{p,F} = \prod_{v|p} U_v$,

where $U_v = [K_F \begin{pmatrix} 1 & 0 \\ 0 & \varpi_v \end{pmatrix} K_F]$, with non-zero $U_{p,F}$ -eigenvalue, appearing in the overconvergent cohomology spaces $H^*(K_F, \mathcal{D}_\kappa^r)$, defined using the locally symmetric spaces X_{K_F} appearing in 3.3. Note that the dimension of X_{K_F} (as a real manifold) is $2r_1 + 3r_2$.

4.3. Base change

For F_v/\mathbb{Q}_l we have a map of spherical Hecke algebras

$$\mathbb{T}(\mathrm{GL}_2(F_v), \mathrm{GL}_2(\mathcal{O}_{F_v})) \rightarrow \mathbb{T}(\mathrm{GL}_2(\mathbb{Q}_l), \mathrm{GL}_2(\mathbb{Z}_l))$$

induced by unramified local Langlands and the map $\rho \mapsto \rho|_{W_{F_v}}$ on the Galois side. We make this map explicit.

We write $\mathbb{T}(\mathrm{GL}_2(\mathbb{Q}_l), \mathrm{GL}_2(\mathbb{Z}_l)) = \mathbb{Z}_p[T_l^1, T_l^2]$, where T_l^1 and T_l^2 correspond to the double cosets of $\begin{pmatrix} 1 & 0 \\ 0 & l \end{pmatrix}$ and $\begin{pmatrix} l & 0 \\ 0 & l \end{pmatrix}$ respectively. If

$$\lambda : \mathbb{T}(\mathrm{GL}_2(\mathbb{Q}_l), \mathrm{GL}_2(\mathbb{Z}_l)) \rightarrow \overline{\mathbb{Q}}_p$$

is a homomorphism we associate to it the semisimple unramified Weil group representation where Frob_l (a geometric Frobenius element) has characteristic polynomial $X^2 - \lambda(T_l^1)X + l\lambda(T_l^2)$. Likewise, if

$$\lambda_v : \mathbb{T}(\mathrm{GL}_2(F_v), \mathrm{GL}_2(\mathcal{O}_{F_v})) \rightarrow \overline{\mathbb{Q}}_p$$

we associate the unramified representation where Frob_v has characteristic polynomial $X^2 - \lambda(T_v^1)X + q_v\lambda(T_v^2)$ where $q_v = l^{f_v}$ is the cardinality of the residue field of F_v and l is the rational prime under v .

It follows that the map $\mathbb{T}(\mathrm{GL}_2(F_v), \mathrm{GL}_2(\mathcal{O}_{F_v})) \rightarrow \mathbb{T}(\mathrm{GL}_2(\mathbb{Q}_l), \mathrm{GL}_2(\mathbb{Z}_l))$ takes T_v^1 to $\mathrm{tr}(\mathrm{Frob}_l^{f_v})$ (a homogeneous degree f_v polynomial in T_l^1 and lT_l^2) and T_v^2 to $(T_l^2)^{f_v}$.

We thereby obtain the map of Hecke algebras $\sigma_F^S : \mathbb{T}_{\mathrm{GL}_2/F}^S \rightarrow \mathbb{T}_{\mathrm{GL}_2/\mathbb{Q}}^S$.

We denote by $j : \mathscr{W}_0 \rightarrow \mathscr{W}_F$ the map of weight spaces induced by the norm map $T_{F,0} \rightarrow T_0$.

Theorem 4.3.1. *Let F/\mathbb{Q} be a cyclic extension of number fields. Let g be the number of places of F dividing p , let e, f be the inertial and residual degrees of a place dividing p . There is a canonical finite morphism*

$$i : \mathcal{E}_{\mathrm{cusp}, F\text{-ncm}}^{S,1} \rightarrow \mathcal{X}_F^S$$

lying over $j : \mathscr{W}_0 \rightarrow \mathscr{W}_F$ and compatible with the map $\sigma_F^S : \mathbb{T}_{\mathrm{GL}_2/F}^S \rightarrow \mathbb{T}_{\mathrm{GL}_2/\mathbb{Q}}^S$.

Proof. First we apply the finite map $\mathcal{E}_{\mathrm{cusp}, F\text{-ncm}}^{S,1} \rightarrow \mathcal{E}_{\mathrm{cusp}, F\text{-ncm}}^{S,fg}$. So it suffices to show that there is a canonical finite morphism

$$i : \mathcal{E}_{\mathrm{cusp}, F\text{-ncm}}^{S,fg} \rightarrow \mathcal{X}_F^S$$

with the properties specified by the theorem. The subset $\mathcal{X}^{cl} \subset \mathcal{E}^{S,fg}$ is defined to be the points arising from classical cusp forms of weight $k \geq 2$, level K and U_p^{fg} -slope $< \frac{k-1}{e}$ and which moreover do not have CM by

an imaginary quadratic subfield of F . Note that \mathcal{X}^{cl} is Zariski dense in $(\mathcal{E}_{cusp,F-ncm}^{S,fg})^{rig}$, and hence it is Zariski dense in $\mathcal{E}_{cusp,F-ncm}^{S,fg}$.

By Thm. 4.1.1 (we excluded CM points so the condition of the theorem is satisfied), for each point $x \in \mathcal{X}^{cl}$ we have a cuspidal automorphic representation π_x of $\mathrm{GL}_2(\mathbb{A}_F)$ which is regular algebraic of weight $(k - 2, 0)_{\tau:F \rightarrow \mathbb{C}}$ and whose Hecke eigenvalues are giving by pulling back the Hecke eigenvalues for x by the map σ_F^S . Moreover, for each $v|p$ one of the U_v -eigenvalues on the Iwahori invariants of $\pi_{x,v}$ has p -adic valuation equal to f_v times the slope of the classical form giving rise to x , so there is a $U_{p,F}$ -eigenvalue of $\pi_{x,p}^{I_F}$ with p -adic valuation $< \frac{k-1}{e} = v_p(\varpi_v^{k-1})$. Now we can apply [16, Thm. 3.2.5], together with [16, Thm. 4.3.3] and Prop. 3.3.1 to show that the system of Hecke eigenvalues arising from π_x appears in \mathcal{X}_F^S . Finally, we conclude by applying Thm. 3.2.1 to the eigenvariety data

$$\mathfrak{D}_1 = (\mathbb{A}_{\mathcal{W}_0}^1, \mathcal{H}_{\mathrm{GL}_2/\mathbb{Q}}, \mathbb{T}_{\mathrm{GL}_2/\mathbb{Q}}^S, \psi_{\mathrm{GL}_2/\mathbb{Q}}), \quad \mathfrak{D}_2 = (\mathbb{A}_{\mathcal{W}_F}^1, \mathcal{H}_{\mathrm{GL}_2/F}, \mathbb{T}_{\mathrm{GL}_2/F}^S, \psi_{\mathrm{GL}_2/F})$$

where the map $j : \mathbb{A}_{\mathcal{W}_0}^1 \rightarrow \mathbb{A}_{\mathcal{W}_F}^1$ is induced by $j : \mathcal{W}_0 \rightarrow \mathcal{W}_F$ and we have already defined the map $\sigma_F^S : \mathbb{T}_{\mathrm{GL}_2/F}^S \rightarrow \mathbb{T}_{\mathrm{GL}_2/\mathbb{Q}}^S$. \square

Corollary 4.3.2. *Let F/\mathbb{Q} be a cyclic extension of number fields. Then $(\mathcal{X}_F^S)^{rig} \subset \mathcal{X}_F^S$ is a strict inclusion. Moreover, the same is true if we restrict to the non-ordinary locus (i.e. where the $U_{p,F}$ -eigenvalue is not a unit), and the dimension of the Zariski closed subset $\mathcal{X}_F^S \setminus (\mathcal{X}_F^S)^{rig}$ (and its non-ordinary locus) is at least $[F : \mathbb{Q}]$. Note that the Leopoldt conjecture is known for F, p , so \mathcal{W}_F has dimension $1 + [F : \mathbb{Q}] + r_2$.*

Proof. It follows from [6, Cor. A1] that there is a non-ordinary irreducible component of $\mathcal{E}_{cusp}^{S,1}$ which has a characteristic p point. Since the component is non-ordinary, the non-CM points are Zariski dense. Applying Theorem 4.3.1, we deduce that there is an irreducible component of \mathcal{X}_F^S which contains a characteristic p point. We also see (from the proof of Theorem 4.3.1), that this component contains a classical point x arising from a cuspidal automorphic representation π with a $U_{p,F}$ -eigenvalue of $\pi_p^{I_F}$ with p -adic valuation $< v_p(\varpi_v^{k-1})$. The representation π contributes to the cohomology of X_{K_F} precisely in degrees $r_1 + r_2, \dots, r_1 + 2r_2$ (see [17, 3.6]) — moreover this contribution accounts for the entire generalised eigenspace for the associated system of Hecke eigenvalues at places away from S , by (for example) the Jacquet–Shalika classification theorem [21, Thm. 4.4]. It follows from [16, Thm. 3.2.5] that we have $l(x) = r_2$ and Proposition 3.3.2 therefore implies that the irreducible component under consideration has dimension

$\geq 1 + [F : \mathbb{Q}]$ and so the characteristic p locus in this irreducible component has dimension $\geq [F : \mathbb{Q}]$. \square

Remark 4.3.3. Using other known cases of Langlands functoriality it is possible to produce more examples of p -adic functoriality which show the existence of a large characteristic p locus in extended eigenvarieties. For example, using solvable base change and Dieulefait’s results on base change for GL_2 [12, 13], one can extend the above corollary to eigenvarieties for GL_2 over a solvable extension F' of a totally real field F . One could also consider the symmetric square lifting and show the existence of large characteristic p loci in extended eigenvarieties for GL_3/\mathbb{Q} .

The reader may also wonder, in view of §3.4, whether Corollary 4.3.2 remains true if we had included \mathcal{A}_p^+ in the construction of the eigenvariety. This is true; it follows from the fact that, if we denote this eigenvariety by $\mathcal{X}_F^{S, \mathcal{A}_p^+}$, then by the construction of these eigenvarieties there is a canonical finite surjective map $\mathcal{X}_F^{S, \mathcal{A}_p^+} \rightarrow \mathcal{X}_F^S$. Similar remarks apply if one adds or removes other Hecke operators (using Proposition 3.4.4 if one wishes to change controlling operator). One can also prove a version of Theorem 4.3.1 using the eigenvarieties incorporating the full Atkin–Lehner algebra \mathcal{A}_p^+ — the norm map between tori induces a map between the Atkin–Lehner Hecke algebras for F and for \mathbb{Q} , which is easily seen to be compatible with base change functoriality for automorphic representations which are unramified principal series or unramified twist of Steinberg at p .

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