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# Completeness on the worm domain and the Müntz–Szász problem for the Bergman space

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In this paper we are concerned with the problem of completeness in the Bergman space of the worm domain $\mathcal{W}_\mu$ and its truncated version $\mathcal{W}'_\mu$. We determine some orthogonal systems and show that they are not complete, while showing that the union of two particular such systems is complete.

In order to prove our completeness result we introduce the *Müntz–Szász problem* for the 1-dimensional Bergman space of the disk
$\{\zeta : |\zeta - 1| < 1\}$ and find a sufficient condition for its solution.

## Introduction

The Diederich–Fornaess worm domain was introduced in [9] and is defined for a given $\mu > 0$ as

$$(1) \qquad \mathcal{W}_\mu = \left\{(z_1, z_2) \in \mathbb{C}^2 : \left|z_1 - e^{i \log|z_2|^2}\right|^2 < 1 - \varphi\big(\log|z_2|^2\big)\right\},$$

where $\varphi : [-A, A] \to [0, 1]$ is a smooth, convex, even function that vanishes identically in $[-\mu, \mu]$, with $A > \mu$, $\varphi(A) = 1$, and increasing on $[\mu, A]$. As a result, $\mathcal{W}_\mu$ is smooth, pseudoconvex and strictly pseudoconvex at all points $(z_1, z_2) \in \partial\mathcal{W}_\mu$ with $z_1 \neq 0$. See [6] for a thorough discussion of basic properties of the worm. The worm turned out to be of fundamental importance in the theory of geometric analysis in several complex variables, see [15], [1], [2], [7], [8], [16, 17], [18], [19, 20] and references therein.

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For computational purposes, $\mathcal{W}_\mu$ is often truncated to the non-smooth bounded domain

$$(2)\qquad \mathcal{W}'_\mu = \left\{(z_1,z_2)\in\mathbb{C}^2 : \left|z_1 - e^{i\log|z_2|^2}\right|^2 < 1,\ \left|\log|z_2|^2\right| < \mu\right\},$$

that is, one replaces the function $\varphi$ with the characteristic function of the complement of the interval $[-\mu,\mu]$.

In the discussion that follows, we let $\Omega$ denote either domain $\mathcal{W}_\mu$ or $\mathcal{W}'_\mu$.

In this study we are concerned with the question of finding orthogonal and complete sets in $A^2(\Omega)$. For details about the notions of orthogonality and completeness in a Hilbert space, see Section 1. As is well known, when $\mu \geq \pi$, $\Omega$ has non-trivial *Nebenhülle*. Here the Nebenhülle is understood to be the interior of the connected component that contains $\overline{\Omega}$ of the intersection of all domains of holomorphy containing $\overline{\Omega}$. We first show that this easily implies that the closure in $A^2(\Omega)$ of the holomorphic polynomials is a proper subset of $A^2(\Omega)$.

Thus we are led to consider sets of suitable "monomials" in $\Omega$ that can be defined as the holomorphic continuation of non-integral powers $z_1^\eta$, when $z=(z_1,z_2)$ is initially restricted to $\Delta\times\{z_2 : |z_2|=1\}$, where we denote by $\Delta$ the disk $\{\zeta\in\mathbb{C} : |\zeta-1|<1\}$.

We determine some orthogonal sets

$$\{H_{2k,j}\,,k,j\in\mathbb{Z},k\geq 0\},\quad\text{and}\quad\{H_{2k+1,j}\,,j\in\mathbb{Z},k\geq 0\}$$

(see Corollary 1.5), and show that their union determines a complete set in $A^2(\mathcal{W}'_\mu)$ when $\mu>0$ (Theorem 3.1). We also show that each of the two systems, however, is not complete (Proposition 3.2).

In order to prove our completeness result, Theorem 3.1, we prove a result of independent interest, Theorem 2.1. We naturally use the name *the Müntz–Szász problem for the Bergman space* for the question of characterizing the sequences $\{\lambda_j\}$ in the right half-plane for which the sets of powers $\{\zeta^{\lambda_j-1}\}$ form a complete set in $A^2(\Delta)$.

The classical Müntz–Szász theorem deals with the completeness of sets of powers $\{t^{\lambda_j-\frac12}\}$ in $L^2([0,1])$, where again $\lambda_j$ is in the right half plane. The solution was provided by C. Müntz [21] and by O. Szász [25] in two separate papers, where they showed that the set $\{t^{\lambda_j-\frac12}\}$ is complete $L^2([0,1])$ if and only if $\sum_{j=1}^{+\infty}(1+|\lambda_j|^2)^{-1}\operatorname{Re}\lambda_j=\infty$ (see also [22] or [24] for a more accessible reference).

We find a sufficient condition for the solution of the Müntz–Szász problem for the Bergman space $A^2(\Delta)$ and use it to prove our completeness result for $A^2(\mathcal{W}'_\mu)$. The Müntz–Szász problem for the Bergman space has been further studied in [23].

Finally we show that the complete set $\{H_{k,j},\, k,j \in \mathbb{Z}, k \geq 0\}$, is not a Schauder basis (Theorem 3.3). The definition of Schauder basis is recalled in Section 1.

## 1. Orthogonal sets in $A^2(\mathcal{W}'_\mu)$

Let $\mu > 0$ and consider the domain $\mathcal{W}'_\mu$. The problem we address here is to find a possibly complete orthonormal system for $A^2(\mathcal{W}'_\mu)$ and consequentely have a way to obtain an expression for the Bergman kernel.

For the reader's convenience, let us recall a few definitions, for which we refer to [27]. A sequence of vectors $\{v_n\}_n$ in a Banach space $V$ is

- a *Schauder basis* if for each $w \in V$ there exists a unique scalar sequence $\{c_n\}_n$ such that $\sum_n c_n v_n$ converges to $w$ with respect to the norm topology.

Now suppose $V$ to be a separable Hilbert space. A sequence $\{v_n\}_n$ in $V$ is

- an *orthogonal system* if $\langle v_m, v_n \rangle = 0$ whenever $m \neq n$;
- an *orthonormal system* if it is orthogonal and $\|v_n\| = 1$ for all $n$;
- a *complete system* if 0 is the only vector in $V$ that is orthogonal to $v_n$ for all $n$; or, equivalently, if the linear span of $\{v_n\}_n$ is dense in $V$.

A complete orthonormal system is automatically a Schauder basis and it is called an *orthornormal basis*. On the other hand, a sequence in $V$ that is not orthogonal may be complete without being a Schauder basis; in other words, the aforementioned sequences $\{c_n\}_n$ may exist for all $w \in V$ without being unique.

In the analysis on the worm domains $\mathcal{W}_\mu$ and $\mathcal{W}'_\mu$ a special role is played by the functions

$$E_\eta(z) = e^{\eta L(z)}\,, \tag{3}$$

where

$$L(z) = \log\left(z_1 e^{-i\log|z_2|^2}\right) + i\log|z_2|^2\,, \tag{4}$$

and log denotes the principal branch of the logarithm, so that

$$E_\eta(z_1, z_2) = \big(z_1 e^{-i\log|z_2|^2}\big)^\eta e^{i\eta\log|z_2|^2} .$$

The function $L$ is well defined and holomorphic in a domain containing $\cup_\mu \mathcal{W}'_\mu$ (see [18], Lemma 1.2 and Proposition 1.3). Moreover, we point out that the fiber of $\mathcal{W}'_\mu$ over each $z_1 \in D(0,2)\setminus\{0\}$ is not connected and that $L(z)$ is locally constant in $z_2$, but not constant. The same happens with $E_\eta(z)$ for $\eta \in \mathbb{C}\setminus\mathbb{Z}$, while $E_k(z) = z_1^k$ for all $k\in\mathbb{Z}$, $z \in \mathcal{W}'_\mu$. Hence the functions $E_\eta$ are the analytic continuation to $\mathcal{W}'_\mu$ of the monomial $z_1^\eta$ defined in $\mathcal{W}'_{\pi/2}$ using the principal branch of the logarithm.

It is well known that the functions that are holomorphic in a neighborhood of the closure $\overline{\mathcal{W}_\mu}$ are not dense in $A^2(\mathcal{W}_\mu)$. Since a proof of this fact does not explicitly appear in the literature, we prove the following result that applies to both domains $\mathcal{W}_\mu$ and $\mathcal{W}'_\mu$.

**Proposition 1.1.** *Let $\mu \ge 2\pi$ and let $A^2(\overline{\mathcal{W}_\mu})$ denote the closure in $A^2(\mathcal{W}_\mu)$ of the functions that are holomorphic in a neighborhood of $\mathcal{W}_\mu$. Then, if $f \in A^2(\overline{\mathcal{W}_\mu})$, then $f$ is holomorphic on $\widehat{\mathcal{W}_\mu}$, where*

$$\widehat{\mathcal{W}_\mu} \supseteq \bigcup_{-\mu\le a\le\mu-2\pi} \big\{(z_1,z_2) : a < \log|z_2|^2 < a+2\pi,\ |z_1 - e^{ia}| < 1\big\}. \tag{5}$$

*Therefore $A^2(\overline{\mathcal{W}_\mu}) \subsetneqq A^2(\mathcal{W}_\mu)$.*

*The same conclusions hold true with $\mathcal{W}'_\mu$ in place of $\mathcal{W}_\mu$.*

In particular, the polynomials are not dense in either $A^2(\mathcal{W}_\mu)$ or $A^2(\mathcal{W}'_\mu)$. By contrast, D. Catlin [5] showed that, for every smoothly bounded pseudoconvex domain $\Omega$, the holomorphic functions in $C^\infty(\overline{\Omega})$ are dense in $A^2(\Omega)$.

*Proof.* It suffices to prove the result in the case of $\mathcal{W}'_\mu$, since the same argument can be repeated *verbatim* for $\mathcal{W}_\mu$.

Suppose $f$ is holomorphic in a neighborhood of $\overline{\mathcal{W}'_\mu}$ and let

$$\begin{aligned}\mathcal{E}_a = &\big\{(0,z_2) :\ a \le \log|z_2|^2 \le a+2\pi\big\}\\ &\bigcup \big\{(z_1,z_2) :\ \log|z_2|^2 = a \text{ or } a+2\pi,\ |z_1 - e^{ia}| \le 1\big\}.\end{aligned}$$

Set

$$F_a(z_1,z_2) = \frac{1}{2\pi i}\int_\gamma \frac{f(z_1,\zeta)}{\zeta - z_2}\, d\zeta ,$$

where $\gamma$ is the oriented boundary of the annulus $\{z_2 \in \mathbb{C} : a < \log|z_2|^2 < a + 2\pi\}$. Then $F_a$ is holomorphic on the set

$$\widetilde{\mathcal{E}}_a = \big\{(z_1, z_2) :\ a < \log|z_2|^2 < a + 2\pi,\ |z_1 - e^{ia}| < 1\big\}\,.$$

However, $F_a(0, z_2) = f(0, z_2)$, since $f$ is holomorphic in a neighborhood of $\{(0, z_2) : \big|\log|z_2|^2\big| \le \mu\}$ and $-\mu \le a \le \mu - 2\pi$ implies $a + 2\pi \le \mu$. It follows that $F_a$ is a holomorphic extension of $f$ to the set $\widetilde{\mathcal{E}}_a$ and thus $f$ extends holomorphically to an open set $\widehat{\mathcal{W}'_\mu}$ containing the right-hand side of (5).

Now suppose that $\{f_n\}$ are holomorphic in a a neighborhood of $\overline{\mathcal{W}'_\mu}$ and that $f_n \to f$ in $A^2(\mathcal{W}'_\mu)$. Then $f_n \to f$ uniformly on the compact sets $\{\log|z_2|^2 = a,\ a + 2\pi;\ |z_1 - e^{ia}| \le 1 - \delta\}$, for $-\mu < a < \mu - 2\pi$. By Cauchy's formula, $\{f_n\}$ is Cauchy in uniform norm also on the sets

$$\big\{a \le \log|z_2|^2 \le a + 2\pi,\ |z_1 - e^{ia}| \le 1 - \delta\big\}\,.$$

Therefore $f$ extends holomorphically to the set on the right-hand side of (5).

Finally, the functions $E_\eta$ with $\eta$ not an integer cannot be extended holomorphically to any of the sets $\widetilde{\mathcal{E}}_a$, so that $A^2(\overline{\mathcal{W}'_\mu}) \subsetneq A^2(\mathcal{W}'_\mu)$.

It is immediate to check that the arguments above apply to the case of $\mathcal{W}_\mu$ as well. □

Thus, we are led to consider the set of "monomials" of the form $\{E_{\eta_j}(z)z_2^j\}_{j\in\mathbb{Z}}$ and ask whether these are orthogonal, and/or complete, for some choice of values $\eta_j \in \mathbb{C}$.

We denote by $dA$ the Lebesgue measure in the complex plane.

**Lemma 1.2.** *Let* $\operatorname{Re}\alpha, \operatorname{Re}\beta > -1$. *Then*

$$\int_\Delta \zeta^\alpha \overline{\zeta^\beta}\, dA(\zeta) = \pi \frac{\Gamma(\alpha + \overline{\beta} + 2)}{\Gamma(\alpha + 2)\Gamma(\overline{\beta} + 2)}\,.$$

*In particular,* $\zeta^\alpha$ *and* $\zeta^\beta$ *are never orthogonal to each other in* $A^2(\Delta)$.

*Proof.* We have

$$\begin{aligned}
\int_\Delta \zeta^\alpha \overline{\zeta^\beta}\, dA(\zeta) &= \int_0^2 \int_{-\cos^{-1}(r/2)}^{\cos^{-1}(r/2)} \big(re^{i\theta}\big)^\alpha \big(re^{-i\theta}\big)^{\overline{\beta}}\, d\theta\, r\, dr \\
&= \int_0^2 r^{\alpha+\overline{\beta}+1} \int_{-\cos^{-1}(r/2)}^{\cos^{-1}(r/2)} e^{i\theta(\alpha-\overline{\beta})}\, d\theta dr \\
&= \frac{2}{i(\alpha-\overline{\beta})} \int_0^2 r^{\alpha+\overline{\beta}+1} \sinh\big(\cos^{-1}(r/2) i(\alpha-\overline{\beta})\big)\, dr \\
&= \frac{4}{i(\alpha-\overline{\beta})} \int_0^{\pi/2} \big(2\cos s\big)^{\alpha+\overline{\beta}+1} \sin s \sinh\big(is(\alpha-\overline{\beta})\big)\, ds \\
&= \frac{2^{\alpha+\overline{\beta}+3}}{\alpha+\overline{\beta}+2} \int_0^{\pi/2} \big(\cos s\big)^{\alpha+\overline{\beta}+2} \cosh\big(is(\alpha-\overline{\beta})\big)\, ds \\
&= \frac{2^{\alpha+\overline{\beta}+3}}{\alpha+\overline{\beta}+2} \int_0^{\pi/2} \big(\cos s\big)^{\alpha+\overline{\beta}+2} \cos\big(s(\alpha-\overline{\beta})\big)\, ds\,.
\end{aligned}$$

Now we use [14, (9) p. 391] and, denoting by $B$ the beta function, we obtain that

$$\begin{aligned}
\int_\Delta \zeta^\alpha \overline{\zeta^\beta}\, dA(\zeta) &= \frac{\pi}{(\alpha+\overline{\beta}+2)(\alpha+\overline{\beta}+3)} \cdot \frac{1}{B(\alpha+2, \overline{\beta}+2)} \\
&= \pi \frac{\Gamma(\alpha+\overline{\beta}+2)}{\Gamma(\alpha+2)\Gamma(\overline{\beta}+2)}\,,
\end{aligned}$$

as we wished to prove. □

For a given bounded domain $\Omega$ in $\mathbb{C}^2$ that is rotationally invariant[1] in the second variable $z_2$, such as $\mathcal{W}_\mu$ and $\mathcal{W}'_\mu$, using the Fourier expansion in $z_2$, the Bergman space $A^2(\Omega)$ decomposes as an orthogonal sum

$$A^2(\Omega) = \bigoplus_{j\in\mathbb{Z}} \mathcal{H}^j\,. \tag{6}$$

Here

$$\mathcal{H}^j = \big\{F \in A^2(\Omega) :\, F(z_1,z_2) = f(z_1,|z_2|) z_2^j\big\}\,,$$

where $f$ is holomorphic in $z_1$ and locally constant in $|z_2|$. The orthogonal projection of $A^2$ onto $\mathcal{H}^j$ is given by

$$Q_j F(z_1,z_2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(z_1, e^{it} z_2) e^{-ijt}\, dt\,.$$

[1]These are called *Hartogs* domains.

Then we set

$$f(z_1, |z_2|) = \frac{Q_j F(z_1, z_2)}{z_2^j},$$

and observe that the right-hand side is holomorphic in $\Omega$, but depends only on the modulus of $z_2$. Hence $f$ is locally constant in $|z_2|$. In general, it will be constant only if the fiber over a point $(z_1, z_2)$ in $\Omega$ with $z_1$ fixed, is a connected set in the $z_2$-plane.

Let us go back to the case of $\mathcal{W}'_\mu$ and let $F, G \in \mathcal{H}^j$,

$$F(z_1, z_2) = f(z_1, |z_2|) z_2^j \quad \text{and} \quad G(z_1, z_2) = g(z_1, |z_2|) z_2^j.$$

We have

$$\begin{aligned}
(7) \qquad & \langle F, G\rangle_{A^2(\mathcal{W}'_\mu)} \\
&= \int_{|\log|z_2|^2|<\mu} \int_{|z_1 - e^{i\log|z_2|^2}|<1} f(z_1, |z_2|)\overline{g(z_1, |z_2|)} |z_2|^{2j}\, dA(z_1) dA(z_2) \\
&= 2\pi \int_{|\log r^2|<\mu} \int_{|z_1 - e^{i\log r^2}|<1} f(z_1, r)\overline{g(z_1, r)}\, dA(z_1)\, r^{2j+1} dr \\
&= \pi \int_{|s|<\mu} \int_{|z_1 - e^{is}|<1} f(z_1, e^{s/2})\overline{g(z_1, e^{s/2})}\, dA(z_1)\, e^{s(j+1)} ds \\
&= \pi \int_\Delta \int_{|s|<\mu} f(\zeta e^{is}, e^{s/2})\overline{g(\zeta e^{is}, e^{s/2})}\, e^{s(j+1)} ds\, dA(\zeta).
\end{aligned}$$

Let

$$(8) \qquad \nu = \pi/2\mu.$$

be the reciprocal of the winding number of $\mathcal{W}'_\mu$, as also defined in [1] and let $h$ be the entire function

$$h(z) = \frac{\sinh[\mu(j+1+iz)]}{j+1+iz}.$$

Define

$$(9) \qquad \gamma_{\alpha\beta} = h(\alpha - \overline{\beta}).$$

**Proposition 1.3.** *Let $\mu > 0$. For $\alpha \in \mathbb{C}$ and $j \in \mathbb{Z}$ let $F_{\alpha,j}(z_1, z_2) = E_\alpha(z) z_2^j$. Then $F_{\alpha,j} \in A^2(\mathcal{W}'_\mu)$ if and only if $\operatorname{Re}\alpha > -1$. Moreover, if $\operatorname{Re}\alpha, \operatorname{Re}\beta > -1$,* *then*

$$\langle F_{\alpha,j},\, F_{\beta,j}\rangle_{A^2(\mathcal{W}'_\mu)} = (2\pi)^2\gamma_{\alpha\beta}\frac{\Gamma(\alpha+\overline{\beta}+2)}{\Gamma(\alpha+2)\Gamma(\overline{\beta}+2)}\,.$$

*In particular,* $\langle F_{\alpha,j},\, F_{\beta,j}\rangle_{A^2(\mathcal{W}'_\mu)} = 0$ *if and only if*

$$\alpha-\overline{\beta} = 2k\nu + i(j+1) \qquad \textit{with } k\in\mathbb{Z}\setminus\{0\}\,. \tag{10}$$

*Proof.* We compute $\langle F_{\alpha,j}, F_{\beta,j}\rangle_{A^2(\mathcal{W}'_\mu)}$. Starting from (7) we obtain

$$\begin{aligned}
&\langle F_{\alpha,j},\, F_{\beta,j}\rangle_{A^2(\mathcal{W}'_\mu)} \\
&= \pi\int_\Delta\int_{|s|<\mu} E_\alpha(\zeta e^{is}, e^{s/2})\overline{E_\beta(\zeta e^{is}, e^{s/2})}\, e^{s(j+1)} ds\, dA(\zeta) \\
&= \pi\int_\Delta \zeta^\alpha\overline{\zeta^\beta}\int_{|s|<\mu} e^{is(\alpha-\overline{\beta})}\, e^{s(j+1)} ds\, dA(\zeta) \\
&= 2\pi\gamma_{\alpha\beta}\int_\Delta \zeta^\alpha\overline{\zeta^\beta}\, dA(\zeta)\,,
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
\gamma_{\alpha\beta} &:= \tfrac{1}{2}\int_{|s|<\mu} e^{s(j+1+i(\alpha-\overline{\beta}))}\, ds \\
&= \begin{cases} \frac{\sinh(\mu(j+1+i(\alpha-\overline{\beta})))}{j+1+i(\alpha-\overline{\beta})} & \text{if } j+1+i(\alpha-\overline{\beta})\neq 0 \\ \mu & \text{if } j+1+i(\alpha-\overline{\beta}) = 0\,, \end{cases}
\end{aligned}$$

as claimed.

Therefore, $\gamma_{\alpha\beta}=0$ if and only if

$$\mu(j+1+i(\alpha-\overline{\beta})) = k\pi i \qquad \text{for } k\in\mathbb{Z}\setminus\{0\}\,,$$

that is,

$$\alpha-\overline{\beta} = 2k\nu + i(j+1) \qquad \text{for } k\in\mathbb{Z}\setminus\{0\}\,. \tag{12}$$

Notice that, when $\alpha=\beta$, the previous computation gives

$$\begin{aligned}
\|E_\alpha(z)z_2^j\|^2_{A^2(\mathcal{W}'_\mu)} &= 2\pi\gamma_{\alpha,\alpha}\int_\Delta |\zeta^\alpha|^2\, dA(\zeta) \\
&= 2\pi\gamma_{\alpha,\alpha}\int_{|\zeta|<1} e^{2[\operatorname{Re}\alpha\log|\zeta-1|-\operatorname{Im}\alpha\arg(\zeta-1)]}\, dA(\zeta) \\
&= 2\pi\gamma_{\alpha,\alpha}\int_{|\zeta|<1} |\zeta-1|^{2\operatorname{Re}\alpha} e^{-2\operatorname{Im}\alpha\arg(\zeta-1)}\, dA(\zeta)\,,
\end{aligned}$$

which is finite if and only if $\operatorname{Re}\alpha > -1$. This proves the first part of the statement. The second part now follows from Lemma 1.2. $\square$

The following corollaries now follow at once.

**Corollary 1.4.** *Then, for* $\operatorname{Re}\alpha > -1$*, we have that* $F_{\alpha,j} \in A^2(\mathcal{W}'_\mu)$ *and*

$$\|F_{\alpha,j}\|^2_{A^2(\mathcal{W}'_\mu)} = (2\pi)^2 \frac{\sinh[\mu(j+1-2\operatorname{Im}\alpha)]}{j+1-2\operatorname{Im}\alpha} \frac{\Gamma(2+2\operatorname{Re}\alpha)}{|\Gamma(2+\alpha)|^2}\,.$$

For $c_0 > -1$, and $\ell = 0,1,2,\dots$ we set

$$H_{\ell,j}(z_1,z_2) = E_{c_0+\nu\ell+i(j+1)/2}(z) z_2^j\,. \tag{13}$$

**Corollary 1.5.** *For* $\mu > 0$*, each of the two sets*

$$\begin{aligned} &\{H_{2k,j}\,,\ j\in\mathbb{Z},\ k=0,1,2,\dots\}, \quad \textit{and} \\ &\{H_{2k+1,j}\,,\ j\in\mathbb{Z},\ k=0,1,2,\dots\}, \end{aligned} \tag{14}$$

*is an orthogonal system in* $A^2(\mathcal{W}'_\mu)$.

## 2. The Müntz–Szász problem for the Bergman space

In endeavoring to establish whether the system $\{H_{\ell,j}\}$ is complete we are led to consider the Müntz–Szász problem for the Bergman space.

Recall that we set $\Delta = \{\zeta : |\zeta - 1| < 1\}$. We consider a set of functions $\{\zeta^{\lambda_k}\}$, $k = 1,2,\dots$ and would like to find a necessary and sufficient condition for this set to be a *complete set* in $A^2(\Delta)$, that is, its linear span to be dense in $A^2(\Delta)$.

**Theorem 2.1.** *Let* $\mathcal{S}$ *be the subset of* $A^2(\Delta)$ *whose elements are the functions* $\zeta^{\lambda_k}$ *for* $k = 0,1,2,\dots,$ *where* $\lambda_k = ak + c_0 + ib,$ $0 < a < 1,$ $c_0 > -1$ *and* $b \in \mathbf{R}$*. Then* $\mathcal{S}$ *is a complete set in* $A^2(\Delta)$.

*Proof.* Consider the biholomorphic map $C : \mathcal{U} \to \Delta$ given by

$$C(\omega) = \frac{2i}{i+\omega}$$

of the upper half plane $\mathcal{U}$ onto $\Delta$. Then

$$T : A^2(\Delta) \ni f \mapsto (f \circ C)C' \in A^2(\mathcal{U})$$

is a surjective isometry. Next, by the Paley–Wiener theorem, the Fourier transform $\mathcal{F}$, given by

$$(\mathcal{F}g)(\xi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} g(t)e^{-i\xi t}\,dt\,,$$

provides a surjective isometry of $A^2(\mathcal{U})$ onto $L^2\big((0,+\infty), d\xi/\xi\big)$, see e.g. [3]. Therefore, $\{\zeta^{\lambda_k}\}$ will be complete in $A^2(\Delta)$ if and only if $\big\{\mathcal{F}\big(T\zeta^{\lambda_k}\big)\big\}$ is complete in $L^2\big((0,+\infty), d\xi/\xi\big)$.

Now,

$$T(\zeta^\lambda)(\omega) = -\frac{(2i)^{\lambda+1}}{(i+\omega)^{\lambda+2}}\,,$$

while

$$\begin{aligned}\mathcal{F}\Big((u+i)^{-(\lambda+2)}\Big)(\xi) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{(u+i)^{\lambda+2}}e^{-i\xi u}\,du\\ &= \frac{1}{i^{\lambda+2}\Gamma(\lambda+2)}\xi^{\lambda+1}e^{-\xi}\chi_{(0,+\infty)}(\xi)\,,\end{aligned}$$

(see also [11], Lemma 1).

Therefore

$$\mathcal{F}\big(T(\zeta^\lambda)\big)(\xi) = -\frac{(2\xi)^{\lambda+1}e^{-\xi}}{i\Gamma(\lambda+2)}\chi_{(0,+\infty)}(\xi)\,.$$

Hence the set $\{\zeta^{\lambda_k}\}$ is complete in $A^2(\Delta)$ if and only if the set $\{\xi^{\lambda_k+1}e^{-\xi}\}$ is complete in $L^2\big((0,+\infty), d\xi/\xi\big)$, that is, the set $\{\xi^{\lambda_k+\frac{1}{2}}e^{-\xi/2}\}$ is complete in $L^2\big((0,+\infty)\big)$.

Next, we consider the transformation $\xi \mapsto \xi^a = t$ of $(0,+\infty)$ onto itself and the induced isometry $\Lambda$ of $L^2\big((0,+\infty)\big)$ onto itself given by

$$(\Lambda\psi)(t) = \sqrt{\tfrac{1}{a}}\psi(t^{1/a})\Big(t^{\frac{1}{2}(\frac{1}{a}-1)}\Big)\,.$$

Under such a transformation, since $\lambda_k = ak + c_0 + ib$, we see that $\{\xi^{\lambda_k+\frac{1}{2}}e^{-\xi/2}\}$ is complete in $L^2\big((0,+\infty)\big)$ if and only if $\{t^k t^\alpha e^{-\frac{1}{2}t^{1/a}}\}$ is complete in $L^2\big((0,+\infty)\big)$, where

$$\alpha = \frac{c_0+1}{a} - \frac{1}{2} + i\frac{b}{2a}\,.$$

We know from [26, Thm. 5.7.1] that the system

$$\{t^{n+c}e^{-t/2} : n = 0,1,2,\dots\},$$

with $c > -1/2$, is complete in $L^2\big((0,+\infty)\big)$. Thus, if $\psi \in L^2\big((0,+\infty)\big)$ is orthogonal to $t^k t^\alpha e^{-\frac{1}{2}t^{1/a}}$ for all $k = 0, 1, 2, \ldots$ it follows that

$$\int_0^{+\infty} t^{k+\operatorname{Re}\alpha} e^{-t/2} \overline{t^{-i\operatorname{Im}\alpha} e^{\frac{1}{2}(t-t^{1/a})}\psi(t)}\, dt = 0$$

for $k = 0, 1, 2, \ldots$. Since $0 < a < 1$, $e^{\frac{1}{2}(t-t^{1/a})}$ is bounded and also $\operatorname{Re}\alpha > -\frac{1}{2}$, we obtain that $t^{-i\operatorname{Im}\alpha} e^{\frac{1}{2}(t-t^{1/a})}\psi = 0$, that is, $\psi = 0$. This concludes the proof. □

## 3. Complete sets in $A^2(\mathcal{W}'_\mu)$

From our Müntz–Szász Theorem 2.1 for the Bergman space $A^2(\Delta)$, we obtain the following density result in $A^2(\mathcal{W}'_\mu)$.

**Theorem 3.1.** *Let $\mu > \pi/2$. Let $H_{\ell,j}(z_1, z_2)$ be as in* (13). *Then $\{H_{\ell,j}\}_{\ell,j\in\mathbb{Z},\,\ell\geq 0}$, is a complete set in $A^2(\mathcal{W}'_\mu)$.*

Notice that the set $\{H_{\ell,j}\}$, $\ell, j \in \mathbb{Z}$, $\ell \geq 0$, is the union of the two sets in (14).

*Proof.* We wish to show that if $F \in A^2(\mathcal{W}'_\mu)$ is orthogonal to $H_{\ell,j}$, for $\ell, j \in \mathbb{Z}$, $\ell \geq 0$, then $F$ is identically zero. It suffices to show that, for each $j \in \mathbb{Z}$ fixed, any function $F \in \mathcal{H}^j$ orthogonal to $H_{\ell,j}$ for all $\ell \geq 0$, is identically zero.

Writing $F(z_1, z_2) = f(z_1, |z_2|)z_2^j$, from (11) we then have

$$\begin{aligned}(15)\quad 0 &= \langle F, H_{\ell,j}\rangle_{A^2(\mathcal{W}'_\mu)}\\ &= \pi \int_\Delta \int_{|s|<\mu} f(\zeta e^{is}, e^{s/2}) \overline{E_{c_0+\nu\ell+i(j+1)/2}(\zeta e^{is}, e^{s/2})}\, e^{s(j+1)} ds\, dA(\zeta)\\ &= \pi \int_\Delta \int_{|s|<\mu} f(\zeta e^{is}, e^{s/2})\, e^{s[(j+1)/2+i(c_0+\nu\ell)]}\, ds\, \overline{\zeta^{c_0+\nu\ell+i(j+1)/2}} dA(\zeta)\,,\end{aligned}$$

for $\ell = 0, 1, \ldots$. Notice that the function

$$\begin{aligned}(16)\qquad Tf(\zeta, w) &= \int_{|s|<\mu} f(\zeta e^{is}, e^{s/2}) e^{s[(j+1)/2+ic_0]}\, e^{iw}\, ds\\ &= \mathcal{F}\big(f(\zeta e^{is}, e^{s/2}) e^{s[(j+1)/2+ic_0]}\chi_{\{|s|<\mu\}}\big)(w)\end{aligned}$$

is analytic in $\zeta \in \Delta$, and by the Paley–Wiener theorem [22], is an entire function in $w$ of exponential type at most $\mu$. Moreover, the function

$$w \mapsto \pi \int_\Delta Tf(\zeta, w)\overline{\zeta^{c_0+\nu\ell+i(j+1)/2}} dA(\zeta)$$

is again an entire function of exponential type at most $\mu$ and by (15) it vanishes at the points $w_\ell = \nu\ell$. Observe that

$$\limsup_{r\to+\infty} \frac{\exp\{2\sum_{\nu\ell<r} 1/(\nu\ell)\}}{r^{2\mu/\pi}} = \left(\frac{1}{\nu}\right)^{\frac{2}{\nu}} \lim_{r\to+\infty} r^{1/\nu} = +\infty\,. \tag{17}$$

By a classical result of Fuchs [13], we know that an entire function of type $\mu$ whose zero set $\{w_\ell = \nu\ell\}$ satisfies (17) must vanish identically, that is,

$$\int_\Delta Tf(\zeta, w)\overline{\zeta^{c_0+\nu\ell+i(j+1)/2}} dA(\zeta) = 0\,,$$

for $\ell = 0, 1, \ldots$. Since $\mu > \pi/2$ we have $\nu < 1$ and by Theorem 2.1 it now follows that $Tf(\cdot, w) = 0$, hence,

$$\int_{|s|<\mu} f(\zeta e^{is}, e^{s/2})\, e^{s[(j+1)/2+ic_0]} e^{isw}\, ds = 0\,, \tag{18}$$

for all $\zeta \in \Delta$ and $w \in \mathbb{C}$. This implies that $f$ vanishes identically and we are done. □

Notice that, had we considered either of the orthogonal systems mentioned in Corollary 1.5, we would have ended up with the points $\{w_{2\ell}\}$ only, or with $\{w_{2\ell+1}\}$. The analog of Condition (17) would not have been satisfied and we could not have proved completeness using this approach. In fact, we are going to show in the next proposition that each of the two systems is incomplete.

It is also worth mentioning that the worm domains $\mathcal{W}'_\mu$ are increasingly badly behaved as $\mu$ becomes large. On the other hand, the proof of our density result breaks down when $\mu \le \pi/2$. This is somewhat surprising, since when $\mu \le \pi/2$ the fibers over $z_1$ are connected, the geometry of the domain is much simpler and in principle it should be easier to obtain such results on $\mathcal{W}'_\mu$ when $\mu \le \pi/2$.

**Proposition 3.2.** *Let $\{H_{\ell,j}\}_{\ell,j\in\mathbb{Z},\,\ell\geq 0}$ be as in Theorem 3.1. Then, for each $m$ fixed,*

$$\|H_{2m+1,j}\|^2_{A^2(\mathcal{W}'_\mu)} > \sum_{j'=-\infty}^{+\infty}\sum_{k=0}^{+\infty}\frac{1}{\|H_{2k,j'}\|^2_{A^2(\mathcal{W}'_\mu)}}\left|\left\langle H_{2m+1,j},\, H_{2k,j'}\right\rangle\right|^2, \tag{19}$$

*and, analogously, for each $k$ fixed,*

$$\|H_{2k,j}\|^2_{A^2(\mathcal{W}'_\mu)} > \sum_{j'=-\infty}^{+\infty}\sum_{m=0}^{+\infty}\frac{1}{\|H_{2m+1,j'}\|^2_{A^2(\mathcal{W}'_\mu)}}\left|\left\langle H_{2k,j},\, H_{2m+1,j'}\right\rangle\right|^2.$$

*Hence, neither system $\{H_{2k,j}\}$ nor $\{H_{2m+1,j}\}$ is complete in $A^2(\mathcal{W}'_\mu)$.*

*Proof.* By orthogonality, it suffices to consider the case $j'=j$ and, dropping the index $j$, we write $F_k = H_{2k,j}$ and $G_m = H_{2m+1,j}$. By Proposition 1.3 we have that

$$\begin{aligned}
\langle G_m,\, F_k\rangle_{A^2(\mathcal{W}'_\mu)} &= (2\pi)^2\frac{\sin\left[\mu\big(2k-(2m+1)\big)\nu\right]}{\big(2k-(2m+1)\big)\nu}\\
&\quad\times\frac{\Gamma\big(2c_0+2+\big(2(k+m)+1\big)\nu\big)}{\Gamma\big(c_0+2+2k\nu+i\frac{j+1}{2}\big)\Gamma\big(c_0+2+(2m+1)\nu-i\frac{j+1}{2}\big)},\\
\|F_k\|^2_{A^2(\mathcal{W}'_\mu)} &= (2\pi)^2\mu\frac{\Gamma\big(2c_0+2+4k\nu\big)}{\left|\Gamma\big(c_0+2+2k\nu+i\frac{j+1}{2}\big)\right|^2},\\
\|G_m\|^2_{A^2(\mathcal{W}'_\mu)} &= (2\pi)^2\mu\frac{\Gamma\big(2c_0+2+2(2m+1)\nu\big)}{\left|\Gamma\big(c_0+2+(2m+1)\nu-i\frac{j+1}{2}\big)\right|^2}.
\end{aligned}$$

Therefore, (19) is equivalent to

$$\begin{aligned}
&\mu\Gamma\big(2c_0+2+2(2m+1)\nu\big)\\
&\quad> \sum_{k=0}^{+\infty}\left[\frac{\sin\left[\mu\big(2k-(2m+1)\big)\nu\right]}{\big(2k-(2m+1)\big)\nu}\right]^2\frac{\Gamma\big(2c_0+2+\big(2(k+m)+1\big)\nu\big)^2}{\mu\Gamma\big(2c_0+2+4k\nu\big)},
\end{aligned}\tag{20}$$

which in turn is implied by

(21)
$$\begin{aligned}1 &> \sum_{k=0}^{+\infty} \frac{1}{\big[\mu\big(2k-(2m+1)\big)\nu\big]^2} \frac{\Gamma\big(2c_0+2+\big(2(k+m)+1\big)\nu\big)^2}{\Gamma\big(2c_0+2+2(2m+1)\nu\big)\Gamma\big(2c_0+2+4k\nu\big)} \\ &= \sum_{k=0}^{+\infty} \frac{1}{\pi^2\big(k-\frac{2m+1}{2}\big)^2} \frac{\Gamma\big(2c_0+2+\big(2(k+m)+1\big)\nu\big)^2}{\Gamma\big(2c_0+2+2(2m+1)\nu\big)\Gamma\big(2c_0+2+4k\nu\big)}\,.\end{aligned}$$

Now, on the one hand the right-hand side in (21) is less than or equal to

$$\sum_{k=0}^{+\infty} \frac{1}{\pi^2\big(k-\frac{2m+1}{2}\big)^2}\,,$$

since for all $x, y > 0$, $c \geq 0$,

$$\begin{aligned}\Gamma(c+x+y)^2 &= \left(\int_0^{+\infty} t^{x+y+c-1}e^{-t}dt\right)^2 \\ &\leq \left(\int_0^{+\infty} t^{2x+c-1}e^{-t}dt\right)\left(\int_0^{+\infty} t^{2y+c-1}e^{-t}dt\right) \\ &= \Gamma(c+2x)\Gamma(c+2y)\,.\end{aligned}$$

On the other hand, we claim that

$$\pi^2 > \sum_{k=0}^{+\infty} \frac{1}{\big(k-\frac{2m+1}{2}\big)^2}.$$

Indeed, setting $h(w) = \pi\cot(\pi w)$ and $Q(w) = \big(w-\frac{2m+1}{2}\big)^2$, we have

$$\operatorname{Res}\left(\frac{h}{Q}, k\right) = \frac{1}{Q(k)} = \frac{1}{\big(k-\frac{2m+1}{2}\big)^2}$$

for all $k \in \mathbb{Z}$ and

$$\operatorname{Res}\left(\frac{h}{Q}, \frac{2m+1}{2}\right) = \lim_{w\to\frac{2m+1}{2}} \frac{\pi}{\sin(\pi w)} \frac{\cos(\pi w)}{w-\frac{2m+1}{2}} = -\pi^2.$$

The fact that

$$0=\lim_{n\to+\infty}\int_{\partial D(0,n+\frac12)}\frac{h(w)}{Q(w)}dw$$
$$=2\pi i\lim_{n\to+\infty}\left[\operatorname{Res}\left(\frac{h}{Q},\frac{2m+1}{2}\right)+\sum_{k=-n}^{n}\operatorname{Res}\left(\frac{h}{Q},k\right)\right]$$

implies that

$$\pi^2=\sum_{k=-\infty}^{+\infty}\frac{1}{\left(k-\frac{2m+1}{2}\right)^2}>\sum_{k=0}^{+\infty}\frac{1}{\left(k-\frac{2m+1}{2}\right)^2},$$

as claimed. This concludes the proof. □

Finally, we show that the complete system of Theorem 3.1 is not a Schauder basis for $A^2(\mathcal{W}'_\mu)$, for all $\mu\geq\pi/2$. For the definition of Schauder basis, see Section 1.

**Theorem 3.3.** *Let $\mu\geq\pi/2$, and let $H_{\ell,j}(z_1,z_2)=E_{c_0+\nu\ell+i(j+1)/2}(z_1,z_2)z_2^j$, $\ell,j\in\mathbb{Z}$, $\ell\geq0$. For each $j\in\mathbb{Z}$ fixed, the function $H_{0,j}$ is in the $A^2(\mathcal{W}'_\mu)$-closure of* span$\{H_{\ell,j},\ \ell=1,2,\dots\}$. *In particular, this violates the uniqueness requirement in the definition of Schauder basis.*

*Proof.* We first assume that $\mu>\pi/2$.

Let $Q=Q_n$ be a polynomial of degree $n$ of one complex variable, without constant term, $Q(w)=\sum_{\ell=1}^n c_\ell w^\ell$. Then, arguing as in (7) we have

(22)
$$\|H_{0,j}-\sum_{\ell=1}^{n}c_\ell H_{\ell,j}\|^2_{A^2(\mathcal{W}'_\mu)}$$
$$=\pi\int_{|s|<\mu}\int_\Delta\big|\zeta^{c_0+i(j+1)/2}e^{is(c_0+i(j+1)/2)}$$
$$-\sum_{\ell=1}^{n}c_\ell\zeta^{c_0+\nu\ell+i(j+1)/2}e^{is(c_0+i(j+1)/2+\nu\ell)}\big|^2\,dA(\zeta)\,e^{s(j+1)}ds$$
$$=\pi\int_{|s|<\mu}\int_\Delta\big|\zeta^{c_0+i(j+1)/2}\big|^2\left|1-\sum_{\ell=1}^{n}c_\ell\zeta^{\nu\ell}e^{is\nu\ell}\right|^2\,dA(\zeta)\,ds$$

$$\begin{aligned}
&\leq C \int_{|s|<\mu} \int_{\Delta} \left| 1 - \sum_{\ell=1}^{n} c_\ell (\zeta^\nu e^{is\nu})^\ell \right|^2 dA(\zeta)\, ds \\
&= C \frac{1}{\nu^2} \int_{|s|<\mu} \int_{\Omega_s} \left| 1 - \sum_{\ell=1}^{n} c_\ell w^\ell \right|^2 \left| w^{\frac{1}{\nu}-1} \right|^2 dA(w)\, ds \,,
\end{aligned}$$

where we have set $w = \zeta^\nu e^{is\nu}$. Since $\mu > \pi/2$, $\nu < 1 - 2\delta$ for some $\delta > 0$, so that $-\frac{\pi}{2} < \nu s < \frac{\pi}{2}$ and $0 < \nu\frac{\pi}{2} < \frac{\pi}{2}(1-2\delta)$. Hence,

$$\begin{aligned}
\Omega_s &\subseteq \left\{ w = \rho e^{it} : \, 0 < \rho < 2^\nu, \, \nu\big(s - \tfrac{\pi}{2}\big) < t < \nu\big(s + \tfrac{\pi}{2}\big) \right\} \\
&\subseteq \left\{ w = \rho e^{it} : \, 0 < \rho < 2^\nu, \, |t| < \pi(1-\delta) \right\} \\
&=: S \,.
\end{aligned}$$

Plugging this into (22) we obtain that

$$\begin{aligned}
\|H_{0,j} - \sum_{\ell=1}^{n} c_\ell H_{\ell,j}\|^2_{A^2(\mathcal{W}'_\mu)} &\leq C \int_{|s|<\mu} \int_S \left| 1 - Q_n(w) \right|^2 |w|^{2(\frac{1}{\nu}-1)}\, dA(w)\, ds \\
&= C \int_S \left| 1 - Q_n(w) \right|^2 |w|^{2(\frac{1}{\nu}-1)}\, dA(w) \,.
\end{aligned}$$

Setting $d\omega(w) = |w|^{2(\frac{1}{\nu}-1)} dA(w)$, the conclusion will follow if we show that there exist polynomials $P_n = 1 - Q_n$ such that $P_n(0) = 1$ and $\|P_n\|_{A^2(S,d\omega)} \to 0$ as $n \to +\infty$.

In order to prove that such polynomials exist, let $\Delta_+$ be the half disk $\{z \in \mathbb{C} : \, |z| < 1, \operatorname{Re} z > 0\}$ and $p(z) = (z - \frac{1}{2})^2 + \frac{3}{4}$. Then $p(0) = 1$, and $|p(z)| \leq 1$ for $z \in \overline{\Delta_+}$, as it is elementary to check. Therefore,

$$F(w) = p((2^{-\nu} w)^{1/[2(1-\delta)]})$$

is a function holomorphic on $S$ such that

- $F$ is continuous on $\overline{S}$;
- $F(0) = 1$;
- $|F(w)| \leq 1$ on $\overline{S}$.

Observe that $\omega(S) < +\infty$. Given $\varepsilon > 0$, let $K$ be a compact subset of $S$ such that $\omega(S \setminus K) < \varepsilon$ and let $n$ be a positive integer such that $|F^n(w)| \leq \varepsilon$

for $w \in K$. Then

$$\int_S |F^n(w)|^2 \, d\omega(w) \le C\varepsilon \, .$$

By Mergelyan's approximation theorem (see [24] e.g.), we can find polynomials $p_n$ such that $|F^n(w) - p_n(w)| \le \varepsilon$ for $w \in \overline{S}$. Finally, we set $P_n = \frac{1}{p_n(0)} p_n$ and the conclusion follows easily.

Finally, let $\mu = \pi/2$, so that $\nu = 1$. Set $\mathcal{D} = \cup_{|s|<\pi/2} \{z_1 : |z_1 - e^{is}| < 1\}$. We have,

$$\begin{aligned}
(23) \qquad & \Big\| H_{0,j} - \sum_{\ell=1}^{n} c_\ell H_{\ell,j} \Big\|^2_{A^2(\mathcal{W}'_\mu)} \\
&= \pi \int_{|s|<\mu} \int_{|z_1 - e^{is}|<1} \left| z_1^{2c_0 + i(j+1)} \right| \Big| 1 - \sum_{\ell=1}^{n} c_\ell z_1^\ell \Big|^2 \, dA(z_1) \, e^{s(j+1)} \, ds \\
&\le C \int_{\mathcal{D}} \Big| 1 - \sum_{\ell=1}^{n} c_\ell z_1^\ell \Big|^2 \, dA(z_1) \, .
\end{aligned}$$

We observe that $\mathcal{D}$ is a Jordan domain, having the origin as a boundary point. By [12] we know that the polynomials are dense in $A^2(\mathcal{D})$ and by [4] it follows that there exists no bounded boundary evaluation point on the space of polynomials. Hence, the right hand side of (23) can by made arbitrarily small and the conclusion now follows. We leave the simple details to the reader. □

**Remark 3.4.** It follows from Theorems 3.1 and 3.3 that the set

$$\{H_{\ell,j}\}_{\ell,j \in \mathbb{Z},\, \ell \ge 0}$$

is complete, but not a Schauder basis. It would be of interest to show that the set $\{H_{\ell,j}\}_{\ell,j\in\mathbb{Z},\,\ell\ge 0}$ is however a *frame* for $A^2(\mathcal{W}')$, that is, there exist constants $c_1, c_2 > 0$ such that

$$c_1 \|f\|^2_{A^2(\mathcal{W}')} \le \sum_{\ell,j \in \mathbb{Z},\, \ell \ge 0} \left| \langle f, H_{\ell,j} \rangle_{A^2(\mathcal{W}')} \right|^2 \le c_2 \|f\|^2_{A^2(\mathcal{W}')} \, .$$

Indeed, the theory of frames in Hilbert function spaces constitute a fundamental tool, especially in sampling and reconstruction of functions — see [10] where frames were introduced in the context of nonharmonic Fourier series, and [27] for applications of the theory of frames to the present setting.

We also recall that, in a separable Hilbert space, a frame that is also a basis is called a *Riesz basis*. Hence, in particular, the complete set $\{H_{\ell,j}\}_{\ell,j\in\mathbb{Z},\,\ell\geq 0}$ of Theorem 3.1 is not a Riesz basis either.

## Concluding remarks

Thanks to work of several authors, the worm domain has become an important object of study. In particular, we are beginning to understand the Bergman kernel and projection on some versions of the worm. But the original smooth worm $\mathcal{W}_\mu$ is particularly resistive to analysis. It does not have the built-in symmetries of some of the non-smooth worms. In particular, we do not have a useful complete orthogonal basis for the Bergman space on $\mathcal{W}_\mu$. In addition to the alternative approach mentioned in [18, §5], this paper has offered some first steps towards addressing that problem.

As an additional remark, we point out that the results obtained in this work can be generalized to the case of worm domains in $\mathbb{C}^n$ defined and studied in [2].

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