Localized energy for wave equations with degenerate trapping

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Localized energy estimates have become a fundamental tool when studying wave equations in the presence of asymptotically flat background geometry. Trapped rays necessitate a loss when compared to the estimate on Minkowski space. A loss of regularity is a common way to incorporate such. When trapping is sufficiently weak, a logarithmic loss of regularity suffices. Here, by studying a warped product manifold introduced by Christianson and Wunsch, we encounter the first explicit example of a situation where an estimate with an algebraic loss of regularity exists and this loss is sharp. Due to the global-in-time nature of the estimate for the wave equation, the situation is more complicated than for the Schrödinger equation. An initial estimate with sub-optimal loss is first obtained, where extra care is required due to the low frequency contributions. An improved estimate is then established using energy functionals that are inspired by WKB analysis. Finally, it is shown that the loss cannot be improved by any power by saturating the estimate with a quasimode.

1. Introduction

When studying wave equations on asymptotically flat backgrounds, (integrated) local energy estimates have become a fundamental tool. In fact, in a number of scenarios, it has been shown that these local energy estimates imply other known measures of dispersion, such as pointwise decay estimates [28], [19] and Strichartz estimates [18]. When there are null geodesics that remain in a compact set for all times, trapping is said to occur, and trapping is a known obstruction to local energy estimates [24], [25]. So any estimates in the presence of trapping must have a loss when compared to those estimates available on Minkowski space. This loss is often realized as a loss of regularity. In many situations where the trapping is sufficiently hyperbolic (unstable), an estimate with a minimal loss (say, a logarithmic loss of regularity) can be recovered. See, e.g., [3], [5], [9], [12, 13], [15], [23], [29], [32]. On the other hand, in presence of elliptic (stable) trapped rays, nearly everything, say everything but a logarithmic amount of decay, is lost [2]. Explicit examples where an algebraic loss of regularity is necessary and sufficient to prove integrated local energy estimates have not previously appeared.

A similar story exists for the Schrödinger equation, where the analog of the local energy estimate is the local smoothing estimate [10], [26], [31]. There, as the speed of propagation for the Schrödinger equation is proportional to the frequency, the estimate gives a 1/2-degree of smoothing. When trapped rays exist, it is known that the full 1/2-degree of smoothing cannot be recovered [11]. And as above, until recently, examples where the trapping caused a minimal loss and examples where the trapping disallowed all but a minimal amount of the smoothing were known, but nothing explicitly had been established in between.

In [8], the authors considered the Schrödinger equation in the presence of degenerate trapping on a product manifold or a surface of revolution. In this example, when $2m$ denotes the degeneracy of the trapping, they succeeded in showing that a loss of $\frac{m-1}{2(m+1)}$ derivatives when compared to the estimate on Euclidean space was necessary and sufficient. This provided the first example of local smoothing with a sharp algebraic loss. Subsequent studies include [7], [4], [6] which address, respectively, the loss caused by inflection points in the generating function, the case of infinitely degenerate trapping, and Strichartz estimates with degenerate trapping.

Here we consider the wave equation on the same geometric background. The local smoothing estimate is interesting even locally in time, and in fact, [8] proved such for times in the unit interval. For the wave equation, however, local in time estimates follow trivially from uniform energy bounds. Thus

for the wave equation only global in time estimates are considered, and this brings some new low frequency analysis into play.

We shall now describe the geometric setting. We consider the manifold $\mathbb{R} \times \mathbb{R} \times \mathbb{S}^2$ equipped with the Lorentzian metric

$$ds^2 = -dt^2 + dx^2 + a(x)^2 \, d\sigma_{\mathbb{S}^2}^2.$$

The generating function of this surface of revolution is given by $a(x) = (x^{2m} + 1)^{1/2m}$. This constructs a two ended surface, which is asymptotically Minkowski in both x-directions. Due to the critical point of $a(x)$ at $x = 0$, a surface of trapped null geodesics is formed. The case of $m = 1$ is the nondegenerate case, which is the well-studied case of hyperbolic trapping mentioned above, and is a simplified model for the trapping that occurs on, e.g., Schwarzschild spaces. The analysis of [15] can be directly mimicked to provide the local energy estimates with minimal loss. It is the degenerate cases $m \geq 2$ that interest us here.

In this geometric setup and in these coordinates, we note that

$$(1.1) \qquad \Box_{\mathfrak{g}} u = -\partial_t^2 u + a(x)^{-2} \partial_x \Big[a(x)^2 \partial_x u \Big] + \frac{1}{a(x)^2} \slashed{\Delta}_{\mathbb{S}^2} u$$

where $\slashed{\Delta}_{\mathbb{S}^2}$ denotes the Laplacian on $\mathbb{S}^2$. Using the product structure of the metric, we will separate space and time in the volume form and indicate $dV = a(x)^2 \, dx \, d\sigma_{\mathbb{S}^2}$. We will use $dV \, dt$ when the volume form of the full space-time is desired.

As this metric is static, there is a natural coercive energy, which is conserved when $\Box_g u = 0$:

$$E[u](t) = \int (\partial_t u)^2 + (\partial_x u)^2 + \frac{1}{a(x)^2} |\slashed{\nabla}_0 u|^2 \, dV = \int |\partial u|^2 \, dV.$$

Here we are using $\partial u = (\partial_t u, \partial_x u, \slashed{\nabla}_0 u)$, where $\slashed{\nabla}_0$ denotes derivatives tangential to $\mathbb{S}^2$, so that $|\partial u|^2 = |\partial u|_{\mathfrak{g}}^2 = (\partial_t u)^2 + (\partial_x u)^2 + \frac{1}{a(x)^2} |\slashed{\nabla}_0 u|^2$. The conservation of energy can be proved by multiplying $\Box_{\mathfrak{g}} u$ by $\partial_t u$ and integrating by parts. The same method, more generally, yields:

$$(1.2) \qquad E[u](t) \lesssim E[u](0) + \left| \int_0^t \int \Box_{\mathfrak{g}} u \partial_t u \, dV \, dt \right|.$$

We now describe the spaces that shall be used to measure the local energy:

$$\|u\|_{LE} = \sup_{j \geq 0} 2^{-j/2} \|u\|_{L^2 L^2([0,T] \times \{\langle x \rangle \approx 2^j\})}, \quad \|u\|_{LE^1} = \|(\partial u, \langle x \rangle^{-1} u)\|_{LE}.$$

Here $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$. Forcing terms will be frequently measured in the corresponding dual norm:

$$\|F\|_{LE^*} = \sum_{j \geq 0} 2^{j/2} \|F\|_{L^2 L^2([0,T] \times \{\langle x \rangle \approx 2^j\})}.$$

Here $L^2 L^2$ indicates the full space-time L^2 norm (where to mimic what is commonly seen on Minkowski space, the first L^2 is in t and the second L^2 is over the spatial variables (x, θ, ϕ)). The norms in t will be taken over $[0, T]$, but all constants will be independent of T, which yields the desired global estimates. We shall use notations such as LE_R to indicate the LE norm restricted to a single dyadic annulus with $2^j \approx R$ and $LE_{>R}$ to indicate the LE norm with the restriction that $2^j > R$.

On $(1 + 3)$-dimensional Minkowski space, the uniform energy bound and the (integrated) local energy estimate read as

$$\|\partial u\|_{L^\infty L^2} + \|u\|_{LE^1} \lesssim \|\partial u(0, \cdot)\|_{L^2} + \|\Box u\|_{L^1 L^2 + LE^*}.$$

Such estimates originated in the works [20–22] and can be proved by pairing $\Box u$ with $C\partial_t u + \frac{r}{r+2^j} \partial_r u + \frac{1}{r+2^j} u$, integrating over a space-time slab, integrating by parts, and using a Hardy inequality. See, e.g., [27], [16]. And in fact, these estimates hold on any stationary, Lorentzian, asymptotically flat spaces provided that there are no trapped rays and there are no eigenvalues nor resonances on the real line or in the lower half of the complex plane. See [17], which can also be referred to for a more complete history of such estimates.

As was done in preceding works such as [8], [15], our first goal is to develop an estimate that provides the local energy estimate away from the trapped set. The loss here will be manifest through a coefficient that vanishes where there is trapping. In [8], this follows from a relatively standard integration by parts argument. But as we must now consider global-in-time estimates, some new low frequency contributions must be considered. However, using a refinement of the exterior estimate of [17], which is inspired by that of [14], we shall obtain:

Theorem 1.1. [1] *On the geometry described above, if, for each t, $u(t,x)$ vanishes for sufficiently large $|x|$, we have*

(1.3)
$$\sup_t E[u](t)^{1/2} + \|\partial_x u\|_{LE} + \left\| \frac{|x|^m}{\langle x \rangle^m} \partial_t u \right\|_{LE}$$
$$+ \left\| \frac{|x|^m}{\langle x \rangle^m} \frac{1}{a(x)} |\nabla_0 u| \right\|_{LE} + \|\langle x \rangle^{-1} u\|_{LE}$$
$$\lesssim E[u](0)^{1/2} + \|\Box_{\mathfrak{g}} u\|_{L^1 L^2 + (|x|/\langle x \rangle)^m LE^*}$$

and, upon setting $D_\omega = -i\sqrt{\frac{1}{a^2}\Delta_{\mathbb{S}^2}}$,

(1.4)
$$\sup_t E[u](t)^{1/2} + \|\partial_x u\|_{LE} + \left\| \frac{|x|^m}{\langle x \rangle^m} \partial_t u \right\|_{LE}$$
$$+ \left\| \frac{|x|^m}{\langle x \rangle^m} \frac{1}{a(x)} |\nabla_0 u| \right\|_{LE} + \|\langle x \rangle^{-1} u\|_{LE}$$
$$\lesssim E[u](0)^{1/2} + \|\langle D_\omega \rangle^{\frac{m-1}{2(m+1)}} \Box_g u\|_{LE^*}^{1/2} \|\langle D_\omega \rangle^{-\frac{m-1}{2(m+1)}} \partial u\|_{LE}^{1/2}$$
$$+ \|\Box_{\mathfrak{g}} u\|_{LE^*}.$$

We note that when $m = 1$ this is the direct analog of [15, Theorem 1.2]. See also [1] where the case of $m = 1$ was considered. Theorem 1.1 is already essential as it provides lossless local energy estimates away from the trapping at $x = 0$ and everywhere for $\partial_x u$ and for $\langle x \rangle^{-1} u$. Moreover, in the sequel, this estimate will allow us to localize our analysis to a small neighborhood of the trapping.

We then seek to improve the corresponding loss. One option would be to use the resolvent bounds that were developed in [8]. As an alternative, we shall adapt the techniques of [15], which are based on energy functionals that are inspired by WKB theory. We hope that this alternate method may provide more flexibility when considering more general cases of degenerate trapping. As a brief motivation, see [30], which adapted the methods of [15] to the setting of Kerr backgrounds with sufficiently small angular momenta. Using these methods, we prove

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[1]For a normed space $X = \{x : \|x\|_X < \infty\}$, we have $aX = \{ax : \|x\|_X < \infty\} = \{y : \|a^{-1}y\|_X < \infty\}$. And thus, we denote $\|y\|_{aX} = \|a^{-1}y\|_X$. Moreover, if X and Y are normed spaces, we recall that $\|f\|_{X+Y} = \inf_{f=f_1+f_2}(\|f_1\|_X + \|f_2\|_Y)$.

Theorem 1.2.   *Let $m \geq 2$. On the geometry described above, suppose $\Box_{\mathfrak{g}} u = 0$. Then we have*

$$(1.5) \qquad \qquad \|u\|_{LE^1}^2 \lesssim E[\langle D_\omega \rangle^{\frac{m-1}{2(m+1)}} u](0).$$

For clarity of exposition, we have only stated this for the homogeneous case. A forcing term, which is measured in $\langle D_\omega \rangle^{-\frac{m-1}{m+1}} LE^*$ may be included. This is done, e.g., in [15] for the nondegenerate case, and the argument here would be similar.

Our last task is to show that this estimate is sharp in the sense that the estimate fails to hold if the loss is decreased by any power. Here we will rely on the quasimode that was constructed in [8]. Saturating the estimate for the wave equation is a bit different than saturating the local-in-time estimate for the Schrödinger equation. Rather than saturating the amount of smoothing available, here we must saturate the integrability. The spectral parameter that arises here is squared due to the wave equation being second order in time. By choosing the correct root, the solution that is constructed from the quasimode has appropriate growth to do exactly this. We, in fact, prove:

Theorem 1.3.   *There exist compactly supported functions ψ_0 and ψ_1 and a $T > 0$ so that the solution ψ to*

$$\Box_{\mathfrak{g}} \psi = 0, \quad (\psi, \partial_t \psi)|_{t=0} = (\psi_0, \psi_1)$$

satisfies

$$\int_0^T \|\beta(|x|)a(x)^{-1}|\overline{\nabla}_0 \psi|\|_{L^2}^2 \, dt \gtrsim E[\langle D_\omega \rangle^{\frac{m-1}{2(m+1)}} \psi](0),$$

where β is a smooth cutoff function that is identically 1 on $(-\infty, 1/2)$ and vanishes on $(1, \infty)$.

The article is organized as follows. In the next section, we shall prove Theorem 1.1. To do so, the analysis is broken into a low frequency regime and a range of frequencies that is bounded away from zero. The next section is devoted to refining the analysis to prove the sharp estimate as stated in Theorem 1.2. The arguments here are adaptations of those developed in [15]. Finally, in the last section, we prove Theorem 1.3, which shows that the loss of regularity cannot be improved by any power.

2. A lossy estimate — Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. Some of this section is inspired by [17], but as we are examining an explicit metric that is stationary and has a product structure, the methods can be significantly simplified.

Throughout this article, we fix $\beta(\rho)$ to be a smooth, monotone cutoff that is 1 for $\rho < 1/2$ and 0 for $\rho > 1$. To prove Theorem 1.1, we shall analyze $u_{<\tau} = \beta(D_t/\tau)u$ and $u_{>\tau} = (1 - \beta(D_t/\tau))u$ separately. The parameter $0 < \tau \ll 1$ will be chosen later. In particular, we shall establish the following estimate that handles any frequency range that is bounded away from 0.

Proposition 2.1. *For any $\tau > 0$, we have*

$$
(2.1) \qquad \|\partial_x u_{>\tau}\|_{LE}^2 + \left\| \frac{|x|^m}{\langle x \rangle^m} \partial_t u_{>\tau} \right\|_{LE}^2
$$

$$
+ \left\| \frac{|x|^m}{\langle x \rangle^m} \frac{1}{a(x)} |\nabla_0 u_{>\tau}| \right\|_{LE}^2 + \|\langle x \rangle^{-1} u_{>\tau}\|_{LE}^2
$$

$$
\lesssim E[u](0) + \|\Box_{\mathfrak{g}} u\|_{L^1 L^2 + (|x|/\langle x \rangle)^m LE^*} \|(\partial u, a(x)^{-1} u)\|_{L^\infty L^2 \cap (\langle x \rangle/|x|)^m LE}
$$

and

$$
(2.2) \qquad \|\partial_x u_{>\tau}\|_{LE}^2 + \left\| \frac{|x|^m}{\langle x \rangle^m} \partial_t u_{>\tau} \right\|_{LE}^2
$$

$$
+ \left\| \frac{|x|^m}{\langle x \rangle^m} \frac{1}{a(x)} |\nabla_0 u_{>\tau}| \right\|_{LE}^2 + \|\langle x \rangle^{-1} u_{>\tau}\|_{LE}^2
$$

$$
\lesssim E[u](0) + \|\langle D_\omega \rangle^{\frac{m-1}{2(m+1)}} \Box_{\mathfrak{g}} u\|_{LE^*} \|\langle D_\omega \rangle^{-\frac{m-1}{2(m+1)}} \partial u\|_{LE} + \|\Box_{\mathfrak{g}} u\|_{LE^*}^2.
$$

It is here that the effects of trapping are observed, as is evidenced by the coefficients that vanish at the location of the trapping.

The above will be combined with the following estimate for sufficiently small frequencies.

Proposition 2.2. *For any $\tau > 0$ sufficiently small, we have*

$$
(2.3) \qquad\qquad \|u_{<\tau}\|_{LE^1}^2 \lesssim E[u](0) + \|\Box_{\mathfrak{g}} u\|_{L^1 L^2 + LE^*}^2.
$$

The vanishing at the trapping is irrelevant here as trapping is a high frequency phenomenon.

As the metric is stationary, it is trivial to commute the frequency cutoff with $\Box_{\mathfrak{g}}$. Upon combining (1.2), (2.1), and (2.3), Theorem 1.1 results.

All of the results in this section come from a multiplier method and an associated integration by parts. At this point, we will record the following abstract calculation that shall be used many times in the sequel. Suppose $w, g \in C^2$, $f \in C^1$, and for each t, $w(t, x)$ vanishes for large enough $|x|$. Then

$$
\begin{aligned}
(2.4) \quad & -\int_0^T \int \Box_{\mathfrak{g}} w \left\{ f(x) \partial_x w + g(x) w \right\} dV \, dt \\
& = \int \partial_t w \left(f(x) \partial_x w + g(x) w \right) dV \Big|_0^T \\
& \quad + \int_0^T \int \left(f'(x) + g(x) - \frac{1}{2} \left\{ a(x)^{-2} \partial_x (a(x)^2 f(x)) \right\} \right) (\partial_x w)^2 \, dV \, dt \\
& \quad + \int_0^T \int \left(f(x) \frac{a'(x)}{a(x)} + g(x) - \frac{1}{2} \left\{ a(x)^{-2} \partial_x (a(x)^2 f(x)) \right\} \right) \\
& \qquad \times \frac{1}{a(x)^2} |\nabla_0 w|^2 \, dV \, dt \\
& \quad + \int_0^T \int \left(-g(x) + \frac{1}{2} \left\{ a(x)^{-2} \partial_x (a(x)^2 f(x)) \right\} \right) (\partial_t w)^2 \, dV \, dt \\
& \quad - \frac{1}{2} \int_0^T \int \left(a(x)^{-2} \partial_x [a(x)^2 \partial_x g] \right) w^2 \, dV \, dt.
\end{aligned}
$$

2.1. Exterior estimates

We first establish an estimate away from $x = 0$, which is where the trapping occurs. This estimate shows that the local energy estimates necessarily hold near the infinite ends with a lower order error term that is supported on a compact region. An estimate analogous to this was first established in [14] for the Schrödinger equation and in [17] for more general wave equations. Here the warped product structure simplifies the choice of multiplier, but some care must be taken to accommodate having two ends.

Though our proof does not significantly differ from [17], we have provided a sharper statement that allows for a difference in the radius outside of which you hope to estimate your solution and the radius at which you are cutting away and permitting an error term. This difference in radii provides a degree of smallness that allows us to simplify the low frequency analysis.

Proposition 2.3. *For any parameters R and R_1 satisfying $(1/2)R \geq R_1 \geq 2$, we have the following:*

$$(2.5) \qquad \|u\|^2_{LE^1_{|x|>R}} \lesssim E[u](0) + \int_0^T \int |\Box_{\mathfrak{g}} u| \left(|\partial u| + \frac{1}{a(x)}|u| \right) dV \, dt$$
$$+ R^{-1}R_1^{-1}\|u\|^2_{LE_{|x|\approx R_1}}.$$

Proof. We shall use (2.4) with

$$f(x) = (1 - \beta(|x|/R_1))h(x), \quad h(x) = \frac{x}{|x| + \rho}, \quad \rho \geq R,$$

$$g(x) = \frac{1}{2}a(x)^{-2}h(x)\partial_x\left[(1 - \beta(|x|/R_1))a(x)^2\right].$$

We can then compute

$$h'(x) = \frac{\rho}{(|x| + \rho)^2}, \quad h''(x) = -\frac{2\rho\,\mathrm{sgn}(x)}{(|x| + \rho)^3}.$$

We examine the coefficients of each term in the right side of (2.4). To start, using that β is a monotonically decreasing function, we have

$$(2.6) \qquad f'(x) + g(x) - \frac{1}{2}\left\{a(x)^{-2}\partial_x(a(x)^2 f(x))\right\}$$
$$= \frac{1}{2}(1 - \beta(|x|/R_1))h'(x) - R_1^{-1}\beta'(|x|/R_1)\mathrm{sgn}(x)h(x)$$
$$\geq \frac{1}{2}(1 - \beta(|x|/R_1))\frac{\rho}{(|x| + \rho)^2}.$$

And moreover, the right side is $\approx 1/\rho$ on $|x| \approx \rho$.

For the angular derivatives, we have

$$(2.7) \qquad f(x)\frac{a'(x)}{a(x)} + g(x) - \frac{1}{2}\left\{a(x)^{-2}\partial_x(a(x)^2 f(x))\right\}$$
$$= (1 - \beta(|x|/R_1))\left(\frac{a'(x)}{a(x)}h(x) - \frac{1}{2}h'(x)\right).$$

We record that $\frac{a'(x)}{a(x)}h(x) - \frac{1}{2}h'(x) \geq \frac{1}{2}\frac{|x|}{(|x|+\rho)^2}$ on the support of $1 - \beta(|x|/R_1)$. And this coefficient is also $\approx 1/\rho$ on $|x| \approx \rho$.

For the time derivatives, we simply get

$$(2.8) \qquad - g(x) + \frac{1}{2}\left\{a(x)^{-2}\partial_x(a(x)^2 f(x))\right\} = \frac{1}{2}(1 - \beta(|x|/R_1))h'(x),$$

which is everywhere non-negative and $\approx 1/\rho$ on $|x| \approx \rho$.

It remains to examine the lower order term. We compute

$$(2.9) \quad -\frac{1}{4}\Big\{a(x)^{-2}\partial_x\Big[a(x)^2\partial_x\Big(h(x)a(x)^{-2}\partial_x(a(x)^2)\Big)\Big]\Big\}$$
$$= -\frac{1}{2}\frac{x^{2m-1}}{1+x^{2m}}h''(x) + \frac{2m-1}{(1+x^{2m})^2}x^{2m-2}\left(\frac{h(x)}{x} - h'(x)\right)$$
$$+ \frac{2m-1}{(1+x^{2m})^3}x^{2m-3}m(x^{2m}-1)h(x).$$

Here we observe that $-\frac{x^{2m-1}}{1+x^{2m}}h''(x)$ is everywhere non-negative and is $\gtrsim \rho^{-3}$ when $|x| \approx \rho$. Moreover, we have that $h(x)/x \geq h'(x)$, which gives that the second term in the right side of (2.9) is nonnegative. And finally, the last term of (2.9) is easily seen to be non-negative on the support of $(1 - \beta(|x|/R_1))$. We now account for the error term that results when derivatives land on the cutoff. Here we see that

$$(2.10) \quad \Bigg| -\frac{1}{2}a(x)^{-2}\partial_x[a(x)^2\partial_x g(x)]$$
$$+ \frac{1}{4}(1 - \beta(|x|/R_1))\Big\{a(x)^{-2}\partial_x\Big[a(x)^2\partial_x\Big(h(x)a(x)^{-2}\partial_x(a(x)^2)\Big)\Big]\Big\}\Bigg|$$
$$\lesssim \rho^{-1}R_1^{-2}\mathbf{1}_{|x|\approx R_1}.$$

Using each of these analyses in (2.4), we have established

$$-\int_0^T \int \Box_{\mathfrak{g}}u\Big\{f(x)\partial_x u + g(x)u\Big\}\,dV\,dt - \int \partial_t u\Big(f(x)\partial_x u + g(x)u\Big)\,dV\Big|_0^T$$
$$\gtrsim \rho^{-1}\int_0^T \int_{|x|\approx\rho}|\partial u|^2\,dV\,dt + \rho^{-3}\int_0^T \int_{|x|\approx\rho}u^2\,dV\,dt$$
$$- \rho^{-1}R_1^{-2}\int_0^T \int_{|x|\approx R_1}u^2\,dV\,dt.$$

By the Schwarz inequality, we may bound, independently of ρ,

$$\int f(x)\partial_t u\partial_x u\,dV \lesssim E[u](t),$$

to which we can, in turn, apply (1.2). The term

$$\frac{1}{2}\int a(x)^{-2}h(x)\partial_x((1 - \beta(|x|/R_1))a(x)^2)u\partial_t u\,dV$$

is handled similarly when combined with the following variant of a Hardy inequality:

$$(2.11) \qquad \int a(x)^{-2} u^2 \, dV \lesssim \int (\partial_x u)^2 \, dV.$$

To apply such, notice that

$$a(x)^{-2} h(x) \partial_x (\beta(|x|) a(x)^2) \lesssim a(x)^{-1}.$$

To prove (2.11), we integrate by parts to see

$$\int a(x)^{-2} u^2 \, dV = -2 \int x u \partial_x u \, dx \, d\sigma \lesssim \int \left| \frac{x}{a(x)} \right| \left| \frac{u}{a(x)} \right| |\partial_x u| \, dV$$

$$\lesssim \left(\int a(x)^{-2} u^2 \, dV \right)^{1/2} \left(\int (\partial_x u)^2 \, dV \right)^{1/2}.$$

Applying these bounds to the time boundary terms and combining with (1.2), we have

$$\sup_{t \in [0,T]} E[u](t) + \rho^{-1} \int_0^T \int_{|x| \approx \rho} |\partial u|^2 \, dV \, dt + \rho^{-3} \int_0^T \int_{|x| \approx \rho} u^2 \, dV \, dt$$

$$\lesssim E[u](0) + \int_0^T \int |\Box_\mathfrak{g} u| \left(|\partial u| + \frac{|u|}{a(x)} \right) dV \, dt + \rho^{-1} R_1^{-1} \|u\|_{LE_{R_1}}^2.$$

Taking the supremum over $\rho \geq R$ yields (2.5). $\qquad\qquad \Box$

2.2. High frequency estimate (Proposition 2.1)

In this section, we prove a local energy estimate for any range of time-frequencies that is bounded away from 0. To begin, we establish the estimate on a large ball, which will be supplemented with (2.5).

Lemma 2.4. *For any $R > 0$ sufficiently large, we have*

$$(2.12) \qquad \frac{1}{R} \left\| (1 + x^{2m})^{-1/2} \partial_x u \right\|^2_{L^2 L^2_{|x|<R/2}}$$

$$+ \frac{1}{R^{4m+1}} \left\| |x|^m (1 + x^{2m})^{-1/2} a(x)^{-1} |\overline{\nabla}_0 u| \right\|^2_{L^2 L^2_{|x|<R/2}}$$

$$+ \frac{1}{R} \left\| |x|^m (1 + x^{2m})^{-1/2} \partial_t u \right\|^2_{L^2 L^2_{|x|<R/2}} + \frac{1}{R^{4m+1}} \left\| \langle x \rangle^{-1} u \right\|^2_{L^2 L^2_{|x|<R/2}}$$

$$\lesssim E[u](0) + \int_0^T \int_{|x|<R} |\Box_{\mathfrak{g}} u| \left(|\partial u| + |\langle x \rangle^{-1} u| \right) dV \, dt + \|u\|^2_{LE^1_{|x|\approx R}}.$$

Proof. We apply (2.4) with all of the $w = \beta(|x|/R)u$, $f(x) = \frac{x}{Ra(x/R^2)}$, and $g(x) = \frac{1}{2} f'(x) + \frac{\delta}{R^{4m}} \frac{a'(x)}{a(x)} f(x)$, and we shall now examine the coefficients of the last four terms in the right side of (2.4). Here $\delta > 0$ is a small parameter that will be fixed later.

For the coefficient of $(\partial_x w)^2$, noting that $\frac{a'(x)}{a(x)} f(x) \geq 0$, we have

$$f'(x) + g(x) - \frac{1}{2} \left\{ a(x)^{-2} \partial_x (a(x)^2 f(x)) \right\} \geq f'(x) - \frac{a'(x)}{a(x)} f(x)$$

$$= \frac{1}{R} \frac{1 - \left(\frac{x}{R} \right)^{4m}}{(1 + x^{2m}) \left(1 + \frac{x^{2m}}{R^{4m}} \right)^{1 + \frac{1}{2m}}}.$$

In particular, this is nonnegative on the support of $\beta(|x|/R)$, and

$$(2.13) \qquad \int_0^T \int \left(f'(x) + g(x) - \frac{1}{2} \left\{ a(x)^{-2} \partial_x (a(x)^2 f(x)) \right\} \right) (\partial_x w)^2 \, dV \, dt$$

$$\gtrsim \frac{1}{R} \left\| (1 + x^{2m})^{-1/2} \partial_x u \right\|^2_{L^2 L^2_{|x|<R/2}}$$

since $\beta(|x|/R) \equiv 1$ on $|x| < R/2$. Here and throughout this proof, all implicit constants are independent of R.

For the angular derivatives, we have

$$f(x) \frac{a'(x)}{a(x)} + g(x) - \frac{1}{2} \left\{ a(x)^{-2} \partial_x (a(x)^2 f(x)) \right\} = \frac{\delta}{R^{4m}} \frac{a'(x)}{a(x)} f(x)$$

$$= \frac{\delta}{R^{4m+1}} \frac{x^{2m}}{1 + x^{2m}} \left(1 + \left(\frac{x}{R^2} \right)^{2m} \right)^{-\frac{1}{2m}}.$$

And hence,

(2.14)
$$\int_0^T \int \left(f(x)\frac{a'(x)}{a(x)} + g(x) - \frac{1}{2}\left\{ a(x)^{-2}\partial_x(a(x)^2 f(x)) \right\} \right)$$
$$\times \frac{1}{a(x)^2}|\nabla_0 w|^2 \, dV \, dt$$
$$\gtrsim \frac{\delta}{R^{4m+1}} \left\| |x|^m(1+x^{2m})^{-1/2}a(x)^{-1}|\nabla_0 u| \right\|^2_{L^2 L^2_{|x|<R/2}}.$$

And for the time derivatives, it follows that

$$-g(x) + \frac{1}{2}\left\{ a(x)^{-2}\partial_x(a(x)^2 f(x)) \right\} = \left(1 - \frac{\delta}{R^{4m}} \right)\frac{a'(x)}{a(x)}f(x)$$

and

(2.15)
$$\int_0^T \int \left(-g(x) + \frac{1}{2}\left\{ a(x)^{-2}\partial_x(a(x)^2 f(x)) \right\} \right)(\partial_t w)^2 \, dV \, dt$$
$$\gtrsim \frac{1}{R} \left\| |x|^m(1+x^{2m})^{-1/2}\partial_t u \right\|^2_{L^2 L^2_{|x|<R/2}}.$$

It remains to examine the lower order term, whose coefficient is

$$-\frac{1}{2}a(x)^{-2}\partial_x(a(x)^2\partial_x g).$$

We first compute

$$-\frac{1}{4}a(x)^{-2}\partial_x(a(x)^2\partial_x f'(x))$$
$$= \frac{(2m+1)\left(\frac{x}{R^2}\right)^{2m}\left[x^{2m} - 1 - 2\left(\frac{x}{R^2}\right)^{2m} + 2m(1+x^{2m})\left(1 - \left(\frac{x}{R^2}\right)^{2m} \right) \right]}{4Rx^2(1+x^{2m})\left(1 + \left(\frac{x}{R^2}\right)^{2m} \right)^{3+\frac{1}{2m}}},$$

which can be observed to be nonnegative on $|x| < R$ for any R sufficiently large. Moreover

$$-\frac{1}{4}\int_0^T \int \left(a(x)^{-2}\partial_x[a(x)^2\partial_x f'(x)] \right)w^2 \, dV \, dt$$
$$\gtrsim \frac{1}{R^{4m+1}} \left\| |x|^{m-1}(1+x^{2m})^{-1/2}\beta(|x|/R)u \right\|^2_{L^2 L^2}.$$

On the other hand,

$$-\frac{1}{2R^{4m}}a(x)^{-2}\partial_x\left[a(x)^2\partial_x\left(\frac{a'(x)}{a(x)}f(x)\right)\right]$$

$$=\frac{1}{2}\frac{x^{2m-2}}{R^{4m+1}(1+x^{2m})}\frac{1}{\left(1+\left(\frac{x}{R^2}\right)^{2m}\right)^{2+\frac{1}{2m}}}\left\{4m^2\frac{x^{2m}-1}{(x^{2m}+1)^2}\left(1+\left(\frac{x}{R^2}\right)^{2m}\right)^2\right.$$

$$-\left(\frac{x}{R^2}\right)^{2m}\frac{1}{x^{2m}+1}\left(1-x^{2m}+2\left(\frac{x}{R^2}\right)^{2m}\right)$$

$$+\left.\frac{2m\left[1+(x^{4m}+5)\left(\frac{x}{R^2}\right)^{2m}+3\left(\frac{x}{R^2}\right)^{4m}+x^{2m}\left(-1+2\left(\frac{x}{R^2}\right)^{2m}+\left(\frac{x}{R^2}\right)^{4m}\right)\right]}{(1+x^{2m})^2}\right\}.$$

On the support of $\beta(|x|/R)$ for R sufficiently large, it follows that

$$\left|-\frac{\delta}{2R^{4m}}a(x)^{-2}\partial_x\left[a(x)^2\partial_x\left(\frac{a'(x)}{a(x)}f(x)\right)\right]\right|$$

$$\lesssim\delta\frac{|x|^{2m-2}}{R^{4m+1}(1+x^{2m})}\quad\text{for }|x|<R.$$

Thus, if $\delta>0$ is chosen sufficiently small, it follows that

$$(2.16)\qquad -\frac{1}{2}\int_0^T\int\left(a(x)^{-2}\partial_x[a(x)^2\partial_x g]\right)w^2\,dV\,dt$$

$$\gtrsim\frac{1}{R^{4m+1}}\left\||x|^{m-1}(1+x^{2m})^{-1/2}u\right\|^2_{L^2L^2_{|x|<R/2}}.$$

We next provide a simple argument to show that the vanishing of the coefficient at the origin in (2.16) can be removed. Indeed, integration by parts gives

$$\int\beta(|x|)u^2a(x)^2\,dx=-\int\beta'(|x|)|x|u^2a^2(x)\,dx$$

$$-2\int\beta(|x|)xu\partial_x ua^2(x)\,dx$$

$$-2\int\beta(|x|)\frac{xa'(x)}{a(x)}u^2\,a(x)^2\,dx.$$

Applying the Schwarz inequality to the second term and bootstrapping yields

$$(2.17) \qquad \int \beta(|x|)u^2 \, dV \lesssim \int |\beta'(|x|)||x|u^2 \, dV + \int \beta(|x|)x^2(\partial_x u)^2 \, dV$$
$$+ \int \beta(|x|)\frac{x^{2m}}{(1+x^{2m})}u^2 \, dV.$$

Upon inclusion of a factor of R^{-4m-1}, each of the terms on the right side can be controlled by the right sides of (2.13) and (2.16).

Using (2.13), (2.15), (2.14), (2.16), and (2.17) in (2.4) gives

$$(2.18) \qquad -\int_0^T \int \Box_{\mathfrak{g}} w \Big\{ f(x)\partial_x w + g(x)w \Big\} \, dV \, dt$$
$$- \int \partial_t w \Big(f(x)\partial_x w + g(x)w \Big) dV \Big|_0^T$$
$$\gtrsim \frac{1}{R} \Big\| (1+x^{2m})^{-1/2}\partial_x u \Big\|_{L^2 L^2_{|x|<R/2}}^2$$
$$+ \frac{1}{R} \Big\| |x|^m(1+x^{2m})^{-1/2}\partial_t u \Big\|_{L^2 L^2_{|x|<R/2}}^2$$
$$+ \frac{\delta}{R^{4m+1}} \Big\| |x|^m(1+x^{2m})^{-1/2}a(x)^{-1}|\nabla_0 u| \Big\|_{L^2 L^2_{|x|<R/2}}^2$$
$$+ \frac{1}{R^{4m+1}} \Big\| \langle x \rangle^{-1}\beta(|x|/R)u \Big\|_{L^2 L^2}^2.$$

Noting that $f(x)$ is bounded (independent of R) and $|g(x)| \lesssim 1/R$ on the support of $\beta(|x|/R)$, we can apply the Schwarz inequality and (2.11) to bound each of the time-boundary terms by the energy at that time. Thus, (1.2) gives

$$(2.19) \qquad \frac{1}{R} \Big\| (1+x^{2m})^{-1/2}\partial_x u \Big\|_{L^2 L^2_{|x|<R/2}}^2$$
$$+ \frac{\delta}{R^{4m+1}} \Big\| |x|^m(1+x^{2m})^{-1/2}a(x)^{-1}|\nabla_0 u| \Big\|_{L^2 L^2_{|x|<R/2}}^2$$
$$+ \frac{1}{R} \Big\| |x|^m(1+x^{2m})^{-1/2}\partial_t u \Big\|_{L^2 L^2_{|x|<R/2}}^2 + \frac{1}{R^{4m+1}} \Big\| \langle x \rangle^{-1}\beta(|x|/R)u \Big\|_{L^2 L^2}^2$$
$$\lesssim E[u](0) + \int_0^T \int |\Box_{\mathfrak{g}} w| \Big(|\partial_t w| + |\partial_x w| + |\langle x \rangle^{-1}w| \Big) \, dV \, dt.$$

We finally examine the nonhomogeneous term and notice

$$(2.20) \qquad \int_0^T \int |[\Box_{\mathfrak{g}}, \beta(|x|/R)]u|\left(|\partial w| + |\langle x \rangle^{-1} w|\right) dV\, dt \lesssim \|u\|_{LE_R^1}^2,$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

We may now establish our main high frequency estimate.

Proof of Proposition 2.1. We sum (2.12) (with R replaced by $2R$) and (2.5) (with $R_1 = R/2$) multiplied by a large constant to see that

$$(2.21) \qquad \frac{1}{R}\left\||(1 + x^{2m})^{-1/2}\partial_x u_{>\tau}\right\|_{L^2 L^2_{|x|<R}}^2$$

$$+ \frac{1}{R}\left\||x|^m(1 + x^{2m})^{-1/2}\partial_t u_{>\tau}\right\|_{L^2 L^2_{|x|<R}}^2$$

$$+ \frac{1}{R^{4m+1}}\left\||x|^m(1 + x^{2m})^{-1/2}a(x)^{-1}|\nabla_0 u_{>\tau}|\right\|_{L^2 L^2_{|x|<R}}^2$$

$$+ \frac{1}{R^{4m+1}}\left\|\langle x \rangle^{-1}\beta(|x|/2R)u_{>\tau}\right\|_{L^2 L^2}^2 + \|u_{>\tau}\|_{LE^1_{|x|>R}}^2$$

$$\lesssim E[u](0) + \int_0^T \int |\Box_{\mathfrak{g}} u|\left(|\partial u| + |\langle x \rangle^{-1} u|\right) dV\, dt$$

$$+ R^{-2}\|u_{>\tau}\|_{LE_{|x|\approx R}}^2.$$

By multiplying (2.5) by a large enough constant, the $\|u\|_{LE^1_{|x|\approx R}}^2$ error term from (2.12) may be bootstrapped. Since

$$R^{-2}\|u_{>\tau}\|_{LE_{|x|\approx R}}^2 \lesssim \tau^{-2} R^{-3}\|\partial_t u_{>\tau}\|_{L^2 L^2_{|x|\approx R}}^2,$$

we see that the error term may be bootstrapped provided that R is chosen sufficiently large (depending on τ). And from what results, the desired estimate follows immediately. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

2.3. Low frequency estimate (Proposition 2.2)

Here we establish a local energy estimate for sufficiently low time-frequencies. We will again use (2.5), but we modify the interior estimate that we couple with it.

Lemma 2.5. *For any $R > 0$, we have*

$$
\begin{aligned}
(2.22) \quad & \|\langle x \rangle^{-1/2} \partial_x u\|_{L^2 L^2_{|x|<R/2}}^2 + \|\langle x \rangle^{-1/2} a(x)^{-1} \not\nabla_0 u\|_{L^2 L^2_{|x|<R/2}}^2 \\
& + \|\langle x \rangle^{-1/2} \partial_t u\|_{L^2 L^2_{|x|<R/2}}^2 + \|\langle x \rangle^{-m-\frac{3}{2}} u\|_{L^2 L^2_{|x|<R/2}}^2 \\
& \lesssim E[u](0) + \int_0^T \int |\Box_{\mathfrak{g}} u| \left(|\partial_t u| + \frac{1}{a(x)}|u| \right) dV \, dt \\
& + \|\langle x \rangle^{-1/2} \partial_t u\|_{L^2 L^2_{|x|<R}}^2 + \|u\|_{LE^1_{|x|\approx R}}^2.
\end{aligned}
$$

Proof. Here we apply (2.4) with $f \equiv 0$ and $g(x) = 1/a(x)$. This gives

$$
\begin{aligned}
(2.23) \quad & \int_0^T \int \frac{1}{a(x)} \left[(\partial_x w)^2 + \frac{1}{a(x)^2} |\not\nabla_0 w|^2 \right] dV \, dt \\
& + \frac{2m-1}{2} \int_0^T \int \frac{x^{2m-2}}{(1+x^{2m})^2} \frac{1}{a(x)} w^2 \, dV \, dt \\
& = - \int \frac{1}{a(x)} w \partial_t w \, dV \Big|_0^T - \int_0^T \int \frac{1}{a(x)} w \Box_{\mathfrak{g}} w \, dV \, dt \\
& + \int_0^T \int \frac{1}{a(x)} (\partial_t w)^2 \, dV \, dt.
\end{aligned}
$$

We may again apply (2.17) to eliminate the vanishing in the coefficient of the third term in the left side. Also applying the Schwarz inequality, (2.11), and (1.2), we obtain

$$
\begin{aligned}
& \int_0^T \int \frac{1}{a(x)} \left[(\partial_x w)^2 + \frac{1}{a(x)^2} |\not\nabla_0 w|^2 \right] dV \, dt \\
& + \int_0^T \int \langle x \rangle^{-2m-2} \frac{1}{a(x)} w^2 \, dV \, dt \\
& \lesssim E[w](0) + \int_0^T \int |\Box_{\mathfrak{g}} w| \left(|\partial_t w| + \frac{1}{a(x)}|w| \right) dV \, dt \\
& + \int_0^T \int \frac{1}{a(x)} (\partial_t w)^2 \, dV \, dt.
\end{aligned}
$$

The proof concludes by letting $w = \beta(|x|/R)u$ and using (2.20). $\qquad \square$

We now combine the preceding, (2.22), with the exterior estimate (2.5) in order to establish a local energy estimate for sufficiently small frequencies.

Proof of Proposition 2.2. We supplement (2.22) (applied to $u_{<\tau}$) with (1.2) and (2.5) with $R_1 = 2$. This gives

$$
\begin{aligned}
&\|\partial u_{<\tau}\|_{L^\infty L^2}^2 + \|\langle x \rangle^{-1/2} \partial u_{<\tau}\|_{L^2 L^2_{|x|<R}}^2 \\
&+ \|\langle x \rangle^{-m-\frac{3}{2}} u_{<\tau}\|_{L^2 L^2_{|x|<R}}^2 + \|u_{<\tau}\|_{LE^1_{|x|>R}}^2 \\
&\lesssim E[u](0) + \|\Box_{\mathfrak{g}} u\|_{L^1 L^2 + LE^*} \|(\partial u_{<\tau}, a(x)^{-1} u_{<\tau})\|_{L^\infty L^2 \cap LE} \\
&+ \|\langle x \rangle^{-1/2} \partial_t u_{<\tau}\|_{L^2 L^2_{|x|<2R}}^2 + R^{-1} \|u_{<\tau}\|_{LE_{|x|\approx 1}}^2.
\end{aligned}
$$

Here we have also applied the Schwarz inequality to the inhomogeneous term. By choosing R sufficiently large, we may bootstrap the last term in the right side. The remaining error term can then be absorbed provided that τ is sufficiently small (depending on R). $\qquad\square$

3. An improved estimate — Proof of Theorem 1.2

In order to prove Theorem 1.2, we define $P = -\partial_t^2 + \partial_x^2 + a(x)^{-2} \slashed{\nabla}_0 \cdot \slashed{\nabla}_0$ and shall prove:

Proposition 3.1. *Suppose that $v(t, \cdot)$ is supported in $\{|x| < 1\}$, that $v(t,x) \equiv 0$ for $t \le 0$, and $Pv = g$ where $g(t, \cdot)$ is supported in $\{|x| < 1\}$ and $g(t,x) \equiv 0$ for $t \le 0$. Then, for $\delta > 0$,*

$$
(3.1) \quad \|\partial_x v\|_{LE} + \|v\|_{\langle D_\omega \rangle^{\frac{m-1}{2(m+1)}} LE^1} \lesssim \|g\|_{\langle D_\omega \rangle^{-\frac{m-1}{2(m+1)}} LE^* + (|x|/\langle x \rangle)^{(m-1+\delta)/2} LE^*}.
$$

We will first show that this suffices to prove Theorem 1.2. To begin, we shall argue that if $v(t, \cdot)$ is supported in $\{|x| < \varepsilon\}$ with $\varepsilon > 0$ sufficiently small then we may replace P by $\Box_{\mathfrak{g}}$ in the above proposition. Indeed, if $\Box_{\mathfrak{g}} v = g$, then $Pv = g - 2\frac{a'(x)}{a(x)} \partial_x v$. Then (3.1) gives

$$
\begin{aligned}
\|\partial_x v\|_{LE} + \|v\|_{\langle D_\omega \rangle^{\frac{m-1}{2(m+1)}} LE^1} &\lesssim \|g\|_{\langle D_\omega \rangle^{-\frac{m-1}{2(m+1)}} LE^* + (|x|/\langle x \rangle)^{\frac{m-1+\delta}{2}} LE^*} \\
&+ \||x|^{\frac{3m-1-\delta}{2}} \partial_x v\|_{LE^*}.
\end{aligned}
$$

Due to the restricted support of v, this last term is $\varepsilon^{\frac{3m-1-\delta}{2}} \mathcal{O}(\|\partial_x v\|_{LE})$ and may be bootstrapped provided ε is small enough.

We next argue that the $L^1 L^2$ norm may be added in the right side. This follows trivially from (1.3) for the ∂_x components. Suppose that ϕ solves $\Box_{\mathfrak{g}} \phi = f$ backwards in time and that $\phi(t,x), f(t,x) \equiv 0$ for $t \ge T$. Then by

(1.4) and (3.1), it follows that

$$\|\partial\phi\|_{L^\infty L^2} + \|(|x|/\langle x\rangle)^m \partial\phi\|_{LE} \lesssim \|\langle D_\omega\rangle^{\frac{m-1}{2(m+1)}} f\|_{LE^*}.$$

Hence,

$$\langle \partial_t v, f \rangle = -\langle \Box_{\mathfrak{g}} v, \partial_t \phi \rangle = -\langle g, \partial_t \phi \rangle$$
$$\leq \|g\|_{L^1 L^2} \|\partial_t \phi\|_{L^\infty L^2} \lesssim \|g\|_{L^1 L^2} \|\langle D_\omega\rangle^{\frac{m-1}{2(m+1)}} f\|_{LE^*},$$

and thus by duality,

$$(3.2) \qquad \|\langle D_\omega\rangle^{-\frac{m-1}{2(m+1)}} \partial_t v\|_{LE} \lesssim \|g\|_{L^1 L^2}.$$

Applying (2.23) with $w = \langle D_\omega\rangle^{-\frac{m-1}{2(m+1)}} v$, (1.2), and (2.11), we have

$$(3.3) \qquad \|\langle D_\omega\rangle^{-\frac{m-1}{2(m+1)}} |\overline{\nabla}_0 v|\|_{LE}^2 \lesssim \|\Box_{\mathfrak{g}} v\|_{L^1 L^2} \|\langle D_\omega\rangle^{-\frac{m-1}{m+1}} v\|_{L^\infty L^2}$$
$$+ \|\langle D_\omega\rangle^{-\frac{m-1}{2(m+1)}} \partial_t v\|_{LE}^2.$$

As (2.11) and (1.2) permit us to bound

$$\|\langle D_\omega\rangle^{-\frac{m-1}{m+1}} v\|_{L^\infty L^2} \lesssim \|a(x)^{-1} u\|_{L^\infty L^2} \lesssim \|g\|_{L^1 L^2},$$

(3.2) and (3.3) give

$$(3.4) \qquad \|\langle D_\omega\rangle^{-\frac{m-1}{2(m+1)}} |\overline{\nabla}_0 v|\|_{LE} \lesssim \|g\|_{L^1 L^2}.$$

We have therefore sharpened (3.1) to say that if $\Box_{\mathfrak{g}} v = g_1 + g_2$ where $g_1 \in (|x|/\langle x\rangle)^{\frac{m-1+\delta}{2}} LE^*$, $g_2 \in L^1 L^2$ and if $v(t,x), g_1(t,x), g_2(t,x)$ vanish identically for $t \leq 0$ and if each is supported where $|x| < \varepsilon$, then

$$(3.5) \qquad \|\partial_x v\|_{LE} + \|\langle D_\omega\rangle^{-\frac{m-1}{2(m+1)}} \partial v\|_{LE}$$
$$\lesssim \inf_{g = g_1 + g_2} \left(\|(\langle x\rangle/|x|)^{\frac{m-1+\delta}{2}} g_1\|_{LE^*} + \|g_2\|_{L^1 L^2} \right).$$

We now proceed to make a few reductions in order to use (3.5) to prove Theorem 1.2. First, it suffices to replace u by $v = (1 - \beta(t))\beta(|x|/\varepsilon)u$ as $\|(1 - \beta(|x|/\varepsilon))u\|_{LE^1}$ is trivially controlled by the left side of (1.3), which provides a better bound, and, using a Hardy inequality, $\|\beta(t)\beta(|x|/\varepsilon)u\|_{LE^1}^2 \lesssim$

$\sup_{0 \leq t \leq 1} E[u](t)$, which is appropriately bounded using (1.2). We now observe that v satisfies

$$\Box_{\mathfrak{g}} v = 2\varepsilon^{-1} \operatorname{sgn}(x)\beta'(|x|/\varepsilon)(1 - \beta(t))\partial_x u$$
$$+ 2\varepsilon^{-1}\operatorname{sgn}(x)\frac{a'(x)}{a(x)}\beta'(|x|/\varepsilon)(1 - \beta(t))u$$
$$+ \varepsilon^{-2}\beta''(|x|/\varepsilon)(1 - \beta(t))u + 2\beta(|x|/\varepsilon)\beta'(t)\partial_t u + \beta(|x|/\varepsilon)\beta''(t)u.$$

Then (3.5) gives that

$$\|\langle D_\omega \rangle^{-\frac{m-1}{2(m+1)}}\partial(1 - \beta(t))\beta(|x|)u\|_{LE} \lesssim_\varepsilon \|\partial_x u\|_{LE} + \|\langle x \rangle^{-1}u\|_{LE}$$
$$+ \sup_{1/2 \leq t \leq 1} E[u](t)^{1/2}$$

and subsequent applications of (1.2) and (1.3) complete the proof of Theorem 1.2.

3.1. WKB Analysis

It now suffices to establish Proposition 3.1. The method that we shall employ is inspired by [15]. We shall develop energy functionals that are based in WKB analysis in order to achieve (3.1).

We begin with $Pv = g$. Upon taking the Fourier transform in t and expanding into spherical harmonics, it suffices to examine

$$(3.6) \qquad \partial_x^2 \phi_{\lambda,\tau} + V_{\lambda,\tau}(x)\phi_{\lambda,\tau} = g_{\lambda,\tau} \quad \text{where} \quad V_{\lambda,\tau} = \tau^2 - \frac{\lambda^2}{a(x)^2}.$$

Here τ is the Fourier dual variable to t, and λ^2 runs over the eigenvalues of $-\Delta_{\mathbb{S}^2}$. And to prove (3.1), if $g_{\lambda,\tau} = g_{1,\lambda,\tau} + g_{2,\lambda,\tau}$ with $g_{1,\lambda,\tau} \in L_x^2$ and $g_{2,\lambda,\tau} \in (|x|/\langle x \rangle)^{\frac{m-1+\delta}{2}} L_x^2$, it will suffice to show

$$(3.7) \qquad \langle \lambda \rangle^{-\frac{m-1}{2(m+1)}}(|\tau| + \lambda)\|\phi_{\lambda,\tau}\|_{L_x^2} + \|\partial_x \phi_{\lambda,\tau}\|_{L_x^2}$$
$$\lesssim \langle \lambda \rangle^{\frac{m-1}{2(m+1)}}\|g_{1,\lambda,\tau}\|_{L_x^2} + \left\|\left(\frac{\langle x \rangle}{|x|}\right)^{\frac{m-1+\delta}{2}}g_{2,\lambda,\tau}\right\|_{L_x^2}.$$

For ease of notation, we shall drop the subscripts in the sequel. The analysis will be broken into four cases depending on the relationship between τ and λ.

In the first three cases, we will prove the stronger estimate

(3.8) $$\|\partial_x \phi\|_{L^2} + (|\tau| + \lambda)\|\phi\|_{L^2} \lesssim \|g\|_{L^2}.$$

Case I: $(\lambda, \tau \lesssim 1)$ Define the positive definite energy functional $E[\phi](x) = (\partial_x \phi)^2 + \phi^2$. Then, since $|V_{\lambda,\tau}| \lesssim 1$ here,

$$\partial_x E[\phi](x) = 2\partial_x \phi(\partial_x^2 \phi + \phi) \lesssim g^2 + E[\phi](x).$$

By Gronwall's inequality and the fact that ϕ is compactly supported, it follows that

$$(\partial_x \phi)^2 + \phi^2 \lesssim \|g\|_{L^2}^2,$$

from which the desired result is immediate upon integrating.

Case II: $(\lambda \ll \tau)$ Here we similarly define $E[\phi](x) = (\partial_x \phi)^2 + V_{\lambda,\tau}(x)\phi^2$. Then,

$$\partial_x E[\phi] \lesssim |g\partial_x \phi| + \phi^2 |\partial_x V_{\lambda,\tau}(x)|.$$

Since $|\partial_x V_{\lambda,\tau}(x)| \lesssim V_{\lambda,\tau}(x)$, this gives

$$\partial_x E[\phi](x) \lesssim E[\phi](x) + g^2.$$

The desired estimate then follows, as above, from Gronwall's inequality and integrating both sides of what results.

Case III: $(\tau \ll \lambda)$ Here, due to the different sign of $V_{\lambda,\tau}$ in this regime, we instead consider the boundary value problem

$$\begin{cases} \partial_x^2 \phi + V_{\lambda,\tau}\phi = g, \\ \phi(-1) = \phi(1) = 0. \end{cases}$$

Multiplying both sides of the equation by $-\phi$ and integrating, we find

$$\int_{-1}^{1} (\partial_x \phi)^2 \, dx - \int_{-1}^{1} V_{\lambda,\tau}\phi^2 \, dx = -\int_{-1}^{1} g\phi \, dx.$$

In this case, $V_{\lambda,\tau} \approx -\lambda^2$ on the support of ϕ, and the desired result follows from the Schwarz inequality.

Case IV: $(\tau \approx \lambda \gg 1)$ Here we write

$$V(x) = \lambda^2 \left(1 - \frac{1}{(x^{2m}+1)^{1/m}} + \varepsilon\right) = \lambda^2(b(x)+\varepsilon),$$

where $|\varepsilon| \lesssim 1$. And we shall further subdivide this analysis based on the relationship between λ and ε: $\varepsilon \geq -C\lambda^{-2m/(1+m)}$ and $\varepsilon \leq -C\lambda^{-2m/(1+m)}$ where $C > 0$ is a sufficiently large constant.

Case A: $(\varepsilon \geq -C\lambda^{-2m/(1+m)})$ Here we shall show

(3.9)
$$\lambda\|(\lambda^{-2m/(1+m)} + |b(x)+\varepsilon|)^{1/4}\phi\|_{L_x^\infty}$$
$$+ \|(\lambda^{-2m/(1+m)} + |b(x)+\varepsilon|)^{-1/4}\partial_x\phi\|_{L_x^\infty}$$
$$\lesssim \|(\lambda^{-2m/(1+m)} + |b(x)+\varepsilon|)^{-1/4}g\|_{L_x^1}.$$

And due to the following lemma, this will imply (3.7) in this case.

Lemma 3.2. *For $m > 1$ and $\varepsilon > -C\lambda^{-2m/(1+m)}$, where $C > 0$, we have*

(3.10) $$\|(\lambda^{-2m/(1+m)} + |b(x)+\varepsilon|)^{-1/4}\|_{L^2(\{|x|\leq 1\})} \lesssim \lambda^{(m-1)/2(m+1)}$$

and for any $\delta > 0$

$$\|(\lambda^{-2m/(1+m)} + |b(x)+\varepsilon|)^{-1/4}|x|^{\frac{m-1+\delta}{2}}\|_{L^2(\{|x|\leq 1\})} = \mathcal{O}(1).$$

Proof. We shall begin by examining the case that $b(x) \leq 2C\lambda^{-2m/(1+m)}$. From this, we have $|x| \lesssim \lambda^{-1/(1+m)}$ and

$$\int_{\{x\,:\,b(x)\leq 2C\lambda^{-2m/(1+m)}\}} \frac{1}{(\lambda^{-2m/(1+m)} + |b(x)+\varepsilon|)^{-1/2}}\,dx$$
$$\lesssim \int_{|x|\lesssim\lambda^{-1/(1+m)}} \lambda^{m/(1+m)}\,dx$$
$$\approx \lambda^{(m-1)/(m+1)},$$

which gives the first estimate. Similarly,

$$\int_{\{x\,:\,b(x)\leq 2C\lambda^{-2m/(1+m)}\}} \frac{|x|^{m-1+\delta}}{(\lambda^{-2m/(1+m)} + |b(x)+\varepsilon|)^{-1/2}}\,dx$$
$$\lesssim \int_{|x|\lesssim\lambda^{-1/(1+m)}} \lambda^{\frac{1-\delta}{m+1}}\,dx = \mathcal{O}(1).$$

We next consider when $b(x) \geq 2C\lambda^{-2m/(1+m)}$. In this setting, we have that $|b(x) + \varepsilon| \geq |b(x)|/2$ and that $|x| \gtrsim \lambda^{-1/(1+m)}$. Then we have

$$\int_{\{x \,:\, b(x) \geq 2C\lambda^{-2m/(1+m)}\}} \frac{1}{(\lambda^{-2m/(1+m)} + |b(x) + \varepsilon|)^{-1/2}} \, dx$$
$$\lesssim \int_{c\lambda^{-1/(1+m)}}^{1} b(x)^{-1/2} \, dx.$$

Note that

$$b(x)^{-1/2} = \frac{(x^{2m} + 1)^{1/2m}}{((x^{2m} + 1)^{1/m} - 1)^{1/2}}.$$

Moreover, we have

$$(3.11) \quad (x^{2m} + 1)^{1/m} - 1 = \int_0^{|x|} \frac{2y^{2m-1}}{(y^{2m} + 1)^{1-\frac{1}{m}}} \, dy \gtrsim \int_0^{|x|} y^{2m-1} \, dy \approx |x|^{2m}.$$

We also note for later purposes that the quantity on the left is $\mathcal{O}(|x|^{2m})$ on the support of ϕ. In particular, on $|x| \leq 1$, $b(x)^{-1/2} \lesssim |x|^{-m}$. This gives that

$$\|(\lambda^{-2m/(1+m)} + |b(x) + \varepsilon|)^{-1/4}\|_{L^2(\{x \,:\, b(x) \geq 2C\lambda^{-2m/(1+m)}\})}^2$$
$$\lesssim \int_{c\lambda^{-1/(1+m)}}^{1} |x|^{-m} \, dx \lesssim \lambda^{\frac{m-1}{m+1}}$$

as desired for the first estimate, and

$$\|(\lambda^{-2m/(1+m)} + |b(x) + \varepsilon|)^{-1/4} |x|^{\frac{m-1+\delta}{2}}\|_{L^2(\{x \,:\, b(x) \geq 2C\lambda^{-2m/(1+m)}\})}^2$$
$$\lesssim \int_{c\lambda^{-1/(1+m)}}^{1} |x|^{-1+\delta} \, dx = \mathcal{O}(1).$$

$\square$

We now proceed to proving (3.9).

Proof of (3.9). Here we consider two subcases.
     Case i: $(b(x) \geq 2C\lambda^{-2m/(1+m)})$ We define

$$E[\phi] = \lambda^2 (b(x) + \varepsilon)^{1/2} \phi^2 + (b(x) + \varepsilon)^{-1/2} (\partial_x \phi)^2$$
$$+ \frac{1}{2} b'(x)(b(x) + \varepsilon)^{-3/2} \phi \partial_x \phi.$$

Since $\varepsilon \geq -C\lambda^{-2m/(1+m)}$, it follows that $1 \ll \lambda^2(b(x) + \varepsilon)^{(m+1)/m}$. Since (3.11) shows that $|b'(x)|^2 \lesssim b(x)^{(2m-1)/m}$, on $|x| \leq 1$, we have

$$\frac{1}{2}(b'(x))^2(b(x) + \varepsilon)^{-5/2} \leq \lambda^2(b(x) + \varepsilon)^{1/2}$$

provided C is sufficiently large. Thus, by observing that

$$\frac{1}{2}b'(x)(b(x) + \varepsilon)^{-3/2}\phi\partial_x\phi \geq -\frac{1}{4}(b(x) + \varepsilon)^{-1/2}(\partial_x\phi)^2$$
$$-\frac{1}{4}(b'(x))^2(b(x) + \varepsilon)^{-5/2}\phi^2,$$

it follows that $E[\phi]$ is positive.

We can compute

$$(3.12) \quad \frac{dE}{dx} = \left(\frac{1}{2}b''(x)(b(x) + \varepsilon)^{-3/2} - \frac{3}{4}(b'(x))^2(b(x) + \varepsilon)^{-5/2}\right)\phi\partial_x\phi$$
$$+ \left(2(b(x) + \varepsilon)^{-1/2}\partial_x\phi + \frac{1}{2}b'(x)(b(x) + \varepsilon)^{-3/2}\phi\right)g.$$

From this and that $b'(x)(b(x) + \varepsilon)^{-3/2}\lambda^{-1} = \mathcal{O}(1)$ on $|x| \leq 1$, which is a consequence of (3.11), it follows that

$$(3.13) \quad \frac{dE}{dx} \lesssim \lambda^{-1}b''(x)(b(x) + \varepsilon)^{-3/2}E + \lambda^{-1}(b'(x))^2(b(x) + \varepsilon)^{-5/2}E$$
$$+ (b(x) + \varepsilon)^{-1/4}gE^{1/2}.$$

Using (3.11), we observe that

$$\lambda^{-1}b''(x)(b(x) + \varepsilon)^{-3/2} + \lambda^{-1}(b'(x))^2(b(x) + \varepsilon)^{-5/2}$$
$$\lesssim (b(x) + \varepsilon)^{-1/2-1/m}\lambda^{-1}$$
$$\lesssim (b(x) + \varepsilon)^{-1/2}\lambda^{-(m-1)/(m+1)}.$$

So,

$$\frac{d\,E^{1/2}}{dx} \lesssim \lambda^{-(m-1)/(m+1)}(b(x) + \varepsilon)^{-1/2}E^{1/2} + (b(x) + \varepsilon)^{-1/4}g.$$

Upon integrating (if $x > 0$, we apply the negative of the integral from 1 to x, and if $x < 0$, we integrate from -1 to x) and applying Gronwall's inequality,

since

$$\lambda^{-(m-1)/(m+1)} \int (b(x) + \varepsilon)^{-1/2} \, dx = \mathcal{O}(1)$$

by (3.10), it follows that

$$E^{1/2} \lesssim \int (b(x) + \varepsilon)^{-1/4} g \, dx.$$

This estimate implies (3.9).

Case ii: $(b(x) \leq 2C\lambda^{-2m/(1+m)})$ Defining

$$E[\phi] = \lambda^{(m+2)/(m+1)} \phi^2 + \lambda^{m/(m+1)} (\partial_x \phi)^2,$$

it follows that

$$\begin{aligned} \frac{dE}{dx} &= 2\lambda^{(m+2)/(m+1)} \phi \partial_x \phi + 2\lambda^{m/(m+1)} \partial_x \phi (g - \lambda^2 (b + \varepsilon) \phi) \\ &\lesssim \lambda^{1/(m+1)} E + \lambda^{m/2(m+1)} |g| E^{1/2}. \end{aligned}$$

Dividing through by $E^{1/2}$, integrating, and applying Gronwall's inequality, where (3.10) shows that the integral in the resulting exponential is $\mathcal{O}(1)$, we have

$$E^{1/2} \lesssim \|\lambda^{m/2(m+1)} g\|_{L^1},$$

which gives (3.9) in this range. $\qquad\square$

Case B: $(\varepsilon \leq -C\lambda^{-2m/(1+m)})$ In this remaining case, we will show

$$(3.14) \qquad \begin{aligned} &\lambda \|(\lambda^{-2/3} |\varepsilon|^{(2m-1)/3m} + |b(x) + \varepsilon|)^{1/4} \phi\|_{L_x^\infty} \\ &\quad + \|(\lambda^{-2/3} |\varepsilon|^{(2m-1)/3m} + |b(x) + \varepsilon|)^{-1/4} \partial_x \phi\|_{L_x^\infty} \\ &\lesssim \|(\lambda^{-2/3} |\varepsilon|^{(2m-1)/3m} + |b(x) + \varepsilon|)^{-1/4} g\|_{L_x^1}, \end{aligned}$$

which, as above, will imply (3.7) when combined with the lemma:

Lemma 3.3. *For $m > 1$ and $\varepsilon \leq -C\lambda^{-2m/(1+m)}$, where $C > 0$, we have*

$$(3.15) \quad \|((\lambda^{-2/3} |\varepsilon|^{(2m-1)/3m} + |b(x) + \varepsilon|)^{-1/4}\|_{L^2(\{|x| \leq 1\})} \lesssim \lambda^{(m-1)/2(m+1)}$$

and for any $\delta > 0$

$$(3.16) \quad \|((\lambda^{-2/3} |\varepsilon|^{(2m-1)/3m} + |b(x) + \varepsilon|)^{-1/4} |x|^{\frac{m-1+\delta}{2}}\|_{L^2(\{|x| \leq 1\})} = \mathcal{O}(1).$$

Proof. Here, as we will below, we consider three separate cases. For convenience, we will abbreviate $\alpha = C\lambda^{-2/3}|\varepsilon|^{\frac{2m-1}{3m}}$.

Case i: $(b + \varepsilon \geq C\lambda^{-2/3}|\varepsilon|^{(2m-1)/3m})$ Using that $\varepsilon < 0$ here, we have $b'(x) \gtrsim (b(x))^{\frac{2m-1}{2m}} \gtrsim (b(x) + \varepsilon)^{\frac{2m-1}{2m}}$. Thus,

$$\int_{b+\varepsilon \geq \alpha} \frac{dx}{(b(x) + \varepsilon)^{1/2}} \lesssim \int_{\substack{b+\varepsilon \geq \alpha \\ x > 0}} \frac{b'(x)}{(b(x) + \varepsilon)^{\frac{1}{2} + \frac{2m-1}{2m}}}\, dx$$
$$\lesssim (\alpha)^{-\frac{m-1}{2m}} \lesssim \lambda^{(m-1)/(m+1)},$$

which establishes (3.15). For (3.16), we similarly argue

$$\int_{b+\varepsilon \geq \alpha} \frac{|x|^{m-1+\delta}}{(b(x) + \varepsilon)^{1/2}}\, dx \lesssim \int_{\substack{b+\varepsilon \geq \alpha \\ x > 0}} \frac{(b')^{\frac{m-1+\delta}{2m-1}}}{(b(x) + \varepsilon)^{1/2}}\, dx$$
$$\lesssim \int_{\substack{b+\varepsilon \geq \alpha \\ x > 0}} \frac{b'(x)}{(b(x) + \varepsilon)^{1 - \frac{\delta}{2m}}}\, dx \lesssim |\varepsilon|^{\delta/2m}.$$

Case ii: $(|b + \varepsilon| \leq C\lambda^{-2/3}|\varepsilon|^{(2m-1)/3m})$ We note that $|b + \varepsilon| \leq \alpha$ if and only if

$$\left[\frac{1}{[1 - (|\varepsilon| - \alpha)]^m} - 1\right]^{1/2m} \leq |x| \leq \left[\frac{1}{[1 - (|\varepsilon| + \alpha)]^m} - 1\right]^{1/2m}.$$

The length of this interval satisfies:

$$(3.17) \qquad \left[\frac{1}{[1 - (|\varepsilon| + \alpha)]^m} - 1\right]^{1/2m} - \left[\frac{1}{[1 - (|\varepsilon| - \alpha)]^m} - 1\right]^{1/2m}$$
$$= \frac{1}{2m} \int_{\frac{1}{[1-(|\varepsilon|-\alpha)]^m} - 1}^{\frac{1}{[1-(|\varepsilon|+\alpha)]^m} - 1} x^{-1 + \frac{1}{2m}}\, dx$$
$$\lesssim |\varepsilon|^{-1 + \frac{1}{2m}} \left(\frac{1}{[1 - (|\varepsilon| + \alpha)]^m} - \frac{1}{[1 - (|\varepsilon| - \alpha)]^m}\right)$$
$$\lesssim |\varepsilon|^{-1 + \frac{1}{2m}} \alpha \approx \lambda^{-\frac{2}{3}}|\varepsilon|^{-\frac{2m-1}{6m}}.$$

Then

$$\int_{|b+\varepsilon| \leq \alpha} \lambda^{1/3}|\varepsilon|^{-(2m-1)/6m}\, dx \lesssim \lambda^{-1/3}|\varepsilon|^{-(2m-1)/3m} \lesssim \lambda^{(m-1)/(m+1)},$$

which establishes (3.15).

Similarly, for (3.16), we have

$$\int_{|b+\varepsilon|\leq\alpha}\lambda^{1/3}|\varepsilon|^{-(2m-1)/6m}|x|^{m-1+\delta}\,dx$$

$$\lesssim \lambda^{1/3}|\varepsilon|^{-\frac{2m-1}{6m}}\left\{\left[\frac{1}{[1-(|\varepsilon|+\alpha)]^m}-1\right]^{\frac{m+\delta}{2m}}-\left[\frac{1}{[1-(|\varepsilon|-\alpha)]^m}-1\right]^{\frac{m+\delta}{2m}}\right\}$$

$$\lesssim \lambda^{1/3}|\varepsilon|^{-(5m-1)/6m}|\varepsilon|^{\delta/2m}\alpha$$

$$\approx \lambda^{-1/3}|\varepsilon|^{-(m+1)/6m}|\varepsilon|^{\delta/2m}\lesssim |\varepsilon|^{\delta/2m}.$$

Case iii: $(b+\varepsilon\leq -C\lambda^{-2/3}|\varepsilon|^{(2m-1)/3m})$ In this region, we have $|\varepsilon|>C\lambda^{-2m/1+m}$, $|b+\varepsilon|\geq C\lambda^{-2/3}|\varepsilon|^{\frac{2m-1}{3m}}$, and $|x|\lesssim|\varepsilon|^{1/2m}$. Thus, since $m\geq 2$,

$$\int_{|b(x)+\varepsilon|\geq C\lambda^{-2/3}|\varepsilon|^{\frac{2m-1}{3m}}}\frac{dx}{|b(x)+\varepsilon|^{1/2}}\lesssim \lambda^{1/3}|\varepsilon|^{-\frac{2m-1}{6m}}|\varepsilon|^{1/2m}\lesssim \lambda^{\frac{m-1}{m+1}},$$

which gives (3.15). For (3.16), by factoring the denominator, we observe

$$\int_{b(x)+\varepsilon\leq-\alpha}\frac{|x|^{m-1+\delta}}{|b(x)+\varepsilon|^{1/2}}\,dx$$

$$\lesssim \frac{1}{|\varepsilon|^{1/4}}\int_{b(x)+\varepsilon\leq-\alpha}\frac{|x|^{m-1+\delta}}{(|\varepsilon|^{1/2}-(b(x))^{1/2})^{1/2}}\,dx$$

$$\lesssim \frac{1}{|\varepsilon|^{1/4}}\int_{b(x)+\varepsilon\leq-\alpha}\frac{b'(x)}{b(x)^{1/2}(|\varepsilon|^{1/2}-(b(x))^{1/2})^{1/2}}\,dx=\mathcal{O}(1).$$

$$\square$$

Proof of (3.14). We will split the proof into three cases. For the first case, we argue in a fashion similar to that of Case i of (3.9). And in the second case, the argument mirrors that of Case ii of (3.9). In the remaining case, the sign of the potential term $b(x)+\varepsilon$ changes, and we use arguments common to elliptic equations.

Case i: $(b+\varepsilon\geq 2C\lambda^{-2/3}|\varepsilon|^{(2m-1)/3m})$ Here, as in Case $A(i)$, we have $|b'(x)|^2\lesssim b(x)^{(2m-1)/m}$. Then we see that

$$|b'(x)|^2\lesssim |b(x)+\varepsilon|^{(2m-1)/m}+|\varepsilon|^{(2m-1)/m}$$

$$\leq |b(x)+\varepsilon|^3\Big(|b(x)+\varepsilon|^{-(m+1)/m}+|\varepsilon|^{(2m-1)/m}|b(x)+\varepsilon|^{-3}\Big).$$

Thus, in this case, we have

(3.18) $$|b'(x)|^2\lesssim \lambda^2|b(x)+\varepsilon|^3.$$

If we define,

$$E[\phi] = \lambda^2 (b(x) + \varepsilon)^{1/2} \phi^2 + (b(x) + \varepsilon)^{-1/2} (\partial_x \phi)^2$$
$$+ \frac{1}{2} b'(x)(b(x) + \varepsilon)^{-3/2} \phi \partial_x \phi,$$

it follows from (3.18) that $E[\phi]$ is positive provided C is sufficiently large.

We then compute

$$\frac{dE}{dx} \lesssim \lambda^{-1} b''(x)(b(x) + \varepsilon)^{-3/2} E + \lambda^{-1} (b'(x))^2 (b(x) + \varepsilon)^{-5/2} E$$
$$+ (b(x) + \varepsilon)^{-1/4} g E^{1/2},$$

and an application of Gronwall's inequality will yield (3.14) provided that we can show

(3.19)
$$\lambda^{-1} \int_{b(x) + \varepsilon \geq 2C\lambda^{-2/3} |\varepsilon|^{(2m-1)/3m}} \left(b''(x)(b(x) + \varepsilon)^{-\frac{3}{2}} + (b'(x))^2 (b(x) + \varepsilon)^{-\frac{5}{2}} \right) dx$$
$$= \mathcal{O}(1).$$

By symmetry, it suffices to demonstrate this where $x > 0$.

We first note that

$$(b'(x))^2 (b(x) + \varepsilon)^{-5/2} \lesssim b'(x)(b(x) + \varepsilon)^{-(3m+1)/2m}$$
$$+ |\varepsilon|^{\frac{2m-1}{2m}} b'(x)(b(x) + \varepsilon)^{-5/2}.$$

Thus using the change of variables $u = b(x) + \varepsilon$, we can evaluate

$$\lambda^{-1} \int_{b(x) + \varepsilon \geq 2C\lambda^{-2/3} |\varepsilon|^{(2m-1)/3m}} (b'(x))^2 (b(x) + \varepsilon)^{-5/2} \, dx$$
$$\lesssim \lambda^{-1} (\lambda^{-2/3} |\varepsilon|^{\frac{2m-1}{3m}})^{-\frac{m+1}{2m}} + \lambda^{-1} |\varepsilon|^{\frac{2m-1}{2m}} (\lambda^{-2/3} |\varepsilon|^{\frac{2m-1}{3m}})^{-3/2} = \mathcal{O}(1).$$

For the other term in (3.19), we argue similarly. Here,

$$b''(x) \lesssim b'(x)|x|^{-1} \lesssim b'(x)(b(x) + \varepsilon)^{-1/2m}$$

where the last step follows as $\varepsilon < 0$ in this case. Hence

$$\lambda^{-1} b''(x)(b(x) + \varepsilon)^{-3/2} \lesssim \lambda^{-1} b'(x)(b(x) + \varepsilon)^{-(3m+1)/2m},$$

and the integral of the right side was evaluated in the previous step, which completes the proof of (3.19).

Case ii: ($|b + \varepsilon| \leq C\lambda^{-2/3}|\varepsilon|^{(2m-1)/3m}$). We define

$$E[\phi] = \lambda^2 \lambda^{-\frac{1}{3}} |\varepsilon|^{\frac{2m-1}{6m}} \phi^2 + \lambda^{\frac{1}{3}} |\varepsilon|^{-\frac{2m-1}{6m}} (\partial_x \phi)^2$$

and compute

$$\frac{dE}{dx} = 2\lambda^2 \left(\lambda^{-\frac{1}{3}} |\varepsilon|^{\frac{2m-1}{6m}} - \lambda^{\frac{1}{3}} |\varepsilon|^{-\frac{2m-1}{6m}} (b(x) + \varepsilon) \right) \phi \partial_x \phi + 2\lambda^{\frac{1}{3}} |\varepsilon|^{-\frac{2m-1}{6m}} \partial_x \phi g.$$

Thus,

(3.20) $$\frac{d}{dx} E^{1/2} \lesssim \lambda^{\frac{2}{3}} |\varepsilon|^{\frac{2m-1}{6m}} E^{1/2} + \lambda^{1/6} |\varepsilon|^{-\frac{2m-1}{12m}} g.$$

An application of Gronwall's inequality to (3.20) yields (3.14) in the given range when combined with (3.17).

Case iii: ($b + \varepsilon \leq -C\lambda^{-2/3}|\varepsilon|^{(2m-1)/3m}$) We define $x_\pm$ so that $[x_-, x_+] = \{x : b + \varepsilon \leq -\alpha\}$. We then multiply the equation (3.6) by $-\lambda\phi$ and integrate over $[x_-, x_+]$ to obtain

$$\int_{x_-}^{x_+} \lambda(\partial_x \phi)^2 + \lambda^3 |b(x) + \varepsilon| \phi^2 \, dx = \int_{x_-}^{x_+} -\lambda\phi g \, dx + \lambda\phi\partial_x\phi \Big|_{x_-}^{x_+}.$$

But by the previous case, we have

(3.21) $$\left| \lambda\phi\partial_x\phi \Big|_{x_-}^{x_+} \right| \lesssim \|(\lambda^{-2/3}|\varepsilon|^{(2m-1)/3m} + |b(x) + \varepsilon|)^{-1/4} g\|_{L_x^1}^2.$$

Thus, we have shown

(3.22) $$\int_{x_-}^{x_+} \lambda(\partial_x \phi)^2 + \lambda^3 |b(x) + \varepsilon| \phi^2 \, dx$$
$$\lesssim \lambda \|(\lambda^{-2/3}|\varepsilon|^{(2m-1)/3m} + |b(x) + \varepsilon|)^{1/4} \phi\|_{L^\infty}$$
$$\times \|(\lambda^{-2/3}|\varepsilon|^{(2m-1)/3m} + |b(x) + \varepsilon|)^{-1/4} g\|_{L_x^1}$$
$$+ \|(\lambda^{-2/3}|\varepsilon|^{(2m-1)/3m} + |b(x) + \varepsilon|)^{-1/4} g\|_{L_x^1}^2.$$

By the Fundamental Theorem of Calculus and (3.21), we have

$$\lambda^2 (\lambda^{-2/3}|\varepsilon|^{(2m-1)/3m} + |b(x) + \varepsilon|)^{1/2} \phi^2(x)$$
$$\lesssim \|(\lambda^{-2/3}|\varepsilon|^{(2m-1)/3m} + |b(x) + \varepsilon|)^{-1/4} g\|_{L_x^1}^2$$
$$+ \int_{x_-}^{x_+} \lambda^2 (\lambda^{-2/3}|\varepsilon|^{(2m-1)/3m} + |b(x) + \varepsilon|)^{-1/2} |b'(x)| \phi^2$$
$$+ 2\lambda^2 (\lambda^{-2/3}|\varepsilon|^{(2m-1)/3m} + |b(x) + \varepsilon|)^{1/2} |\phi\partial_x\phi| \, dx.$$

Using (3.18), it follows that this is

$$\lesssim \|(\lambda^{-2/3}|\varepsilon|^{(2m-1)/3m} + |b(x) + \varepsilon|)^{-1/4}g\|_{L_x^1}^2$$
$$+ \int_{x_-}^{x_+} \lambda^3 |b(x) + \varepsilon|\phi^2 + \lambda(\partial_x\phi)^2 \, dx.$$

And thus, combined with (3.22), this completes the proof of (3.14).   □

4. Sharpness via quasimodes — Proof of Theorem 1.3

The notation here is much simpler if we work with complex valued solutions to the wave equation. Upon constructing a complex valued solution that saturates the estimate, it immediately follows that either the real part or the imaginary part of the constructed solution is a real valued solution that also saturates the estimate. Moreover, we shall work within the $\phi = \pi/2$ plane, which is a restriction that is preserved by our equation.

The proof is based on the following quasimode that was constructed in [8].

Lemma 4.1 ([8]). *Given $\lambda \gg 1$ and $\alpha, \gamma \in \mathbb{R}$, there exists a function $\tilde{u} \in C^3(\mathbb{R})$ so that supp $\tilde{u} \subseteq [-2\lambda^{-\frac{1}{m+1}}, 2\lambda^{-\frac{1}{m+1}}]$, $\|\tilde{u}\|_{L^2}^2 \approx \lambda^{\frac{m-1}{m+1}}$, and*

$$-\lambda^{-2}\partial_x^2\tilde{u} = \left(E + \frac{x^{2m}}{m}\right)\tilde{u} - R,$$

where $E = (\alpha + i\gamma)\lambda^{-\frac{2m}{m+1}}$ and $\|R\|_{L^2} \leq C\lambda^{-\frac{2m}{m+1}}\|\tilde{u}\|_{L^2}$.

We shall let ψ solve

$$\begin{cases} \Box_{\mathfrak{g}}\psi = 0, \\ \psi(0, x, \theta, \phi) = e^{i\lambda\theta}a(x)^{-1}\tilde{u}(x), \quad \partial_t\psi(0, x, \theta, \phi) = i\tau e^{i\lambda\theta}a(x)^{-1}\tilde{u}(x). \end{cases}$$

It will suffice to prove

$$(4.1) \qquad \int_0^T \|\beta(|x|)a(x)^{-1}|\nabla_0\psi|\|_{L^2}^2 \, dt \gtrsim \|\langle D_\omega\rangle^{\frac{m-1}{2(m+1)}}a(x)^{-1}|\nabla_0\psi_0|\|_{L^2}^2$$
$$+ \|\langle D_\omega\rangle^{\frac{m-1}{2(m+1)}}\psi_1\|_{L^2}^2,$$

as integration by parts and Lemma 4.1 give that

$$\|\langle D_\omega\rangle^{\frac{m-1}{2(m+1)}}\partial_x\psi_0\|_{L^2} \lesssim \|\langle D_\omega\rangle^{\frac{m-1}{2(m+1)}}a(x)^{-1}|\nabla_0\psi_0|\|_{L^2}.$$

To show that ψ satisfies (4.1), we construct the approximate solution

$$v = e^{i\tau t}e^{i\lambda\theta}a(x)^{-1}\tilde{u}(x).$$

Here τ satisfies $\tau^2 = \lambda^2(1 + E)$ with $\Im\tau < 0$. We compute that

$$\Box_{\mathfrak{g}}v = a(x)^{-1}e^{i\tau t}e^{i\lambda\theta}\left[\left(\lambda^2 - \lambda^2\frac{x^{2m}}{m} - \lambda^2 a(x)^{-2} - \frac{a''(x)}{a(x)}\right)\tilde{u} + \lambda^2 R\right]$$

with $(v, \partial_t v)|_{t=0} = (\psi, \partial_t\psi)|_{t=0}$. On the support of $\tilde{u}$, we have $a(x)^{-2} = 1 - \frac{x^{2m}}{m} + \mathcal{O}(\lambda^{-\frac{4m}{m+1}})$ and $\lambda^{-2}a''(x)/a(x) = \mathcal{O}(\lambda^{-\frac{4m}{m+1}})$. Thus, $\Box_{\mathfrak{g}}v = \tilde{R}$ where $\tilde{R} = \lambda^2 a(x)^{-1}e^{i\tau t}e^{i\lambda\theta}(R + \mathcal{O}(\lambda^{-\frac{4m}{m+1}})\tilde{u})$, and in particular

$$(4.2) \qquad \|\tilde{R}\|_{L^2} \lesssim e^{|\Im\tau|t}\lambda^{\frac{2}{m+1}}\|\tilde{u}\|_{L^2}.$$

For $T = \varepsilon\lambda^{\frac{m-1}{m+1}}$ where $\varepsilon > 0$ will be chosen later but is independent of λ, we compute

$$\|\partial_\theta v\|^2_{L^2 L^2} = \int_0^T \|e^{i\tau t}\partial_\theta\psi(0,\,\cdot\,)\|^2_{L^2}\,dt$$

$$= \int_0^T e^{2|\Im\tau|t}\|\partial_\theta\psi(0,\,\cdot\,)\|^2_{L^2}\,dt = T\cdot B(2T)\|\partial_\theta\psi(0,\,\cdot\,)\|^2_{L^2}$$

where $B(T) = \frac{e^{T|\Im\tau|}-1}{T|\Im\tau|}$. Hence

$$(4.3) \qquad \|\partial_\theta v\|^2_{L^2 L^2} = \varepsilon B(2T)\lambda^{\frac{m-1}{m+1}}\|\partial_\theta\psi(0,\,\cdot\,)\|^2_{L^2}.$$

We now estimate the error $w(t, x, \theta) = \psi(t, x, \theta) - v(t, x, \theta)$, which solves

$$\Box_{\mathfrak{g}}w = \tilde{R}, \quad (w, \partial_t w)|_{t=0} = (0, 0).$$

Thus, by (1.2),

$$\|(a(x))^{-1}\partial_\theta w\|_{L^\infty L^2} \leq \int_0^T \|\tilde{R}\|_{L^2}\,ds$$

$$\lesssim \int_0^T e^{|\Im\tau|t}\lambda^{-\frac{m-1}{m+1}}\|\partial_\theta\psi(0,\,\cdot\,)\|_{L^2}\,ds$$

$$= T\cdot B(T)\lambda^{-\frac{m-1}{m+1}}\|\partial_\theta\psi(0,\,\cdot\,)\|_{L^2}.$$

And hence

$$(4.4) \qquad \|(a(x))^{-1}\partial_\theta w\|_{L^2 L^2} \lesssim B(T) T^{3/2} \lambda^{-\frac{m-1}{m+1}} \|\partial_\theta \psi(0,\,\cdot\,)\|_{L^2}$$
$$= \varepsilon^{3/2} B(T) \lambda^{\frac{m-1}{2(m+1)}} \|\partial_\theta \psi(0,\,\cdot\,)\|_{L^2}.$$

Using (4.3), (4.4), and that v is supported on $[-1/2, 1/2]$ if λ is sufficiently large, it follows

$$\|\beta(|x|)\partial_\theta \psi\|^2_{L^2 L^2} \geq \frac{1}{2} \|\beta(|x|)\partial_\theta v\|^2_{L^2 L^2} - \|\beta(|x|)\partial_\theta w\|^2_{L^2 L^2}$$
$$\geq \frac{1}{2} \|\partial_\theta v\|^2_{L^2 L^2} - \|a(x)^{-1}\partial_\theta w\|^2_{L^2 L^2}$$
$$\geq \varepsilon \left(\frac{1}{2} B(2T) - C^2 \varepsilon^2 (B(T))^2 \right) \lambda^{\frac{m-1}{m+1}} \|\partial_\theta \psi(0,\,\cdot\,)\|^2_{L^2}.$$

If we choose $\varepsilon = \frac{1}{2C}$, we can compute

$$\frac{1}{2} B(2T) - C^2 \varepsilon^2 (B(T))^2 = \frac{1}{2} \sum_{k=2}^{\infty} \frac{(k-2)2^{k-2} + 1}{(k-1)!} (|\Im \tau| T)^{k-2} \geq \frac{1}{2},$$

which completes the proof.

Acknowledgements

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