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Non-vanishing theorem for lc pairs admitting a Calabi–Yau pair

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We prove the non-vanishing conjecture for lc pairs (X, Δ) when X is of Calabi–Yau type.

1. Introduction

Throughout this paper we will work over the complex number field. In this paper we deal with varieties of Calabi–Yau type.

Definition 1.1. Let X be a normal projective variety. Then, X is *of Calabi–Yau type* if there is an $\mathbb{R}$-divisor $C \geq 0$ such that (X, C) is lc and $K_X + C \equiv 0$.

We also recall statement of the non-vanishing conjecture, which is one of the most important open problems in the birational geometry.

Conjecture 1.2 (Non-vanishing). *Let (X, Δ) be a projective lc pair such that $K_X + \Delta$ is pseudo-effective. Then, there exists an $\mathbb{R}$-divisor $E \geq 0$ such that $K_X + \Delta \sim_{\mathbb{R}} E$.*

It is known by Birkar [2] that Conjecture 1.2 implies the existence of log minimal models.

In this paper, we study the non-vanishing conjecture for lc pairs whose underlying variety is of Calabi–Yau type. The following theorem is the main result of this paper.

Theorem 1.3. *Let X be a normal projective variety. Suppose that X is of Calabi–Yau type.*

Then, for any lc pair (X, Δ), Conjecture 1.2 holds.

We briefly introduce known results on Conjecture 1.2. Currently, Conjecture 1.2 is proved for lc pairs of dimension ≤ 3, but Conjecture 1.2 is only partially solved in higher dimensions. For example, Conjecture 1.2 for lc pairs (X, Δ) of $\dim X \geq 4$ is known when

- (X, Δ) is klt and Δ is big ([4]),
- (X, Δ) is klt and X is rationally connected ([10]), or
- $K_X \equiv 0$ ([9], [5], [14], see also [1] and [18]).

Moreover, the arguments in [10] and [6] show that Conjecture 1.2 holds for any lc pair (X, Δ) such that $\dim X = 4$ and X is uniruled, though it is not written explicitly in their papers. Lazić and Peternell proved Conjecture 1.2 for terminal 4-folds under the assumption that $\chi(X, \mathcal{O}_X) \neq 0$ and K_X has a singular metric with algebraic singularities and semipositive curvature current ([17, Theorem B]).

We note that the case $K_X \equiv 0$ mentioned above is a special case of Theorem 1.3. Indeed, when $K_X \equiv 0$ in Theorem 1.3, the statement follows from [5, Corollary 3.3] or the abundance theorem for numerically trivial lc pairs, which is proved by Gongyo [9] (see also [14]). From a viewpoint of Conjecture 1.2, Theorem 1.3 can be regarded as a generalization of the result of [9].

The contents of this paper are as follows: In Section 2, we collect some notations, definitions and two important theorems. In Section 3, we prove Theorem 1.3.

2. Preliminaries

In this section, we collect notations, definitions and two important theorems.

Singularities of pairs. A *pair* (X, Δ) consists of a normal variety X and a boundary $\mathbb{R}$-divisor Δ, that is, an $\mathbb{R}$-divisor whose coefficients belong to $[0, 1]$, on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier.

Let (X, Δ) be a pair. For any prime divisor D over X, $a(D, X, \Delta)$ denotes the discrepancy of D with respect to (X, Δ). In this paper, we use the definitions of Kawamata log terminal (klt, for short) pair, log canonical (lc, for short) pair and divisorially log terminal (dlt, for short) pair as in [16] or [4].

Next, we define log birational models, log minimal models and Mori fiber spaces. Our definition of log minimal models and Mori fiber spaces are non-standard because it is different from the traditional one (see, for example, [16]) and also slightly different from [3, Definition 2.1 and Definition 2.2].

Definition 2.1 (Log birational model). Let $\pi\colon X \to Z$ be a projective morphism from a normal variety to a variety and let (X, Δ) be an lc pair. Let $\pi'\colon X' \to Z$ be a projective morphism from a normal variety to Z and let $\phi\colon X \dashrightarrow X'$ be a birational map over Z. Let E be the reduced ϕ^{-1}-exceptional divisor on X', that is, $E = \sum E_j$ where E_j are ϕ^{-1}-exceptional prime divisors on X'. Then, the pair $(X', \Delta' = \phi_*\Delta + E)$ is called a *log birational model* of (X, Δ) over Z.

Definition 2.2 (Log minimal model and Mori fiber space). Notations as in Definition 2.1, a log birational model (X', Δ') of (X, Δ) over Z is a *weak log canonical model* (*weak lc model*, for short) if

- $K_{X'} + \Delta'$ is nef over Z, and
- for any prime divisor D on X which is exceptional over X', we have

$$a(D, X, \Delta) \leq a(D, X', \Delta').$$

A weak lc model (X', Δ') of (X, Δ) over Z is a *log minimal model* if

- (X', Δ') is $\mathbb{Q}$-factorial, and
- the above inequality on discrepancies is strict.

A log minimal model (X', Δ') of (X, Δ) over Z is called a *good minimal model* if $K_{X'} + \Delta'$ is semi-ample over Z.

On the other hand, a log birational model (X', Δ') of (X, Δ) over Z is called a *Mori fiber space* if X' is $\mathbb{Q}$-factorial and there is a contraction $X' \to W$ over Z with $\dim W < \dim X'$ such that

- the relative Picard number $\rho(X'/W)$ is one and $-(K_{X'} + \Delta')$ is ample over W, and
- for any prime divisor D over X, we have

$$a(D, X, \Delta) \leq a(D, X', \Delta')$$

  and strict inequality holds if D is a divisor on X and exceptional over X'.

As we mentioned before, our definition of log minimal model and Mori fiber space is slightly different from that of [3]. The difference is that we do not assume those models to be dlt. But this difference is not important. More specifically, for any lc pair, the existence of log minimal models (resp. Mori fiber spaces) as in Definition 2.2 is equivalent to the existence of log minimal models (resp. Mori fiber spaces) which are dlt. Indeed, if an lc pair (X, Δ) has a log minimal model (X', Δ') (resp. a Mori fiber space) as in Definition 2.2, we can construct a log minimal model (resp. a Mori fiber space) of (X, Δ) which is dlt (see [12, Remark 2.4]). In our definition, any weak lc model (X', Δ') of a $\mathbb{Q}$-factorial lc pair (X, Δ) constructed with the $(K_X + \Delta)$-MMP is a log minimal model of (X, Δ) even though (X', Δ') may not be dlt.

Next, we define log smooth models.

Definition 2.3 (Log smooth model). Let (X, Δ) be an lc pair and let $f\colon Y \to X$ be a log resolution of $(X, \mathrm{Supp}\Delta)$. Let Γ be a boundary $\mathbb{R}$-divisor on Y such that Γ is a simple normal crossing divisor. Then, the pair (Y, Γ) is a *log smooth model* of (X, Δ) if we can write

$$K_Y + \Gamma = f^*(K_X + \Delta) + F$$

with an effective f-exceptional divisor F such that every f-exceptional prime divisor E satisfying $a(E, X, \Delta) > -1$ is a component of F and $\Gamma - \llcorner \Gamma \lrcorner$.

By definition, $\mathrm{Supp}\,\Gamma = \mathrm{Supp}\, f_*^{-1}\Delta \cup \mathrm{Ex}\,(f)$ and the image of any lc center of (Y, Γ) on X is an lc center of (X, Δ). Any f-exceptional prime divisor E is a component of F if and only if $a(E, X, \Delta) > -1$.

Finally, we recall two important theorems. We freely use these theorems without any mention.

Theorem 2.4 (Dlt blow-up, [15, Theorem 3.1], [7, Theorem 10.4]). *Let X be a normal quasi-projective variety of dimension n and let Δ be an $\mathbb{R}$-divisor such that (X, Δ) is lc. Then, there exists a projective birational morphism $f\colon Y \to X$ from a normal quasi-projective variety Y such that*

(i) *Y is $\mathbb{Q}$-factorial, and*

(ii) *if we set*

$$\Gamma = f_*^{-1}\Delta + \sum_{E:f\text{-exceptional}} E,$$

then the pair (Y, Γ) is dlt and $K_Y + \Gamma = f^*(K_X + \Delta)$.

In the proof of Theorem 1.3, we use a special kind of dlt blow-up (see [12, Corollary 2.11]).

Theorem 2.5 ([3, Theorem 4.1]). *Let (X,Δ) be a $\mathbb{Q}$-factorial lc pair such that $(X,0)$ is klt, and let $\pi\colon X\to Z$ be a projective morphism of normal quasi-projective varieties. If there exists a log minimal model of (X,Δ) over Z, then any $(K_X+\Delta)$-MMP over Z with scaling of an ample divisor terminates.*

3. Proof of Theorem 1.3

In this section, we prove Theorem 1.3.

Lemma 3.1. *Let (X,B) be a projective lc pair and let $\pi\colon (X,B)\to Z$ be a contraction to a normal projective variety Z such that we have $K_X+B\sim_{\mathbb{R}}\pi^*D$ for some D on Z. Then, we can construct the following diagram*

$$\begin{tikzcd}
(X,B) \arrow[r, dashed] \arrow[d, "\pi"'] & (X_0,B_0) \arrow[d, "\pi_0"] \\
Z & Z_0 \arrow[l, "h"]
\end{tikzcd}$$

such that

(1) the morphisms π_0 and h are contractions and h is birational,

(2) (X_0,B_0) is a log birational model of (X,B) and it is a projective $\mathbb{Q}$-factorial lc pair such that $(X_0,0)$ is klt,

(3) $K_{X_0}+B_0\sim_{\mathbb{R}}\pi_0^*h^*D$,

(4) $B_0=B'_0+B''_0$ with $B'_0\geq 0$ and $B''_0\geq 0$ such that $B''_0\sim_{\mathbb{R},Z_0}0$ and any lc center of (X_0,B'_0) dominates Z_0, and

(5) Z_0 is a projective $\mathbb{Q}$-factorial variety such that $(Z_0,0)$ is klt.

Proof. The idea of the proof can be found in [12, Proof of Lemma 4.3]. We prove Lemma 3.1 in two steps.

Step 1. In this step we construct a diagram

$$\begin{tikzcd}
(X,B) \arrow[r, dashed] \arrow[d, "\pi"'] & (\overline{X},\overline{B}) \arrow[d, "\overline{\pi}"] \\
Z & \overline{Z} \arrow[l, "\overline{h}"]
\end{tikzcd}$$

such that $(\overline{X}, \overline{B})$, $\overline{\pi}$ and $\overline{h}$ satisfy (1), (2), (3) and (4) of the lemma.

First, take a dlt blow-up $(W, \Psi) \to (X, B)$ as in [12, Corollary 2.11]. We can decompose $\Psi = \Psi' + \Psi''$ with $\Psi' \geq 0$ and $\Psi'' \geq 0$ such that Ψ'' is vertical over Z and any lc center of (W, Ψ') dominates Z. Moreover, we have $K_W + \Psi' + \Psi'' \sim_{\mathbb{R}, Z} 0$. Since (W, Ψ') is $\mathbb{Q}$-factorial and dlt, by [13, Theorem 1.1], we can run the $(K_W + \Psi')$-MMP over Z with scaling and get a good minimal model $(W, \Psi') \dashrightarrow (\overline{X}, \overline{B}')$ over Z. Let $\overline{B}$ and $\overline{B}''$ be the birational transform of Ψ and Ψ'' on $\overline{X}$, respectively. Then, we have $\overline{B} = \overline{B}' + \overline{B}''$. Let $\overline{\pi} \colon \overline{X} \to \overline{Z}$ be the contraction over Z induced by $K_{\overline{X}} + \overline{B}'$, and let $\overline{h} \colon \overline{Z} \to Z$ be the induced morphism.

We check that the pair $(\overline{X}, \overline{B} = \overline{B}' + \overline{B}'')$ and the morphisms $\overline{\pi} \colon \overline{X} \to \overline{Z}$ and $\overline{h} \colon \overline{Z} \to Z$ satisfy the conditions (1), (2), (3) and (4) of the lemma. By construction, $\overline{\pi}$ and $\overline{h}$ satisfy condition (1). We also have $K_{\overline{X}} + \overline{B} \sim_{\mathbb{R}} \overline{\pi}^* \overline{h}^* D$, which is condition (3). Moreover, since (X, B) is lc and since $K_X + B$ and $K_{\overline{X}} + \overline{B}$ are both $\mathbb{R}$-linearly equivalent to the pullback of D, we see that $(\overline{X}, \overline{B})$ is lc. Since $(W, 0)$ is $\mathbb{Q}$-factorial klt and $(W, \Psi') \dashrightarrow (\overline{X}, \overline{B}')$ is a sequence of steps of the $(K_W + \Psi')$-MMP, $(\overline{X}, 0)$ is $\mathbb{Q}$-factorial klt. Therefore, $(\overline{X}, \overline{B})$ satisfies condition (2). Because we have $K_{\overline{X}} + \overline{B}' + \overline{B}'' \sim_{\mathbb{R}, \overline{Z}} 0$ and $K_{\overline{X}} + \overline{B}' \sim_{\mathbb{R}, \overline{Z}} 0$, we obtain $\overline{B}'' \sim_{\mathbb{R}, \overline{Z}} 0$. Finally, we check that any lc center of $(\overline{X}, \overline{B}')$ dominates $\overline{Z}$. Pick any prime divisor P over $\overline{X}$ such that $a(P, \overline{X}, \overline{B}') = -1$. Then $a(P, W, \Psi') = -1$, and thus P dominates Z. Since $\overline{h} \colon \overline{Z} \to Z$ is birational, we see that P dominates $\overline{Z}$. Therefore, any lc center of $(\overline{X}, \overline{B}')$ dominates $\overline{Z}$. In this way, we see that the pair $(\overline{X}, \overline{B} = \overline{B}' + \overline{B}'')$ satisfies condition (4). So we complete this step.

Step 2. We put $\overline{D} = \overline{h}^* D$. By construction, we have $K_{\overline{X}} + \overline{B} \sim_{\mathbb{R}} \overline{\pi}^* \overline{D}$. In this step, we construct a diagram

$$\begin{tikzcd}
(X, B) \arrow[r, dashed] \arrow[d, "\pi"'] & (\overline{X}, \overline{B}) \arrow[r, dashed] \arrow[d, "\overline{\pi}"'] & (X_0, B_0) \arrow[d, "\pi_0"] \\
Z & \overline{Z} \arrow[l, "\overline{h}"] & Z_0 \arrow[l, "h_0"]
\end{tikzcd}$$

such that (X_0, B_0), π_0 and $h := \overline{h} \circ h_0$ satisfy all the conditions of the lemma.

By construction of $\overline{\pi} \colon (\overline{X}, \overline{B}) \to \overline{Z}$, there exists an $\mathbb{R}$-divisor $\overline{T} \geq 0$ on $\overline{Z}$ such that $\overline{B}'' \sim_{\mathbb{R}} \overline{\pi}^* \overline{T}$. By [8, Corollary 3.2], there exists a klt pair on $\overline{Z}$. Let $h_0 \colon Z_0 \to \overline{Z}$ be a dlt blow-up of the klt pair, which is a small birational morphism. By construction, Z_0 is $\mathbb{Q}$-factorial. Let $\overline{\varphi} \colon \overline{W} \to \overline{X}$ be a log resolution of $(\overline{X}, \operatorname{Supp} \overline{B}')$ such that the induced map $\pi_{\overline{W}} \colon \overline{W} \dashrightarrow Z_0$ is a morphism.

We pick a boundary divisor $\Psi'_{\overline{W}}$ so that $(\overline{W}, \Psi'_{\overline{W}})$ is a log smooth model of $(\overline{X}, \overline{B}')$. Then, we have

$$\begin{aligned} K_{\overline{W}} + \Psi'_{\overline{W}} &= \overline{\varphi}^*(K_{\overline{X}} + \overline{B}') + E_{\overline{W}} \sim_{\mathbb{R}} \overline{\varphi}^* \overline{\pi}^*(\overline{D} - \overline{T}) + E_{\overline{W}} \\ &= (h_0 \circ \pi_{\overline{W}})^*(\overline{D} - \overline{T}) + E_{\overline{W}} \end{aligned}$$

for a $\overline{\varphi}$-exceptional divisor $E_{\overline{W}} \geq 0$. By construction of $\Psi'_{\overline{W}}$, any $\overline{\varphi}$-exceptional prime divisor E_i on $\overline{W}$ is a component of $E_{\overline{W}}$ if and only if $a(E_i, \overline{X}, \overline{B}') > -1$.

We run the $(K_{\overline{W}} + \Psi'_{\overline{W}})$-MMP over Z_0 with scaling of an ample divisor. By the argument of very exceptional divisors (see [3, Theorem 3.4]), $E_{\overline{W}}$ is contracted after finitely many steps. Thus, we get a model $(\overline{W}, \Psi'_{\overline{W}}) \dashrightarrow (X_0, B'_0)$ such that $K_{X_0} + B'_0 \sim_{\mathbb{R}, Z_0} 0$. Let $\pi_0 \colon X_0 \to Z_0$ be the induced morphism. We have the following diagram.

$$\begin{tikzcd} (\overline{X}, \overline{B}') \arrow[d, "\overline{\pi}"'] & (\overline{W}, \Psi'_{\overline{W}}) \arrow[l, "\overline{\varphi}"'] \arrow[r, dashed] \arrow[dr, "\pi_{\overline{W}}"'] & (X_0, B'_0) \arrow[d, "\pi_0"] \\ \overline{Z} & & Z_0 \arrow[ll, "h_0"] \end{tikzcd}$$

Moreover, we have $K_{X_0} + B'_0 \sim_{\mathbb{R}} \pi_0^* h_0^*(\overline{D} - \overline{T})$. Let B''_0 be the birational transform of $\overline{\varphi}^* \overline{B}''$ on X_0, and we put $B_0 = B'_0 + B''_0$. Recall that the divisor $\overline{T}$ on $\overline{Z}$ satisfies $\overline{B}'' \sim_{\mathbb{R}} \overline{\pi}^* \overline{T}$. Hence we have $B''_0 \sim_{\mathbb{R}} \pi_0^* h_0^* \overline{T}$.

From now on, we check that the pair $(X_0, B_0 = B'_0 + B''_0)$ and the morphisms $\pi_0 \colon X_0 \to Z_0$ and $\overline{h} \circ h_0 \colon Z_0 \to Z$ satisfy all the conditions of the lemma. By construction, π_0 and $\overline{h} \circ h_0$ satisfy condition (1), and Z_0 satisfies condition (5). Since $B''_0 \sim_{\mathbb{R}} \pi_0^* h_0^* \overline{T}$, we have

$$K_{X_0} + B_0 = K_{X_0} + B'_0 + B''_0 \sim_{\mathbb{R}} \pi_0^* h_0^*(\overline{D} - \overline{T}) + \pi_0^* h_0^* \overline{T} = \pi_0^* h_0^* \overline{D}.$$

Therefore, we see that $K_{X_0} + B_0$ satisfies condition (3). Next, we check that (X_0, B_0) satisfies condition (2). Note that (X_0, B_0) is lc since $(\overline{X}, \overline{B})$ is lc and $K_{\overline{X}} + \overline{B}$ and $K_{X_0} + B_0$ are both $\mathbb{R}$-linearly equivalent to the pullback of $\overline{D}$. Let E_i be a $\overline{\varphi}$-exceptional prime divisor on $\overline{W}$ such that $a(E_i, \overline{X}, \overline{B}) > -1$. We show that E_i is contracted by the map $\overline{W} \dashrightarrow X_0$. Since we have $a(E_i, \overline{X}, \overline{B}') \geq a(E_i, \overline{X}, \overline{B}) > -1$, we see that E_i is a component of $E_{\overline{W}}$. Then, E_i is contracted by $\overline{W} \dashrightarrow X_0$ since $E_{\overline{W}}$ is contracted by $\overline{W} \dashrightarrow X_0$, and therefore (X_0, B_0) is a log birational model of $(\overline{X}, \overline{B})$. Since $\overline{W}$ is smooth and $(\overline{W}, \Psi'_{\overline{W}}) \dashrightarrow (X_0, B'_0)$ is a sequence of steps of the $(K_{\overline{W}} + \Psi'_{\overline{W}})$-MMP,

$(X_0, 0)$ is $\mathbb{Q}$-factorial klt. Therefore, we see that (X_0, B_0) satisfies condition (2). Finally we check that the pair $(X_0, B_0 = B_0' + B_0'')$ satisfies condition (4). Pick any prime divisor P over X_0 such that $a(P, X_0, B_0') = -1$. Then, we have $a(P, \overline{W}, \Psi'_{\overline{W}}) = -1$, and hence $a(P, \overline{X}, \overline{B}') = -1$ because $(\overline{W}, \Psi'_{\overline{W}})$ is a log smooth model of $(\overline{X}, \overline{B}')$. Because the pair $(\overline{X}, \overline{B}' + \overline{B}'')$ satisfies condition (4), P dominates $\overline{Z}$. Since $h_0\colon Z_0 \to \overline{Z}$ is birational, P dominates Z_0 and hence any lc center of (X_0, B_0') dominates Z_0. Since we have $B_0'' \sim_{\mathbb{R}, Z_0} 0$, the pair $(X_0, B_0 = B_0' + B_0'')$ satisfies condition (4). So we are done. $\square$

Remark 3.2. By construction of the diagram, we see that the divisor B_0'' is reduced, i.e., all coefficients of B_0'' are one (see [12, Lemma 4.3]). But we do not use this fact in this paper.

Lemma 3.3. *Let $\pi\colon (X, B) \to Z$ be a contraction such that*

- *(X, B) is a projective $\mathbb{Q}$-factorial lc pair such that $(X, 0)$ is klt,*
- *$K_X + B \sim_{\mathbb{R}} \pi^* D$ for some D on Z,*
- *$B = B' + B''$ with $B' \geq 0$ and $B'' \geq 0$ such that $B'' \sim_{\mathbb{R}, Z} 0$ and any lc center of (X, B') dominates Z, and*
- *Z is a projective $\mathbb{Q}$-factorial variety such that $(Z, 0)$ is klt.*

Let T be an effective $\mathbb{R}$-divisor on Z such that $B'' \sim_{\mathbb{R}} \pi^* T$. If D is pseudo-effective but $D - eT$ is not pseudo-effective for any $e > 0$, then we can construct the following diagram

$$\begin{tikzcd}
(X, B) \arrow[r, dashed] \arrow[d, "\pi"'] & (\widetilde{X}, \widetilde{B}) \arrow[d, "\widetilde{\pi}"] & \\
Z \arrow[r, dashed] & \widetilde{Z} \arrow[r] & Z^{\vee}
\end{tikzcd}$$

where $\widetilde{B}$ is the birational transform of B on $\widetilde{X}$, such that

- *$(\widetilde{X}, \widetilde{B})$ is projective $\mathbb{Q}$-factorial lc, $(\widetilde{X}, 0)$ is klt, $\widetilde{Z}$ is projective and $\mathbb{Q}$-factorial, $(\widetilde{Z}, 0)$ is klt, and $Z^{\vee}$ is normal and projective,*
- *the maps $X \dashrightarrow \widetilde{X}$ and $Z \dashrightarrow \widetilde{Z}$ are birational contractions,*
- *the morphism $\widetilde{Z} \to Z^{\vee}$ is a contraction such that $\rho(\widetilde{Z}/Z^{\vee}) = 1$ and $\dim Z^{\vee} < \dim \widetilde{Z}$, and*
- *$K_{\widetilde{X}} + \widetilde{B} \sim_{\mathbb{R}} \widetilde{\pi}^* \widetilde{D}$ and $\widetilde{D} \sim_{\mathbb{R}, Z^{\vee}} 0$, where $\widetilde{D}$ is the birational transform of D on $\widetilde{Z}$.*

Proof. We can construct the desired diagram by the same argument as in [12, Step 1 and 2 in the proof of Proposition 5.3]. We write down the details for the reader's convenience.

Let $\{e_n\}_{n\geq 1}$ be a strictly decreasing sequence of positive real numbers such that $e_n < 1$ for any n and $\lim_{n\to\infty} e_n = 0$. By [8, Corollary 3.2], for any $n \geq 1$, we can find a boundary $\mathbb{R}$-divisor Θ_n such that (Z, Θ_n) is klt and

$$K_X + B - e_nB'' \sim_{\mathbb{R}} \pi^*(D - e_nT) \sim_{\mathbb{R}} \pi^*(K_Z + \Theta_n).$$

Since $K_Z + \Theta_n \sim_{\mathbb{R}} D - e_nT$ is not pseudo-effective for any $n \geq 1$, we can run the $(K_Z + \Theta_n)$-MMP with scaling and obtain a Mori fiber space $\widetilde{Z}_n \to Z_n^{\vee}$, and let $Z \dashrightarrow \widetilde{Z}_n$ be the corresponding birational contraction. Let $\widetilde{D}_n$ and $\widetilde{T}_n$ be the birational transforms of D and T on $\widetilde{Z}_n$, respectively. Since $K_Z + \Theta_n \sim_{\mathbb{R}} D - e_nT$ and since D is pseudo-effective, $\widetilde{D}_n - e_n\widetilde{T}_n$ is anti-ample over $Z_n^{\vee}$ and $\widetilde{D}_n$ is nef over $Z_n^{\vee}$. Therefore, $\widetilde{T}_n$ is ample over $Z_n^{\vee}$. Furthermore, by applying the $\mathbb{R}$-boundary divisor version of [12, Lemma 3.6], we have the following diagram

$$\begin{tikzcd}
(X, B - e_nB'') \arrow[r, dashed] \arrow[d, "\pi"'] & (\widetilde{X}_n, \widetilde{B}_n - e_n\widetilde{B}''_n) \arrow[d, "\pi_n"] & \\
Z \arrow[r, dashed] & \widetilde{Z}_n \arrow[r] & Z_n^{\vee}
\end{tikzcd}$$

such that the upper horizontal birational map is a sequence of steps of the $(K_X + B - e_nB'')$-MMP and

$$K_{\widetilde{X}_n} + \widetilde{B}_n - e_n\widetilde{B}''_n \sim_{\mathbb{R}} \pi_n^*(\widetilde{D}_n - e_n\widetilde{T}_n) \quad \text{and} \quad \widetilde{B}''_n \sim_{\mathbb{R}} \pi_n^*\widetilde{T}_n,$$

where $\widetilde{B}_n$ and $\widetilde{B}''_n$ are the birational transforms of B and B'' on $\widetilde{X}_n$, respectively. Now apply the ACC for log canonical thresholds ([11, Theorem 1.1]) to $\widetilde{X}_n$ and apply the ACC for numerically trivial pairs ([11, Theorem 1.5]) to the general fiber of $\widetilde{X}_n \to Z_n^{\vee}$. We see that for some n the pair $(\widetilde{X}_n, \widetilde{B}_n)$ is lc and $K_{\widetilde{X}_n} + \widetilde{B}_n \sim_{\mathbb{R}, Z_n^{\vee}} 0$ ([12, Step 2 in the proof of Proposition 5.3]). Moreover, we have $\widetilde{D}_n \sim_{\mathbb{R}, Z_n^{\vee}} 0$ because we have $K_{\widetilde{X}_n} + \widetilde{B}_n \sim_{\mathbb{R}} \pi_n^*\widetilde{D}_n$. For this n, we put $\widetilde{Z} = \widetilde{Z}_n$, $Z^{\vee} = Z_n^{\vee}$ and $\widetilde{X} = \widetilde{X}_n$. Then,

$$\begin{tikzcd}
(X, B) \arrow[r, dashed] \arrow[d, "\pi"'] & (\widetilde{X}, \widetilde{B}) \arrow[d] & \\
Z \arrow[r, dashed] & \widetilde{Z} \arrow[r] & Z^{\vee}
\end{tikzcd}$$

is the desired diagram. □

Proof of Theorem 1.3. By hypothesis, there is an $\mathbb{R}$-divisor C on X such that (X, C) is lc and $K_X + C \equiv 0$. Then, we have $K_X + C \sim_{\mathbb{R}} 0$ by the abundance theorem for numerically trivial lc pairs. Thus, we may assume $C \neq 0$, and Theorem 1.3 for (X, Δ) is equivalent to Theorem 1.3 for $(X, t\Delta + (1-t)C)$ for any $0 < t < 1$ because $K_X + t\Delta + (1-t)C \sim_{\mathbb{R}} t(K_X + \Delta)$. Therefore, throughout the proof we may freely replace (X, Δ) with $(X, t\Delta + (1-t)C)$.

By taking a dlt blow-up of (X, C) and by replacing (X, Δ) with $(X, t\Delta + (1-t)C)$ for some $0 < t \ll 1$, we can assume X is $\mathbb{Q}$-factorial and $(X, 0)$ is klt.

We prove Theorem 1.3 by induction on the dimension of X.

Step 1. Let $\tau(X, 0; \Delta)$ be the pseudo-effective threshold of Δ with respect to $(X, 0)$, that is,

$$\tau(X, 0; \Delta) = \inf\{\tau \in \mathbb{R}_{\geq 0} \mid K_X + \tau\Delta \text{ is pseudo-effective}\}.$$

Since $C \neq 0$, the divisor K_X is not pseudo-effective, and thus we have $\tau(X, 0; \Delta) > 0$. By replacing (X, Δ) with $(X, \tau(X, 0; \Delta)\Delta)$, we can assume $\tau(X, 0; \Delta) = 1$. We apply [10, Lemma 3.1] in lc setting. By the same argument as in [10, Proof of Lemma 3.1] (see also the proof of Lemma 3.3), the assertion of [10, Lemma 3.1] also holds when the given pair is $\mathbb{Q}$-factorial lc and its underlying variety is klt. Therefore, we can construct a birational contraction $\phi\colon X \dashrightarrow X'$ and a contraction $X' \to Z'$ such that $\dim Z' < \dim X'$, $(X', \phi_*\Delta)$ is lc and $K_{X'} + \phi_*\Delta \sim_{\mathbb{R}, Z'} 0$. We note that the assumption $\tau(X, 0; \Delta) = 1$ is only used for this argument. So we do not use the assumption in the rest of the proof. Since we have $K_X + C \sim_{\mathbb{R}} 0$, the pair (X', ϕ_*C) is lc. Take a log resolution $\psi\colon Y \to X$ of $(X, \mathrm{Supp}(\Delta + C))$ so that the induced map $f\colon Y \dashrightarrow X'$ is a morphism, and let (Y, Δ_Y) and (Y, C_Y) be log smooth models of (X, Δ) and (X, C), respectively.

Since $K_X + C \sim_{\mathbb{R}} 0$, by the negativity lemma, we have $\psi^*(K_X + C) = f^*(K_{X'} + \phi_*C)$. By construction of log smooth models, we see that $K_Y + C_Y - f^*(K_{X'} + \phi_*C)$ is effective, and it is f-exceptional. So we can run the $(K_Y + C_Y)$-MMP over X' with scaling of an ample divisor, and by [3, Theorem 3.5], after finitely many steps the f-exceptional divisor is contracted. Hence, we get a model $f'\colon (Y', C_{Y'}) \to X'$ such that $K_{Y'} + C_{Y'} =$

$f'^*(K_{X'} + \phi_* C) \sim_{\mathbb{R}} 0$. Now we have the following diagram.

$$\begin{tikzcd} Y \arrow[r, dashed] \arrow[d, "\psi"'] \arrow[dr, "f"] & Y' \arrow[d, "f'"] & \\ X \arrow[r, dashed, "\phi"'] & X' \arrow[r] & Z' \end{tikzcd}$$

By construction, the pair $(Y', C_{Y'})$ is lc and $Y \dashrightarrow Y'$ is a sequence of steps of the $(K_Y + t\Delta_Y + (1-t)C_Y)$-MMP for any $0 < t \ll 1$. Fix a sufficiently small $t > 0$ and set $\Gamma_Y = t\Delta_Y + (1-t)C_Y$. Since (Y, Δ_Y) and (Y, C_Y) are log smooth and lc, (Y, Γ_Y) is $\mathbb{Q}$-factorial dlt. Let $\Gamma_{Y'}$ be the birational transform of Γ_Y on Y'. Then, the pair $(Y', \Gamma_{Y'})$ is $\mathbb{Q}$-factorial dlt, and we can write

$$K_{Y'} + \Gamma_{Y'} = f'^*\big(K_{X'} + t\phi_*\Delta + (1-t)\phi_* C\big) + F$$

with an f'-exceptional divisor F. Note that F may not be effective. Run the $(K_{Y'} + \Gamma_{Y'})$-MMP over X' with scaling of an ample divisor. By [3, Theorem 3.5], we reach a model $f'' \colon (Y'', \Gamma_{Y''}) \to X'$ such that

$$K_{Y''} + \Gamma_{Y''} = f''^*\big(K_{X'} + t\phi_*\Delta + (1-t)\phi_* C\big) + F_{Y''}$$

with $F_{Y''} \leq 0$. Now we recall that $(X', \phi_*\Delta)$ and $(X', \phi_* C)$ are lc. Combining it with the above equation, we see that $(Y'', \Gamma_{Y''} - F_{Y''})$ is also lc. By construction, we also have $K_{Y''} + \Gamma_{Y''} - F_{Y''} \sim_{\mathbb{R},\,Z'} 0$. Since $-F_{Y''} \geq 0$ and $(Y'', 0)$ is $\mathbb{Q}$-factorial klt, by [13, Theorem 1.1], we can run the $(K_{Y''} + \Gamma_{Y''})$-MMP over Z' and obtain a good minimal model $(Y'', \Gamma_{Y''}) \dashrightarrow (Y''', \Gamma_{Y'''})$ over Z'. Let $\pi \colon Y''' \to Z$ be the contraction over Z' induced by $K_{Y'''} + \Gamma_{Y'''}$, and let $C_{Y'''}$ be the birational transform of C_Y on Y'''. Note that $\dim Z = \dim Z'$ because the restriction of $K_{Y'''} + \Gamma_{Y'''}$ to any general fiber of $Y''' \to Z'$ is trivial. We also have $K_{Y'''} + \Gamma_{Y'''} \sim_{\mathbb{R},\,Z} 0$ and $K_{Y'''} + C_{Y'''} \sim_{\mathbb{R}} 0$. Furthermore, by construction, the birational map $Y \dashrightarrow Y'''$ is a sequence of steps of the $(K_Y + \Gamma_Y)$-MMP. If $K_{Y'''} + \Gamma_{Y'''}$ is $\mathbb{R}$-linearly equivalent to an effective divisor, then $K_Y + \Gamma_Y$ is $\mathbb{R}$-linearly equivalent to an effective divisor, and so is $K_X + t\Delta + (1-t)C$. Since $K_{Y'''} + C_{Y'''} \sim_{\mathbb{R}} 0$, the pair $(Y''', \Gamma_{Y'''})$ satisfies the hypothesis of Theorem 1.3. So we can replace (X, Δ) and (X, C) by $(Y''', \Gamma_{Y'''})$ and $(Y''', C_{Y'''})$.

In this way, to prove Theorem 1.3, we can assume that there exists a contraction $\pi \colon X \to Z$ to a normal projective variety Z such that $\dim Z < \dim X$ and $K_X + \Delta \sim_{\mathbb{R},\,Z} 0$.

Step 2. We apply Lemma 3.1 to $(X, C) \to Z$. We can construct a diagram

$$\begin{tikzcd} (X, C) \arrow[r, dashed] \arrow[d, "\pi"'] & (X_0, C_0) \arrow[d, "\pi_0"] \\ Z & Z_0 \arrow[l, "h"] \end{tikzcd}$$

such that

- π_0 and h are contractions and h is birational,
- (X_0, C_0) is a log birational model of (X, C) and it is a projective $\mathbb{Q}$-factorial lc pair such that $(X_0, 0)$ is klt,
- $K_{X_0} + C_0 \sim_{\mathbb{R}} 0$,
- $C_0 = C_0' + C_0''$ with $C_0' \geq 0$ and $C_0'' \geq 0$ such that $C_0'' \sim_{\mathbb{R}, Z_0} 0$ and any lc center of (X_0, C_0') dominates Z_0, and
- Z_0 is a projective $\mathbb{Q}$-factorial variety and $(Z_0, 0)$ is klt.

Let $\varphi\colon W \to X$ and $\varphi_0\colon W \to X_0$ be a common resolution. We define the divisor Ψ on W by equation $K_W + \Psi = \varphi^*(K_X + \Delta)$, and set $\Delta_0 = \varphi_{0*}\Psi$. Note that Δ_0 may not be effective but $t\Delta_0 + (1-t)C_0$ is effective for any $0 < t \ll 1$ because (X_0, C_0) is a log birational model of (X, C). By construction, we have $K_{X_0} + \Delta_0 \sim_{\mathbb{R}, Z_0} 0$ and any lc center of $(X_0, t\Delta_0 + (1-t)C_0)$ is an lc center of (X_0, C_0). The pair $(X_0, t\Delta_0 + (1-t)C_0)$ is lc and it satisfies the hypothesis of Theorem 1.3 since $K_{X_0} + C_0 \sim_{\mathbb{R}} 0$. Moreover, it is sufficient to prove Theorem 1.3 for $(X_0, t\Delta_0 + (1-t)C_0)$. Therefore, we can replace $(X, \Delta) \to Z$ and (X, C) by $(X_0, t\Delta_0 + (1-t)C_0) \to Z_0$ and (X_0, C_0), respectively.

In this way, we can assume that

(i) Z is a projective $\mathbb{Q}$-factorial variety and $(Z, 0)$ is klt,

(ii) $C = C' + C''$ for some $C' \geq 0$ and $C'' \geq 0$ such that $C'' \sim_{\mathbb{R}, Z} 0$ and any lc center of (X, C') dominates Z, and

(iii) any lc center of (X, Δ) is an lc center of (X, C).

Step 3. In this step we prove Theorem 1.3 for (X, Δ) when $C'' = 0$. In this case we have $C = C'$.

By conditions (ii) and (iii) in Step 2, all lc centers of (X, Δ) and those of (X, C) dominate Z. Therefore, by [8, Corollary 3.2], there exists an $\mathbb{R}$-divisor Θ (resp. G) on Z such that (Z, Θ) is klt (resp. (Z, G) is klt) and

$K_X + \Delta \sim_{\mathbb{R}} \pi^*(K_Z + \Theta)$ (resp. $K_X + C \sim_{\mathbb{R}} \pi^*(K_Z + G)$). Then, there is an $\mathbb{R}$-divisor $E \geq 0$ such that $K_Z + \Theta \sim_{\mathbb{R}} E$ by the induction hypothesis. Thus, we see that $K_X + \Delta \sim_{\mathbb{R}} \pi^* E$ and so we are done.

Step 4. By Step 3, we can assume that $C'' \neq 0$. Then, the divisor $K_X + C' \sim_{\mathbb{R}} -C''$ is not pseudo-effective, and hence

$$\bigl(K_X + t\Delta + (1-t)C\bigr) - (1-t)C'' = t(K_X + \Delta) + (1-t)(K_X + C')$$

is not pseudo-effective for any $0 < t \ll 1$. We fix a sufficiently small $t > 0$ and set $\Delta_{(t)} = t\Delta + (1-t)C$, $C'_{(t)} = C' + tC''$ and $C''_{(t)} = (1-t)C''$. Then $C'_{(t)} + C''_{(t)} = C$. Since (X, C) is lc, any lc center of $(X, C'_{(t)})$ is an lc center of (X, C'), and thus any lc center of $(X, C'_{(t)})$ dominates Z. Moreover, by construction of $\Delta_{(t)}$, any lc center of $(X, \Delta_{(t)})$ is an lc center of (X, C). Therefore, we see that $\Delta_{(t)}$, $C'_{(t)}$ and $C''_{(t)}$ satisfy the conditions (ii) and (iii) in Step 2 in this proof. We also have $\Delta_{(t)} - C''_{(t)} = t\Delta + (1-t)C'$, and therefore any lc center of $(X, \Delta_{(t)} - C''_{(t)})$ is an lc center of (X, C'). So any lc center of $(X, \Delta_{(t)} - C''_{(t)})$ dominates Z. Since we have $K_X + \Delta_{(t)} - C''_{(t)} = K_X + t\Delta + (1-t)C - (1-t)C''$, the divisor $K_X + \Delta_{(t)} - C''_{(t)}$ is not pseudo-effective. In this way, we can replace Δ, C' and C'' by $\Delta_{(t)}$, $C'_{(t)}$ and $C''_{(t)}$ respectively, and we may assume that $\Delta - C'' \geq 0$, any lc center of $(X, \Delta - C'')$ dominates Z and $K_X + \Delta - C''$ is not pseudo-effective.

We put $\tau = \tau(X, \Delta - C''; C'')$, where the right hand side is the pseudo-effective threshold of C'' with respect to $(X, \Delta - C'')$. We have $0 < \tau \leq 1$ by construction. Put $\overline{\Delta} = \Delta - C'' + \tau C''$, $\overline{C}' = C' + (1-\tau)C''$ and $\overline{C}'' = \tau C''$. Then $\overline{\Delta} - \overline{C}'' = \Delta - C''$ and $\overline{C}' + \overline{C}'' = C$. In particular, $\overline{\Delta} - \overline{C}'' \geq 0$ and any lc center of $(X, \overline{\Delta} - \overline{C}'')$ dominates Z. Note that any lc center of $(X, \overline{C}')$ is an lc center of (X, C') since $\tau > 0$ and (X, C) is lc, and hence any lc center of $(X, \overline{C}')$ dominates Z. Note also that any lc center of $(X, \overline{\Delta})$ is an lc center of (X, C). Thus, the pair $(X, \overline{\Delta})$ and the divisors $\overline{C}'$ and $\overline{C}''$ satisfy the conditions (ii) and (iii) in Step 2. Because it is sufficient to prove Theorem 1.3 for $(X, \overline{\Delta})$, we can replace Δ, C' and C'' by $\overline{\Delta}$, $\overline{C}'$ and $\overline{C}''$, respectively.

In this way, by replacing those divisors, we can assume that

- $\Delta - C'' \geq 0$ and any lc center of $(X, \Delta - C'')$ dominates Z, and
- $K_X + \Delta - eC''$ is not-pseudo-effective for any $e > 0$.

In the rest of the proof we do not use C'.

Step 5. Pick $\mathbb{R}$-divisors D and T on Z such that $K_X + \Delta \sim_{\mathbb{R}} \pi^* D$ and $C'' \sim_{\mathbb{R}} \pi^* T$ respectively. By Step 1, 2 and 4, $(X, \Delta) \to Z$ and $C'' \neq 0$ satisfy

- (X, Δ) is a projective $\mathbb{Q}$-factorial lc pair such that $(X, 0)$ is klt,
- $K_X + \Delta \sim_{\mathbb{R}} \pi^* D$,
- Z is a projective $\mathbb{Q}$-factorial variety such that $(Z, 0)$ is klt,
- $\Delta - C'' \geq 0$, $C'' \geq 0$, $C'' \sim_{\mathbb{R}} \pi^* T$ and any lc center of $(X, \Delta - C'')$ dominates Z, and
- $K_X + \Delta - eC''$ is not pseudo-effective for any $e > 0$.

Therefore, we can apply Lemma 3.3 and we can obtain the following diagram

$$\begin{tikzcd} (X,\Delta) \arrow[r, dashed] \arrow[d, "\pi"'] & (\widetilde{X},\widetilde{\Delta}) \arrow[d, "\widetilde{\pi}"] & \\ Z \arrow[r, dashed] & \widetilde{Z} \arrow[r] & Z^{\vee} \end{tikzcd}$$

where $\widetilde{\Delta}$ is the birational transform of Δ on $\widetilde{X}$, such that

- $(\widetilde{X}, \widetilde{\Delta})$ is a projective $\mathbb{Q}$-factorial lc pair, $\widetilde{Z}$ is projective and $\mathbb{Q}$-factorial, and $Z^{\vee}$ is a normal projective variety,
- the maps $X \dashrightarrow \widetilde{X}$ and $Z \dashrightarrow \widetilde{Z}$ are birational contractions,
- the morphism $\widetilde{Z} \to Z^{\vee}$ is a contraction such that $\rho(\widetilde{Z}/Z^{\vee}) = 1$ and $\dim Z^{\vee} < \dim \widetilde{Z}$, and
- $K_{\widetilde{X}} + \widetilde{\Delta} \sim_{\mathbb{R}} \widetilde{\pi}^* \widetilde{D}$ and $\widetilde{D} \sim_{\mathbb{R},\, Z^{\vee}} 0$, where $\widetilde{D}$ is the birational transform of D on $\widetilde{Z}$.

Take a log resolution $Y_1 \to X$ of $(X, \mathrm{Supp}\,(\Delta + C))$ such that the induced map $Y_1 \dashrightarrow \widetilde{X}$ is a morphism. Let (Y_1, Δ_{Y_1}) and (Y_1, C_{Y_1}) be log smooth models of (X, Δ) and (X, C) respectively. Then, we can apply the argument of Step 1 to $Y_1 \to \widetilde{X} \to Z^{\vee}$ because $(\widetilde{X}, \widetilde{\Delta})$ is lc and $K_{\widetilde{X}} + \widetilde{\Delta} \sim_{\mathbb{R},\, Z^{\vee}} 0$. Thus, we can get a contraction $Y_1''' \to Z_1$ over $Z^{\vee}$ and lc pairs $(Y_1''', \Gamma_{Y_1'''})$ and $(Y_1''', C_{Y_1'''})$ such that $\dim Z_1 = \dim Z^{\vee}$, $K_{Y_1'''} + \Gamma_{Y_1'''} \sim_{\mathbb{R},\, Z_1} 0$ and $K_{Y_1'''} + C_{Y_1'''} \sim_{\mathbb{R}} 0$. Here $C_{Y_1'''}$ is the birational transform of C_{Y_1} on Y_1''', and $\Gamma_{Y_1'''}$ is the birational transform of $t\Delta_{Y_1} + (1-t)C_{Y_1}$ on Y_1''' for a sufficiently small $t > 0$. As in Step 1, we see that it is sufficient to prove Theorem 1.3 for $(Y_1''', \Gamma_{Y_1'''})$. So we may replace $(X, \Delta) \to Z$ and (X, C) by $(Y_1''', \Gamma_{Y_1'''}) \to Z_1$ and $(Y_1''', C_{Y_1'''})$, respectively. For details, see the second paragraph of Step 1.

We replace $(X,\Delta) \to Z$ by $(Y_1''', \Gamma_{Y_1'''}) \to Z_1$. After replacing it, the dimension of Z strictly decreases because we have $\dim Z_1 = \dim Z^{\vee} < \dim Z$. This is crucial to the proof.

Step 6. From now on, we repeat the argument of Step 2-5.

By the same argument as in Step 2, we can assume $(X,\Delta) \to Z$ and (X, C) satisfy conditions (i), (ii) and (iii) in Step 2. Then, there are two possibilities:

- Theorem 1.3 holds for (X,Δ) (see Step 3), or
- we can find a contraction $Y_2''' \to Z_2$ with $\dim Z_2 < \dim Z$ and lc pairs $(Y_2''', \Gamma_{Y_2'''})$ and $(Y_2''', C_{Y_2'''})$ such that $K_{Y_2'''} + \Gamma_{Y_2'''} \sim_{\mathbb{R}, Z_2} 0$, $K_{Y_2'''} + C_{Y_2'''} \sim_{\mathbb{R}} 0$ and Theorem 1.3 for (X,Δ) is implied from Theorem 1.3 for $(Y_2''', \Gamma_{Y_2'''})$ (see Step 4 and 5).

If we are in the first case, we stop the argument. If we are in the second case, we replace $(X,\Delta) \to Z$ by $(Y_2''', \Gamma_{Y_2'''}) \to Z_2$ and repeat the argument of Step 2-5. Each time we replace $(X,\Delta) \to Z$ in the argument of Step 5, the dimension of Z strictly decreases. Therefore, this process eventually stops. Thus, we can prove Theorem 1.3, and so we are done. $\square$

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