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# Homological dimension of simple pro-$p$-Iwahori–Hecke modules

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Let $G$ be a split connected reductive group defined over a nonarchimedean local field of residual characteristic $p$, and let $\mathcal{H}$ be the pro-$p$-Iwahori–Hecke algebra over $\overline{\mathbb{F}}_p$ associated to a fixed choice of pro-$p$-Iwahori subgroup. We explore projective resolutions of simple right $\mathcal{H}$-modules. In particular, subject to a mild condition on $p$, we give a classification of simple right $\mathcal{H}$-modules of finite projective dimension, and consequently show that "most" simple modules have infinite projective dimension.

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## 1. Introduction

Let $p$ be a prime number. The mod-$p$ representations of $p$-adic reductive groups have been the subject of intense recent study, culminating in the work of Abe–Henniart–Herzig–Vignéras [3]. The aforementioned article classifies irreducible admissible representations of a $p$-adic reductive group $G$ in terms of supersingular representations, and it is expected that this classification will be useful in extending the mod-$p$ Local Langlands Correspondence beyond the case $G = \mathrm{GL}_2(\mathbb{Q}_p)$.

In a parallel vein, there has been substantial progress in recent years in understanding the category of modules over the *pro-p-Iwahori–Hecke algebra* $\mathcal{H} = \overline{\mathbb{F}}_p[I(1)\backslash G/I(1)]$, where $I(1)$ is a fixed pro-$p$-Iwahori subgroup of $G$. We refer the reader to [21] for a description of this algebra, and to [22], [13], [2] for a classification of its simple modules. The interest in this algebra comes from its link with the category of smooth mod-$p$ representations of $G$, and thus with the mod-$p$ Local Langlands Program. To wit, there exists an equivalence between the category of $\mathcal{H}$-modules and the category of smooth mod-$p$ $G$-representations generated by their $I(1)$-invariant vectors, when $G = \mathrm{PGL}_2(\mathbb{Q}_p)$ or $\mathrm{SL}_2(\mathbb{Q}_p)$ ([12], [9]). Moreover, if one replaces $\mathcal{H}$ by a certain related differential graded Hecke algebra $\mathcal{H}^\bullet$, one obtains an equivalence between the (unbounded) derived category of differential graded $\mathcal{H}^\bullet$-modules and the (unbounded) derived category of $G$-representations (subject to some restrictions on $I(1)$, see [17]).

Since we have a good understanding of the structure of simple $\mathcal{H}$-modules, it therefore becomes imperative to better understand their homological properties. In the case when $G$ is split, a first step towards achieving this was taken by Ollivier and Schneider in [14], where a functorial resolution of any $\mathcal{H}$-module was constructed by making use of coefficient systems on the Bruhat–Tits building of $G$. The authors use this resolution to show, among other things, that the algebra $\mathcal{H}$ has infinite global dimension (at least generically), and possesses a simple module of infinite projective dimension.

Our goal in this note will be to give a classification of those simple $\mathcal{H}$-modules of finite projective dimension. We now give an overview of the contents herein, and of our main result.

We assume henceforth that the group $G$ is defined and split over a fixed nonarchimedean local field $F$ of residual characteristic $p$. After recalling the necessary notation in Section 2, we investigate the algebras $\mathcal{H}_\mathcal{F}$ and $\mathcal{H}_\mathcal{F}^\dagger$, which are certain "small" subalgebras of $\mathcal{H}$ associated to a facet $\mathcal{F}$ in the semisimple Bruhat–Tits building of $G$. The Ollivier–Schneider resolution is constructed from algebras of this form, and the projective dimensions of $\mathcal{H}$-modules are controlled by their restrictions to $\mathcal{H}_{\mathcal{F}}$ and $\mathcal{H}_{\mathcal{F}}^{\dagger}$. Thus, it will be important for our purposes to understand how to transfer homological properties of modules from one algebra to the other. We take this up in Section 3. Once this is complete, we recall more precisely in Section 4 the resolution of $\mathcal{H}$-modules constructed in [14], and record a useful associated spectral sequence.

With the preliminaries in place, we review in Section 5 Abe's classification of simple $\mathcal{H}$-modules in terms of parabolic coinduction and supersingular modules of Levi components (see [2]). Lemma 5.1 shows that every simple module admits a resolution by a certain Čech complex, with each term being a direct sum of parabolically coinduced representations. Using the results already obtained, along with an analog of the Mackey formula for $\mathcal{H}$-modules (Lemma 3.6), we obtain the following result (Proposition 6.1):

**Proposition.** *Suppose the root system of $G$ is of type $A_n$, $p \nmid |\pi_1(G)_{\mathrm{tor}}|$ and $\mathfrak{m}$ is a simple subquotient of a "principal series $\mathcal{H}$-module" (that is, a simple subquotient of the $\mathcal{H}$-module $\mathrm{Ind}_B^G(\chi)^{I(1)}$, where $B$ is a Borel subgroup of $G$ and $\chi$ is a smooth character of $B$). Then $\mathfrak{m}$ has finite projective dimension over $\mathcal{H}$ which is bounded above by the rank of $G$.*

Our next goal will be to generalize the above proposition to an arbitrary simple module $\mathfrak{m}$ and an arbitrary $G$. In Section 7, we recall the classification of simple supersingular right $\mathcal{H}$-modules due to Ollivier and Vignéras ([13] and [22]). Note that there is no such classification for supersingular *representations* of $G$. Using several reductions, we completely determine when such a module has finite projective dimension in Theorem 7.7. This result will be required as input into our main theorem.

By Abe's classification (cf. [2]), every simple right $\mathcal{H}$-module is a subquotient of the parabolic coinduction of a simple supersingular $\mathcal{H}_M$-module, where $M$ is a Levi subgroup of $G$ and $\mathcal{H}_M$ is the associated pro-$p$-Iwahori–Hecke algebra. Thus, our next task in Section 8 is to understand how projective dimension behaves under parabolic coinduction and taking subquotients. The parabolic coinduction functor has an exact left adjoint (cf. [20]), and a straightforward argument with Ext spaces shows that finiteness of projective dimension is preserved under parabolic coinduction (Lemma 8.1). Passing to subquotients is less straightforward, and occupies the remainder of the section.

With all the pieces in place, we obtain our main result:

**Theorem.** *Suppose $p \nmid |\pi_1(G)_{\mathrm{tor}}|$, and let $M$ denote a standard Levi subgroup of $G$. Let $\mathfrak{n}$ be a simple supersingular right $\mathcal{H}_M$-module, and $\mathfrak{m}$ a simple subquotient of the parabolic coinduction of $\mathfrak{n}$ from $\mathcal{H}_M$ to $\mathcal{H}$. Then the following are equivalent:*

- *$\mathfrak{m}$ has finite projective dimension over $\mathcal{H}$;*
- *$\mathfrak{n}$ has finite projective dimension over $\mathcal{H}_M$;*
- *the root system of $M$ is of type $A_1 \times \cdots \times A_1$, and the characteristic function of $(I(1) \cap M)\alpha^\vee(x)(I(1) \cap M)$ acts trivially on $\mathfrak{n}$ for all $x$ in the residue field of $F$ and all simple roots $\alpha$ of $M$.*

*Moreover, when $G$ is semisimple and $\mathfrak{m}$ satisfies the above conditions, the resolution of Ollivier–Schneider is a projective resolution of $\mathfrak{m}$, and the projective dimension of $\mathfrak{m}$ is equal to the rank of $G$.*

(In fact, we can strengthen the final statement somewhat; see below.)

This theorem shows that "most" simple $\mathcal{H}$-modules have infinite projective dimension, and in particular, that simple supersingular modules are generically of infinite projective dimension. On the other hand, simple subquotients of "principal series $\mathcal{H}$-modules" have finite projective dimension.

In the final section (Section 9), we present some complementary results. First, we show how the above theorem adapts to simple modules over the *Iwahori–Hecke algebra* $\mathcal{H}'$, defined with respect to an Iwahori subgroup containing $I(1)$. More precisely, the analog of the result above goes through mostly unchanged, except that the last condition is replaced by the simpler condition "the root system of $M$ is of type $A_1 \times \cdots \times A_1$" (see Subsection 9.1 for more details).

Next, we specialize in Subsection 9.2 to the case $G = \mathrm{PGL}_2(\mathbb{Q}_p)$ or $\mathrm{SL}_2(\mathbb{Q}_p)$ with $p > 2$. Recall that in this case, we have an equivalence between the category of $\mathcal{H}$-modules and the category of smooth $G$-representations generated by their $I(1)$-invariant vectors. By using this equivalence of categories, we demonstrate how to construct projective resolutions for certain irreducible smooth $G$-representations in the aforementioned representation category. However, these resolutions will not be projective in the entire category of smooth $G$-representations.

Finally, we examine in Subsection 9.3 what can be said when an $\mathcal{H}$-module has a central character. We prove that if such a module has finite projective dimension, the resolution of Ollivier–Schneider actually gives a projective resolution in the full subcategory of $\mathcal{H}$-modules with the given central character. This implies, in particular, the following fact: if $p \nmid |\pi_1(G/Z)|$, where $Z$ denotes the connected center of $G$, and $\mathfrak{m}$ is a simple $\mathcal{H}$-module which has finite projective dimension over $\mathcal{H}$, then the projective dimension of $\mathfrak{m}$ is in fact equal to the rank of $G$ (without any semisimplicity hypotheses).

## 2. Notation

### 2.1. General notation

Let $p, F$ and $G$ be as in the introduction, so that $G$ is a split connected reductive group over a fixed nonarchimedean local field $F$ of residual characteristic $p$. We let $k_F$ denote the residue field of $F$, and $q$ its order.

We will abuse notation throughout and conflate algebraic groups with their groups of $F$-points. Denote by $Z$ the connected center of $G$, and let $r_{\mathrm{ss}}$ and $r_Z$ denote the semisimple rank of $G$ and the rank of $Z$, respectively. Fix a maximal split torus $T$ of $G$ and let

$$\langle -, - \rangle : X^*(T) \times X_*(T) \longrightarrow \mathbb{Z}$$

denote the natural perfect pairing. The group $T$ acts on the standard apartment $(X_*(T)/X_*(Z)) \otimes_{\mathbb{Z}} \mathbb{R}$ of the semisimple Bruhat–Tits building of $G$ by translation via $\nu$, where $\nu : T \longrightarrow X_*(T)$ is the homomorphism defined by

$$\langle \chi, \nu(\lambda) \rangle = -\mathrm{val}(\chi(\lambda))$$

for all $\chi \in X^*(T)$ and $\lambda \in T$, and where $\mathrm{val} : F^\times \longrightarrow \mathbb{Z}$ is the normalized valuation.

In the standard apartment, we fix a chamber $C$ and a hyperspecial vertex $x$ such that $x \in \overline{C}$. Given a facet $\mathcal{F}$ in the semisimple Bruhat–Tits building, we let $\mathcal{P}_{\mathcal{F}}$ denote the parahoric subgroup associated to $\mathcal{F}$, and let $\mathcal{P}_{\mathcal{F}}^\dagger$ denote the stabilizer of $\mathcal{F}$ in $G$; we have $\mathcal{P}_{\mathcal{F}} \subset \mathcal{P}_{\mathcal{F}}^\dagger$. We set $I := \mathcal{P}_C$, an Iwahori subgroup, and let $I(1)$ denote the pro-$p$-Sylow subgroup of $I$.

We identify the root system $\Phi \subset X^*(T)$ with the set of affine roots which are 0 on $x$. Denote by $\Phi^+ \subset \Phi$ the subset of affine roots which are positive on $C$; we have $\Phi = \Phi^+ \sqcup -\Phi^+$. We let $\Pi$ denote the basis of $\Phi$ defined by $\Phi^+$, and define $B = T \ltimes U$ to be the Borel subgroup containing $T$ defined by $\Phi^+$, where $U$ is the unipotent radical of $B$. A *standard parabolic subgroup* $P = M \ltimes N$ is any parabolic subgroup containing $B$. It will be tacitly assumed that all Levi subgroups $M$ appearing are standard; that is, they are Levi factors of standard parabolic subgroups and contain $T$.

Given a standard Levi subgroup $M$, we let $\Pi_M$ (resp. $\Phi_M$, resp. $\Phi_M^+$) denote the simple roots (resp. root system, resp. positive roots) defined by $M$. Reciprocally, given a subset $J \subset \Pi$, we let $M_J$ denote the standard Levi subgroup it defines.

### 2.2. Weyl groups

Denote by $W_0 := N_G(T)/T$ the Weyl group of $G$, with length function $\ell : W_0 \longrightarrow \mathbb{Z}_{\geq 0}$ (defined with respect to $\Pi$). For a standard Levi subgroup $M$, we let $W_{M,0} \subset W_0$ denote the corresponding Weyl group, and set

$$W_0^M := \{w \in W_0 : w(\Pi_M) \subset \Phi^+\}.$$

Every element $w$ of $W_0$ can be written as $w = vu$ for unique $v \in W_0^M, u \in W_{M,0}$, satisfying $\ell(w) = \ell(v) + \ell(u)$ (see [6, Ch. IV, Exercices du §1, (3)]).

We set

$$\Lambda := T/(T \cap I), \quad \widetilde{\Lambda} := T/(T \cap I(1)),$$
$$W := N_G(T)/(T \cap I), \quad \widetilde{W} := N_G(T)/(T \cap I(1)).$$

Note that the map $\nu$ descends to $\Lambda$ and $\widetilde{\Lambda}$. For any standard Levi subgroup $M$, we let $W_M$ denote the subgroup of $W$ generated by $W_{M,0}$ and $\Lambda$ (that is, $W_M$ is the preimage of $W_{M,0}$ under the surjection $W \twoheadrightarrow W_0$).

The group $(T \cap I)/(T \cap I(1))$ identifies with the group of $k_F$-points of $T$, and we denote this group by $T(k_F)$. Given any subset $X$ of $W$, we let $\widetilde{X}$ denote its preimage in $\widetilde{W}$ under the natural projection $\widetilde{W} \twoheadrightarrow W$, so that $\widetilde{X}$ is an extension of $X$ by $T(k_F)$. For typographical reasons we write $\widetilde{X}_\Box$ as opposed to $\widetilde{X_\Box}$ if the symbol $X$ has some decoration $\Box$. We will usually denote generic elements of $\widetilde{W}$ by $\widetilde{w}$, and given an element $w \in W$ we often let $\widehat{w} \in \widetilde{W}$ denote a specified choice of lift.

Since $x$ is hyperspecial we have $W_0 \cong (N_G(T) \cap \mathcal{P}_x)/(T \cap \mathcal{P}_x)$, which gives a section to the surjection $W \twoheadrightarrow W_0$. We will always view $W_0$ as a subgroup of $W$ via this section. This gives the decomposition

$$W \cong W_0 \ltimes \Lambda.$$

In particular, we see that any $\widetilde{w} \in \widetilde{W}$ may be written as $\widehat{\overline{w}}\lambda$, where $\overline{w}$ is the image of $\widetilde{w}$ in $W_0 \subset W$, $\widehat{\overline{w}} \in \widetilde{W}$ is a fixed choice of lift, and $\lambda \in \widetilde{\Lambda}$. Moreover, the length function $\ell$ on $W_0$ extends to $W$ and $\widetilde{W}$ (see [21, Cor. 5.10]).

We also have a decomposition

$$W \cong W_{\mathrm{aff}} \rtimes \Omega.$$

Here, $W_{\mathrm{aff}}$ is the affine Weyl group, generated by the set $S$ of simple affine reflections fixing the walls of $C$ (chosen as in [14, §4.3]). Every element of $S$ is of the form $s_\alpha$, where $s_\alpha$ is the reflection in the hyperplane defined by the kernel of an affine root $\alpha$. Moreover, the pair $(W_{\mathrm{aff}}, S)$ is a Coxeter system, and the restriction of $\ell$ to $W_{\mathrm{aff}}$ agrees with the length function of $W_{\mathrm{aff}}$ as a Coxeter group. The group $\Omega$ is the subgroup of elements stabilizing $C$; equivalently, $\Omega$ is the subgroup of length 0 elements of $W$. It is a finitely generated abelian group, and we write

$$\Omega \cong \Omega_{\mathrm{tor}} \times \Omega_{\mathrm{free}}$$

where $\Omega_{\mathrm{tor}}$ is the (finite) torsion subgroup and $\Omega_{\mathrm{free}}$ is the free part. Since the group $Z/(Z \cap I)$ embeds as a finite-index subgroup of $\Omega_{\mathrm{free}}$, we have $\Omega_{\mathrm{free}} \cong \mathbb{Z}^{\oplus r_Z}$.

### 2.3. Hecke algebras

Let $\mathcal{H}$ denote the pro-$p$-Iwahori–Hecke algebra of $G$ with respect to $I(1)$ over $\overline{\mathbb{F}}_p$:

$$\mathcal{H} := \mathrm{End}_G\left(\text{c-ind}_{I(1)}^G(\mathbf{1})\right),$$

where $\mathbf{1}$ denotes the trivial $I(1)$-module over $\overline{\mathbb{F}}_p$ (see [21] for details). For any standard Levi subgroup $M$, let $\mathcal{H}_M$ denote the analogously defined pro-$p$-Iwahori–Hecke algebra of $M$ with respect to $I_M(1) := I(1) \cap M$ (which is *not* a subalgebra of $\mathcal{H}$ in general). For any facet $\mathcal{F} \subset \overline{C}$, we let

$$\mathcal{H}_{\mathcal{F}} := \mathrm{End}_{\mathcal{P}_{\mathcal{F}}}\left(\text{c-ind}_{I(1)}^{\mathcal{P}_{\mathcal{F}}}(\mathbf{1})\right), \qquad \mathcal{H}_{\mathcal{F}}^{\dagger} := \mathrm{End}_{\mathcal{P}_{\mathcal{F}}^{\dagger}}\left(\text{c-ind}_{I(1)}^{\mathcal{P}_{\mathcal{F}}^{\dagger}}(\mathbf{1})\right);$$

extending functions on $\mathcal{P}_{\mathcal{F}}$ by zero to $G$ gives a $\mathcal{P}_{\mathcal{F}}$-equivariant injection

$$\text{c-ind}_{I(1)}^{\mathcal{P}_{\mathcal{F}}}(\mathbf{1}) \hookrightarrow \text{c-ind}_{I(1)}^{G}(\mathbf{1}),$$

which induces

$$\begin{aligned}\mathcal{H}_{\mathcal{F}} = \mathrm{End}_{\mathcal{P}_{\mathcal{F}}}\left(\text{c-ind}_{I(1)}^{\mathcal{P}_{\mathcal{F}}}(\mathbf{1})\right) \hookrightarrow \mathrm{Hom}_{\mathcal{P}_{\mathcal{F}}}\left(\text{c-ind}_{I(1)}^{\mathcal{P}_{\mathcal{F}}}(\mathbf{1}), \text{c-ind}_{I(1)}^{G}(\mathbf{1})\right)\\ \cong \mathrm{End}_G\left(\text{c-ind}_{I(1)}^{G}(\mathbf{1})\right) = \mathcal{H}\end{aligned}$$

(and similarly for $\mathcal{H}_{\mathcal{F}}^{\dagger}$). We therefore view $\mathcal{H}_{\mathcal{F}}$ and $\mathcal{H}_{\mathcal{F}}^{\dagger}$ as subalgebras of $\mathcal{H}$, and $\mathcal{H}_{\mathcal{F}}$ as a subalgebra of $\mathcal{H}_{\mathcal{F}}^{\dagger}$ (see [14, §§3.3.1 and 4.9] for more details).

We view $\mathcal{H}_M$ as the convolution algebra of $\overline{\mathbb{F}}_p$-valued, $I_M(1)$-biinvariant functions on $M$. The group $\widetilde{W}_M$ gives a full set of coset representatives for $I_M(1)\backslash M/I_M(1)$, and for $\widetilde{w} \in \widetilde{W}_M$, we let $\mathrm{T}_{\widetilde{w}}^M$ denote the characteristic function of $I_M(1)\widetilde{w}I_M(1)$ (and drop the superscript when $M = G$). For standard properties of the algebras $\mathcal{H}_M$ (quadratic relations, Bernstein basis, definition of the elements $\mathrm{T}_{\widetilde{w}}^{M,*}$, etc.), we defer to [21]. Let us only recall the braid relations: if $\widetilde{w}, \widetilde{w}' \in \widetilde{W}_M$ satisfy $\ell_M(\widetilde{w}\widetilde{w}') = \ell_M(\widetilde{w}) + \ell_M(\widetilde{w}')$, where $\ell_M$ is the length function on $\widetilde{W}_M$, then

$$\mathrm{T}_{\widetilde{w}}^M \mathrm{T}_{\widetilde{w}'}^M = \mathrm{T}_{\widetilde{w}\widetilde{w}'}^M.$$

## 3. Preliminary results

We first record some simple results concerning the algebras $\mathcal{H}_{\mathcal{F}}$ and $\mathcal{H}_{\mathcal{F}}^{\dagger}$. Given a right module $\mathfrak{m}$ over an associative unital ring $\mathcal{R}$, we let $\mathrm{pd}_{\mathcal{R}}(\mathfrak{m})$ and $\mathrm{id}_{\mathcal{R}}(\mathfrak{m})$ denote the projective and injective dimensions of $\mathfrak{m}$ over $\mathcal{R}$, respectively.

**Remark 3.1.** Let $n \in \mathbb{Z}_{\geq 0}$. Recall that an associative unital (left and right) noetherian ring $\mathcal{R}$ is called *$n$-Gorenstein* if $\mathrm{id}_{\mathcal{R}}(\mathcal{R}) \leq n$, where $\mathcal{R}$ is viewed as either a left or right $\mathcal{R}$-module ([8, Def. 9.1.9]). Fix a facet $\mathcal{F} \subset \overline{C}$. By [14, Thm. 3.14, Prop. 5.5, Remarks following Lem. 5.2] the algebras $\mathcal{H}_{\mathcal{F}}$, $\mathcal{H}_{\mathcal{F}}^{\dagger}$, and $\mathcal{H}$ are all $n$-Gorenstein, where $n = 0, r_Z$, and $r_{\mathrm{ss}} + r_Z$, respectively. We shall use the following fact several times ([8, Thm. 9.1.10]): let $\mathcal{R} \in \{\mathcal{H}_{\mathcal{F}}, \mathcal{H}_{\mathcal{F}}^{\dagger}, \mathcal{H}\}$, so that $\mathcal{R}$ is $n$-Gorenstein with $n$ as above, and let $\mathfrak{m}$ be a right module over $\mathcal{R}$. Then the following are equivalent:

- $\mathrm{id}_{\mathcal{R}}(\mathfrak{m}) < \infty$;
- $\mathrm{pd}_{\mathcal{R}}(\mathfrak{m}) < \infty$;
- $\mathrm{id}_{\mathcal{R}}(\mathfrak{m}) \leq n$;
- $\mathrm{pd}_{\mathcal{R}}(\mathfrak{m}) \leq n$.

**Remark 3.2.** We will employ the Eckmann–Shapiro lemma extensively, so we briefly recall it here (see [4, Cor. 2.8.4] for more details). Let $\mathcal{A}$ and $\mathcal{B}$ be two unital rings. Suppose $\mathcal{A}$ is a subring of $\mathcal{B}$, and that $\mathcal{B}$ is projective as a left (resp. right) $\mathcal{A}$-module. If $\mathfrak{m}$ is a right $\mathcal{B}$-module and $\mathfrak{n}$ is a right

$\mathcal{A}$-module, we then have:

- $\mathrm{Ext}^i_{\mathcal{A}}(\mathfrak{n}, \mathfrak{m}|_{\mathcal{A}}) \cong \mathrm{Ext}^i_{\mathcal{B}}(\mathfrak{n} \otimes_{\mathcal{A}} \mathcal{B}, \mathfrak{m})$;
- $\mathrm{Ext}^i_{\mathcal{A}}(\mathfrak{m}|_{\mathcal{A}}, \mathfrak{n}) \cong \mathrm{Ext}^i_{\mathcal{B}}(\mathfrak{m}, \mathrm{Hom}_{\mathcal{A}}(\mathcal{B}, \mathfrak{n}))$.

**Lemma 3.3.** *Let $\mathcal{F}, \mathcal{F}'$ denote two facets such that $\mathcal{F}' \subset \overline{\mathcal{F}}$, $\mathcal{F} \subset \overline{C}$, and let $\mathfrak{m}$ be a right $\mathcal{H}_{\mathcal{F}'}$-module. If $\mathfrak{m}$ is projective, then the $\mathcal{H}_{\mathcal{F}}$-module $\mathfrak{m}|_{\mathcal{H}_{\mathcal{F}}}$ is projective.*

*Proof.* Given $\mathcal{F}' \subset \overline{\mathcal{F}}$, we let $W_{\mathcal{F}} \subset W_{\mathcal{F}'}$ denote the subgroups of $W_{\mathrm{aff}}$ generated by those elements of $S$ which fix pointwise each respective facet (see [14, §4.3] for more details). The group $W_{\mathcal{F}}$ identifies with a parabolic subgroup of $W_{\mathcal{F}'}$, and we let $W^{\mathcal{F}}$ denote the set of minimal coset representatives of $W_{\mathcal{F}'}/W_{\mathcal{F}}$ (cf. definition of $W_0^M$). For each $v \in W^{\mathcal{F}} \subset W_{\mathcal{F}'}$, we fix a lift $\widehat{v} \in \widetilde{W}_{\mathcal{F}'}$. Then any element $\widetilde{w} \in \widetilde{W}_{\mathcal{F}'}$ may be written uniquely as $\widetilde{w} = \widehat{v}\widetilde{u}$, with $v \in W^{\mathcal{F}}, \widetilde{u} \in \widetilde{W}_{\mathcal{F}}$, satisfying $\ell(\widetilde{w}) = \ell(\widehat{v}) + \ell(\widetilde{u})$.

By the comments preceding [14, Lem. 4.20], the algebra $\mathcal{H}_{\mathcal{F}}$ has a basis given by $\{\mathrm{T}_{\widetilde{w}}\}_{\widetilde{w} \in \widetilde{W}_{\mathcal{F}}}$ (and similarly for $\mathcal{H}_{\mathcal{F}'}$). Using the above remarks, we see that $\mathcal{H}_{\mathcal{F}}$ identifies with a parabolic subalgebra of $\mathcal{H}_{\mathcal{F}'}$, and $\mathcal{H}_{\mathcal{F}'}$ is free as a right $\mathcal{H}_{\mathcal{F}}$-module, with basis $\{\mathrm{T}_{\widehat{v}}\}_{v \in W^{\mathcal{F}}}$. Therefore, by the Eckmann–Shapiro Lemma,

$$\mathrm{Ext}^i_{\mathcal{H}_{\mathcal{F}}}(\mathfrak{m}|_{\mathcal{H}_{\mathcal{F}}}, \mathfrak{n}) \cong \mathrm{Ext}^i_{\mathcal{H}_{\mathcal{F}'}}(\mathfrak{m}, \mathrm{Hom}_{\mathcal{H}_{\mathcal{F}}}(\mathcal{H}_{\mathcal{F}'}, \mathfrak{n})) = 0 \text{ for } i > 0. \qquad \square$$

**Lemma 3.4.** *Assume that $p \nmid |\Omega_{\mathrm{tor}}|$. Let $\mathcal{F}$ denote a facet such that $\mathcal{F} \subset \overline{C}$, and let $\mathfrak{m}$ be a right $\mathcal{H}^{\dagger}_{\mathcal{F}}$-module. Then $\mathfrak{m}|_{\mathcal{H}_{\mathcal{F}}}$ is projective over $\mathcal{H}_{\mathcal{F}}$ if and only if $\mathrm{pd}_{\mathcal{H}^{\dagger}_{\mathcal{F}}}(\mathfrak{m}) \leq r_Z$.*

*Proof.* Let $\Omega_{\mathcal{F}}$ denote the subgroup of $\Omega$ stabilizing $\mathcal{F}$, and write $\Omega_{\mathcal{F}} \cong \Omega_{\mathcal{F},\mathrm{tor}} \times \Omega_{\mathcal{F},\mathrm{free}}$, where $\Omega_{\mathcal{F},\mathrm{tor}}$ is finite and $\Omega_{\mathcal{F},\mathrm{free}}$ is free of rank $r_Z$ (note that $Z/(Z \cap I)$ embeds as a finite-index subgroup of $\Omega_{\mathcal{F},\mathrm{free}}$). The algebra $\mathcal{H}^{\dagger}_{\mathcal{F}}$ is generated by $\mathcal{H}_{\mathcal{F}}$ and $\{\mathrm{T}_{\widetilde{\omega}}\}_{\widetilde{\omega} \in \widetilde{\Omega}_{\mathcal{F}}}$ ([14, Lem. 4.20]), and we let $\mathcal{H}_{\mathcal{F},\mathrm{free}}$ denote the subalgebra of $\mathcal{H}^{\dagger}_{\mathcal{F}}$ generated by $\mathcal{H}_{\mathcal{F}}$ and $\{\mathrm{T}_{\widetilde{\omega}}\}_{\widetilde{\omega} \in \widetilde{\Omega}_{\mathcal{F},\mathrm{free}}}$. We fix a set of generators $\{\omega_i\}_{i=1}^{r_Z}$ for $\Omega_{\mathcal{F},\mathrm{free}}$, and let $\{\widehat{\omega}_i\}_{i=1}^{r_Z} \subset \widetilde{\Omega}_{\mathcal{F},\mathrm{free}}$ denote a fixed set of lifts. Using the braid relations, we see that $\mathcal{H}_{\mathcal{F},\mathrm{free}}$ is free over $\mathcal{H}_{\mathcal{F}}$ with basis $\{\mathrm{T}_{\widehat{\omega}_1^{\ell_1} \cdots \widehat{\omega}_{r_Z}^{\ell_{r_Z}}}\}_{\ell_i \in \mathbb{Z}}$. Moreover, this gives $\mathcal{H}_{\mathcal{F},\mathrm{free}}$ the structure of a(n iterated) skew Laurent polynomial algebra over $\mathcal{H}_{\mathcal{F}}$ (see [10, §§1.2 and 1.4.3] for the relevant definition). Therefore, if $\mathfrak{m}|_{\mathcal{H}_{\mathcal{F}}}$ is projective, [10, Prop.

7.5.2(ii)] gives

$$\mathrm{pd}_{\mathcal{H}_{\mathcal{F},\mathrm{free}}}(\mathfrak{m}|_{\mathcal{H}_{\mathcal{F},\mathrm{free}}}) \leq \mathrm{pd}_{\mathcal{H}_{\mathcal{F}}}(\mathfrak{m}|_{\mathcal{H}_{\mathcal{F}}}) + r_Z = r_Z.$$

Now, fix a set of lifts $\{\widehat{\omega}\}_{\omega\in\Omega_{\mathcal{F},\mathrm{tor}}} \subset \widetilde{\Omega}_{\mathcal{F},\mathrm{tor}}$ containing 1. Once again using the braid relations, we see that $\mathcal{H}_{\mathcal{F}}^{\dagger}$ is free over $\mathcal{H}_{\mathcal{F},\mathrm{free}}$, with basis given by the elements $\{\mathrm{T}_{\widehat{\omega}}\}_{\omega\in\Omega_{\mathcal{F},\mathrm{tor}}}$. This gives $\mathcal{H}_{\mathcal{F}}^{\dagger}$ the structure of a crossed product algebra: $\mathcal{H}_{\mathcal{F}}^{\dagger} \cong \mathcal{H}_{\mathcal{F},\mathrm{free}} * \Omega_{\mathcal{F},\mathrm{tor}}$ (see [10, §1.5.8]). Since $p \nmid |\Omega_{\mathcal{F},\mathrm{tor}}|$ by assumption, [10, Thm. 7.5.6(ii)] implies

$$\mathrm{pd}_{\mathcal{H}_{\mathcal{F}}^{\dagger}}(\mathfrak{m}) = \mathrm{pd}_{\mathcal{H}_{\mathcal{F},\mathrm{free}}}(\mathfrak{m}|_{\mathcal{H}_{\mathcal{F},\mathrm{free}}}) \leq r_Z.$$

To prove the converse, recall that $\mathcal{H}_{\mathcal{F}}^{\dagger}$ is free over $\mathcal{H}_{\mathcal{F}}$, and therefore any projective resolution of $\mathfrak{m}$ restricts to a projective resolution of $\mathfrak{m}|_{\mathcal{H}_{\mathcal{F}}}$. Hence, if $\mathrm{pd}_{\mathcal{H}_{\mathcal{F}}^{\dagger}}(\mathfrak{m}) \leq r_Z$, we then obtain $\mathrm{pd}_{\mathcal{H}_{\mathcal{F}}}(\mathfrak{m}|_{\mathcal{H}_{\mathcal{F}}}) \leq r_Z < \infty$, and $\mathfrak{m}|_{\mathcal{H}_{\mathcal{F}}}$ must be projective by Remark 3.1. □

**Lemma 3.5.** *Let $\mathcal{F}$ denote a facet such that $\mathcal{F} \subset \overline{C}$, let $\mathfrak{m}$ be a right $\mathcal{H}_{\mathcal{F}}^{\dagger}$-module, and let $n \in \mathbb{Z}_{\geq 0}$. Then $\mathrm{pd}_{\mathcal{H}_{\mathcal{F}}^{\dagger}}(\mathfrak{m}) \leq n$ if and only if $\mathrm{pd}_{\mathcal{H}_{\mathcal{F}}^{\dagger}}(\mathfrak{m}(\epsilon_{\mathcal{F}})) \leq n$, where $\epsilon_{\mathcal{F}}$ denotes the orientation character of $\mathcal{P}_{\mathcal{F}}^{\dagger}$ (see [14, §§3.1 and 3.3.1]).*

*Proof.* This follows from the fact that the functor $\mathfrak{m} \longmapsto \mathfrak{m}(\epsilon_{\mathcal{F}})$ is exact on the category of $\mathcal{H}_{\mathcal{F}}^{\dagger}$-modules, and preserves projectives. □

The following result will be used in a subsequent section. We use notation and terminology from [2, §4.1]. For a standard Levi subgroup $M$, we let $\mathcal{H}_M^-$ denote the subalgebra of $\mathcal{H}_M$ consisting of functions supported on $M$-negative elements. Recall that, if $w \in W_{M,0} \subset W_M$ has a fixed lift $\widehat{w} \in \widetilde{W}_M$ and $\lambda \in \widetilde{\Lambda}$, the element $\lambda\widehat{w} \in \widetilde{W}_M$ is *$M$-negative* if $\langle \alpha, \nu(\lambda)\rangle \geq 0$ for every $\alpha \in \Phi^+ \smallsetminus \Phi_M^+$. By Lemma 4.6 of *loc. cit.*, we have an injective algebra morphism

$$\begin{aligned} j_M^- : \mathcal{H}_M^- &\hookrightarrow \mathcal{H} \\ \mathrm{T}_{\widetilde{w}}^{M,*} &\longmapsto \mathrm{T}_{\widetilde{w}}^* \end{aligned}$$

for $M$-negative $\widetilde{w} \in \widetilde{W}_M$, and we view $\mathcal{H}$ as a right $\mathcal{H}_M^-$-module via $j_M^-$. Note that, while $\mathcal{H}_M$ depends only on the Levi subgroup $M$, $\mathcal{H}_M^-$ and $j_M^-$ depend on the choice of positive roots, and therefore on the choice of hyperspecial vertex $x \in \overline{C}$.

**Lemma 3.6 (Mackey formula).** *Let $M$ be a standard Levi subgroup, and $\mathfrak{n}$ a right $\mathcal{H}_M$-module. We then have an isomorphism of right $\mathcal{H}_x$-modules*

$$\begin{aligned}\mathrm{Hom}_{\mathcal{H}_M^-}(\mathcal{H},\mathfrak{n})|_{\mathcal{H}_x} &\stackrel{\sim}{\longrightarrow} \mathrm{Hom}_{\mathcal{H}_M^-\cap\mathcal{H}_x}(\mathcal{H}_x,\mathfrak{n}|_{\mathcal{H}_M^-\cap\mathcal{H}_x})\\ \varphi &\longmapsto \varphi|_{\mathcal{H}_x}.\end{aligned}$$

*Proof.* The restriction map is clearly well-defined and $\mathcal{H}_x$-equivariant. It therefore suffices to check that it is an isomorphism of vector spaces.

Recall that the choice of hyperspecial vertex $x$ gives an identification $W_0 \cong W_x$, where $W_x$ denotes the subgroup of $W_{\mathrm{aff}}$ generated by those elements of $S$ which fix $x$. For each $v \in W_0^M$, we fix a lift $\widehat{v} \in \widetilde{W}_0^M$. Then, given any element $\widetilde{w} \in \widetilde{W}_0 \cong \widetilde{W}_x$, we may write $\widetilde{w} = \widehat{v}\widetilde{u}$ for unique $v \in W_0^M, \widetilde{u} \in \widetilde{W}_{M,0}$, satisfying $\ell(\widetilde{w}) = \ell(\widehat{v}) + \ell(\widetilde{u})$. One sees easily that $\{\mathrm{T}_{\widetilde{u}}\}_{\widetilde{u}\in\widetilde{W}_{M,0}}$ gives a basis for $\mathcal{H}_M^- \cap \mathcal{H}_x$ and $\{\mathrm{T}_{\widetilde{w}}\}_{\widetilde{w}\in\widetilde{W}_0}$ gives a basis for $\mathcal{H}_x$, so that $\mathcal{H}_M^- \cap \mathcal{H}_x$ identifies with a parabolic subalgebra of $\mathcal{H}_x$. Therefore, the braid relations and the factorization $\widetilde{w} = \widehat{v}\widetilde{u}$ imply that $\mathcal{H}_x$ is free as a right $\mathcal{H}_M^- \cap \mathcal{H}_x$-module, with basis $\{\mathrm{T}_{\widehat{v}}\}_{v\in W_0^M}$, and the map

$$\begin{aligned}\mathrm{Hom}_{\mathcal{H}_M^-\cap\mathcal{H}_x}(\mathcal{H}_x,\mathfrak{n}|_{\mathcal{H}_M^-\cap\mathcal{H}_x}) &\longrightarrow \bigoplus_{v\in W_0^M}\mathfrak{n}\\ \psi &\longmapsto (\psi(\mathrm{T}_{\widehat{v}}))_{v\in W_0^M}\end{aligned}$$

is an isomorphism of vector spaces.

Consider now the diagram of vector spaces

$$\begin{array}{ccc}\mathrm{Hom}_{\mathcal{H}_M^-}(\mathcal{H},\mathfrak{n})|_{\mathcal{H}_x} & \longrightarrow & \mathrm{Hom}_{\mathcal{H}_M^-\cap\mathcal{H}_x}(\mathcal{H}_x,\mathfrak{n}|_{\mathcal{H}_M^-\cap\mathcal{H}_x})\\ \big\downarrow & & \big\downarrow\wr\\ \bigoplus_{v\in W_0^M}\mathfrak{n} & & \bigoplus_{v\in W_0^M}\mathfrak{n}\end{array}$$

where the right vertical map is the map of the previous paragraph, and the left vertical map is given by $\varphi \longmapsto (\varphi(\mathrm{T}_{\widehat{v}}))_{v\in W_0^M}$. By [2, Lem. 4.10], the left vertical map is also an isomorphism of vector spaces, and composing its inverse with the horizontal restriction map and the right vertical map gives the identity on $\bigoplus_{v\in W_0^M}\mathfrak{n}$. Therefore the restriction map is an isomorphism. □

## 4. Resolutions and spectral sequences

We now consider resolutions and projective dimensions of $\mathcal{H}$-modules. For $0 \leq i \leq r_{\mathrm{ss}}$, we fix a finite set $\mathscr{F}_i$ of representatives of the $G$-orbits of $i$-dimensional facets in the Bruhat–Tits building of $G$, subject to the condition that every element of $\mathscr{F}_i$ is contained in $\overline{C}$. Given a right $\mathcal{H}$-module $\mathfrak{m}$, [14, Thm. 3.12] implies that we have a resolution

$$\begin{aligned}(1)\qquad & 0 \longrightarrow \bigoplus_{\mathcal{F}\in\mathscr{F}_{r_{\mathrm{ss}}}} \mathfrak{m}|_{\mathcal{H}_{\mathcal{F}}^{\dagger}}(\epsilon_{\mathcal{F}}) \otimes_{\mathcal{H}_{\mathcal{F}}^{\dagger}} \mathcal{H} \longrightarrow \cdots \\ & \longrightarrow \bigoplus_{\mathcal{F}\in\mathscr{F}_{0}} \mathfrak{m}|_{\mathcal{H}_{\mathcal{F}}^{\dagger}}(\epsilon_{\mathcal{F}}) \otimes_{\mathcal{H}_{\mathcal{F}}^{\dagger}} \mathcal{H} \longrightarrow \mathfrak{m} \longrightarrow 0,\end{aligned}$$

which gives a hyper-Ext spectral sequence

$$\begin{aligned}(2)\qquad E_1^{i,j} &= \mathrm{Ext}^j_{\mathcal{H}}\left(\bigoplus_{\mathcal{F}\in\mathscr{F}_i} \mathfrak{m}|_{\mathcal{H}_{\mathcal{F}}^{\dagger}}(\epsilon_{\mathcal{F}}) \otimes_{\mathcal{H}_{\mathcal{F}}^{\dagger}} \mathcal{H}, \mathfrak{n}\right) \\ &\cong \bigoplus_{\mathcal{F}\in\mathscr{F}_i} \mathrm{Ext}^j_{\mathcal{H}_{\mathcal{F}}^{\dagger}}\left(\mathfrak{m}|_{\mathcal{H}_{\mathcal{F}}^{\dagger}}(\epsilon_{\mathcal{F}}), \mathfrak{n}|_{\mathcal{H}_{\mathcal{F}}^{\dagger}}\right) \Longrightarrow \mathrm{Ext}^{i+j}_{\mathcal{H}}(\mathfrak{m},\mathfrak{n}).\end{aligned}$$

(The isomorphism on the left-hand side of the spectral sequence above follows from Proposition 4.21 of *loc. cit.* and the Eckmann–Shapiro lemma.)

**Remark 4.1.** We have the following variation on the above. Let $G_{\mathrm{aff}}$ denote the subgroup of $G$ generated by all parahoric subgroups, and let $\mathcal{H}_{\mathrm{aff}}$ denote the subalgebra of $\mathcal{H}$ consisting of elements with support in $G_{\mathrm{aff}}$. We view the algebras $\mathcal{H}_{\mathcal{F}}$ as subalgebras of $\mathcal{H}_{\mathrm{aff}}$.

The group $G_{\mathrm{aff}}$ acts on the Bruhat–Tits building of $G$, with a transitive action on chambers. For $0 \leq i \leq r_{\mathrm{ss}}$, we fix a finite set $\mathscr{F}_i^{\mathrm{aff}}$ of representatives of the $G_{\mathrm{aff}}$-orbits of $i$-dimensional facets, subject to the condition that every element of $\mathscr{F}_i^{\mathrm{aff}}$ is contained in $\overline{C}$. Note that the sets $\mathscr{F}_i$ and $\mathscr{F}_i^{\mathrm{aff}}$ are different in general. The results of [14] easily modify to this setting, and given a right $\mathcal{H}_{\mathrm{aff}}$-module $\mathfrak{m}$, we obtain a resolution

$$\begin{aligned}(3)\qquad & 0 \longrightarrow \bigoplus_{\mathcal{F}\in\mathscr{F}_{r_{\mathrm{ss}}}^{\mathrm{aff}}} \mathfrak{m}|_{\mathcal{H}_{\mathcal{F}}} \otimes_{\mathcal{H}_{\mathcal{F}}} \mathcal{H}_{\mathrm{aff}} \longrightarrow \cdots \\ & \longrightarrow \bigoplus_{\mathcal{F}\in\mathscr{F}_{0}^{\mathrm{aff}}} \mathfrak{m}|_{\mathcal{H}_{\mathcal{F}}} \otimes_{\mathcal{H}_{\mathcal{F}}} \mathcal{H}_{\mathrm{aff}} \longrightarrow \mathfrak{m} \longrightarrow 0,\end{aligned}$$

which gives a spectral sequence

$$E_1^{i,j} = \mathrm{Ext}^j_{\mathcal{H}_{\mathrm{aff}}}\left(\bigoplus_{\mathcal{F}\in\mathscr{F}_i^{\mathrm{aff}}} \mathfrak{m}|_{\mathcal{H}_\mathcal{F}} \otimes_{\mathcal{H}_\mathcal{F}} \mathcal{H}_{\mathrm{aff}}, \mathfrak{n}\right) \cong \bigoplus_{\mathcal{F}\in\mathscr{F}_i^{\mathrm{aff}}} \mathrm{Ext}^j_{\mathcal{H}_\mathcal{F}}(\mathfrak{m}|_{\mathcal{H}_\mathcal{F}}, \mathfrak{n}|_{\mathcal{H}_\mathcal{F}}) \Longrightarrow \mathrm{Ext}^{i+j}_{\mathcal{H}_{\mathrm{aff}}}(\mathfrak{m}, \mathfrak{n}). \tag{4}$$

In particular, the resolution (3) shows that $\mathcal{H}_{\mathrm{aff}}$ is $r_{\mathrm{ss}}$-Gorenstein (cf. Propositions 1.2(i) and 4.21 of *loc. cit.*).

The following simple observation will be useful later.

**Lemma 4.2.** *Let $\mathfrak{m}$ denote a right $\mathcal{H}$-module. Then $\mathrm{pd}_\mathcal{H}(\mathfrak{m}) < \infty$ if and only if $\mathrm{pd}_{\mathcal{H}_\mathcal{F}^\dagger}(\mathfrak{m}|_{\mathcal{H}_\mathcal{F}^\dagger}) < \infty$ for every facet $\mathcal{F} \subset \overline{C}$. Moreover, if $p \nmid |\Omega_{\mathrm{tor}}|$, then $\mathrm{pd}_\mathcal{H}(\mathfrak{m}) < \infty$ if and only if $\mathfrak{m}|_{\mathcal{H}_\mathcal{F}}$ is projective over $\mathcal{H}_\mathcal{F}$ for every facet $\mathcal{F} \subset \overline{C}$.*

The same claims obviously apply to $\mathcal{H}_{\mathrm{aff}}$ and $\mathcal{H}_\mathcal{F}$ (without the need for the "$p \nmid |\Omega_{\mathrm{tor}}|$" assumption).

*Proof.* [14, Prop. 4.21] shows that $\mathcal{H}$ is free over $\mathcal{H}_\mathcal{F}^\dagger$ as either a left or right module. The Eckmann–Shapiro lemma then gives

$$\mathrm{Ext}^i_{\mathcal{H}_\mathcal{F}^\dagger}\left(\mathfrak{m}|_{\mathcal{H}_\mathcal{F}^\dagger}, \mathfrak{n}\right) \cong \mathrm{Ext}^i_\mathcal{H}\left(\mathfrak{m}, \mathrm{Hom}_{\mathcal{H}_\mathcal{F}^\dagger}(\mathcal{H}, \mathfrak{n})\right),$$

which gives one implication. On the other hand, if $\mathrm{pd}_{\mathcal{H}_\mathcal{F}^\dagger}(\mathfrak{m}|_{\mathcal{H}_\mathcal{F}^\dagger}) < \infty$ for all $\mathcal{F} \subset \overline{C}$, then the $E_1^{i,j}$ term in (2) vanishes for $i$ and $j$ sufficiently large (this uses Lemma 3.5). The spectral sequence (2) then shows that $\mathrm{Ext}^i_\mathcal{H}(\mathfrak{m}, \mathfrak{n}) = 0$ for all $\mathcal{H}$-modules $\mathfrak{n}$ and $i$ sufficiently large (independent of $\mathfrak{n}$), so that $\mathrm{pd}_\mathcal{H}(\mathfrak{m}) < \infty$.

If we assume $p \nmid |\Omega_{\mathrm{tor}}|$, then Remark 3.1 and Lemma 3.4 imply that $\mathrm{pd}_{\mathcal{H}_\mathcal{F}^\dagger}(\mathfrak{m}|_{\mathcal{H}_\mathcal{F}^\dagger}) < \infty$ if and only if $\mathfrak{m}|_{\mathcal{H}_\mathcal{F}}$ is projective over $\mathcal{H}_\mathcal{F}$. □

## 5. Simple modules

We recall the classification of simple $\mathcal{H}$-modules from [2]. Let $P = M \ltimes N$ denote a standard parabolic subgroup, and let $\mathfrak{n}$ denote a simple supersingular right $\mathcal{H}_M$-module (see [22, Def. 6.10] and Section 7 below). Set

$$\Pi(\mathfrak{n}) := \Pi_M \sqcup \left\{\alpha \in \Pi : \begin{array}{l} \diamond\ \langle\beta, \alpha^\vee\rangle = 0 \text{ for all } \beta \in \Pi_M \\ \diamond\ \mathrm{T}^M_{\alpha^\vee(x)} \text{ acts trivially on } \mathfrak{n} \text{ for all } x \in F^\times \end{array}\right\},$$

and let $P(\mathfrak{n}) = M(\mathfrak{n}) \ltimes N(\mathfrak{n})$ denote the corresponding standard parabolic subgroup. Given any standard parabolic subgroup $Q = L \ltimes V$ satisfying $P \subset Q \subset P(\mathfrak{n})$, [2, Prop. 4.16] shows that $\mathfrak{n}$ extends uniquely to an $\mathcal{H}_L$-module satisfying certain properties. We denote this extension by $\mathfrak{n}^{e_L}$. Given any right $\mathcal{H}_L$-module $\mathfrak{v}$, we define the parabolic coinduction

$$I_L(\mathfrak{v}) := \mathrm{Hom}_{\mathcal{H}_L^-}(\mathcal{H}, \mathfrak{v}),$$

where $\mathcal{H}$ is viewed as a right $\mathcal{H}_L^-$-module via $j_L^-$, and set

$$I(P, \mathfrak{n}, Q) := I_L(\mathfrak{n}^{e_L}) \Bigg/ \left( \sum_{Q \subsetneq Q' \subset P(\mathfrak{n})} I_{L'}(\mathfrak{n}^{e_{L'}}) \right).$$

By [2, Thm. 4.22], the $\mathcal{H}$-modules $I(P, \mathfrak{n}, Q)$ are simple and $I(P, \mathfrak{n}, Q) \cong I(P', \mathfrak{n}', Q')$ if and only if $P = P', Q = Q'$, and $\mathfrak{n} \cong \mathfrak{n}'$. Moreover, the $I(P, \mathfrak{n}, Q)$ exhaust all isomorphism classes of simple right $\mathcal{H}$-modules (for varying $P, \mathfrak{n}$, and $Q$). Using [2, Cor. 4.26] and the fact that $I_M(\mathfrak{n})$ is multiplicity-free, we get that the Jordan–Hölder factors of $I_L(\mathfrak{n}^{e_L})$ are given by

$$\{I(P, \mathfrak{n}, Q') : Q \subset Q' \subset P(\mathfrak{n})\}.$$

We now give a result on the structure of $I(P, \mathfrak{n}, Q)$. In the proof below, if $J$ is a subset of $\Pi$ such that $\Pi_M \subset J \subset \Pi(\mathfrak{n})$, we use $I_J(\mathfrak{n})$ to denote $I_{M_J}(\mathfrak{n}^{e_{M_J}})$.

**Lemma 5.1.** *The Čech complex $\mathcal{C}_\bullet$ of $I(P, \mathfrak{n}, Q)$*

$$0 \longrightarrow I_{M(\mathfrak{n})}(\mathfrak{n}^{e_{M(\mathfrak{n})}}) \xrightarrow{\partial_r} \bigoplus_{\substack{L \subset L' \subset M(\mathfrak{n}) \\ |\Pi_{L'} \smallsetminus \Pi_L| = r-1}} I_{L'}(\mathfrak{n}^{e_{L'}}) \xrightarrow{\partial_{r-1}} \cdots$$

$$\cdots \xrightarrow{\partial_2} \bigoplus_{\substack{L \subset L' \subset M(\mathfrak{n}) \\ |\Pi_{L'} \smallsetminus \Pi_L| = 1}} I_{L'}(\mathfrak{n}^{e_{L'}}) \xrightarrow{\partial_1} I_L(\mathfrak{n}^{e_L}) \xrightarrow{\partial_0} I(P, \mathfrak{n}, Q) \longrightarrow 0$$

*is exact, where $r = |\Pi(\mathfrak{n}) \smallsetminus \Pi_L|$.*

We will abbreviate the resolution above by $\mathcal{C}_\bullet \longrightarrow I(P, \mathfrak{n}, Q) \longrightarrow 0$.

*Proof.* The differentials are given as follows. We fix a numbering $\Pi(\mathfrak{n}) \smallsetminus \Pi_L = \{\alpha_1, \ldots, \alpha_r\}$, and let $(f_{J'})_{J'} \in \bigoplus_{\substack{\Pi_L \subset J' \subset \Pi(\mathfrak{n}) \\ |J' \smallsetminus \Pi_L| = j}} I_{J'}(\mathfrak{n}) = \mathcal{C}_j$ for $1 \leq j \leq r$. Let $J''$ denote a subset of $\Pi$ such that $\Pi_L \subset J'' \subset \Pi(\mathfrak{n})$ and $|J'' \smallsetminus \Pi_L| =$

$j-1$, and write $\Pi(\mathfrak{n}) \smallsetminus J'' = \{\alpha_{\ell_1}, \ldots, \alpha_{\ell_{r-j+1}}\}$ with $\ell_1 < \cdots < \ell_{r-j+1}$. We then have

$$(\mathrm{pr}_{J''} \circ \partial_j)((f_{J'})_{J'}) = \sum_{k=1}^{r-j+1} (-1)^k f_{J'' \cup \{\alpha_{\ell_k}\}}.$$

The standard argument shows that this gives a complex.

Exactness at $\mathcal{C}_0$ is clear. Suppose now that $(f_{J'})_{J'} \in \ker(\partial_j)$ for $1 \leq j \leq r$, and set $J'_0 := \Pi_L \cup \{\alpha_1, \ldots, \alpha_j\}$. We have

$$(\mathrm{pr}_{J'_0 \smallsetminus \{\alpha_j\}} \circ \partial_j)((f_{J'})_{J'}) = \sum_{k=1}^{r-j+1} (-1)^k f_{(J'_0 \smallsetminus \{\alpha_j\}) \cup \{\alpha_{j-1+k}\}} = 0,$$

which implies $f_{J'_0} \in I_{J'_0}(\mathfrak{n}) \cap \left(\sum_{k=2}^{r-j+1} I_{(J'_0 \smallsetminus \{\alpha_j\}) \cup \{\alpha_{j-1+k}\}}(\mathfrak{n})\right)$. We clearly have

$$\begin{aligned}\sum_{k=2}^{r-j+1} I_{J'_0 \cup \{\alpha_{j-1+k}\}}(\mathfrak{n}) &\subset \sum_{k=2}^{r-j+1} \left(I_{J'_0}(\mathfrak{n}) \cap I_{(J'_0 \smallsetminus \{\alpha_j\}) \cup \{\alpha_{j-1+k}\}}(\mathfrak{n})\right) \\ &\subset I_{J'_0}(\mathfrak{n}) \cap \left(\sum_{k=2}^{r-j+1} I_{(J'_0 \smallsetminus \{\alpha_j\}) \cup \{\alpha_{j-1+k}\}}(\mathfrak{n})\right),\end{aligned}$$

and comparing the Jordan–Hölder factors of both sides shows both inclusions must be equalities (this uses the fact that $I_M(\mathfrak{n})$ is multiplicity-free). Write

$$f_{J'_0} = \sum_{k=2}^{r-j+1} (-1)^{k-1} g_{J'_0 \cup \{\alpha_{j-1+k}\}},$$

with $g_{J'_0 \cup \{\alpha_{j-1+k}\}} \in I_{J'_0 \cup \{\alpha_{j-1+k}\}}(\mathfrak{n})$, and define $(g_K)_K \in \bigoplus_{\substack{\Pi_L \subset K \subset \Pi(\mathfrak{n}) \\ |K \smallsetminus \Pi_L| = j+1}} I_K(\mathfrak{n}) = \mathcal{C}_{j+1}$ by

$$g_K = \begin{cases} g_{J'_0 \cup \{\alpha_{j-1+k}\}} & \text{if } K = J'_0 \cup \{\alpha_{j-1+k}\}, k \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

This gives

$$\mathrm{pr}_{J'_0}((f_{J'})_{J'} - \partial_{j+1}((g_K)_K)) = f_{J'_0} - \sum_{k=1}^{r-j} (-1)^k g_{J'_0 \cup \{\alpha_{j+k}\}} = 0.$$

Replacing $(f_{J'})_{J'}$ by $(f_{J'})_{J'} - \partial_{j+1}((g_K)_K)$, we may assume $f_{\Pi_L \cup \{\alpha_1, \ldots, \alpha_j\}} = 0$.

We now fix $j \leq s \leq r-1$, and assume that $f_{J'} = 0$ for every $J'$ with $J' \smallsetminus \Pi_L \subset \{\alpha_1, \ldots, \alpha_s\}$. Let $J_0'$ be such that $J_0' \smallsetminus \Pi_L \subset \{\alpha_1, \ldots, \alpha_{s+1}\}$ and $\alpha_{s+1} \in J_0'$, and write $\Pi(\mathfrak{n}) \smallsetminus J_0' = \{\alpha_{\ell_1}, \ldots, \alpha_{\ell_{r-j}}\}$ as above. By assumption, we have

$$(\mathrm{pr}_{J_0' \smallsetminus \{\alpha_{s+1}\}} \circ \partial_j)((f_{J'})_{J'}) = \pm f_{J_0'} + \sum_{\substack{k=1 \\ \ell_k > s+1}}^{r-j} (-1)^{k+1} f_{(J_0' \smallsetminus \{\alpha_{s+1}\}) \cup \{\alpha_{\ell_k}\}} = 0,$$

and as above we may write

$$f_{J_0'} = \sum_{\substack{k=1 \\ \ell_k > s+1}}^{r-j} (-1)^k g_{J_0' \cup \{\alpha_{\ell_k}\}}$$

with $g_{J_0' \cup \{\alpha_{\ell_k}\}} \in I_{J_0' \cup \{\alpha_{\ell_k}\}}(\mathfrak{n})$. We define $(g_K)_K \in \bigoplus_{\substack{\Pi_L \subset K \subset \Pi(\mathfrak{n}) \\ |K \smallsetminus \Pi_L| = j+1}} I_K(\mathfrak{n}) = \mathcal{C}_{j+1}$ by

$$g_K = \begin{cases} g_{J_0' \cup \{\alpha_{\ell_k}\}} & \text{if } K = J_0' \cup \{\alpha_{\ell_k}\}, \ell_k > s+1, \\ 0 & \text{otherwise.} \end{cases}$$

Fix $J_1'$ with $J_1' \smallsetminus \Pi_L \subset \{\alpha_1, \ldots, \alpha_{s+1}\}$ and $|J_1' \smallsetminus \Pi_L| = j$. One easily checks that

$$\mathrm{pr}_{J_1'}((f_{J'})_{J'} - \partial_{j+1}((g_K)_K)) = \begin{cases} 0 & \text{if } J_1' = J_0', \\ f_{J_1'} & \text{if } J_1' \neq J_0'. \end{cases}$$

Therefore, replacing $(f_{J'})_{J'}$ by $(f_{J'})_{J'} - \partial_{j+1}((g_K)_K)$ (several times if necessary), we may assume that $f_{J'} = 0$ for every $J'$ with $J' \smallsetminus \Pi_L \subset \{\alpha_1, \ldots, \alpha_{s+1}\}$. By induction on $s$ we get that $(f_{J'})_{J'} \in \mathrm{im}(\partial_{j+1})$. □

**Remark 5.2.** The above proof remains valid for an arbitrary (i.e., not necessarily split) connected reductive group.

Now let $\psi : T \longrightarrow \overline{\mathbb{F}}_p^\times$ be a smooth character of $T$. We use the same notation $\psi$ to denote the corresponding right $\mathcal{H}_T$-module, with action given by $v \cdot \mathrm{T}_\lambda^T = \psi(\lambda)^{-1} v$, for $v \in \psi, \lambda \in \widetilde{\Lambda}$. Note that the $\mathcal{H}_T$-module $\psi$ is supersingular (see Section 7). We will call $I_T(\psi)$ a *principal series module*. In this setting, we have $\Pi(\psi) = \{\alpha \in \Pi : \psi \circ \alpha^\vee(x) = 1 \text{ for all } x \in F^\times\}$, and we let $P(\psi)$ denote the corresponding standard parabolic subgroup.

**Lemma 5.3.** *Let $\psi : T \longrightarrow \overline{\mathbb{F}}_p^\times$ be a smooth character, and let $Q = L \ltimes V$ denote a standard parabolic subgroup with $B \subset Q \subset P(\psi)$.*

1) *The right $\mathcal{H}_x$-module $I_L(\psi^{e_L})|_{\mathcal{H}_x}$ is projective.*

2) *The right $\mathcal{H}_x$-module $I(B,\psi,Q)|_{\mathcal{H}_x}$ is projective.*

*Proof.* (1) Using the Mackey formula (Lemma 3.6), we have

$$I_L(\psi^{e_L})|_{\mathcal{H}_x} \cong \mathrm{Hom}_{\mathcal{H}_L^- \cap \mathcal{H}_x}(\mathcal{H}_x, \psi^{e_L}|_{\mathcal{H}_L^- \cap \mathcal{H}_x}).$$

The construction of [2, Prop. 4.16] shows that $\psi^{e_L}|_{\mathcal{H}_L^- \cap \mathcal{H}_x}$ is a twist of the trivial character (cf. (5)), and therefore it is an injective $\mathcal{H}_L^- \cap \mathcal{H}_x$-module (this follows from [14, Prop. 6.19] or [11, Thm. 5.2]). It follows that $\mathrm{Hom}_{\mathcal{H}_L^- \cap \mathcal{H}_x}(\mathcal{H}_x, \psi^{e_L}|_{\mathcal{H}_L^- \cap \mathcal{H}_x})$ is injective as an $\mathcal{H}_x$-module, and the result follows from Remark 3.1.

(2) Restricting the Čech resolution of Lemma 5.1 to $\mathcal{H}_x$ and using part (1) gives a projective resolution

$$\mathcal{C}_\bullet|_{\mathcal{H}_x} \longrightarrow I(B,\psi,Q)|_{\mathcal{H}_x} \longrightarrow 0,$$

and therefore $I(B,\psi,Q)|_{\mathcal{H}_x}$ has finite projective dimension. We conclude by using Remark 3.1. □

## 6. Principal series modules – Type $A_n$

As a first step, we examine the simple subquotients of principal series modules in type $A_n$. We will generalize this result in the coming sections.

**Proposition 6.1.** *Assume the root system of $G$ is of type $A_n$, and assume $p \nmid |\Omega_{\mathrm{tor}}|$. Let $\mathfrak{m}$ be a simple subquotient of a principal series module for $\mathcal{H}$. Then $\mathrm{pd}_{\mathcal{H}}(\mathfrak{m}) \leq n + r_Z$. Moreover, when $G$ is semisimple,* (1) *is a projective resolution of $\mathfrak{m}$, and the bound on projective dimension is sharp.*

*Proof.* Since the root system of $G$ is of type $A_n$, every vertex in the Bruhat–Tits building is hyperspecial. Given the vertex $x \in \overline{C}$, we may write $\mathfrak{m} \cong I(B,\psi,Q)$ for some character $\psi$ of $T$ and standard parabolic subgroup $Q$. Note that $B, \psi$, and $Q$ implicitly depend on the vertex $x$, which defines a set of positive roots $\Phi^+$. Choosing another hyperspecial vertex $x' \in \overline{C}$, we obtain an isomorphism $\mathfrak{m} \cong I(B',\psi',Q')$, where $B'$ is the Borel subgroup containing $T$ and defined by $x'$, possibly different from $B$ (this isomorphism follows from considering central characters). Therefore, Lemma 5.3 shows that $\mathfrak{m}|_{\mathcal{H}_y}$ is projective for every vertex $y \in \overline{C}$, and Lemma 3.3 shows that $\mathfrak{m}|_{\mathcal{H}_{\mathcal{F}}}$ is projective for every facet $\mathcal{F} \subset \overline{C}$. Lemma 4.2 and Remark 3.1 then give the bound on the projective dimension.

When $G$ is semisimple, we have $r_Z = 0$, so that $\mathfrak{m}|_{\mathcal{H}^\dagger_{\mathcal{F}}}(\epsilon_{\mathcal{F}})$ is projective over $\mathcal{H}^\dagger_{\mathcal{F}}$ for every facet $\mathcal{F} \subset \overline{C}$. Since induction preserves projectivity, (1) is a projective resolution. Finally, [14, Cor. 6.12] gives the sharpness of the bound. $\square$

**Example 6.2.** We give an example to show that the condition $p \nmid |\Omega_{\mathrm{tor}}|$ in Proposition 6.1 is necessary. Assume that $F$ is a finite extension of $\mathbb{Q}_2$, and let $G = \mathrm{PGL}_2(F)$, so that $\Omega_{\mathrm{tor}} = \Omega \cong \mathbb{Z}/2\mathbb{Z}$. Let $\mathfrak{m}$ denote either the trivial or sign character of $\mathcal{H}$ (see (5) below). The action of $\mathcal{H}^\dagger_C \cong \overline{\mathbb{F}}_2[\widetilde{\Omega}]$ on $\mathfrak{m}$ factors through a block isomorphic to the algebra $\overline{\mathbb{F}}_2[\Omega] \cong \overline{\mathbb{F}}_2[X]/(X^2)$, and therefore the module $\mathfrak{m}|_{\mathcal{H}^\dagger_C}$ has infinite projective dimension (see Example 7.3 below). By Lemma 4.2, $\mathfrak{m}$ has infinite projective dimension over $\mathcal{H}$.

## 7. Supersingular modules

We again suppose $G$ is an arbitrary split connected reductive group, and turn our attention to the supersingular modules. We fix a set $\{\widehat{s_\alpha}\}_{s_\alpha \in S} \subset \widetilde{W}_{\mathrm{aff}}$ of lifts of elements of $S$ as in [14, §4.8]. The affine algebra $\mathcal{H}_{\mathrm{aff}}$ is then generated by the elements $\{\mathrm{T}_t\}_{t \in T(k_F)}$ and $\{\mathrm{T}_{\widehat{s_\alpha}}\}_{s_\alpha \in S}$, with quadratic relations given by

$$\mathrm{T}_{\widehat{s_\alpha}}^2 = \mathrm{T}_{\widehat{s_\alpha}} \left( \sum_{x \in k_F^\times} \mathrm{T}_{\alpha^\vee(x)} \right).$$

(If $\alpha$ is an affine root of the form $\alpha(\lambda) = \beta(\lambda) + k$ with $\beta \in \Phi, k \in \mathbb{Z}$, and $\lambda \in T$, we define $\alpha^\vee := \beta^\vee$.) Moreover, each irreducible component $\Pi'$ of $\Pi$ gives rise to a subset $S'$ of $S$ and a subalgebra of $\mathcal{H}_{\mathrm{aff}}$, generated by $\{\mathrm{T}_t\}_{t \in T(k_F)}$ and $\{\mathrm{T}_{\widehat{s_\alpha}}\}_{s_\alpha \in S'}$; these are called the *irreducible components of* $\mathcal{H}_{\mathrm{aff}}$ (see the discussion preceding [22, Lem. 6.8]).

Recall from [22, Prop. 2.2] that the characters of $\mathcal{H}_{\mathrm{aff}}$ are parametrized by pairs $(\xi, J)$, where $\xi : T(k_F) \longrightarrow \overline{\mathbb{F}}_p^\times$ is a character, and $J \subset S_\xi$, where

$$S_\xi := \left\{ s_\alpha \in S : \xi \circ \alpha^\vee(x) = 1 \text{ for all } x \in k_F^\times \right\}.$$

If $\chi$ is parametrized by $(\xi, J)$, then we have

$$\chi(\mathrm{T}_t) = \xi(t),$$
$$\chi(\mathrm{T}_{\widehat{s_\alpha}}) = \begin{cases} 0 & \text{if } s_\alpha \notin J, \\ -1 & \text{if } s_\alpha \in J. \end{cases}$$

In particular, the *trivial and sign characters of* $\mathcal{H}_{\mathrm{aff}}$ are parametrized by $(\mathbf{1}, \emptyset)$ and $(\mathbf{1}, S)$, respectively, where $\mathbf{1}$ denotes the trivial character of $T(k_F)$. These characters extend to characters of $\mathcal{H}$: they are given by

$$\mathrm{T}_{\widetilde{w}} \longmapsto q^{\ell(\widetilde{w})} \qquad \text{and} \qquad \mathrm{T}_{\widetilde{w}} \longmapsto (-1)^{\ell(\widetilde{w})}, \tag{5}$$

respectively (using the convention that $0^0 = 1$). Given any subalgebra $\mathcal{A}$ of $\mathcal{H}$ (e.g., $\mathcal{H}_{\mathcal{F}}$, $\mathcal{H}_{\mathcal{F}}^{\dagger}$, etc.), we define the *trivial and sign characters of* $\mathcal{A}$ to be the restrictions of the above characters to $\mathcal{A}$. Finally, if $\chi$ is a character of $\mathcal{H}_{\mathrm{aff}}$ parametrized by $(\xi, J)$ and $\xi' : T(k_F) \longrightarrow \overline{\mathbb{F}}_p^{\times}$ is a character such that $S_{\xi'} = S$, we define the *twist of* $\chi$ *by* $\xi'$ to be the character of $\mathcal{H}_{\mathrm{aff}}$ parametrized by $(\xi\xi', J)$.

There is a notion of supersingularity for characters of $\mathcal{H}_{\mathrm{aff}}$. We will not give the actual definition here, but merely point out that [22, Thm. 6.15] implies that a character $\chi$ of $\mathcal{H}_{\mathrm{aff}}$ is supersingular if and only if its restriction to each irreducible component of $\mathcal{H}_{\mathrm{aff}}$ is different from a twist of the trivial or sign character. When $G = T$ (so that $r_{\mathrm{ss}} = 0$), every character of $\mathcal{H}_{\mathrm{aff}}$ is supersingular.

Fix a supersingular character $\chi$. Let $\widetilde{\Omega}_{\chi}$ denote the subgroup of $\widetilde{\Omega}$ such that, for every $\widetilde{\omega} \in \widetilde{\Omega}_{\chi}$, the element $\mathrm{T}_{\widetilde{\omega}}$ fixes $\chi$ under conjugation (note that $\mathrm{T}_{\widetilde{\omega}}$ is invertible and preserves $\mathcal{H}_{\mathrm{aff}}$ under conjugation). We have $T(k_F) \subset \widetilde{\Omega}_{\chi}$, and $\widetilde{\Omega}_{\chi}$ is of finite index in $\widetilde{\Omega}$. We let $\mathcal{H}_{\mathrm{aff},\chi}$ denote the subalgebra of $\mathcal{H}$ generated by $\mathcal{H}_{\mathrm{aff}}$ and $\{\mathrm{T}_{\widetilde{\omega}}\}_{\widetilde{\omega} \in \widetilde{\Omega}_{\chi}}$; the braid relations imply that $\mathcal{H}_{\mathrm{aff},\chi}$ is free over $\mathcal{H}_{\mathrm{aff}}$, and $\mathcal{H}$ is free over $\mathcal{H}_{\mathrm{aff},\chi}$ (as either left or right modules).

Now let $\tau$ denote an irreducible finite-dimensional representation of $\widetilde{\Omega}_{\chi}$ (with $\widetilde{\Omega}_{\chi}$ acting on the *right*), such that the action of $t \in T(k_F)$ on $\tau$ is equal to the scalar $\chi(\mathrm{T}_t)$. We endow $\chi \otimes_{\overline{\mathbb{F}}_p} \tau$ with a right action of $\mathcal{H}_{\mathrm{aff},\chi}$: the element $\mathrm{T}_{\widetilde{w}} \in \mathcal{H}_{\mathrm{aff}}$ acts by the scalar $\chi(\mathrm{T}_{\widetilde{w}})$, while the elements $\mathrm{T}_{\widetilde{\omega}}$ for $\widetilde{\omega} \in \widetilde{\Omega}_{\chi}$ act via the representation $\tau$. Then every simple supersingular right $\mathcal{H}$-module is isomorphic to

$$\left(\chi \otimes_{\overline{\mathbb{F}}_p} \tau\right) \otimes_{\mathcal{H}_{\mathrm{aff},\chi}} \mathcal{H},$$

for a supersingular character $\chi$ of $\mathcal{H}_{\mathrm{aff}}$ and a finite-dimensional irreducible representation $\tau$ of $\widetilde{\Omega}_{\chi}$, which agrees with $\chi$ on $T(k_F)$ (see [22, Thm. 6.18]).

**Lemma 7.1.** *Let* $\chi$ *be a supersingular character of* $\mathcal{H}_{\mathrm{aff}}$*, and let* $\tau$ *be an irreducible finite-dimensional representation of* $\widetilde{\Omega}_{\chi}$ *whose restriction to* $T(k_F)$ *agrees with* $\chi$ *(in the sense above). Set* $\mathfrak{m} := (\chi \otimes_{\overline{\mathbb{F}}_p} \tau) \otimes_{\mathcal{H}_{\mathrm{aff},\chi}} \mathcal{H}$*. If* $\mathrm{pd}_{\mathcal{H}}(\mathfrak{m}) < \infty$*, then* $\mathrm{pd}_{\mathcal{H}_{\mathrm{aff}}}(\chi) < \infty$*.*

*Proof.* By the proof of [22, Prop. 6.17], we have

$$\mathfrak{m}|_{\mathcal{H}_{\mathrm{aff}}} \cong \left( \bigoplus_{\widetilde{\omega} \in \widetilde{\Omega}_\chi \backslash \widetilde{\Omega}} \chi^{\widetilde{\omega}} \right)^{\oplus \dim_{\overline{\mathbb{F}}_p}(\tau)},$$

where $\chi^{\widetilde{\omega}}$ denotes the character of $\mathcal{H}_{\mathrm{aff}}$ given by first conjugating an element by $\mathrm{T}_{\widetilde{\omega}}$ and then applying $\chi$. Since $\mathcal{H}$ is free over $\mathcal{H}_{\mathrm{aff}}$, the Eckmann–Shapiro lemma

$$\mathrm{Ext}^i_{\mathcal{H}}\left(\mathfrak{m}, \mathrm{Hom}_{\mathcal{H}}(\mathcal{H}_{\mathrm{aff}}, \mathfrak{n})\right) \cong \bigoplus_{\widetilde{\omega} \in \widetilde{\Omega}_\chi \backslash \widetilde{\Omega}} \mathrm{Ext}^i_{\mathcal{H}_{\mathrm{aff}}}(\chi^{\widetilde{\omega}}, \mathfrak{n})^{\oplus \dim_{\overline{\mathbb{F}}_p}(\tau)}$$

gives the claim. □

Our next goal will be to determine which supersingular characters $\chi$ of $\mathcal{H}_{\mathrm{aff}}$ have finite projective dimension. We require a bit of notation. Given a character $\xi : T(k_F) \longrightarrow \overline{\mathbb{F}}_p^\times$, we let $e_\xi \in \mathcal{H}$ denote the associated idempotent:

$$e_\xi := |T(k_F)|^{-1} \sum_{t \in T(k_F)} \xi(t) \mathrm{T}_{t^{-1}}. \tag{6}$$

For $w \in W$, we let $w.\xi$ denote the left action of $W$ on $\xi$ by conjugating the argument (note that the action factors through the projection $W \twoheadrightarrow W_0$), and let $e_{w.\xi}$ denote the corresponding idempotent.

We first consider several examples which will arise below.

**Example 7.2.** Let $\mathcal{A}$ denote the unital four-dimensional algebra over $\overline{\mathbb{F}}_p$ generated by two primitive orthogonal idempotents $e_1 \neq 0, 1$ and $e_2 := 1 - e_1$, and an element T which satisfies

$$e_1\mathrm{T} = \mathrm{T}e_2, \quad e_2\mathrm{T} = \mathrm{T}e_1, \quad \mathrm{T}^2 = 0.$$

The algebra $\mathcal{A}$ decomposes into principal indecomposable (right) modules as

$$\mathcal{A} = e_1\mathcal{A} \oplus e_2\mathcal{A},$$

and if we let $\sigma_j$ denote the (one-dimensional) socle of $e_j\mathcal{A}$, then $e_j\mathcal{A}$ is a nonsplit extension of $\sigma_{3-j}$ by $\sigma_j$. Splicing these two short exact sequences together gives an infinite projective resolution

$$\cdots \longrightarrow e_{3-j}\mathcal{A} \longrightarrow e_j\mathcal{A} \longrightarrow e_{3-j}\mathcal{A} \longrightarrow \sigma_j \longrightarrow 0,$$

which may be used to compute

$$\dim_{\overline{\mathbb{F}}_p}\left(\mathrm{Ext}^i_{\mathcal{A}}(\sigma_j,\sigma_j)\right) = \begin{cases} 1 & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd,} \end{cases}$$

$$\dim_{\overline{\mathbb{F}}_p}\left(\mathrm{Ext}^i_{\mathcal{A}}(\sigma_j,\sigma_{3-j})\right) = \begin{cases} 0 & \text{if } i \text{ is even,} \\ 1 & \text{if } i \text{ is odd.} \end{cases}$$

Therefore both $\sigma_1$ and $\sigma_2$ are of infinite projective dimension.

**Example 7.3.** Let $\mathcal{B}$ denote the two-dimensional algebra $\overline{\mathbb{F}}_p[X]/(X^2)$, and let $\sigma$ denote the (one-dimensional) socle of $\mathcal{B}$. Then $\sigma$ is the unique simple $\mathcal{B}$-module, and $\mathcal{B}$ is a nonsplit extension of $\sigma$ by $\sigma$. Splicing this short exact sequence with itself gives an infinite projective resolution

$$\cdots \longrightarrow \mathcal{B} \longrightarrow \mathcal{B} \longrightarrow \mathcal{B} \longrightarrow \sigma \longrightarrow 0,$$

which may be used to compute

$$\dim_{\overline{\mathbb{F}}_p}\left(\mathrm{Ext}^i_{\mathcal{B}}(\sigma,\sigma)\right) = 1 \text{ for all } i \geq 0.$$

Therefore $\sigma$ is of infinite projective dimension.

**Example 7.4.** Let $\mathcal{C}$ denote a 0-Hecke algebra over $\overline{\mathbb{F}}_p$ corresponding to an irreducible root system of rank 2 (that is, a 0-Hecke algebra of type $A_2$, $B_2$ or $G_2$). The algebra $\mathcal{C}$ is generated by two elements $\mathrm{T}$ and $\mathrm{T}'$, subject to the relations

$$\mathrm{T}^2 = -\mathrm{T},\ (\mathrm{T}')^2 = -\mathrm{T}',\ \underbrace{\mathrm{T}\mathrm{T}'\mathrm{T}\cdots}_{m \text{ times}} = \underbrace{\mathrm{T}'\mathrm{T}\mathrm{T}'\cdots}_{m \text{ times}},$$

where $m = 3, 4,$ or $6$. By [11], the algebra $\mathcal{C}$ has 4 simple modules, each one-dimensional, given by the characters

$$\mathrm{T} \longmapsto \varepsilon, \qquad \mathrm{T}' \longmapsto \varepsilon',$$

with $\varepsilon, \varepsilon' \in \{0, -1\}$. Moreover, Theorems 5.1 and 5.2 of *loc. cit.* imply that the only simple modules which are projective are the trivial and sign characters (i.e., corresponding to $(\varepsilon, \varepsilon') = (0,0)$ and $(-1,-1)$, respectively). Since

$\mathcal{C}$ is a Frobenius algebra, the remaining simple modules must have infinite projective dimension (cf. Remark 3.1).

**Lemma 7.5.** *Let $\chi$ be a supersingular character of $\mathcal{H}_{\mathrm{aff}}$, parametrized by the pair $(\xi, J)$. If $\mathrm{pd}_{\mathcal{H}_{\mathrm{aff}}}(\chi) < \infty$, then $S_\xi = S$.*

*Proof.* We may assume $S \neq \emptyset$, and suppose that $S_\xi \neq S$. Choose $s \in S \smallsetminus S_\xi$, and let $\mathcal{F}$ denote the codimension 1 facet of $C$ which is fixed by $s$. Consider first the case $s.\xi \neq \xi$. The action of $\mathcal{H}_{\mathcal{F}}$ on $\chi$ factors through the block $(e_\xi + e_{s.\xi})\mathcal{H}_{\mathcal{F}}$, and the latter algebra is of the form considered in Example 7.2 (take $e_1 = e_\xi, e_2 = e_{s.\xi}, \mathrm{T} = (e_\xi + e_{s.\xi})\mathrm{T}_{\widehat{s}}$). Since $(e_\xi + e_{s.\xi})\mathcal{H}_{\mathcal{F}}$ is a block of $\mathcal{H}_{\mathcal{F}}$, we obtain

$$\mathrm{Ext}^i_{\mathcal{H}_{\mathcal{F}}}(\chi|_{\mathcal{H}_{\mathcal{F}}}, \chi|_{\mathcal{H}_{\mathcal{F}}}) = \mathrm{Ext}^i_{(e_\xi+e_{s.\xi})\mathcal{H}_{\mathcal{F}}}\left(\chi|_{(e_\xi+e_{s.\xi})\mathcal{H}_{\mathcal{F}}}, \chi|_{(e_\xi+e_{s.\xi})\mathcal{H}_{\mathcal{F}}}\right) \neq 0$$

for infinitely many $i$. Therefore $\mathrm{pd}_{\mathcal{H}_{\mathcal{F}}}(\chi|_{\mathcal{H}_{\mathcal{F}}}) = \infty$, and Lemma 4.2 gives $\mathrm{pd}_{\mathcal{H}_{\mathrm{aff}}}(\chi) = \infty$.

Assume now that $s.\xi = \xi$. Then the action of $\mathcal{H}_{\mathcal{F}}$ factors through the block $e_\xi\mathcal{H}_{\mathcal{F}}$, and the latter algebra is of the form considered in Example 7.3 (take $e_\xi$ as the unit and $X = e_\xi\mathrm{T}_{\widehat{s}}$). Once again, $e_\xi\mathcal{H}_{\mathcal{F}}$ is a block of $\mathcal{H}_{\mathcal{F}}$, and

$$\mathrm{Ext}^i_{\mathcal{H}_{\mathcal{F}}}(\chi|_{\mathcal{H}_{\mathcal{F}}}, \chi|_{\mathcal{H}_{\mathcal{F}}}) = \mathrm{Ext}^i_{e_\xi\mathcal{H}_{\mathcal{F}}}\left(\chi|_{e_\xi\mathcal{H}_{\mathcal{F}}}, \chi|_{e_\xi\mathcal{H}_{\mathcal{F}}}\right) \neq 0$$

for infinitely many $i$. As above, we conclude that $\mathrm{pd}_{\mathcal{H}_{\mathrm{aff}}}(\chi) = \infty$. □

**Lemma 7.6.** *Let $\chi$ be a supersingular character of $\mathcal{H}_{\mathrm{aff}}$, parametrized by the pair $(\xi, J)$, and assume $S_\xi = S$. If $\mathrm{pd}_{\mathcal{H}_{\mathrm{aff}}}(\chi) < \infty$, then the root system of $G$ is of type $A_1 \times \cdots \times A_1$ (possibly empty product).*

*Proof.* Assume that the root system of $G$ is not of type $A_1 \times \cdots \times A_1$ (so that, in particular, the semisimple rank of $G$ is at least 2). Then there exists an irreducible component of the affine Dynkin diagram of $G$ which is not of type $\widetilde{A}_1$. The character $\chi$ corresponds to a labeling of the vertices of this component, with the vertex corresponding to the simple affine root $\alpha$ being labeled by $\chi(\mathrm{T}_{\widehat{s_\alpha}}) \in \{0, -1\}$. Since $\chi$ is supersingular and $S_\xi = S$, there must exist two adjacent vertices with distinct labels. Let $\mathcal{F}$ denote the codimension 2 facet of $C$ which is fixed by the simple reflections corresponding to these two vertices. By assumption, the action of $\mathcal{H}_{\mathcal{F}}$ on $\chi$ factors through the algebra $e_\xi\mathcal{H}_{\mathcal{F}}$, and the latter algebra is of the form considered in Example 7.4 (this requires the assumption that the chosen irreducible component is not of type $\widetilde{A}_1$). By construction of the facet $\mathcal{F}$, $\chi|_{e_\xi\mathcal{H}_{\mathcal{F}}}$ is neither the trivial nor the sign character, and since $e_\xi\mathcal{H}_{\mathcal{F}}$ is a product of blocks of $\mathcal{H}_{\mathcal{F}}$, we obtain $\mathrm{pd}_{\mathcal{H}_{\mathcal{F}}}(\chi|_{\mathcal{H}_{\mathcal{F}}}) = \infty$. Lemma 4.2 then gives $\mathrm{pd}_{\mathcal{H}_{\mathrm{aff}}}(\chi) = \infty$. □

We may now determine when a supersingular module has finite projective dimension.

**Theorem 7.7.** *Let $\mathfrak{m} = (\chi \otimes_{\overline{\mathbb{F}}_p} \tau) \otimes_{\mathcal{H}_{\mathrm{aff},\chi}} \mathcal{H}$ be a simple supersingular $\mathcal{H}$-module, and assume $\chi$ is parametrized by $(\xi, J)$. Then $\mathrm{pd}_{\mathcal{H}}(\mathfrak{m}) < \infty$ if and only if the root system of $G$ is of type $A_1 \times \cdots \times A_1$ (possibly empty product) and $S_\xi = S$.*

Note that when the product $A_1 \times \cdots \times A_1$ is empty, we have $G = T$ and the condition $S_\xi = S$ is vacuous.

*Proof.* Lemmas 7.1, 7.5, and 7.6 give the desired conditions on $G$ and $S_\xi$. We prove the other implication.

Assume that the root system of $G$ is of type $A_1 \times \cdots \times A_1$ and $S_\xi = S$. Given any facet $\mathcal{F} \in \mathscr{F}_i^{\mathrm{aff}}$, the second assumption implies that the action of $\mathcal{H}_{\mathcal{F}}$ on $\chi$ factors through $e_\xi\mathcal{H}_{\mathcal{F}}$, which is a 0-Hecke algebra of type $A_1 \times \cdots \times A_1$ and is therefore semisimple. By Lemma 4.2, we conclude that $\mathrm{pd}_{\mathcal{H}_{\mathrm{aff}}}(\chi) < \infty$.

We claim that $\widetilde{\Omega}_\chi$ is the subgroup generated by $T(k_F)$ and (the image of) $Z/(Z \cap I(1))$. Note first that the set $S$ admits a partition into commuting subsets

$$S = \bigsqcup_{\alpha\in\Pi} \left\{s_\alpha,\ s_\alpha\alpha^\vee(\varpi) =: s'_\alpha\right\},$$

where $\varpi$ is a fixed choice of uniformizer of $F$. The group $\Omega$ acts on $S$ by conjugation, and writing an element of $\Omega$ with respect to the decomposition $W = W_0 \ltimes \Lambda$ shows that $\Omega$ acts on each set $\{s_\alpha,\ s'_\alpha\}$. Next, by definition of supersingularity, the values $\chi(\mathrm{T}_{\widehat{s_\alpha}}), \chi(\mathrm{T}_{\widehat{s'_\alpha}}) \in \{0, -1\}$ are distinct for every choice of $\alpha \in \Pi$. Therefore, if $\widetilde{\omega} \in \widetilde{\Omega}_\chi$, we have

$$\chi(\mathrm{T}_{\widehat{s_\alpha}}) = \chi^{\widetilde{\omega}}(\mathrm{T}_{\widehat{s_\alpha}}) = \chi(\mathrm{T}_{\widetilde{\omega}\widehat{s_\alpha}\widetilde{\omega}^{-1}}) \quad \text{and} \quad \chi(\mathrm{T}_{\widehat{s'_\alpha}}) = \chi^{\widetilde{\omega}}(\mathrm{T}_{\widehat{s'_\alpha}}) = \chi(\mathrm{T}_{\widetilde{\omega}\widehat{s'_\alpha}\widetilde{\omega}^{-1}})$$

for all $\alpha \in \Pi$, which implies $\widetilde{\omega}\widehat{s_\alpha}\widetilde{\omega}^{-1} = \widehat{s_\alpha}t_\alpha$ and $\widetilde{\omega}\widehat{s'_\alpha}\widetilde{\omega}^{-1} = \widehat{s'_\alpha}t'_\alpha$ for some $t_\alpha, t'_\alpha \in T(k_F)$ by the two comments above. Letting $\omega \in \Omega$ denote the image of $\widetilde{\omega}$ under projection, the previous sentence implies that $\omega$ commutes with all of $W_{\mathrm{aff}}$, and therefore must lie in $Z/(Z \cap I)$. This gives the claim.

By choosing a splitting of

$$1 \longrightarrow T(k_F) \longrightarrow \widetilde{\Omega}_\chi \longrightarrow Z/(Z\cap I) \cong \mathbb{Z}^{\oplus r_Z} \longrightarrow 1,$$

we see that $\mathcal{H}_{\mathrm{aff},\chi}$ is a Laurent polynomial algebra over $\mathcal{H}_{\mathrm{aff}}$, with basis given by the elements $\{\mathrm{T}_{\widehat{z}}\}$, where $\widehat{z}$ is in the image of the splitting. [10, Prop. 7.5.2] now implies

$$\begin{aligned}\mathrm{pd}_{\mathcal{H}_{\mathrm{aff},\chi}}\left(\chi\otimes_{\overline{\mathbb{F}}_p}\tau\right) &\leq \mathrm{pd}_{\mathcal{H}_{\mathrm{aff}}}\left((\chi\otimes_{\overline{\mathbb{F}}_p}\tau)|_{\mathcal{H}_{\mathrm{aff}}}\right) + r_Z\\ &= \mathrm{pd}_{\mathcal{H}_{\mathrm{aff}}}\left(\chi^{\oplus\dim_{\overline{\mathbb{F}}_p}(\tau)}\right) + r_Z < \infty.\end{aligned}$$

Finally, by the Eckmann–Shapiro lemma, we get

$$\mathrm{Ext}^i_{\mathcal{H}}(\mathfrak{m},\mathfrak{n}) \cong \mathrm{Ext}^i_{\mathcal{H}_{\mathrm{aff},\chi}}\left(\chi\otimes_{\overline{\mathbb{F}}_p}\tau, \mathfrak{n}|_{\mathcal{H}_{\mathrm{aff},\chi}}\right),$$

which shows that $\mathrm{pd}_{\mathcal{H}}(\mathfrak{m}) < \infty$. □

## 8. Parabolic coinduction

We first show how to transfer information about projective dimension using parabolic coinduction.

**Lemma 8.1.** *Let $P = M \ltimes N$ denote a standard parabolic subgroup of $G$, and let $\mathfrak{n}$ be a right $\mathcal{H}_M$-module. Then we have $\mathrm{pd}_{\mathcal{H}_M}(\mathfrak{n}) < \infty$ if and only if $\mathrm{pd}_{\mathcal{H}}(I_M(\mathfrak{n})) < \infty$.*

*Proof.* Remark 3.1 implies that it suffices to prove the claim for injective dimensions. By [20, Prop. 4.1] the parabolic coinduction functor $I_M(-)$ from the category of right $\mathcal{H}_M$-modules to the category of right $\mathcal{H}$-modules admits a left adjoint, which we denote by $L_M(-)$ (N.B.: in *loc. cit.*, the functors $I_M(-)$ and $L_M(-)$ are denoted $\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M}(-)$ and $\mathbb{L}^{\mathcal{H}}_{\mathcal{H}_M}(-)$, respectively). Moreover, Sections 3 and 4 of *loc. cit.* show that $L_M(-)$ is given by localization at a central non-zero divisor of $\mathcal{H}^-_M$, and is therefore exact. Hence, we obtain

$$\mathrm{Ext}^i_{\mathcal{H}}(\mathfrak{m}, I_M(\mathfrak{n})) \cong \mathrm{Ext}^i_{\mathcal{H}_M}(L_M(\mathfrak{m}),\mathfrak{n}),$$

which shows that $\mathrm{id}_{\mathcal{H}_M}(\mathfrak{n}) < \infty$ implies $\mathrm{id}_{\mathcal{H}}(I_M(\mathfrak{n})) < \infty$.

Now let $\mathfrak{v}$ be an arbitrary $\mathcal{H}_M$-module. Using the explicit description of the functors $I_M(-)$ and $L_M(-)$, we see that $L_M(I_M(\mathfrak{v})) \cong \mathfrak{v}$ (see also [2, Prop. 4.12]). This gives

$$\mathrm{Ext}^i_{\mathcal{H}}\left(I_M(\mathfrak{v}), I_M(\mathfrak{n})\right) \cong \mathrm{Ext}^i_{\mathcal{H}_M}\left(\mathfrak{v}, \mathfrak{n}\right),$$

which shows that $\mathrm{id}_{\mathcal{H}}(I_M(\mathfrak{n})) < \infty$ implies $\mathrm{id}_{\mathcal{H}_M}(\mathfrak{n}) < \infty$. □

**Remark 8.2.** Let $\psi : T \longrightarrow \overline{\mathbb{F}}_p^\times$ be a smooth character of $T$, and use the same letter to denote the associated right $\mathcal{H}_T$-module. Then $\psi$ is naturally a module over the block $e_{\psi|_{T(k_F)}}\mathcal{H}_T \cong \overline{\mathbb{F}}_p[X_1^{\pm 1}, \ldots, X_{r_{\mathrm{ss}}+r_Z}^{\pm 1}]$. This algebra has global dimension $r_{\mathrm{ss}} + r_Z$ ([10, Thm. 7.5.3(iv)]), and the above lemma implies that $\mathrm{pd}_{\mathcal{H}}(I_T(\psi)) < \infty$. By using a Koszul resolution of the $\mathcal{H}_T$-module $\psi$, we easily obtain

$$\dim_{\overline{\mathbb{F}}_p}\left(\mathrm{Ext}^i_{\mathcal{H}_T}(\psi, \psi)\right) = \binom{r_{\mathrm{ss}} + r_Z}{i},$$

and therefore the proof above shows

$$\mathrm{Ext}^{r_{\mathrm{ss}}+r_Z}_{\mathcal{H}}\left(I_T(\psi), I_T(\psi)\right) \neq 0.$$

Hence $r_{\mathrm{ss}} + r_Z \leq \mathrm{pd}_{\mathcal{H}}(I_T(\psi)) < \infty$. Using Remark 3.1, this shows that the bound on the self-injective dimension of $\mathcal{H}$ obtained in [14] is sharp, i.e., $\mathrm{id}_{\mathcal{H}}(\mathcal{H}) = r_{\mathrm{ss}} + r_Z$.

Before we proceed, we require a simple lemma.

**Lemma 8.3.** *Let $M$ be a standard Levi subgroup of $G$, and let $\Omega_M \cong \Omega_{M,\mathrm{tor}} \times \Omega_{M,\mathrm{free}}$ denote the length 0 subgroup of $W_M$ (relative to the length function on $W_M$). If $p \nmid |\Omega_{\mathrm{tor}}|$, then $p \nmid |\Omega_{M,\mathrm{tor}}|$.*

*Proof.* By [5, §1.1 and Prop. 1.10] (see also [21, Prop. 3.36]), the group $\Omega$ is isomorphic to the quotient of $X_*(T)$ by the subgroup generated by the coroots of $G$ (and likewise for $\Omega_M$). Therefore, we have a surjection $\Omega_M \twoheadrightarrow \Omega$, which is easily seen to be injective when restricted to $\Omega_{M,\mathrm{tor}}$. The result follows. □

**Proposition 8.4.** *Assume $p \nmid |\Omega_{\mathrm{tor}}|$. Let $P = M \ltimes N$ be a standard parabolic subgroup of $G$, and let $\mathfrak{n}$ be a simple supersingular right $\mathcal{H}_M$-module. Let $Q = L \ltimes V$ denote another standard parabolic subgroup such that $P \subset Q \subset P(\mathfrak{n})$. Then $\mathrm{pd}_{\mathcal{H}_M}(\mathfrak{n}) < \infty$ if and only if $\mathrm{pd}_{\mathcal{H}_L}(\mathfrak{n}^{e_L}) < \infty$.*

*Proof.* By Lemma 8.3, we may suppose that $Q = G$. The set $\Pi$ then admits an orthogonal decomposition $\Pi = \Pi_M \sqcup \Pi_2$, where $\langle \beta, \alpha^\vee \rangle = 0$ for every $\beta \in \Pi_M, \alpha \in \Pi_2$, and such that $\mathrm{T}^M_{\alpha^\vee(x)}$ acts trivially on $\mathfrak{n}$ for every $x \in F^\times, \alpha \in \Pi_2$. Moreover, we obtain a partition $S = S_M \sqcup S_2$ on the set of affine reflections (cf. [2, §4.4]). Since this partition comes from the orthogonal decomposition of $\Pi$, the elements of $S_M$ commute with the elements of $S_2$.

Assume first that $\mathrm{pd}_{\mathcal{H}_M}(\mathfrak{n}) = \infty$. By Lemmas 4.2 and 8.3, there must exist some facet $\mathcal{F}_M$ in the semisimple Bruhat–Tits building of $M$ (in the closure of the chamber corresponding to $I \cap M$) such that $\mathfrak{n}|_{\mathcal{H}_{M,\mathcal{F}_M}}$ is not projective. Let $S_{\mathcal{F}_M} \subset S_M$ denote the set of simple reflections which fix $\mathcal{F}_M$ pointwise, so that $\mathcal{H}_{M,\mathcal{F}_M}$ is generated by $\{\mathrm{T}^M_{\widehat{s}}\}_{s \in S_{\mathcal{F}_M}}$ and $\{\mathrm{T}^M_t\}_{t \in T(k_F)}$. Now view $S_{\mathcal{F}_M}$ as a subset of $S$, and let $\mathcal{F}$ denote the facet of $C$ fixed pointwise by $S_{\mathcal{F}_M}$. The algebra $\mathcal{H}_{\mathcal{F}}$ is generated by $\{\mathrm{T}_{\widehat{s}}\}_{s \in S_{\mathcal{F}_M}}$ and $\{\mathrm{T}_t\}_{t \in T(k_F)}$. One easily checks that the elements $\widehat{s}$ for $s \in S_{\mathcal{F}_M}$ are all $M$-negative (see [2, §4.1]; this uses the fact that we have an *orthogonal* decomposition). Therefore, we have $\mathcal{H}_{M,\mathcal{F}_M} \subset \mathcal{H}_M^-$, and the map $j_M^- : \mathcal{H}_M^- \hookrightarrow \mathcal{H}$ induces an algebra isomorphism $\mathcal{H}_{M,\mathcal{F}_M} \xrightarrow{\sim} \mathcal{H}_{\mathcal{F}}$. By [2, Prop. 4.16], the algebra $\mathcal{H}_{\mathcal{F}}$ acts on $\mathfrak{n}^{e_G}$ through the isomorphism $j_M^-$, and we conclude that $\mathfrak{n}^{e_G}|_{\mathcal{H}_{\mathcal{F}}}$ is not projective. Hence $\mathrm{pd}_{\mathcal{H}}(\mathfrak{n}^{e_G}) = \infty$.

Assume now that $\mathrm{pd}_{\mathcal{H}_M}(\mathfrak{n}) < \infty$, and write $\mathfrak{n} = (\chi \otimes_{\overline{\mathbb{F}}_p} \tau) \otimes_{\mathcal{H}_{M,\mathrm{aff},\chi}} \mathcal{H}_M$ as in Section 7. By Theorem 7.7, we have $S_{M,\xi} = S_M$, where $(\xi, J)$ parametrizes the character $\chi$ of $\mathcal{H}_{M,\mathrm{aff}}$. Since $\mathfrak{n}$ extends to $\mathcal{H}$ by assumption, we have $S_\xi = S$ (cf. [2, Prop. 4.16]).

Fix a facet $\mathcal{F} \subset \overline{C}$, and let $S_{\mathcal{F}} \subset S$ denote the set of simple reflections which fix $\mathcal{F}$ pointwise. By definition of the extension $\mathfrak{n}^{e_G}$, the action of $\mathcal{H}_{\mathcal{F}}$ on $\mathfrak{n}^{e_G}$ factors through $e_\xi \mathcal{H}_{\mathcal{F}}$. The latter algebra decomposes as a tensor product

$$e_\xi \mathcal{H}_{\mathcal{F}} \cong \mathcal{H}_1 \otimes_{\overline{\mathbb{F}}_p} \mathcal{H}_2,$$

where $\mathcal{H}_1$ is the algebra generated by $\{e_\xi \mathrm{T}_{\widehat{s}}\}_{s \in S_M \cap S_{\mathcal{F}}}$ and $\mathcal{H}_2$ is the algebra generated by $\{e_\xi \mathrm{T}_{\widehat{s}}\}_{s \in S_2 \cap S_{\mathcal{F}}}$. As above, $\mathcal{H}_1$ is isomorphic via $j_M^-$ to a finite Hecke algebra of the form $e_\xi \mathcal{H}_{M,\mathcal{F}_M}$, where $\mathcal{F}_M$ is the facet in the semisimple Bruhat–Tits building of $M$ fixed by $S_M \cap S_{\mathcal{F}}$. Again using [2, Prop. 4.16], the $e_\xi \mathcal{H}_{\mathcal{F}}$-module $\mathfrak{n}^{e_G}|_{e_\xi \mathcal{H}_{\mathcal{F}}}$ decomposes as a(n external) tensor product

$$\mathfrak{n}^{e_G}|_{e_\xi \mathcal{H}_{\mathcal{F}}} \cong \mathfrak{n}|_{e_\xi \mathcal{H}_{M,\mathcal{F}_M}} \otimes_{\overline{\mathbb{F}}_p} \chi_{\mathrm{triv},2},$$

where $\chi_{\mathrm{triv},2}$ denotes the trivial character of $\mathcal{H}_2$. Since $\mathrm{pd}_{\mathcal{H}_M}(\mathfrak{n}) < \infty$, the restriction of $\mathfrak{n}$ to $\mathcal{H}_{M,\mathcal{F}_M}$ must be projective (this again uses Lemmas 4.2 and 8.3). Since $\chi_{\mathrm{triv},2}$ is projective over $\mathcal{H}_2$ (cf. [11, Thm. 5.2]), we see that $\mathfrak{n}^{e_G}|_{\mathcal{H}_\mathcal{F}}$ is projective. Using Lemma 4.2 one final time gives $\mathrm{pd}_{\mathcal{H}}(\mathfrak{n}^{e_G}) < \infty$. □

We now arrive at our main result.

**Theorem 8.5.** *Assume $p \nmid |\Omega_{\mathrm{tor}}|$. Let $P = M \ltimes N$ be a standard parabolic subgroup of $G$, and let $\mathfrak{n}$ be a simple supersingular right $\mathcal{H}_M$-module. Let $Q = L \ltimes V$ denote another standard parabolic subgroup such that $P \subset Q \subset P(\mathfrak{n})$. Then $\mathrm{pd}_{\mathcal{H}_M}(\mathfrak{n}) < \infty$ if and only if $\mathrm{pd}_{\mathcal{H}}(I(P, \mathfrak{n}, Q)) < \infty$.*

*Proof.* Since parabolic coinduction is exact and transitive (cf. [20, Cor. 1.10]), we have

$$I(P, \mathfrak{n}, Q) = I_{M(\mathfrak{n})}\left(I_{\mathcal{H}_{M(\mathfrak{n})}}(P \cap M(\mathfrak{n}), \mathfrak{n}, Q \cap M(\mathfrak{n}))\right),$$

where $I_{\mathcal{H}_{M(\mathfrak{n})}}(P \cap M(\mathfrak{n}), \mathfrak{n}, Q \cap M(\mathfrak{n}))$ is a simple $\mathcal{H}_{M(\mathfrak{n})}$-module defined in the same manner as $I(P, \mathfrak{n}, Q)$. Therefore, by Lemmas 8.1 and 8.3 we may assume $P(\mathfrak{n}) = G$, so that $\Pi$ admits an orthogonal decomposition $\Pi = \Pi_M \sqcup \Pi_2$ and $S$ admits a partition into commuting subsets $S = S_M \sqcup S_2$ (as in Proposition 8.4).

Assume first that $\mathrm{pd}_{\mathcal{H}_M}(\mathfrak{n}) < \infty$. By Lemma 8.1 and the proposition above, we have $\mathrm{pd}_{\mathcal{H}}(I_{L'}(\mathfrak{n}^{e_{L'}})) < \infty$ for every parabolic subgroup $Q' = L' \ltimes V'$ such that $P \subset Q' \subset G$. Moreover, the Čech resolution of Lemma 5.1 gives rise to a hyper-Ext spectral sequence

$$E_1^{i,j} = \bigoplus_{\substack{L \subset L' \subset G \\ |\Pi_{L'} \smallsetminus \Pi_L| = i}} \mathrm{Ext}^j_{\mathcal{H}}\left(I_{L'}(\mathfrak{n}^{e_{L'}}), \mathfrak{v}\right) \Longrightarrow \mathrm{Ext}^{i+j}_{\mathcal{H}}\left(I(P, \mathfrak{n}, Q), \mathfrak{v}\right).$$

Since $E_1^{i,j}$ vanishes for $i$ and $j$ sufficiently large (independent of $\mathfrak{v}$), we conclude $\mathrm{pd}_{\mathcal{H}}(I(P, \mathfrak{n}, Q)) < \infty$.

Fix now a parabolic subgroup $Q' = L' \ltimes V'$ such that $P \subset Q' \subset G$ and consider the coinduced module $I_{L'}(\mathfrak{n}^{e_{L'}}) = \mathrm{Hom}_{\mathcal{H}^-_{L'}}(\mathcal{H}, \mathfrak{n}^{e_{L'}})$. By [2, Lem. 4.10], we have an isomorphism of vector spaces

$$\begin{aligned} I_{L'}(\mathfrak{n}^{e_{L'}}) &\xrightarrow{\sim} \bigoplus_{v \in W_0^{L'}} \mathfrak{n} \\ \varphi &\longmapsto (\varphi(\mathrm{T}_{\widehat{v}}))_{v \in W_0^{L'}}. \end{aligned}$$

In the above, if $v = s_{\alpha_1} \cdots s_{\alpha_k}$ is a reduced expression for $v \in W_0$ with $s_{\alpha_i} \in S \cap W_0$, we define $\widehat{v} := \widehat{s_{\alpha_1}} \cdots \widehat{s_{\alpha_k}}$; [19, Props. 8.8.3 and 9.3.2] imply this element is well-defined.

The orthogonal decomposition of $\Pi$ implies that $W_0$ is a product of two finite Weyl groups $W_0 \cong W_{M,0} \times W_{2,0}$ (corresponding to the sets of generators $S_M \cap W_0$ and $S_2 \cap W_0$, respectively). This easily implies that the set $W_0^{L'}$ is contained in $W_{2,0}$, and therefore commutes with $S_M$. Using [19, Prop. 9.3.2] again, we see that in fact the elements $\widehat{v}$ and $\widehat{s}$ commute, where $v \in W_0^{L'}$ and $s \in S_M$. This implies that the isomorphism above is equivariant for the operators $\{\mathrm{T}_{\widehat{s}}\}_{s\in S_M}$, where $\mathrm{T}_{\widehat{s}}$ acts on the right-hand side as $\mathrm{T}_{\widehat{s}}^M$ via $j_M^-$. Moreover, by the fact that $\mathrm{T}_{\alpha^\vee(x)}^M$ acts trivially on $\mathfrak{n}$ for all $\alpha \in \Pi_2$ and $x \in k_F^\times$, we see that the actions of $\mathrm{T}_t$ on the left-hand side and $\mathrm{T}_t^M$ on the right-hand side agree, so that the isomorphism is equivariant for the operators $\{\mathrm{T}_t\}_{t\in T(k_F)}$.

Assume now that $\mathrm{pd}_{\mathcal{H}_M}(\mathfrak{n}) = \infty$. As in the proof of Proposition 8.4, there exist facets $\mathcal{F}_M$ in the semisimple Bruhat–Tits building of $M$ and $\mathcal{F} \subset \overline{C}$ such that $\mathfrak{n}|_{\mathcal{H}_{M,\mathcal{F}_M}}$ is not projective and such that $j_M^-$ induces an algebra isomorphism $\mathcal{H}_{M,\mathcal{F}_M} \xrightarrow{\sim} \mathcal{H}_{\mathcal{F}}$. By the discussion of the preceding paragraph, the isomorphism $I_{L'}(\mathfrak{n}^{e_{L'}}) \cong \bigoplus_{v\in W_0^{L'}} \mathfrak{n}$ is an isomorphism of $\mathcal{H}_{\mathcal{F}}$-modules, where $\mathcal{H}_{\mathcal{F}}$ acts on the right-hand side as $\mathcal{H}_{M,\mathcal{F}_M}$ via $j_M^-$. Consequently, we see that $I(P,\mathfrak{n},Q)|_{\mathcal{H}_{\mathcal{F}}}$ is isomorphic to a direct sum of copies of $\mathfrak{n}$ (where we again view $\mathfrak{n}$ as a $\mathcal{H}_{\mathcal{F}}$-module via $j_M^-$), and so $I(P,\mathfrak{n},Q)|_{\mathcal{H}_{\mathcal{F}}}$ is not projective. By Lemma 4.2, we conclude that $\mathrm{pd}_{\mathcal{H}}(I(P,\mathfrak{n},Q)) = \infty$. □

**Corollary 8.6.** *Assume $p \nmid |\Omega_{\mathrm{tor}}|$, and let $\mathfrak{m}$ be a simple right $\mathcal{H}$-module. Write $\mathfrak{m} \cong I(P,\mathfrak{n},Q)$, with $\mathfrak{n} \cong (\chi \otimes_{\overline{\mathbb{F}}_p} \tau) \otimes_{\mathcal{H}_{M,\mathrm{aff},\chi}} \mathcal{H}_M$ and $\chi$ parametrized by $(\xi, J)$. Then the following are equivalent:*

- $\mathrm{pd}_{\mathcal{H}}(\mathfrak{m}) < \infty$*;*
- $\mathrm{pd}_{\mathcal{H}_M}(\mathfrak{n}) < \infty$*;*
- *the root system of $M$ is of type $A_1 \times \cdots \times A_1$ (possibly empty product) and $S_{M,\xi} = S_M$.*

*Moreover, when $G$ is semisimple and* $\mathrm{pd}_{\mathcal{H}}(\mathfrak{m}) < \infty$*, (1) is a projective resolution of $\mathfrak{m}$, and* $\mathrm{pd}_{\mathcal{H}}(\mathfrak{m}) = r_{\mathrm{ss}}$*.*

*Proof.* The first assertion follows from combining Theorems 8.5 and 7.7. Assuming that $G$ is semisimple gives $r_Z = 0$, and Lemma 4.2 shows that $\mathrm{pd}_{\mathcal{H}}(\mathfrak{m}) < \infty$ is equivalent to $\mathfrak{m}|_{\mathcal{H}_{\mathcal{F}}^\dagger}(\epsilon_{\mathcal{F}})$ being projective for every facet $\mathcal{F} \subset \overline{C}$. Since induction preserves projectivity, we get that (1) is a projective resolution, and [14, Cor. 6.12] gives the exact value of $\mathrm{pd}_{\mathcal{H}}(\mathfrak{m})$. □

**Remark 8.7.** Since simple $\mathcal{H}$-modules are finite-dimensional, they possess a central character. Using this fact and a slightly stronger restriction on $p$ than in the corollary above, we can actually prove that $\mathrm{pd}_{\mathcal{H}}(\mathfrak{m}) = r_{\mathrm{ss}} + r_Z$ whenever $\mathrm{pd}_{\mathcal{H}}(\mathfrak{m}) < \infty$; see Proposition 9.5 below.

**Remark 8.8.** Theorem 8.5 gives a slightly different proof of Proposition 6.1. We have chosen to keep the proof of Proposition 6.1 intact, in the hopes that the techniques used therein (especially Lemma 3.6) may find application elsewhere.

## 9. Complements

### 9.1. Iwahori–Hecke modules

As a special case of the above results, we now discuss projective dimensions of Iwahori–Hecke modules.

Let $\mathcal{H}'$ denote the *Iwahori–Hecke algebra* over $\overline{\mathbb{F}}_p$, defined with respect to the subgroup $I$ of $G$:

$$\mathcal{H}' := \mathrm{End}_G\left(\text{c-ind}_I^G(\mathbf{1})\right),$$

where $\mathbf{1}$ now denotes the trivial $I$-module over $\overline{\mathbb{F}}_p$. Given an algebra related to $\mathcal{H}$, we denote with a prime the analogously defined algebra for $\mathcal{H}'$ (so that $\mathcal{H}'_M$ denotes the Iwahori–Hecke algebra of a Levi subgroup $M$ with respect to $I \cap M$, $\mathcal{H}'_{\mathcal{F}}$ is the subalgebra of $\mathcal{H}'$ defined by $\mathrm{End}_{\mathcal{P}_{\mathcal{F}}}(\text{c-ind}_I^{\mathcal{P}_{\mathcal{F}}}(\mathbf{1}))$, etc.). Recall the (primitive) central idempotent $e_{\mathbf{1}} \in \mathcal{H}$ defined in equation (6):

$$e_{\mathbf{1}} = |T(k_F)|^{-1} \sum_{t \in T(k_F)} \mathrm{T}_t.$$

Using $e_{\mathbf{1}}$ we may identify $\mathcal{H}'$ with the subalgebra $e_{\mathbf{1}}\mathcal{H}$ (which is a block of $\mathcal{H}$). Likewise, we make the identifications $\mathcal{H}'_M = e_{\mathbf{1}}\mathcal{H}_M, \mathcal{H}'_{\mathcal{F}} = e_{\mathbf{1}}\mathcal{H}_{\mathcal{F}}$, etc.. See [21] for more details.

Now let $\mathfrak{m}$ denote a right $\mathcal{H}$-module. If $\mathfrak{m} \cdot e_{\mathbf{1}} = \mathfrak{m}$, then we may naturally view $\mathfrak{m}$ as an $\mathcal{H}'$-module, and conversely every $\mathcal{H}'$-module arises in this way. In particular, let $P = M \ltimes N$ be a standard parabolic subgroup, and $\mathfrak{n} := (\chi \otimes_{\overline{\mathbb{F}}_p} \tau) \otimes_{\mathcal{H}_{M,\mathrm{aff},\chi}} \mathcal{H}_M$ a simple supersingular right $\mathcal{H}_M$-module. One easily sees that the simple right $\mathcal{H}$-module $\mathfrak{m} := I(P, \mathfrak{n}, Q)$ satisfies $\mathfrak{m} \cdot e_{\mathbf{1}} = \mathfrak{m}$ if and only if $\mathfrak{n} \cdot e_{\mathbf{1}} = \mathfrak{n}$, if and only if $\chi$ is parametrized by $(\mathbf{1}, J)$ for some $J \subset S_M$. Corollary 8.6 now takes the following form, noting that $\mathrm{pd}_{\mathcal{H}}(\mathfrak{m}) < \infty$ is equivalent to $\mathrm{pd}_{\mathcal{H}'}(\mathfrak{m}) < \infty$ for right $\mathcal{H}$-modules satisfying $\mathfrak{m} \cdot e_{\mathbf{1}} = \mathfrak{m}$.

**Corollary 9.1.** *Assume* $p \nmid |\Omega_{\mathrm{tor}}|$, *and let* $\mathfrak{m}$ *be a simple right* $\mathcal{H}'$*-module. Write* $\mathfrak{m} \cong I(P, \mathfrak{n}, Q)$, *with* $\mathfrak{n} \cong (\chi \otimes_{\overline{\mathbb{F}}_p} \tau) \otimes_{\mathcal{H}_{M,\mathrm{aff},\chi}} \mathcal{H}_M$ *and* $\chi$ *parametrized by* $(\mathbf{1}, J)$*. Then the following are equivalent:*

- $\mathrm{pd}_{\mathcal{H}'}(\mathfrak{m}) < \infty$*;*
- $\mathrm{pd}_{\mathcal{H}'_M}(\mathfrak{n}) < \infty$*;*
- *the root system of* $M$ *is of type* $A_1 \times \cdots \times A_1$ *(possibly empty product).*

*Moreover, when* $G$ *is semisimple and* $\mathrm{pd}_{\mathcal{H}'}(\mathfrak{m}) < \infty$*, the "*$\mathcal{H}'$ *version" of* (1) *is a projective resolution of* $\mathfrak{m}$*, and* $\mathrm{pd}_{\mathcal{H}'}(\mathfrak{m}) = r_{\mathrm{ss}}$*.*

### 9.2. Projective resolutions of $G$-representations

In this subsection, we take $p > 2$, and let $G$ be equal to either $\mathrm{PGL}_2(\mathbb{Q}_p)$ or $\mathrm{SL}_2(\mathbb{Q}_p)$. In this case (cf. [12] and [9]), the category $\mathfrak{Mod}\text{-}\mathcal{H}$ of right $\mathcal{H}$-modules is equivalent to the category $\mathfrak{Rep}_{\overline{\mathbb{F}}_p}^{I(1)}(G)$ of $\overline{\mathbb{F}}_p$-representations of $G$ generated by their $I(1)$-invariant vectors. Explicitly, this equivalence is given by the pair of adjoint functors

$$\begin{aligned} \mathfrak{Mod}\text{-}\mathcal{H} &\cong \mathfrak{Rep}_{\overline{\mathbb{F}}_p}^{I(1)}(G) \\ \mathfrak{m} &\longmapsto \mathfrak{m} \otimes_{\mathcal{H}} \text{c-ind}_{I(1)}^{G}(\mathbf{1}) \\ \pi^{I(1)} &\longleftarrow\!\shortmid \pi. \end{aligned}$$

Let $\pi$ be a smooth irreducible representation of $G$ which is either an irreducible subquotient of a principal series representation, or a supersingular representation which satisfies $\pi^I \neq 0$ (see [7], [1]). The nonzero vector space $\pi^{I(1)}$ then becomes a simple right $\mathcal{H}$-module, and Corollary 8.6 implies that it has projective dimension 1 over $\mathcal{H}$ (for supersingular representations satisfying $\pi^I \neq 0$, we have $\pi^{I(1)} = \pi^I$, and thus we may instead apply Corollary 9.1). We note that, if $\pi'$ is an irreducible supersingular representation of $G$ for which the associated $\mathcal{H}$-module $(\pi')^{I(1)}$ satisfies $S_\xi = \emptyset$, then $(\pi')^{I(1)}$ has infinite projective dimension in $\mathfrak{Mod}\text{-}\mathcal{H}$, and consequently $\pi'$ has infinite projective dimension in $\mathfrak{Rep}_{\overline{\mathbb{F}}_p}^{I(1)}(G)$.

Let $\pi$ be as above, and let $x$ and $x'$ denote the two vertices in the closure of the chamber $C$. By Corollary 8.6, we obtain a projective resolution of $\pi^{I(1)}$ given by

$$0 \longrightarrow \pi^{I(1)}(\epsilon_C) \otimes_{\mathcal{H}_C^\dagger} \mathcal{H} \longrightarrow \pi^{I(1)} \otimes_{\mathcal{H}_x} \mathcal{H} \longrightarrow \pi^{I(1)} \longrightarrow 0$$

for $G = \mathrm{PGL}_2(\mathbb{Q}_p)$, and by

$$0 \longrightarrow \pi^{I(1)} \otimes_{\mathcal{H}_C} \mathcal{H} \longrightarrow \left(\pi^{I(1)} \otimes_{\mathcal{H}_x} \mathcal{H}\right) \oplus \left(\pi^{I(1)} \otimes_{\mathcal{H}_{x'}} \mathcal{H}\right) \longrightarrow \pi^{I(1)} \longrightarrow 0$$

for $G = \mathrm{SL}_2(\mathbb{Q}_p)$ (here we identify the semisimple buildings of the groups $\mathrm{PGL}_2(\mathbb{Q}_p)$ and $\mathrm{SL}_2(\mathbb{Q}_p)$). One easily checks that for any facet $\mathcal{F} \subset \overline{C}$, we have a $G$-equivariant isomorphism

$$\pi^{I(1)}(\epsilon_{\mathcal{F}}) \otimes_{\mathcal{H}_{\mathcal{F}}^\dagger} \text{c-ind}_{I(1)}^G(\mathbf{1}) \cong \text{c-ind}_{\mathcal{P}_{\mathcal{F}}^\dagger}^G \left(\langle \mathcal{P}_{\mathcal{F}}^\dagger.\pi^{I(1)}\rangle \otimes_{\overline{\mathbb{F}}_p} \epsilon_{\mathcal{F}}\right),$$

where $\langle \mathcal{P}_{\mathcal{F}}^\dagger.\pi^{I(1)}\rangle$ denotes the $\mathcal{P}_{\mathcal{F}}^\dagger$-subrepresentation of $\pi$ generated by $\pi^{I(1)}$. Applying the equivalence of categories above, we thus obtain a resolution

$$0 \longrightarrow \text{c-ind}_{\mathcal{P}_C^\dagger}^G \left(\pi^{I(1)} \otimes_{\overline{\mathbb{F}}_p} \epsilon_C\right) \longrightarrow \text{c-ind}_{\mathcal{P}_x}^G \left(\langle \mathcal{P}_x.\pi^{I(1)}\rangle\right) \longrightarrow \pi \longrightarrow 0 \tag{7}$$

when $G = \mathrm{PGL}_2(\mathbb{Q}_p)$, and

$$\begin{aligned} 0 \longrightarrow \text{c-ind}_I^G \left(\pi^{I(1)}\right) \longrightarrow \text{c-ind}_{\mathcal{P}_x}^G \left(\langle \mathcal{P}_x.\pi^{I(1)}\rangle\right) \oplus \text{c-ind}_{\mathcal{P}_{x'}}^G \left(\langle \mathcal{P}_{x'}.\pi^{I(1)}\rangle\right) \\ \longrightarrow \pi \longrightarrow 0 \end{aligned} \tag{8}$$

when $G = \mathrm{SL}_2(\mathbb{Q}_p)$ (note that in both cases, $\pi^{I(1)}$ is naturally a representation of $\mathcal{P}_C^\dagger$). Collecting everything gives the following result.

**Proposition 9.2.** *Let $G = \mathrm{PGL}_2(\mathbb{Q}_p)$ (resp. $G = \mathrm{SL}_2(\mathbb{Q}_p)$), with $p > 2$. Let $\pi$ denote an irreducible subquotient of a principal series representation of $G$, or an irreducible supersingular representation of $G$ which satisfies $\pi^I \neq 0$. Then* (7) *(resp.* (8)*) is a projective resolution of $\pi$ in the abelian category $\mathfrak{Rep}_{\overline{\mathbb{F}}_p}^{I(1)}(G)$.*

**Remark 9.3.** Let $\pi$ be equal to the trivial representation of $G$. The terms in the resolutions (7) and (8) take the form $\text{c-ind}_{\mathcal{P}_{\mathcal{F}}^\dagger}^G(\epsilon_{\mathcal{F}})$, and we have

$$\begin{aligned} \mathrm{Hom}_G\left(\text{c-ind}_{\mathcal{P}_{\mathcal{F}}^\dagger}^G(\epsilon_{\mathcal{F}}), \tau\right) &\cong \mathrm{Hom}_{\mathcal{P}_{\mathcal{F}}^\dagger}(\epsilon_{\mathcal{F}}, \tau|_{\mathcal{P}_{\mathcal{F}}^\dagger}) \\ &\cong \tau^{\mathcal{P}_{\mathcal{F}}^\dagger, \epsilon_{\mathcal{F}}} \\ &:= \left\{v \in \tau : g.v = \epsilon_{\mathcal{F}}(g)v \text{ for all } g \in \mathcal{P}_{\mathcal{F}}^\dagger\right\}, \end{aligned}$$

where $\tau$ is a smooth $G$-representation. Since the coefficient field has characteristic $p$, the functor $\tau \longmapsto \tau^{\mathcal{P}_{\mathcal{F}}^\dagger, \epsilon_{\mathcal{F}}}$ will not be exact in general, and therefore the resolutions (7) and (8) will not give projective resolutions in the entire category $\mathfrak{Rep}_{\overline{\mathbb{F}}_p}(G)$ of smooth $G$-representations.

**Remark 9.4.** By direct inspection, one easily checks that if $\pi$ is an irreducible subquotient of a principal series representation of $\mathrm{PGL}_2(\mathbb{Q}_p)$ or $\mathrm{SL}_2(\mathbb{Q}_p)$ and $y$ a vertex in the closure of $\overline{C}$, then

$$\langle \mathcal{P}_y . \pi^{I(1)} \rangle \cong \pi^{\mathcal{P}_y(1)}$$

as representations of $\mathcal{P}_y$, where $\mathcal{P}_y(1)$ denotes the pro-$p$ radical of $\mathcal{P}_y$. Therefore, the projective resolutions (7) and (8) take the form

$$0 \longrightarrow \text{c-ind}_{\mathcal{P}_C^\dagger}^G \left(\pi^{I(1)} \otimes_{\overline{\mathbb{F}}_p} \epsilon_C\right) \longrightarrow \text{c-ind}_{\mathcal{P}_x}^G \left(\pi^{\mathcal{P}_x(1)}\right) \longrightarrow \pi \longrightarrow 0$$

when $G = \mathrm{PGL}_2(\mathbb{Q}_p)$, and

$$0 \longrightarrow \text{c-ind}_I^G \left(\pi^{I(1)}\right) \longrightarrow \text{c-ind}_{\mathcal{P}_x}^G \left(\pi^{\mathcal{P}_x(1)}\right) \oplus \text{c-ind}_{\mathcal{P}_{x'}}^G \left(\pi^{\mathcal{P}_{x'}(1)}\right) \longrightarrow \pi \longrightarrow 0$$

when $G = \mathrm{SL}_2(\mathbb{Q}_p)$. In other words, the representation $\pi$ is the $0^{\text{th}}$ homology of the coefficient system denoted $\underline{\underline{\pi}}$ in [18, §II.2].

### 9.3. Central characters

One can also ask about the behavior of the resolution (1) when the module $\mathfrak{m}$ possesses a central character. We will show that, by passing to an appropriate subcategory of modules with a fixed "central character" (in a sense to be made precise below), (1) becomes a projective resolution whenever $\mathrm{pd}_{\mathcal{H}}(\mathfrak{m}) < \infty$.

Recall that $Z$ denotes the connected center of $G$. Let us fix a splitting of the short exact sequence

$$1 \longrightarrow (Z \cap I)/(Z \cap I(1)) \longrightarrow Z/(Z \cap I(1)) \longrightarrow Z/(Z \cap I) \cong \mathbb{Z}^{\oplus r_Z} \longrightarrow 1,$$

and for $z \in Z/(Z \cap I)$, we let $\widehat{z} \in Z/(Z \cap I(1)) \hookrightarrow \widetilde{W}$ denote its image under the splitting. Since the elements of $Z/(Z \cap I(1))$ (considered as elements of $\widetilde{W}$) have length zero, we see that the $\overline{\mathbb{F}}_p$-vector space spanned by $\{\mathrm{T}_{\widehat{z}}\}_{z \in Z/(Z\cap I)}$ is a subalgebra of the center of $\mathcal{H}$. Denote this algebra by $\mathcal{Y}$, and note that $\mathcal{Y} \subset \mathcal{H}_{\mathcal{F}}^\dagger$ for every facet $\mathcal{F} \subset \overline{C}$.

Fix now a character $\zeta : \mathcal{Y} \longrightarrow \overline{\mathbb{F}}_p$, and define the quotient algebras

$$\mathcal{H}^{\zeta} := \mathcal{H}/\left(\mathrm{T}_{\widehat{z}} - \zeta(\mathrm{T}_{\widehat{z}})\right)_{z\in Z/(Z\cap I)}\mathcal{H},$$
$$\mathcal{H}_{\mathcal{F}}^{\dagger,\zeta} := \mathcal{H}_{\mathcal{F}}^{\dagger}/\left(\mathrm{T}_{\widehat{z}} - \zeta(\mathrm{T}_{\widehat{z}})\right)_{z\in Z/(Z\cap I)}\mathcal{H}_{\mathcal{F}}^{\dagger},$$

where $\mathcal{F}$ is a facet in the closure of $C$ (since $\mathcal{Y}$ is central, the ideals are two-sided). For a fixed facet $\mathcal{F} \subset \overline{C}$, we have

$$W_{\mathcal{F}} \cap (Z/(Z\cap I)) \subset W_{\mathrm{aff}} \cap \Omega = \{1\},$$

and therefore $\widetilde{W}_{\mathcal{F}} \cap \{\widehat{z}\}_{z\in Z/(Z\cap I)} = \{1\}$. This implies that the composition

$$\mathcal{H}_{\mathcal{F}} \hookrightarrow \mathcal{H}_{\mathcal{F}}^{\dagger} \twoheadrightarrow \mathcal{H}_{\mathcal{F}}^{\dagger,\zeta}$$

is injective, and we may view $\mathcal{H}_{\mathcal{F}}$ as a subalgebra of $\mathcal{H}_{\mathcal{F}}^{\dagger,\zeta}$. Similarly to Lemma 3.4, we see that $\mathcal{H}_{\mathcal{F}}^{\dagger,\zeta}$ is free of finite rank over $\mathcal{H}_{\mathcal{F}}$, with basis given by $\{\mathrm{T}_{\widehat{\omega}}\}_{\omega\in\Omega_{\mathcal{F}}/(Z/(Z\cap I))}$, so that $\mathcal{H}_{\mathcal{F}}^{\dagger,\zeta}$ has the structure of a crossed product algebra over $\mathcal{H}_{\mathcal{F}}$.

Now let $\mathfrak{m}$ be a right $\mathcal{H}$-module on which $\mathcal{Y}$ acts by the character $\zeta$, so that $\mathfrak{m}$ is naturally a module over $\mathcal{H}^{\zeta}$. The inclusion $\mathcal{H}_{\mathcal{F}}^{\dagger} \subset \mathcal{H}$ induces an inclusion $\mathcal{H}_{\mathcal{F}}^{\dagger,\zeta} \subset \mathcal{H}^{\zeta}$ for every facet $\mathcal{F} \subset \overline{C}$, which makes $\mathcal{H}^{\zeta}$ into a free (left and right) $\mathcal{H}_{\mathcal{F}}^{\dagger,\zeta}$-module. We easily see that the natural map $\mathcal{H} \twoheadrightarrow \mathcal{H}^{\zeta}$ induces an isomorphism of $\mathcal{H}^{\zeta}$-modules

$$\mathfrak{m}|_{\mathcal{H}_{\mathcal{F}}^{\dagger}}(\epsilon_{\mathcal{F}}) \otimes_{\mathcal{H}_{\mathcal{F}}^{\dagger}} \mathcal{H} \cong \mathfrak{m}|_{\mathcal{H}_{\mathcal{F}}^{\dagger,\zeta}}(\epsilon_{\mathcal{F}}) \otimes_{\mathcal{H}_{\mathcal{F}}^{\dagger,\zeta}} \mathcal{H}^{\zeta}. \tag{9}$$

Viewing (1) as a resolution of $\mathcal{H}^{\zeta}$-modules, we obtain the following result.

**Proposition 9.5.** *Assume $p \nmid |\Omega/(Z/(Z\cap I))|$, and let $\mathfrak{m}$ be a right $\mathcal{H}$-module on which $\mathcal{Y}$ acts by a character $\zeta$. Then $\mathrm{pd}_{\mathcal{H}}(\mathfrak{m}) < \infty$ if and only if $\mathrm{pd}_{\mathcal{H}^{\zeta}}(\mathfrak{m}) < \infty$. In this case,* (1) *is a resolution of $\mathfrak{m}$ by projective $\mathcal{H}^{\zeta}$-modules, and we have $\mathrm{pd}_{\mathcal{H}^{\zeta}}(\mathfrak{m}) = r_{\mathrm{ss}}$ and $\mathrm{pd}_{\mathcal{H}}(\mathfrak{m}) = r_{\mathrm{ss}} + r_Z$.*

Compare [15, Cor. 2.3].

*Proof.* Assume first that $\mathrm{pd}_{\mathcal{H}}(\mathfrak{m}) < \infty$. The condition on $p$ implies $p \nmid |\Omega_{\mathrm{tor}}|$, so that $\mathfrak{m}|_{\mathcal{H}_{\mathcal{F}}}$ is projective over $\mathcal{H}_{\mathcal{F}}$ for every facet $\mathcal{F} \subset \overline{C}$ (Lemma 4.2). Since $\mathcal{H}_{\mathcal{F}}^{\dagger,\zeta}$ is a crossed product algebra over $\mathcal{H}_{\mathcal{F}}$, [10, Thm. 7.5.6(ii)] again implies that $\mathfrak{m}|_{\mathcal{H}_{\mathcal{F}}^{\dagger,\zeta}}(\epsilon_{\mathcal{F}})$ is projective over $\mathcal{H}_{\mathcal{F}}^{\dagger,\zeta}$ (cf. Lemmas 3.4 and 3.5). Since $\mathcal{H}^{\zeta}$ is free as an $\mathcal{H}_{\mathcal{F}}^{\dagger,\zeta}$-module, the induced $\mathcal{H}^{\zeta}$-module $\mathfrak{m}|_{\mathcal{H}_{\mathcal{F}}^{\dagger,\zeta}}(\epsilon_{\mathcal{F}}) \otimes_{\mathcal{H}_{\mathcal{F}}^{\dagger,\zeta}} \mathcal{H}^{\zeta}$

is projective, which gives one implication and the precise result about the resolution (1).

Assume conversely that $\mathrm{pd}_{\mathcal{H}^\zeta}(\mathfrak{m}) < \infty$. By an analog of Lemma 4.2, we get that $\mathfrak{m}|_{\mathcal{H}_\mathcal{F}}$ is projective over $\mathcal{H}_\mathcal{F}$ for every facet $\mathcal{F} \subset \overline{C}$, and using the argument of the first paragraph shows that in fact $\mathrm{pd}_{\mathcal{H}^\zeta}(\mathfrak{m}) \leq r_{\mathrm{ss}}$. Also, [14, Prop. 5.4] shows that the algebras $\mathcal{H}_\mathcal{F}^{\dagger,\zeta}$ are Frobenius algebras, and by adapting the arguments of Section 6 of *loc. cit.* we see that (1) is a resolution of $\mathfrak{m}$ by (Gorenstein) projective $\mathcal{H}^\zeta$-modules (using the identification (9)). Furthermore, an analog of Lemma 6.10 of *loc. cit.* holds, and we obtain

$$\mathrm{Ext}^{r_{\mathrm{ss}}}_{\mathcal{H}^\zeta}(\mathfrak{m}, \mathcal{H}^\zeta) \neq 0.$$

Thus, we have $\mathrm{pd}_{\mathcal{H}^\zeta}(\mathfrak{m}) = r_{\mathrm{ss}}$.

Fix now a set of generators $z_1, \ldots, z_{r_Z}$ for $Z/(Z \cap I)$; then $\mathrm{T}_{\widehat{z}_i} - \zeta(\mathrm{T}_{\widehat{z}_i})$ are central non-zerodivisors of $\mathcal{H}$ which generate the (proper) ideal

$$\left(\mathrm{T}_{\widehat{z}} - \zeta(\mathrm{T}_{\widehat{z}})\right)_{z \in Z/(Z\cap I)} \mathcal{H}.$$

Applying [10, Thm. 7.3.5(i)] shows that $\mathrm{pd}_{\mathcal{H}}(\mathfrak{m}) = \mathrm{pd}_{\mathcal{H}^\zeta}(\mathfrak{m}) + r_Z = r_{\mathrm{ss}} + r_Z$, which gives the claim. □

**Remark 9.6.** Let $\mathfrak{m}$ and $\mathfrak{n}$ be two right $\mathcal{H}$-modules on which $\mathcal{Y}$ acts by a character $\zeta$. In order to obtain quantitative information about projective dimensions, one can compute the Ext groups between $\mathfrak{m}$ and $\mathfrak{n}$, either in the category of $\mathcal{H}^\zeta$-modules or in the category of $\mathcal{H}$-modules. The relation between the two is controlled by the base-change-for-Ext spectral sequence:

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathcal{H}^\zeta}(\mathfrak{m}, \mathrm{Ext}^j_{\mathcal{H}}(\mathcal{H}^\zeta, \mathfrak{n})) \Longrightarrow \mathrm{Ext}^{i+j}_{\mathcal{H}}(\mathfrak{m}, \mathfrak{n}).$$

The five-term exact sequence associated to the above spectral sequence is a module-theoretic version of a short exact sequence used by Paškūnas in [16, Prop. 8.1].

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## References


[1] R. Abdellatif, *Classification des représentations modulo p de* $\mathrm{SL}(2, F)$, Bull. Soc. Math. France **142** (2014), no. 3, 537–589.

[2] N. Abe, *Modulo p parabolic induction of pro-p-Iwahori Hecke algebra*, J. Reine Angew. Math. DOI:10.1515/crelle-2016-0043.

[3] N. Abe, G. Henniart, F. Herzig, and M.-F. Vignéras, *A classification of irreducible admissible* mod $p$ *representations of p-adic reductive groups*, J. Amer. Math. Soc. **30** (2017), no. 2, 495–559.

[4] D. J. Benson, Representations and Cohomology. I, Vol. 30 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, second edition (1998), ISBN 0-521-63653-1. Basic representation theory of finite groups and associative algebras.

[5] M. Borovoi, *Abelian Galois cohomology of reductive groups*, Mem. Amer. Math. Soc. **132** (1998), no. 626, viii+50.

[6] N. Bourbaki, Éléments de Mathématique, Masson, Paris (1981), ISBN 2-225-76076-4. Groupes et algèbres de Lie. Chapitres 4, 5 et 6. [Lie groups and Lie algebras. Chapters 4, 5 and 6].

[7] C. Breuil, *Sur quelques représentations modulaires et p-adiques de* $\mathrm{GL}_2(\mathbf{Q}_p)$. *I*, Compositio Math. **138** (2003), no. 2, 165–188.

[8] E. E. Enochs and O. M. G. Jenda, Relative Homological Algebra, Volume 1, Vol. 30 of de Gruyter Expositions in Mathematics, Walter de Gruyter GmbH & Co. KG, Berlin, extended edition (2011), ISBN 978-3-11-021520-5.

[9] K. Kozioł, *Pro-p-Iwahori Invariants for* $SL_2$ *and L-Packets of Hecke Modules*, Int. Math. Res. Not. IMRN (2016), no. 4, 1090–1125.

[10] J. C. McConnell and J. C. Robson, Noncommutative Noetherian Rings, Vol. 30 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, revised edition (2001), ISBN 0-8218-2169-5. With the cooperation of L. W. Small.

[11] P. N. Norton, 0-*Hecke algebras*, J. Austral. Math. Soc. Ser. A **27** (1979), no. 3, 337–357.

[12] R. Ollivier, *Le foncteur des invariants sous l'action du pro-p-Iwahori de* $\mathrm{GL}_2(F)$, J. Reine Angew. Math. **635** (2009), 149–185.

[13] R. Ollivier, *Compatibility between Satake and Bernstein isomorphisms in characteristic p*, Algebra Number Theory **8** (2014), no. 5, 1071–1111.

[14] R. Ollivier and P. Schneider, *Pro-p Iwahori-Hecke algebras are Gorenstein*, J. Inst. Math. Jussieu **13** (2014), no. 4, 753–809.

[15] E. Opdam and M. Solleveld, *Homological algebra for affine Hecke algebras*, Adv. Math. **220** (2009), no. 5, 1549–1601.

[16] V. Paškūnas, *Extensions for supersingular representations of* $\mathrm{GL}_2(\mathbb{Q}_p)$, Astérisque (2010), no. 331, 317–353.

[17] P. Schneider, *Smooth representations and Hecke modules in characteristic p*, Pacific J. Math. **279** (2015), no. 1-2, 447–464.

[18] P. Schneider and U. Stuhler, *Representation theory and sheaves on the Bruhat-Tits building*, Inst. Hautes Études Sci. Publ. Math. (1997), no. 85, 97–191.

[19] T. A. Springer, Linear Algebraic Groups, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, second edition (2009), ISBN 978-0-8176-4839-8.

[20] M.-F. Vignéras, *The pro-p Iwahori Hecke algebra of a reductive p-adic group, V (parabolic induction)*, Pacific J. Math. **279** (2015), no. 1-2, 499–529.

[21] M.-F. Vignéras, *The pro-p-Iwahori Hecke algebra of a reductive p-adic group I*, Compos. Math. **152** (2016), no. 4, 693–753.

[22] M.-F. Vignéras, *The pro-p-Iwahori Hecke algebra of a reductive p-adic group III (spherical Hecke algebras and supersingular modules)*, J. Inst. Math. Jussieu **16** (2017), no. 3, 571–608.

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