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Bounded gaps between primes in multidimensional Hecke equidistribution problems

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We prove an analogue of the classical Bombieri-Vinogradov estimate for all subsets of the primes whose distribution is determined by Hecke Grössencharaktere. Using this estimate and Maynard's new sieve techniques, we prove the existence of infinitely many bounded gaps between primes in all such subsets of the primes. We present applications to the study of primes represented by norm forms of number fields and the number of $\mathbb{F}_p$-rational points on certain abelian varieties. In particular, for any fixed $0 < \epsilon < \frac{1}{2}$, there exist infinitely many bounded gaps between primes of the form $p = a^2 + b^2$ such that $|a| < \epsilon\sqrt{p}$. Also, we prove the existence of infinitely many bounded gaps between the primes $p \equiv 1 \pmod{10}$ for which $|p + 1 - \#\mathcal{C}(\mathbb{F}_p)| < \epsilon\sqrt{p}$, where $\mathcal{C}/\mathbb{Q}$ is the hyperelliptic curve $y^2 = x^5 + 1$.

1. Introduction and statement of results

Conjectures about primes represented by polynomials of degree greater than one have captivated number theorists for well over a century. It is conjectured that every irreducible polynomial of degree at least one in $\mathbb{Z}[x]$ represents infinitely many primes, but this is known unconditionally for only the degree one case by Dirichlet's work in 1837. The simplest degree two polynomial to study is $x^2 + 1$. A landmark partial result due to Iwaniec [11] states that there are infinitely many integers n such that $n^2 + 1$ is a product of at most two primes. By the work of Lemke Oliver [15], the same conclusion holds for any irreducible polynomial $f(x)$ of degree two such that $f(x) \not\equiv x(x + 1) \pmod{2}$.

By extending the question to consider primes represented by multivariate polynomials, one can prove much stronger results, especially when one considers norm forms of number fields. For example, any positive definite binary quadratic form $ax_1^2 + bx_1x_2 + cx_2^2 \in \mathbb{Z}[x_1, x_2]$ of discriminant D represents a positive proportion of the primes, where the proportion depends on the number of primitive binary quadratic forms of discriminant D. Using sophisticated sieve methods, Friedlander and Iwaniec [7] proved an asymptotic formula for the number of primes of the form $x_1^2 + x_2^4$, and Heath-Brown [10] did the same for primes of the form $x_1^3 + 2x_2^3$.

As an approximation to understanding the distribution of primes of the form $n^2 + 1$, one might ask for the distribution of primes $p = x_1^2 + x_2^2$ where x_1 is small in terms of p. More generally, one can ask how the values $\frac{x_1}{\sqrt{p}}$ are distributed in $[-1, 1]$ as p varies. A classical result of Hecke states that if $[\alpha, \beta] \subset [-1, 1]$ is a fixed subinterval and $\pi(x) := \#\{p \leq x\}$, then

$$(1.1)\quad \lim_{x\to\infty} \frac{1}{\pi(x)} \# \left\{ p \leq x : p = x_1^2 + x_2^2, \frac{x_1}{\sqrt{p}} \in [\alpha, \beta] \right\} = \frac{1}{2} \int_\alpha^\beta \frac{1}{\pi\sqrt{1-t^2}} dt.$$

This equidistribution law is equivalent to the statement that L-functions associated to Hecke Grössencharactere (henceforth referred to as Hecke characters) for the field $\mathbb{Q}(\sqrt{-1})$ have no zeros on the line $\mathrm{Re}(s) = 1$. This bears resemblance to the proof of the prime number theorem for arithmetic progressions $a \pmod q$, with the role of a residue class modulo q replaced with the role of a subinterval of $[-1, 1]$ and the role of Dirichlet L-functions replaced with the role of Hecke L-functions. By taking $[\alpha, \beta]$ to be a small interval centered at 0, Ankeny [1] used the generalized Riemann hypothesis (GRH) to prove that there are infinitely many primes $p = x_1^2 + x_2^2$ with $x = O(\log p)$. Unconditionally, Harman and Lewis [9] proved that there are infinitely many primes $p = x_1^2 + x_2^2$ with $x_1 = O(p^\theta)$ for any $\theta > 0.119$ using sieve methods.

More generally, let K be a number field of degree $[K : \mathbb{Q}] \geq 2$ and discriminant D_K, and let $\mathrm{N} = \mathrm{N}_{K/\mathbb{Q}}$ denote the absolute field norm of K. Duke [5] studied a generalization of the above work by replacing $x^2 + y^2$ with a more general norm form over K of the shape

$$(1.2)\qquad f(\vec{x}) = \mathrm{N}\left(\sum_{j=1}^{[K:\mathbb{Q}]} \alpha_j x_j \right) \mathrm{N}\mathfrak{a}^{-1}, \quad \vec{x} = (x_1, \ldots, x_{[K:\mathbb{Q}]}),$$

where $\mathfrak{a}$ is a nonzero integral ideal of K and $\{\alpha_1, \ldots, \alpha_{[K:\mathbb{Q}]}\}$ is an integral basis which satisfies two properties:

1) $\det |\alpha_j^{(i)}| = D_K^{\frac{1}{2}} \mathrm{N}\mathfrak{a}$, where $1 \leq i, j \leq [K : \mathbb{Q}]$ and $\alpha_j^{(i)}$ is the i-th conjugate of α_j.

2) For some $1 \leq m \leq [K:\mathbb{Q}]$, α_m is totally positive.

Consider the set of primes given by

$$\mathcal{P}_{f,\mathcal{I},K} = \Big\{ p\colon \text{for some } \vec{x} \in \mathbb{Z}^{[K:\mathbb{Q}]}, \quad \text{we have } f(\vec{x}) = p \text{ and } p^{-\frac{1}{[K:\mathbb{Q}]}}\vec{x} \in \prod_{I \in \mathcal{I}} I \Big\}, \tag{1.3}$$

where $\mathcal{I} = \{I_1, \ldots, I_{[K:\mathbb{Q}]}\}$ is a collection of subintervals of $[-1,1]$ (each with positive Lebesgue measure), at least one of which is exactly $[-1,1]$. Hecke's arguments can be extended to show that the primes in $\mathcal{P}_{f,\mathcal{I},K}$ satisfy an equidistribution law which generalizes (1.1). Duke used this equidistribution law to prove that

$$\#\{p \leq x : \text{for some } \vec{x} \in \mathbb{Z}^{[K:\mathbb{Q}]},\ p = f(\vec{x}) \text{ and } |x_j| \leq p^{\frac{1}{[K:\mathbb{Q}]}-\delta} \text{ for all } j \neq m\} \asymp \frac{x^{1-([K:\mathbb{Q}]-1)\delta}}{\log x}, \tag{1.4}$$

where $0 \leq \delta < \frac{1}{3[K:\mathbb{Q}]}$ and $m \in \{1, \ldots, [K:\mathbb{Q}]\}$ are fixed and $f \asymp g$ means that both $f \ll g$ and $g \ll f$. A key ingredient in Duke's arguments is a zero density estimate for Hecke L-functions; this allowed him to circumvent the need for GRH.

In addition to studying the distribution of primes represented by a single multivariate form, one can ask questions about the distribution of primes represented simultaneously by several univariate linear forms $n + h_i$, where $1 \leq i \leq k$. Setting $\mathcal{H}_k = \{h_1, \ldots, h_k\}$, we call $\mathcal{H}_k$ an admissible set if for all primes p there exists an integer n_p such that $\prod_{i=1}^{k}(n_p + h_i)$ and p are coprime. The prime k-tuples conjecture, first conjectured by Hardy and Littlewood, asserts that if $\mathcal{H}_k$ is admissible, then there exists a positive constant $\mathfrak{S} = \mathfrak{S}(\mathcal{H}_k)$ such that as $x \to \infty$,

$$\#\{n \leq x : \#(\{n + h_1, \ldots, n + h_k\} \cap \mathbb{P}) = k\} \sim \mathfrak{S}\frac{x}{(\log x)^k}, \tag{1.5}$$

where $\mathbb{P}$ is the set of all primes. The twin prime conjecture follows when $\mathcal{H}_2 = \{0, 2\}$.

The prime k-tuples conjecture is completely open for $k > 1$, but the last decade has seen many strong approximations to the conjecture. Goldston, Pintz, and Yıldırım [8] proved one such approximation, namely

$$\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} = 0,$$

where p_n is the n-th prime. This is quite remarkable, considering that the average size of $p_{n+1}-p_n$ is $\log p_n$ by the prime number theorem. By fundamentally improving our understanding of primes in arithmetic progressions, Zhang [26] was able to use the framework provided by Goldston, Pintz, and Yıldırım to prove for the very first time that there exist infinitely many bounded gaps between primes:

$$\liminf_{n\to\infty}(p_{n+1}-p_n) \leq 7\times 10^7.$$

Using an approach very different from and more elementary than that of Zhang, Maynard [16] proved that $\liminf_{n\to\infty}(p_{n+1}-p_n) \leq 600$. Furthermore, for any $m\geq 1$, Maynard's work yields the bound

$$\liminf_{n\to\infty}(p_{m+n}-p_n) \ll m^3 e^{4m}.$$

(The underlying improvement to the Selberg sieve which led to this result was proven independently by Tao, who arrived at slightly different conclusions.) By combining and sharpening the methods in [16, 26], the Polymath 8b project [22] proved

$$\begin{aligned}&\liminf_{n\to\infty}(p_{n+1}-p_n) \leq 246 \quad \text{and}\\ &\liminf_{n\to\infty}(p_{m+n}-p_n) \ll m\exp\left(\left(4-\tfrac{28}{157}\right)m\right).\end{aligned}$$

Maynad [17] recently proved a strong quantitative approximation of (1.5): there exists an absolute constant $c\geq 1$ such that for $k>c$ and $\mathcal{H}_k=\{h_1,\ldots,h_k\}$ admissible,

$$\#\{n\leq x : \#(\{n+h_1,\ldots,n+h_k\}\cap\mathbb{P})\geq c^{-1}\log k\} \gg \frac{x}{(\log x)^k}.$$

It is natural to ask to what extent these advances extend to interesting subsets of the primes. The author extended the work on bounded gaps between primes to the context of the Chebotarev density theorem. Let $K/\mathbb{Q}$ be a Galois extension of number fields with Galois group G and absolute discriminant D_K, and let $C \subset G$ be a conjugacy class. Define

$$\mathcal{P}_C = \left\{ p \nmid D_K : \left[\frac{K/\mathbb{Q}}{p} \right] = C \right\},$$

where $[K/\mathbb{Q}, p]$ denotes the Artin symbol, and let $\varphi(q)$ be Euler's totient function. It follows from the author's work in [25] that there are infinitely many positive integers N such that for some $n \in [N, 2N]$, we have that

$$\begin{aligned} &\#(\{n+h_1,\ldots,n+h_k\} \cap \mathcal{P}_C) \\ &\geq \left(\min\left\{ \frac{1}{4}, \frac{1}{|G|} \right\} \frac{|C|}{|G|} \frac{\varphi(D_K)}{D_K} + o_{k\to\infty}(1) \right) \log k. \end{aligned}$$

The author explored applications of this result to the distribution of ranks of quadratic twists of elliptic curves, congruences for the Fourier coefficients of modular forms, and primes represented by binary quadratic forms. Theorem 3.5 of [17] makes this quantitative: there exists a constant $c_K \geq 1$ such that for $k > c_K$ and $\mathcal{H}_k$ admissible,

$$\#\{n \leq x : \#(\{n+h_1,\ldots,n+h_k\} \cap \mathcal{P}_C) \geq c_K^{-1} \log k\} \gg \frac{x}{(\log x)^k}.$$

We extend of the work of Maynard to the setting of Duke's work on (1.3) by showing that the primes in (1.3) exhibit infinitely many bounded gaps.

Theorem 1.1. *Let $\mathcal{P}_{f,\mathcal{I},K}$ be given by* (1.3)*. There exists a constant $c_{f,\mathcal{I},K} > 0$ such that if $k > c_{f,\mathcal{I},K}$ and $\mathcal{H}_k = \{h_1, \ldots, h_k\}$ is an admissible set, then*

$$\#\{n \leq x : \#(\{n+h_1,\ldots,n+h_k\} \cap \mathcal{P}_{f,\mathcal{I},K}) \geq c_{f,\mathcal{I},K}^{-1} \log k\} \gg \frac{x}{(\log x)^k}.$$

By choosing $K = \mathbb{Q}(\sqrt{-1})$ and $\mathcal{I} = \{[-1,1], [-\epsilon, \epsilon]\}$ for some $0 < \epsilon < \frac{1}{2}$, Theorem 1.1 immediately yields the following result.

Corollary 1.2. *Fix $0 < \epsilon < \frac{1}{2}$, and let $\mathcal{P}_\epsilon = \{p : p = x_1^2 + x_2^2, |x_1| \leq \epsilon\sqrt{p}\}$. There exists a constant $c_\epsilon > 0$ such that if $k > c_\epsilon$ and $\mathcal{H}_k = \{h_1, \ldots, h_k\}$ is an admissible set, then*

$$\#\{n \leq x : \#(\{n+h_1,\ldots,n+h_k\} \cap \mathcal{P}_\epsilon) \geq c_\epsilon^{-1} \log k\} \gg \frac{x}{(\log x)^k}.$$

Another generalization of (1.1) lies in the distribution of the Fourier coefficients of holomorphic cuspidal normalized Hecke eigenforms (i.e., newforms) on congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Let

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z} \in \mathbb{Z}[[e^{2\pi i z}]]$$

be a newform of integral weight $\ell \geq 2$ and level N. Suppose further that there exists an imaginary quadratic number field K such that a prime p is inert in K if and only if $a_f(p) = 0$; in this case, we say that f has complex multiplication, or CM, by the ring of integers of K. By Deligne's proof of the Weil conjectures,

$$|a_f(p)| \leq 2p^{\frac{\ell-1}{2}}$$

for all primes p. By the methods leading to Hecke's proof of (1.1), one finds that if f has complex multiplication, then for any fixed subinterval $[\alpha, \beta] \subset [-1, 1]$,

$$\begin{aligned} &\lim_{x\to\infty} \frac{1}{\pi(x)} \# \left\{ p \leq x : \frac{a_f(p)}{2p^{\frac{\ell-1}{2}}} \in [\alpha, \beta] \right\} \\ &= \frac{1}{2} \int_\alpha^\beta \frac{1}{\pi\sqrt{1-t^2}} dt + \begin{cases} \frac{1}{2} & \text{if } 0 \in [\alpha, \beta], \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{1.6}$$

The results of [1] follow by understanding the error term in (1.6) for the weight 2, level 32 newform associated to the congruent number elliptic curve $E : y^2 = x^3 - x$.

We prove that the primes considered in (1.6) exhibit infinitely many bounded gaps.

Theorem 1.3. *Let*

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z} \in \mathbb{Z}[[e^{2\pi i z}]]$$

be a newform of even integral weight $\ell \geq 2$ and level N that has complex multiplication, let $I = [\alpha, \beta] \subset [-1, 1]$, and let

$$\mathcal{P}_{f,I} = \left\{ p \nmid N : \frac{a_f(p)}{2p^{\frac{\ell-1}{2}}} \in I \right\}.$$

There exists a constant $c_{f,I} > 0$ such that if $k > c_{f,I}$ and $\mathcal{H}_k = \{h_1, \ldots, h_k\}$ is an admissible set, then

$$\{n \le x : \#(\{n+h_1, \ldots, n+h_k\} \cap \mathcal{P}_{f,I}) \ge c_{f,I}^{-1} \log k\} \gg \frac{x}{(\log x)^k}.$$

When $f(z)$ in Theorem 1.3 is a weight 2 newform with integral coefficients, $f(z)$ is the newform associated to an elliptic curve $E/\mathbb{Q}$ of conductor N with complex multiplication. In this case, for every prime $p \nmid N$, we have that

$$a_f(p) = p + 1 - \#E(\mathbb{F}_p),$$

where $\mathbb{F}_p$ is the finite field of order p. We can also consider the distribution of the number of $\mathbb{F}_p$-rational points on a class of curves $\mathcal{C}/\mathbb{Q}$ given by

$$\mathcal{C} : ax^\alpha + by^\beta = c, \tag{1.7}$$

where $a, b, c, \alpha, \beta \in \mathbb{Z} \setminus \{0\}$ and $\alpha \ge \beta \ge 2$. (All elliptic curves of this form necessarily have complex multiplication.) Let $d = \gcd(\alpha, \beta)$, let $M = \operatorname{lcm}(\alpha, \beta)$, and let

$$g = \frac{(\alpha-1)(\beta-1) - (d-1)}{2}$$

be the genus of $\mathcal{C}$. Define

$$a_{\mathcal{C}}(p) = p + 1 - \mathcal{N}_d - \#\mathcal{C}(\mathbb{F}_p),$$
$$\mathcal{N}_d = \begin{cases} d & \text{if } -\frac{a}{b} \text{ is a } d\text{-th power modulo } p, \\ 0 & \text{otherwise.} \end{cases}$$

For each $p \equiv 1 \pmod{M}$ with $p \nmid abc$, we have the Hasse bound $|a_{\mathcal{C}}(p)| \le 2g\sqrt{p}$. It follows from the work of Hecke that the sequence $\{\frac{a_{\mathcal{C}}(p)}{2g\sqrt{p}}\}$ is equidistributed in $[-1, 1]$ with respect to a certain probability measure. When $g = 1$, in which case $\mathcal{C}$ is an elliptic curve over $\mathbb{Q}$ with complex multiplication, this measure is that in (1.6). For a discussion of the genus 2 case, see [6]. Duke [5] used Hecke's equidistribution law to study sets of the form

$$\mathcal{P}_{\mathcal{C},I} = \left\{ p : p \equiv 1 \pmod{M},\ p \nmid abc, \text{ and } \frac{a_{\mathcal{C}}(p)}{2g\sqrt{p}} \in I \right\}, \tag{1.8}$$

where I is a subinterval of $[-1, 1]$. In particular, Duke proved that for any fixed $0 \le \delta < (3\varphi(M))^{-1}$, we have

$$\#\{p \le x : p \equiv 1 \pmod{M},\ p \nmid abc,\ |a_{\mathcal{C}}(p)| \le 2gp^{\frac{1}{2}-\delta}\} \gg \frac{x^{1-\delta\varphi(M)/2}}{\log x}. \tag{1.9}$$

Theorem 1.4. *Let $\mathcal{C}$ be a curve given by* (1.7)*. Let $\mathcal{P}_{\mathcal{C},I}$ be defined by* (1.8)*. There exists a constant $c_{\mathcal{C},I} > 0$ such that if $k > c_{\mathcal{C},I}$ and $\mathcal{H}_k = \{h_1, \ldots, h_k\}$ is an admissible set, then*

$$\#\{n \le x : \#(\{n+h_1, \ldots, n+h_k\} \cap \mathcal{P}_{\mathcal{C},I}) \ge c_{\mathcal{C},I}^{-1} \log k\} \gg \frac{x}{(\log x)^k}.$$

Remark. By choosing some small $\epsilon > 0$ and setting $I = [-\epsilon, \epsilon]$, Theorems 1.3 and 1.4 produce corollaries similar to Corollary 1.2.

As an example, consider the genus 2 curve $\mathcal{C} : y^2 = x^5 + 1$. Defining

$$a_{\mathcal{C}}(p) := p + 1 - \#\mathcal{C}(\mathbb{F}_p),$$

we have that if $p \equiv 1 \pmod 5$, then Hasse's bound applied to $\mathcal{C}$ yields $|a_{\mathcal{C}}(p)| \le 4\sqrt{p}$. We expect that for any fixed $t \equiv 6 \pmod{10}$, there are infinitely many primes $p \equiv 1 \pmod 5$ such that $a_{\mathcal{C}}(p) = t$; this is reasonable to expect in light of the Lang-Trotter conjecture for elliptic curves [14]. Assuming the Generalized Riemann Hypothesis for Hecke characters modulo 25 over the field $\mathbb{Q}(e^{2\pi i/5})$, Sarnak [24] proved that

$$\{p \in [x, 2x] : p \equiv 1 \pmod 5,\ |a_{\mathcal{C}}(p)| = O(\log p)\} \gg \sqrt{x},$$

which is analogous to the results in [1]. By Duke's inequality (1.9), we have unconditionally that if $0 < \delta < \frac{1}{12}$, then

$$\#\{p \le x : p \equiv 1 \pmod 5,\ |a_{\mathcal{C}}(p)| \le p^{\frac{1}{2}-\delta}\} \gg \frac{x^{1-2\delta}}{\log x}.$$

Theorem 1.4 tells us that if we fix $0 < \epsilon < \frac{1}{2}$ and define

$$\mathcal{P}_{\mathcal{C},\epsilon} = \{p : p \equiv 1 \pmod 5 \text{ and } |a_{\mathcal{C}}(p)| \le \epsilon\sqrt{p}\},$$

then there exists a constant $c_{\mathcal{C},\epsilon} > 0$ such that if $k > c_{\mathcal{C},\epsilon}$ and $\mathcal{H}_k = \{h_1, \ldots, h_k\}$ is an admissible set, then

$$\begin{aligned}&\#\{n \le x : \#\{h_i \in \mathcal{H}_k : n + h_i \in \mathbb{P},\ |a_{\mathcal{C}}(n+h_i)| \le \epsilon\sqrt{p}\} \ge c_{\mathcal{C},\epsilon}^{-1} \log k\} \\ &\gg \frac{x}{(\log x)^k}.\end{aligned}$$

In order to prove Theorems 1.1, 1.3, and 1.4, we prove a new result, Theorem 3.1, on the distribution of primes in arithmetic progressions in the spirit of the Bombieri-Vinogradov estimate which holds for all of the above Hecke equidistribution problems. We prove Theorem 3.1 in a manner similar to that of Bombieri's original proof of the Bombieri-Vinogradov theorem [2]. Specifically, we use Duke's large sieve inequality for Hecke characters (Theorem 2.1) to prove a zero density estimate (Theorem 2.3) for families of Hecke characters. Once we prove Theorem 2.3 and deduce Theorem 3.1, we prove a general result (Theorem 5.3) on the behavior of gaps between primes which satisfy a given Hecke equidistribution law using Maynard's sieve techniques. At the level of generality presented in this paper, the Bombieri-Vinogradov estimate in Theorem 3.1 appears to be new and may be of independent interest.

2. A zero density estimate for Hecke L-functions

In this section, we prove a zero density estimate for Hecke characters twisted by ray class characters which are averaged over moduli up to Q. This will be an average form of [5, Theorem 2.1] which generalizes the zero density estimates of Montgomery [18, Theorem 1] for Dirichlet characters. The results in this section all follow the method of proof in [5, Section 2].

2.1. The large sieve

Let $K/\mathbb{Q}$ be a number field of degree $n_K := [K:\mathbb{Q}]$ and absolute discriminant D_K. Let $\mathfrak{q}$ be an integral ideal, and let ξ be a narrow class character modulo $\mathfrak{q}$. For a vector $\mathbf{m} = (m_1, m_2, \ldots, m_{n_K-1}) \in \mathbb{Z}^{n_K-1}$, we define

$$\lambda^{\mathbf{m}} = \prod_{j=1}^{n_K-1} \lambda_j^{m_j}, \tag{2.1}$$

where $\{\lambda_1, \ldots, \lambda_{n_K-1}\}$ is a basis for the torsion-free Hecke characters modulo $\mathfrak{q}$. The implied constants in all asymptotic inequalities depend at most on $D_K \mathrm{N}\mathfrak{q}$.

We begin by stating Duke's large sieve inequality [5, Theorem 1.1].

Theorem 2.1. *Let $c(\mathfrak{a})$ be a function on the ideals of*

$$K \quad \text{and} \quad \|c\|^2 = \sum_{\mathrm{N}\mathfrak{a} \le N} |c(\mathfrak{a})|^2.$$

There exists a constant $A > 0$ such that

$$\sum_{\mathrm{N}\mathfrak{q}\le Q}\ \sideset{}{^*}\sum_{\xi \bmod \mathfrak{q}}\ \sum_{\|\mathbf{m}\|_\infty\le T}\int_{-T}^{T}\Big|\sum_{\mathrm{N}\mathfrak{a}\le N} c(\mathfrak{a})\xi\lambda^{\mathbf{m}}(\mathfrak{a})\mathrm{N}\mathfrak{a}^{-it}\Big|^2\,dt$$
$$\ll (N+Q^2T^{n_K})(\log QT)^A\|c\|^2,$$

where * denotes summing over primitive characters, A depends in an effectively computable manner on at most K, and $\|\mathbf{m}\|_\infty = \max_{1\le j\le n_K-1}|m_j|$.

We can write $L(\frac{1}{2}+it,\xi\lambda^{\mathbf{m}})^2$ as a Dirichlet polynomial roughly of the shape

$$\begin{aligned}(2.2)\quad L(\tfrac{1}{2}+it,\xi\lambda^{\mathbf{m}})^2 &= \sum_{\mathfrak{a}} d(\mathfrak{a})\xi\lambda^{\mathbf{m}}(\mathfrak{a})\mathrm{N}\mathfrak{a}^{-\frac{1}{2}-it}g_1(\mathrm{N}\mathfrak{a}/x)\\ &\quad + O\Big(\sum_{\mathfrak{a}} d(\mathfrak{a})\overline{\xi\lambda^{\mathbf{m}}}(\mathfrak{a})\mathrm{N}\mathfrak{a}^{-\frac{1}{2}+it}g_2(\mathrm{N}\mathfrak{a}/y)\Big)+O(1),\end{aligned}$$

where g_1 and g_2 are smooth, compactly supported functions and

$$xy = (D_K\mathrm{N}\mathfrak{f}_\xi(\tfrac{|t|}{2\pi})^{n_K})^2.$$

Here, $\mathfrak{f}_\xi$ is the conductor of ξ. As a straightforward and standard consequence of Theorem 2.1 and (2.2), we obtain the following fourth moment estimate for the Hecke L-functions $L(s,\xi\lambda^{\mathbf{m}})$.

Corollary 2.2. *There exists a constant $A > 0$ such that*

$$\sum_{\mathrm{N}\mathfrak{q}\le Q}\ \sideset{}{^*}\sum_{\xi \bmod \mathfrak{q}}\ \sum_{\|\mathbf{m}\|_\infty\le T}\int_{-T}^{T}|L(\tfrac{1}{2}+it,\xi\lambda^{\mathbf{m}})|^4dt \ll Q^2T^{n_K}(\log QT)^A.$$

2.2. The density estimate

We now prove a zero density estimate which allows us to average over primitive characters with modulus up to a given bound.

Theorem 2.3. *Let $Q \geq 1$, $T \geq 2$,*

$$N(\sigma, T, \xi\lambda^{\mathbf{m}}) := \#\{\rho = \beta + i\gamma : L(\rho, \xi\lambda^{\mathbf{m}}) = 0, \beta \geq \sigma, |\gamma| \leq T\},$$

and

$$N(\sigma, Q, T) := \sum_{\mathrm{N}\mathfrak{q} \leq Q} \sideset{}{^*}\sum_{\xi \bmod \mathfrak{q}} \sum_{\|\mathbf{m}\|_\infty \leq T} N(\sigma, T, \xi\lambda^{\mathbf{m}}).$$

Suppose that $n_K \geq 2$. There exists a constant $A > 0$, depending in an effectively computable manner on at most K, such that for all $\sigma \in [0, 1]$, we have that

$$N(\sigma, Q, T) \ll (Q^2 T^{n_K})^{\frac{3(1-\sigma)}{2-\sigma}} (\log T)^A \ll (Q^2 T^{n_K})^{3(1-\sigma)} (\log QT)^A.$$

Proof. The proof proceeds along the same lines as [5, Theorem 2.1]. Let

$$M_x = M_x(s, \xi\lambda^{\mathbf{m}}) = \sum_{\mathrm{N}\mathfrak{a} \leq x} \mu(\mathfrak{a}) \xi\lambda^{\mathbf{m}}(\mathfrak{a}) \mathrm{N}\mathfrak{a}^{-s},$$

where μ is the Möbius function for K, and let

$$b(\mathfrak{a}) = \sum_{\substack{\mathfrak{d} | \mathfrak{a} \\ \mathrm{N}\mathfrak{a} \leq x}} \mu(\mathfrak{d}).$$

Then for $\Re(s) > 1$, we have that

$$M_x(s) L(s, \xi\lambda^{\mathbf{m}}) = 1 + \sum_{\mathrm{N}\mathfrak{a} > x} b(\mathfrak{a}) \xi\lambda^{\mathbf{m}}(\mathfrak{a}) \mathrm{N}\mathfrak{a}^{-s}.$$

We now choose $y > 0$, and we smooth to get

$$\begin{aligned} &e^{-\frac{1}{y}} + \sum_{\mathrm{N}\mathfrak{a} > x} b(\mathfrak{a}) \xi\lambda^{\mathbf{m}}(\mathfrak{a}) \mathrm{N}\mathfrak{a}^{-s} e^{-\mathrm{N}\mathfrak{a}/y} - M_x(s, \xi\lambda^{\mathbf{m}}) L(s, \xi\lambda^{\mathbf{m}}) \\ &= \frac{1}{2\pi i} \int_{\Re(w) = \frac{1}{2} - \sigma} M_x(s + w, \xi\lambda^{\mathbf{m}}) L(s + w, \xi\lambda^{\mathbf{m}}) \Gamma(w) y^w dw, \end{aligned} \tag{2.3}$$

where $\sigma \in (\frac{1}{2}, 1]$ and $\xi\lambda^{\mathbf{m}}$ is nontrivial. If $\xi\lambda^{\mathbf{m}}$ is trivial, there is another error term whose effect is negligible due to the exponential decay of $\Gamma(2 - s)$ in vertical strips.

Let $\rho=\beta+i\gamma$ be a zero of $L(s,\xi\lambda^{\mathbf{m}})$ with $\beta\geq\sigma$ and $|\gamma|\leq T$. From (2.3), we have that

$$\begin{aligned}\Big|\sum_{\mathrm{N}\mathfrak{a}>x} b(\mathfrak{a})\xi\lambda^{\mathbf{m}}(\mathfrak{a})\mathrm{N}\mathfrak{a}^{-\rho}e^{-\mathrm{N}\mathfrak{a}/y}\Big| \\ +y^{\frac{1}{2}-\sigma}\int_{\gamma-(\log T)^2}^{\gamma+(\log T)^2}|L(\tfrac{1}{2}+it,\xi\lambda^{\mathbf{m}})|\cdot|M_x(\tfrac{1}{2}+it,\xi\lambda^{\mathbf{m}})|dt\gg 1.\end{aligned}$$

There are three possibilities for ρ:

1) $\Big|\sum_{\mathrm{N}\mathfrak{a}>x} b(\mathfrak{a})\xi\lambda^{\mathbf{m}}(\mathfrak{a})\mathrm{N}\mathfrak{a}^{-\rho}e^{-\mathrm{N}\mathfrak{a}/y}\Big|\gg 1.$
2) For some t_ρ such that $|t_\rho-\gamma|<(\log T)^2$, we have $|M_x(\frac{1}{2}+it_\rho,\xi\lambda^{\mathbf{m}})|>x^{\sigma-\frac{1}{2}}$.
3) $\int_{\gamma-(\log T)^2}^{\gamma+(\log T)^2}|L(\frac{1}{2}+it,\xi\lambda^{\mathbf{m}})|dt\gg(y/x)^{\sigma-\frac{1}{2}}$.

For $j=1,2$, and 3, let N_j be the number of zeros ρ satisfying conditions 1, 2, 3, respectively. We choose a subset R_j of zeros from each class for which the associated set of Hecke characters

$$\begin{aligned}\Omega_j=\{\xi\lambda^{\mathbf{m}}(\mathfrak{a})\mathrm{N}\mathfrak{a}^{-i\gamma}:\|\mathbf{m}\|_\infty\leq T,\\ \xi \bmod \mathfrak{q}\text{ primitive, }\mathrm{N}\mathfrak{q}\leq Q,\rho=\beta+i\gamma\in R_j\}\end{aligned}$$

is $\gg(\log T)^2$ well spaced and is such that $N_j\ll|R_j|(\log T)^3$ for $j=1,2,3$. Since

$$\sum_{j=1}^{3}|N_j|\ll(\log T)^3\sum_{j=1}^{3}|R_j|,$$

the theorem follows once we show that $|R_j|\ll(Q^2T^{n_K})^{\frac{3(1-\sigma)}{2-\sigma}}$ for $j=1,2,3$.

For $j=1$, let $\{I_k\}$ be a cover of $[x,y]$ by $O(\log T)$ dyadic intervals of the form $[N_k,2N_k]$. By the Cauchy-Schwarz inequality, Theorem 2.1, and partial summation, we have that

$$\begin{aligned}|R_1| &\ll \sum_{\Omega_1}\left|\sum_{\mathfrak{a}} b(\mathfrak{a})\xi\lambda^{\mathbf{m}}(\mathfrak{a})\mathrm{N}\mathfrak{a}^{-\beta}\right|^2\\ &\ll \log T\sum_k\sum_{\Omega_1}\left|\sum_{\mathrm{N}\mathfrak{a}\in I_k} b(\mathfrak{a})\xi\lambda^{\mathbf{m}}(\mathfrak{a})\mathrm{N}\mathfrak{a}^{-\beta}\right|^2\\ &\ll \log T\sum_k(|I_k|+Q^2T^{n_K}(\log QT)^A)|I_k|^{1-2\sigma}(\log QT)^A\\ &\ll (y^{2-2\sigma}+Q^2T^{n_K}x^{1-2\sigma})(\log QT)^A.\end{aligned}$$

We conclude that $|R_1| \ll (Q^2T^{n_K})^{\frac{3(1-\sigma)}{2-\sigma}}(\log QT)^A$ by setting $y = x^{3/2}$ and $x = (Q^2T^{n_K})^{\frac{1}{2-\sigma}}$. By arguments similar to those in the case of $j = 1$, we conclude that

$$|R_2| \ll (Q^2T^{n_K})^{\frac{3(1-\sigma)}{2-\sigma}}(\log QT)^A.$$

To handle the case of $j = 3$, write $l_\rho = (\gamma - (\log T)^2, \gamma + (\log T)^2)$. We use Hölder's inequality and Theorem 2.2 to obtain

$$\begin{aligned}|R_3|x^{2\sigma-1} &\ll \sum_{\rho\in R_3}\left|\int_{l_\rho}|L(\tfrac{1}{2}+it,\xi\lambda^{\mathbf{m}})|dt\right|^4\\ &\ll (\log T)^A\sum_{\rho\in R_3}\int_{l_\rho}|L(\tfrac{1}{2}+it,\xi\lambda^{\mathbf{m}})|^4dt\\ &\ll (\log T)^A\sum_{\mathrm{N}\mathfrak{q}\le Q}\sum_{\xi\lambda^{\mathbf{m}} \bmod \mathfrak{q}}\sum_{\|\mathbf{m}\|_\infty\le T}\int_{-2T}^{2T}|L(\tfrac{1}{2}+it,\xi\lambda^{\mathbf{m}})|^4dt\\ &\ll Q^2T^{n_K}(\log QT)^A.\end{aligned}$$

Choosing x as before, we see that $|R_3| \ll (Q^2T^{n_K})^{\frac{3(1-\sigma)}{2-\sigma}}(\log QT)^A$. □

3. A Bombieri-Vinogradov estimate

We now prove a mean value theorem of Bombieri-Vinogradov type for prime ideals of a number field $K/\mathbb{Q}$ which satisfy an equidistribution law dictated by Hecke characters. Let $K/\mathbb{Q}$ be a number field of degree $n_K = [K : \mathbb{Q}]$ and discriminant D_K with ring of integers $\mathcal{O}_K$, and let $\mathfrak{q}$ be an integral ideal of $\mathcal{O}_K$. Choose a fixed set of independent Hecke characters $H = \{\lambda_1, \ldots, \lambda_J\}$

modulo $\mathfrak{q}$, so $1 \leq J \leq n_K - 1$. If $J = 1$, suppose that λ_1 has infinite order. For a prime ideal $\mathfrak{p} \nmid \mathfrak{q}$, define

$$\vec{\theta}_H(\mathfrak{p}) = (\theta_1(\mathfrak{p}), \ldots, \theta_J(\mathfrak{p})) \in (\mathbb{R}/\mathbb{Z})^J, \quad \lambda_j(\mathfrak{p}) = e^{2\pi i \theta_j(\mathfrak{p})} \text{ for } 1 \leq j \leq J.$$

Let $\mathfrak{I}$ be a narrow ideal class modulo $\mathfrak{q}$. For a collection of closed subintervals $\mathcal{I} = \{I_1, \ldots, I_J\}$ of $[0, 1]$, we consider the set of primes

(3.1)
$$\mathcal{P}_{\mathcal{I},\mathfrak{I},\mathfrak{q}} = \left\{ p\colon \text{there exists } \mathfrak{p} \subset \mathcal{O}_K \text{ such that } p = \mathrm{N}\mathfrak{p},\ \mathfrak{p} \in \mathfrak{I},\ \vec{\theta}_H(\mathfrak{p}) \in \prod_{I \in \mathcal{I}} I \right\},$$

where $\theta_j(\mathfrak{p})$ is equidistributed in I_j with respect to the Lebesgue measure μ for all $1 \leq j \leq J$. We now define the prime counting function

$$\pi_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(x; q, a) := \#\{p \leq x : p \equiv a \pmod{q}, p \in \mathcal{P}_{\mathcal{I},\mathfrak{I},\mathfrak{q}}\}.$$

Let $\delta_{\mathcal{I},\mathfrak{I},\mathfrak{q}}$ denote the density of $\mathcal{P}_{\mathcal{I},\mathfrak{I},\mathfrak{q}}$ within the set of all primes. One computes

$$\delta_{\mathcal{I},\mathfrak{I},\mathfrak{q}} = \frac{1}{2^r h_K \varphi(\mathfrak{q})} \prod_{I \in \mathcal{I}} \mu(I),$$

where r is a certain positive integer which is no larger than the number of real embeddings of K into $\mathbb{C}$, h_K is the class number of K, and $\varphi(\mathfrak{q})$, which is the ideal-theoretic generalization of Euler's function. (See the proof of [5, Theorem 3.1] for further discussion.) The goal of this section is to prove the following theorem.

Theorem 3.1. *If $0 < \theta < \frac{1}{6n_K}$ is fixed, then for any fixed $D > 0$, we have that*

$$\sideset{}{'}\sum_{q \leq x^\theta} \max_{(a,q)=1} \max_{y \leq x} \left| \pi_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(y; q, a) - \delta_{\mathcal{I},\mathfrak{I},\mathfrak{q}} \frac{\pi(y)}{\varphi(q)} \right| \ll \frac{x}{(\log x)^D},$$

where $\sum'$ denotes summing over moduli q such that $(q, D_K \mathrm{N}\mathfrak{q}) = 1$.

Theorem 3.1 is a consequence of the following proposition.

Proposition 3.2. *Fix $D > 0$, and for any integer $k \geq 0$, define*

$$\Theta_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(x; q, a, k) = \sum_{\substack{p \leq x \\ p \equiv a \pmod q \\ p \in \mathcal{P}_{\mathcal{I},\mathfrak{I},\mathfrak{q}}}} \frac{\log p}{k!} \left(\log \frac{x}{p} \right)^k .$$

If $0 < \theta < \frac{1}{6n_K}$, *then*

$$\sideset{}{'}\sum_{q \leq x^\theta} \max_{(a,q)=1} \max_{y \leq x} \left| \Theta_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(y; q, a, 3n_K) - \delta_{\mathcal{I},\mathfrak{I},\mathfrak{q}} \frac{y}{\varphi(q)} \right| \ll \frac{x}{(\log x)^D}.$$

Assuming Proposition 3.2 (which we will prove later), we prove Theorem 3.1.

Proof of Theorem 3.1. Since $\Theta_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(y; q, a, k)$ is a monotonically increasing function of y, it is straightforward to verify that for any $0 < \lambda \leq 1$,

$$\begin{aligned} \frac{1}{\lambda} \int_{e^{-\lambda} y}^{y} \Theta_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(t; q, a, k-1) \frac{dt}{t} &\leq \Theta_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(y; q, a, k-1) \\ &\leq \frac{1}{\lambda} \int_{y}^{e^{\lambda} y} \Theta_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(t; q, a, k-1) \frac{dt}{t}. \end{aligned}$$

One can evaluate the integrals directly; they are

$$\begin{aligned} &\Theta_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(y; q, a, k) - \Theta_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(e^{-\lambda} y; q, a, k) \quad \text{and} \\ &\Theta_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(e^{\lambda} y; q, a, k) - \Theta_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(y; q, a, k), \end{aligned}$$

respectively. Defining $R_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(y; q, a, k) = \Theta_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(y; q, a, k) - \delta_{\mathcal{I},\mathfrak{I},\mathfrak{q}} \frac{y}{\varphi(q)}$, the two integrals are each

$$(1 + O(\lambda)) \delta_{\mathcal{I},\mathfrak{I},\mathfrak{q}} \frac{y}{\varphi(q)} + O\left(\frac{1}{\lambda} \max_{y \leq ex} |R_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(y; q, a, k)| \right).$$

Thus

$$\max_{y \leq x} |R_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(y; q, a, k-1)| \ll \frac{\lambda x}{\varphi(q)} + \frac{1}{\lambda} \max_{y \leq ex} |R_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(y; q, a, k)|.$$

Using decreasing induction, we conclude that

$$\max_{y \leq x} |R_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(y; q, a, 0)| \ll \frac{\lambda_1 x}{\varphi(q)} + \frac{1}{\lambda_2} \max_{y \leq e x^{\lambda_3}} |R_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(y; q, a, k)|,$$

where

$$\lambda_1 = \lambda \sum_{j=0}^{k-1} \exp\left(\sum_{i=1}^{j} \lambda^i\right), \quad \lambda_2 = \lambda^{k(k+1)/2}, \quad \lambda_3 = \sum_{j=1}^{k} \lambda^j.$$

For any $D > 0$, let $\lambda = (\log x)^{-(D+2)}$. We have $\lambda_3 \le k\lambda$, and if x is sufficiently large, then $\lambda_1 \ll_k \lambda$. Thus

$$\max_{y \le x} |R_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(y;q,a,0)| \ll_k (\log x)^{k(k+1)(D+2)/2} \max_{y \le x e^{k/(\log x)^{D+2}}} |R_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(y;q,a,k)| + \frac{x}{\varphi(q)(\log x)^{D+2}}.$$

Using the bound

$$q^{-1} \ll \varphi(q)^{-1} \ll q^{-1} \log\log q, \tag{3.2}$$

we have that

$$\sum_{q \le Q}{}' \max_{(a,q)=1} \max_{y \le x} |R_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(y;q,a,0)| \tag{3.3}$$

is bounded by

$$O_k\left(\frac{x(\log Q)^2}{(\log x)^{D+2}} + (\log x)^{(D+2)k(k+1)/2} \times \sum_{q \le Q}{}' \max_{(a,q)=1} \max_{y \le x e^{k/(\log x)^{D+2}}} |R_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(y;q,a,k)|\right).$$

Since $e^{k/(\log x)^{D+2}}$ is bounded when $D > 0$, (3.3) is

$$\ll_k \frac{x(\log Q)^2}{(\log x)^{D+2}} + (\log x)^{(D+2)k(k+1)/2} \sum_{q \le Q}{}' \max_{(a,q)=1} \max_{y \le x} |R_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(y;q,a,k)|.$$

We apply Proposition 3.2 with $k = 3n_K$; thus for any $A > 0$, (3.3) is

$$\ll \frac{x}{(\log x)^D} + (\log x)^{2n_K(3n_K+1)(D+2)} \frac{x}{(\log x)^A}$$

when $Q = x^\theta$ and $0 < \theta < \frac{1}{6n_K}$. Choosing $A = D + 2n_K(3n_K+1)(D+2)$, we see that (3.3) is $\ll x(\log x)^{-D}$. Thus we have demonstrated that if $0 <$

$\theta < \frac{1}{6n_K}$, then

$$\sideset{}{'}\sum_{q \le x^{\theta}} \max_{(a,q)=1} \max_{y \le x} \left| \Theta_{\mathcal{I},\mathfrak{J},\mathfrak{q}}(y; q, a, 0) - \delta_{\mathcal{I},\mathfrak{J},\mathfrak{q}} \frac{y}{\varphi(q)} \right| \ll \frac{x}{(\log x)^D}.$$

Theorem 3.1 now follows from a standard partial summation argument. □

4. Proof of Proposition 3.2

We begin with a series of reductions which reduce the proof to a calculation involving the zero density estimate from Theorem 2.3. Unless otherwise specified, all implied constants in this section depend in an effectively computable manner on at most $D_K \mathrm{N}\mathfrak{q}$. (Certain implied constants will depend on J, but $J \le n_K - 1$. Dependence on $\delta_{\mathcal{I},\mathfrak{J},\mathfrak{q}}$ does not affect our calculations because $\delta_{\mathcal{I},\mathfrak{J},\mathfrak{q}} \in [0,1]$.)

4.1. Equidistribution on tori

We need a J-dimensional version of the Erdős-Turán inequality in order to count the primes in $\mathcal{P}_{\mathcal{I},\mathfrak{J},\mathfrak{q}}$. Let $e(t) = \exp(2\pi i t)$ and $\mathbf{m} = (m_1, \ldots, m_J)$. Let $\mathbf{1}_S$ denote the indicator function of a set S, and let $B = \prod_{I \in \mathcal{I}} I$.

Lemma 4.1. *For each integer $k \ge 0$ and all $x \ge 3$, we have the bound*

$$\left| \sum_{\mathrm{N}\mathfrak{p} \le x} \mathbf{1}_B(\vec{\theta}_H(\mathfrak{p})) \frac{\log \mathrm{N}\mathfrak{p}}{k!} \left(\log \frac{x}{\mathrm{N}\mathfrak{p}} \right)^k - \left(\prod_{I \in \mathcal{I}} \mu(I) \right) x \right| \\ \ll \frac{x}{T} + \sum_{\|\mathbf{m}\|_\infty \le T} \left| \sum_{\mathrm{N}\mathfrak{p} \le x} \lambda^{\mathbf{m}}(\mathfrak{p}) \frac{\log \mathrm{N}\mathfrak{p}}{k!} \left(\log \frac{x}{\mathrm{N}\mathfrak{p}} \right)^k - \delta(\lambda^{\mathbf{m}}) x \right|, \tag{4.1}$$

where $\delta(\lambda^{\mathbf{m}}) = 1$ if $\lambda^{\mathbf{m}}$ is the trivial character and $\delta(\lambda^{\mathbf{m}}) = 0$ otherwise.

The proof proceeds along familiar lines. One uses the multidimensional variant of the Beurling–Selberg construction proved by Cochrane [3, Theorem 1] to majorize and minorize $\mathbf{1}_B$ using explicitly defined finite trigonometric polynomials. This leads to a multidimensional version of the Erdös–Turán inequality [3, Theorem 2]. This is adapted for equidistribution with respect to probability measures other than normalized Lebesgue measure by adapting the proof of [3, Theorem 2] along the lines of Murty and Sinha

[21, Theorem 8]. Since the only noticeable deviation from the calculations in [3, 21] is that the bound $\frac{x}{T}$ uses a trivial upper bound for the prime ideal counting function (see Lagarias and Odlyzko [13]), we omit the proof.

4.2. From primes to zeros

Let χ be a Dirichlet character modulo q, let $\lambda^{\mathbf{m}}$ be as in (2.1), and let ξ be a narrow class character modulo $\mathfrak{q}$. We will use the narrow class characters to isolate the prime ideals $\mathfrak{p}$ of K whose norm is a prime p, and we will use the Dirichlet characters modulo q to isolate the primes p which lie in an arithmetic progression $a \pmod q$ with $(a,q)=1$. We make the restriction $(q, D_K\mathrm{N}\mathfrak{q})=1$ so that $\xi\lambda^{\mathbf{m}}\otimes\chi(\mathfrak{a}) = \xi\lambda^{\mathbf{m}}(\mathfrak{a})\chi(\mathrm{N}\mathfrak{a})$.

Define

$$\delta(\xi\lambda^{\mathbf{m}}\otimes\chi) = \begin{cases} 1 & \text{if } \xi\lambda^{\mathbf{m}}\otimes\chi \text{ is trivial} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Lambda_{\xi\lambda^{\mathbf{m}}\otimes\chi}(\mathfrak{a}) = \begin{cases} \xi\lambda^{\mathbf{m}}\otimes\chi(\mathfrak{p}^m)\log\mathrm{N}\mathfrak{p} & \text{if } \mathfrak{a}=\mathfrak{p}^m \text{ for some prime} \\ & \text{ideal } \mathfrak{p} \text{ and } m\geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

By a procedure similar to that in [20, Sections 0 and 1], it follows from (4.1) and orthogonality of characters that

$$\begin{aligned} &\left|\Theta_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(x;q,a,k) - \delta_{\mathcal{I},\mathfrak{I},\mathfrak{q}}\frac{x}{\varphi(q)}\right| \\ &\ll \frac{1}{\varphi(q)}\left(\frac{x}{T} + \sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \xi \bmod \mathfrak{q}}} \sum_{\|\mathbf{m}\|_\infty\leq T} \left|\sum_{\mathrm{N}\mathfrak{p}\leq x} \frac{\Lambda_{\xi\lambda^{\mathbf{m}}\otimes\chi}(\mathfrak{p})}{k!}\left(\log\frac{x}{\mathrm{N}\mathfrak{p}}\right)^k - \delta(\xi\lambda^{\mathbf{m}}\otimes\chi)x\right|\right). \end{aligned}$$

Now,

$$\begin{aligned}
&\left|\sum_{\mathrm{N}\mathfrak{p}\le x}\frac{\Lambda_{\xi\lambda^{\mathbf{m}}\otimes\chi}(\mathfrak{p})}{k!}\left(\log\frac{x}{\mathrm{N}\mathfrak{p}}\right)^k-\sum_{\mathrm{N}\mathfrak{a}\le x}\frac{\Lambda_{\xi\lambda^{\mathbf{m}}\otimes\chi}(\mathfrak{a})}{k!}\left(\log\frac{x}{\mathrm{N}\mathfrak{a}}\right)^k\right|\\
&=\left|\sum_{\substack{v\ge 2\\ \mathrm{N}\mathfrak{p}^v\le x}}\frac{\xi\lambda^{\mathbf{m}}\otimes\chi(\mathfrak{p}^v)\log\mathrm{N}\mathfrak{p}}{k!}\left(\log\frac{x}{\mathrm{N}\mathfrak{p}^v}\right)^k\right|\\
&\ll\sum_{\substack{v\ge 2\\ \mathrm{N}\mathfrak{p}^v\le x}}\frac{\log\mathrm{N}\mathfrak{p}}{k!}\left(\log\frac{x}{\mathrm{N}\mathfrak{p}^v}\right)^k\ll\frac{1}{k!}\sqrt{x}.
\end{aligned}$$

If the character $(\xi\lambda^{\mathbf{m}}\otimes\chi)'$ is induced by a primitive character $\xi\lambda^{\mathbf{m}}\otimes\chi$, where χ has modulus q, then

$$\begin{aligned}
&\sum_{\mathrm{N}\mathfrak{a}\le x}\Lambda_{\xi\lambda^{\mathbf{m}}\otimes\chi}(\mathfrak{a})\left(\log\frac{x}{\mathrm{N}\mathfrak{a}}\right)^k-\sum_{\mathrm{N}\mathfrak{a}\le x}\Lambda_{(\xi\lambda^{\mathbf{m}}\otimes\chi)'}(\mathfrak{a})\left(\log\frac{x}{\mathrm{N}\mathfrak{a}}\right)^k\\
&\ll(\log x)^{k+1}\log q.
\end{aligned}$$

By a standard application of Mellin inversion,

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{x^s}{s^{k+1}}ds=\begin{cases}\frac{1}{k!}(\log x)^k & \text{if } x\ge 1,\\ 0 & \text{if } x<1.\end{cases}\tag{4.2}$$

If $\xi\lambda^{\mathbf{m}}\otimes\chi$ is primitive, then we use (4.2) and the argument principle to obtain

$$\begin{aligned}
\sum_{\mathrm{N}\mathfrak{a}\le x}\frac{\Lambda_{\xi\lambda^{\mathbf{m}}\otimes\chi}(\mathfrak{a})}{k!}\left(\log\frac{x}{\mathrm{N}\mathfrak{a}}\right)^k&=\frac{1}{2\pi i}\int_{2-\infty}^{2+i\infty}\left(-\frac{L'}{L}(s,\xi\lambda^{\mathbf{m}}\otimes\chi)\right)\frac{x^s}{s^{k+1}}ds\\
&=\delta(\xi\lambda^{\mathbf{m}}\otimes\chi)x-\sum_{0<\beta<1}\frac{x^\rho}{\rho^{k+1}}+O((\log x)^{2k}),
\end{aligned}$$

where the sum extends over the nontrivial zeros $\rho=\beta+i\gamma$ of $L(s,\xi\lambda^{\mathbf{m}}\otimes\chi)$.

To simplify future expressions, we choose $T=x^\theta$ for some constant $0<\theta<\frac{1}{2}$ (which will be determined later) and $q\le x$. We now combine all of the above estimates in this subsection to obtain

$$\begin{aligned}&\left|\Theta_{\mathcal{I},\mathfrak{J},\mathfrak{q}}(x;q,a,k)-\delta_{\mathcal{I},\mathfrak{J},\mathfrak{q}}\frac{x}{\varphi(q)}\right|\\ &\ll\frac{1}{\varphi(q)}\left(\frac{x}{T}+\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \xi \bmod \mathfrak{q}}}\sum_{\|\mathbf{m}\|_\infty\leq T}\left(\sum_{\substack{0<\beta<1\\ |\gamma|\leq T}}\frac{x^\beta}{|\rho|^{k+1}}+\sum_{\substack{0<\beta<1\\ |\gamma|>T}}\frac{x^\beta}{|\gamma|^{k+1}}\right)\right)+T^{n_K}\sqrt{x}.\end{aligned}$$

The innermost sums run over the nontrivial zeros $\rho=\beta+i\gamma$ of the L-function associated to the primitive character that induces $\xi\lambda^{\mathbf{m}}\otimes\chi$.

Let $t\in\mathbb{R}$. By [5, Lemma 3.1.3], over all $L(s,\xi\lambda^{\mathbf{m}}\otimes\chi)$ satisfying $\|\mathbf{m}\|_\infty\leq T$, there are $\ll q^{n_K}T^{n_K-1}\log(q(|t|+1))$ zeros ρ of $L(s,\xi\lambda^{\mathbf{m}}\otimes\chi)$ such that $|\mathrm{Im}(\rho)-t|<1$. Thus

$$\begin{aligned}\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \xi \bmod \mathfrak{q}}}\sum_{\|\mathbf{m}\|_\infty\leq T}\sum_{\substack{0<\beta<1\\ |\gamma|>T}}\frac{x^\beta}{|\gamma|^{k+1}}&\ll q^{n_K}T^{n_K-1}x\int_T^\infty\frac{\log(qt)}{t^{k+1}}dt\\ &\ll q^{n_K}T^{n_K-k-1}x\log x.\end{aligned}$$

Consider the contribution of the zeros ρ satisfying $|\rho|<\frac{1}{4}$. Note that if $\mathfrak{f}$ is the conductor of the primitive character $\xi\lambda^{\mathbf{m}}\otimes\chi$, χ has modulus q, and $(q,D_K\mathrm{N}\mathfrak{q})$, then $\mathrm{N}\mathfrak{f}\ll q^{n_K}$. Applying [5, Lemma 3.1.3] again and noting that $q\leq x$, we have that there are $\ll q^{n_K}T^{n_K-1}\log x$ zeros ρ such that $|\rho|<\frac{1}{4}$. From the consideration of the corresponding zeros $1-\rho$ of the L-function which is dual to $L(s,\xi\lambda^{\mathbf{m}}\otimes\chi)$, we deduce that $|\rho|^{k+1}>x^{-\epsilon}$ for any fixed $\epsilon>0$, provided that q is sufficiently large. (See Bombieri [2] and Prachar [23, Chapter 7] for similar arguments.) As such, the contribution arising from zeros ρ satisfying $|\rho|<\frac{1}{4}$ is

$$\ll\sum_{\substack{\chi \bmod q\\ \xi \bmod \mathfrak{q}}}\sum_{\|\mathbf{m}\|_\infty\leq T}\sum_{|\rho|<\frac{1}{4}}x^{\rho+\epsilon}\ll q^{n_K}T^{n_K-1}x^{\frac{1}{4}+\epsilon}\log x.$$

Regarding the zeros $\rho=\beta+i\gamma$ with $|\rho|\geq\frac{1}{4}$, we take only those with $\beta\geq\frac{1}{2}$, as the contribution from the other zeros are safely absorbed into our existing error terms. For those zeros, $|\rho|^{k+1}\gg|\gamma|^{k+1}+1$. Therefore, we finally have (for $q, T \leq x$) that

$$\max_{(a,q)=1} \max_{y \leq x} \left| \Theta_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(x; q, a, k) - \delta_{\mathcal{I},\mathfrak{I},\mathfrak{q}} \frac{x}{\varphi(q)} \right|$$
$$\ll \frac{1}{\varphi(q)} \left(\frac{x}{T} + \sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \xi \bmod \mathfrak{q}}} \sum_{\|\mathbf{m}\|_\infty \leq T} \sum_{\substack{\frac{1}{2} \leq \beta < 1 \\ |\gamma| \leq T}} \frac{x^\beta}{|\gamma|^{k+1} + 1} \right)$$
$$+ q^{n_K} T^{n_K - 1} \sqrt{x} \left(1 + \frac{\sqrt{x}}{T^k} \right) \log x.$$

4.3. Averaging over progressions

It follows from the above display and (3.2) that

$$\sideset{}{'}\sum_{q \leq Q} \max_{(a,q)=1} \max_{y \leq x} \left| \Theta_{\mathcal{I},\mathfrak{I},\mathfrak{q}}(x; q, a, k) - \delta_{\mathcal{I},\mathfrak{I},\mathfrak{q}} \frac{x}{\varphi(q)} \right| \tag{4.3}$$
$$\ll \sideset{}{'}\sum_{q \leq Q} \frac{1}{\varphi(q)} \sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \xi \bmod \mathfrak{q}}} \sum_{\|\mathbf{m}\|_\infty \leq T} \sum_{\substack{\frac{1}{2} \leq \beta < 1 \\ |\gamma| \leq T}} \frac{x^\beta}{|\gamma|^{k+1} + 1}$$
$$+ Q^{n_K + 1} T^{n_K - 1} \sqrt{x} \left(1 + \frac{\sqrt{x}}{T^k} \right) \log x + \frac{x (\log Q)^2}{T}.$$

We dyadically decompose the interval $[1, Q]$ into $O(\log Q)$ intervals of the shape $[2^{n-1}, 2^n)$ and use (3.2) to obtain

$$\sideset{}{'}\sum_{q \leq Q} \frac{1}{\varphi(q)} \sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \xi \bmod \mathfrak{q}}} \sum_{\|\mathbf{m}\|_\infty \leq T} \sum_{\substack{\frac{1}{2} \leq \beta < 1 \\ |\gamma| \leq T}} \frac{x^\beta}{|\gamma|^{k+1} + 1} \tag{4.4}$$
$$\ll (\log Q)^2 \max_{R \leq Q} \frac{1}{R} \sideset{}{'}\sum_{q \leq R} \sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \xi \bmod \mathfrak{q}}} \sum_{\|\mathbf{m}\|_\infty \leq T} \sum_{\substack{\frac{1}{2} \leq \beta < 1 \\ |\gamma| \leq T}} \frac{x^\beta}{|\gamma|^{k+1} + 1}.$$

If χ has modulus $q \leq R$ and $\xi \otimes \chi$ is primitive, then we may realize $\xi \otimes \chi$ as a primitive (narrow) ray class character $\omega \pmod{\mathfrak{a}}$, where $\mathrm{N}\mathfrak{a} \ll R^{n_K}$. Therefore, we embed (4.4) into a sum over *all* primitive ray class characters $\omega \pmod{\mathfrak{a}}$ with $\mathrm{N}\mathfrak{a} \ll R^{n_K}$. Specifically, if $\rho = \beta + i\gamma$ is a zero of $L(s, \omega\lambda^{\mathbf{m}})$

with $\omega \pmod{\mathfrak{a}}$ a primitive ray class character, then (4.4) is

$$\ll \max_{R\le Q}\frac{1}{R}\sum_{\mathrm{N}\mathfrak{a}\le R^{n_K}}\sideset{}{^*}\sum_{\omega \bmod \mathfrak{a}}\sum_{\|\mathbf{m}\|_\infty\le T}\sum_{\substack{\frac{1}{2}\le\beta<1\\ |\gamma|\le T}}\frac{x^\beta}{|\gamma|^{k+1}+1}$$
$$\ll \log x \max_{R\le Q}\max_{\frac{1}{2}\le\sigma<1}\max_{0\le V\le T}\frac{x^\sigma}{R}\frac{N(\sigma,R^{n_K},V)}{V^{k+1}+1}.$$

Let $0<\epsilon<1$, and let

$$k=3n_K,\quad Q=x^{\frac{1-\epsilon}{6n_K}},\quad \text{and}\quad T=x^{\frac{1+\epsilon}{6n_K}}. \tag{4.5}$$

Collecting all of the above estimates, (4.3) is

$$\ll x^{1-\frac{1}{6n_K}}+(\log x)^3\max_{R\le Q}\max_{\frac{1}{2}\le\sigma<1}\max_{0\le V\le T}\frac{x^\sigma}{R}\frac{N(\sigma,R^{n_K},V)}{V^{k+1}+1}, \tag{4.6}$$

which we will bound using Theorem 2.3. (We could take k to be larger if we wanted, but $k=3n_K$ suffices for our purposes.)

4.4. Finishing the proof

By the zero-free region for Hecke L-functions proven by Coleman [4], there exists a constant $c>0$ such that if $\|\mathbf{m}\|_\infty\le T$, the modulus of ω is at most R^{n_K}, and

$$1-\eta(R,x)<\sigma\le 1,\quad \eta(R,x):=c\max\{\log R,(\log x)^{3/4}\}^{-1}, \tag{4.7}$$

then $N(\sigma,R^{n_K},t)$ is either 0 or 1 for all $t\le T$. If $N(\sigma,R^{n_K},t)=1$, then the zero β_1 which is counted is a Landau-Siegel zero, which is both real and simple. If $L(s,\omega\lambda^{\mathbf{m}})$ has a Landau-Siegel zero, then $\lambda^{\mathbf{m}}$ is trivial and ω is a real character. It follows from the field-uniform version of Siegel's theorem in [20, Section 2] that $x^{\beta_1}\ll x(\log x)^{-D-5}$ for any $D>0$ (with an ineffective implied constant). Thus it suffices for us to work in the range $\frac{1}{2}\le\sigma\le 1-\eta(R,x)$.

By (4.5), if $\frac{1}{2}\le\sigma\le 1-\eta(R,x)$, it follows from Theorem 2.3 that

$$\max_{0\le V\le T}\frac{x^\sigma}{R}\frac{N(\sigma,R^{n_K},V)}{V^{k+1}+1}\ll\frac{x^\sigma}{R}N(\sigma,R^{n_K},2)\ll\frac{x^\sigma}{R}R^{6n_K(1-\sigma)}(\log x)^A.$$

Therefore, the second term in (4.6) is bounded by

$$
\begin{aligned}
&(\log x)^{A+1} \max_{R\le Q} \max_{\frac{1}{2}\le\sigma<1-\eta(R,x)} \frac{x^{\sigma}}{R} R^{6n_K(1-\sigma)} \\
&= (\log x)^{A+1} \max_{R\le Q} \frac{x}{R}\left(\frac{R^{6n_K}}{x}\right)^{\eta(R,x)}.
\end{aligned}
$$

By separately considering the cases when

$$
R \le \exp((\log x)^{3/4}) \quad \text{and} \quad \exp((\log x)^{3/4}) < R \le Q,
$$

we find that

$$
(\log x)^{A+1} \max_{R\le Q} \frac{x}{R}\left(\frac{R^{6n_K}}{x}\right)^{\eta(R,x)} \ll \frac{x}{(\log x)^{D+5}}. \tag{4.8}
$$

Combining (4.8) and with the contribution from the Landau-Siegel zero (if it exists), we see that (4.6), and hence (4.3), is bounded by $x(\log x)^{-D}$ for any $D > 0$ once $Q = x^{\theta}$ with $0 \le \theta < \frac{1}{6n_K}$ per (4.5). This completes the proof of Proposition 3.2.

5. Proofs of Theorems 1.1, 1.3, and 1.4

We will use Theorem 3.1 to prove a very general result on bounded gaps between primes in sets of the form (3.1), from which we will deduce the theorems in the introduction. Given a set of integers $\mathfrak{A}$, a set of primes $\mathfrak{P} \subset \mathfrak{A}$, and a linear form $L(n) = n + h$, define

$$
\begin{aligned}
&\mathfrak{A}(x) = \{n \in \mathfrak{A} : x < n \le 2x\}, \quad \mathfrak{A}(x; q, a) = \{n \in \mathfrak{A}(x) : n \equiv a \pmod{q}\}, \\
&L(\mathfrak{A}) = \{L(n) : n \in \mathfrak{A}\}, \quad \varphi_L(q) = \varphi(hq)/\varphi(h), \\
&\mathfrak{P}_{L,\mathfrak{A}}(x, y) = L(\mathfrak{A}(x)) \cap \mathfrak{P}, \quad \mathfrak{P}_{L,\mathfrak{A}}(x; q, a) = L(\mathfrak{A}(x; q, a)) \cap \mathfrak{P}.
\end{aligned}
$$

We consider the 6-tuple $(\mathfrak{A}, \mathcal{L}_k, \mathfrak{P}, B, x, \theta)$, where $\mathcal{H}_k$ is admissible, $\mathcal{L}_k = \{L_i(n) = n + h_i : h_i \in \mathcal{H}_k\}$, $B \in \mathbb{N}$ is constant, x is a large real number, and $0 \le \theta < 1$. We present a very general hypothesis that Maynard states in Section 2 of [17].

Hypothesis 5.1. *With the above notation, consider the 6-tuple* $(\mathfrak{A}, \mathcal{L}_k, \mathfrak{P}, B, x, \theta)$.

1) We have

$$\sum_{q\le x^\theta}\max_a\left|\#\mathfrak{A}(x;q,a)-\frac{\#\mathfrak{A}(x)}{q}\right|\ll\frac{\#\mathfrak{A}(x)}{(\log x)^{100k^2}}.$$

2) For any $L\in\mathcal{L}_k$, we have

$$\sum_{\substack{q\le x^\theta\\(q,B)=1}}\max_{(L(a),q)=1}\left|\#\mathfrak{P}_{L,\mathfrak{A}}(x;q,a)-\frac{\#\mathfrak{P}_{L,\mathfrak{A}}(x)}{\varphi_L(q)}\right|\ll\frac{\#\mathfrak{P}_{L,\mathfrak{A}}(x)}{(\log x)^{100k^2}}.$$

3) For any $q\le x^\theta$, we have $\#\mathfrak{A}(x;q,a)\ll\dfrac{\#\mathfrak{A}(x)}{q}$.

For $(\mathfrak{A},\mathcal{L}_k,\mathfrak{P},B,x,\theta)$ satisfying Hypothesis 5.1, Maynard proves the following in [17].

Theorem 5.2. *Let $\alpha>0$ and $0\le\theta<1$. There is a constant c depending only on θ and α so that the following holds. Let $(\mathfrak{A},\mathcal{L}_k,\mathfrak{P},B,x,\theta)$ satisfy Hypothesis 5.1. Assume that $c\le k\le(\log x)^\alpha$ and $h_i\le x^\alpha$ for all $1\le i\le k$. If there exists $\delta>(\log k)^{-1}$ such that*

$$\frac{1}{k}\frac{\varphi(B)}{B}\sum_{L_i\in\mathcal{L}_k}\#\mathfrak{P}_{L_i,\mathfrak{A}}(x)\ge\delta\frac{\#\mathfrak{A}(x)}{\log x},$$

then

$$\#\{n\in\mathfrak{A}(x):\#(\mathcal{L}_k(n)\cap\mathfrak{P})\ge c^{-1}\delta\log k\}\gg\frac{\#\mathfrak{A}(x)}{(\log x)^k\exp(ck)}.$$

Using Theorem 5.2, we prove the following result.

Theorem 5.3. *Let $\mathcal{P}_{\mathcal{I},\mathfrak{I},\mathfrak{q}}$ be a set of the form* (3.1). *There exists a constant $c_{\mathcal{I},\mathfrak{I},\mathfrak{q}}>0$ such that if $k>c_{\mathcal{I},\mathfrak{I},\mathfrak{q}}$ and $\mathcal{H}_k=\{h_1,\dots,h_k\}$ is admissible, then*

$$\#\{n\le x:\#(\{n+h_1,\dots,n+h_k\}\cap\mathcal{P}_{\mathcal{I},\mathfrak{I},\mathfrak{q}})\ge c_{\mathcal{I},\mathfrak{I},\mathfrak{q}}^{-1}\log k\}\gg\frac{x}{(\log x)^k}.$$

Proof. The proof is essentially the same as Theorem 3.5 in [17]. Let θ be as in Theorem 3.1. Let $\mathfrak{A}=\mathbb{N}$, $\mathfrak{P}=\mathcal{P}_{\mathcal{I},\mathfrak{I},\mathfrak{q}}$, and $B=D_K\mathrm{N}\mathfrak{q}$. Parts (i) and (iii) of Hypothesis 5.1 are trivial to check for the 6-tuple $(\mathbb{N},\mathcal{L}_k,\mathcal{P}_{\mathcal{I},\mathfrak{I},\mathfrak{q}},D_K\mathrm{N}\mathfrak{q},x,\frac{\theta}{2})$. By Theorem 3.1 partial summation, all of Hypothesis 5.1 holds when D and x are sufficiently large in terms of k and θ.

Given a suitable constant $c_{\mathcal{I},\mathfrak{J},\mathfrak{q}} > 0$ (computed as in [16] and [25]), we let $k \geq c_{\mathcal{I},\mathfrak{J},\mathfrak{q}}$. For our choice of $\mathfrak{A}$ and $\mathfrak{P}$, we have the inequality

$$\frac{1}{k}\frac{\varphi(B)}{B}\sum_{L_i\in\mathcal{L}_k}\#\mathfrak{P}_{L_i,\mathfrak{A}}(x) \geq (1+o(1))\frac{\varphi(B)}{B}\delta_{\mathcal{I},\mathfrak{J},\mathfrak{q}}\frac{\#\mathfrak{A}(x)}{\log x}$$

for all sufficiently large x, where the implied constant in $1+o(1)$ depends only $D_K\mathrm{N}\mathfrak{q}$. Theorem 5.3 now follows directly from Theorem 5.2. $\square$

To prove Theorems 1.1, 1.3, and 1.4, it now suffices to show that the sets of primes considered in the respective theorems are all of the form (3.1) for certain sets of independent Hecke characters. For Theorems 1.1 and 1.4, this is accomplished in the proofs of Theorems 3.2 and 3.3 in [5], respectively. For Theorem 1.3, the desired correspondence follows from the discussion in [12, Chapter 12].

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