

A lower bound for the Bogomolny-Schmit constant for random monochromatic plane waves

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Let f be the the Planar Monochromatic Wave, i.e, the unique a.s. continuous stationary centred Gaussian field on \mathbb{R}^2 with covariance $\mathbb{E}[f(x)f(y)] = J_0(|x - y|)$. Its average number of nodal domains per unit area is given by the Bogomolny-Schmit (or Nazarov-Sodin) constant. In this paper, we prove that $\nu_{BS} \geq 1.39x10^{-4}$.

1. Introduction

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the stationary, isotropic planar centered almost surely continuous Gaussian field¹ with covariance $\mathbb{E}[f(x)f(y)] = J_0(|x - y|)$ where $J_0(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{ir \sin(\theta)} d\theta$ is the 0th Bessel function of the first kind. We call this field **the random monochromatic plane wave**. For each $R > 0$, let us denote by $B(0, R)$ the ball of centre 0 and of radius R , and by $N(R, f)$ the number of connected components of $\mathbb{R}^2 \setminus f^{-1}(\{0\})$ included in $B(0, R)$.

In [10], [12], Nazarov and Sodin showed that the limit

$$(1) \quad \nu_{BS} = 4\pi \times \lim_{R \rightarrow \infty} \frac{\mathbb{E}N(R, f)}{\pi R^2}$$

exists, and is positive. However, their method does not give an explicit value for the constant ν_{BS} , sometimes called the *Bogomolny-Schmit constant* (or also Nazarov-Sodin constant). An equivalent definition for ν_{BS} is that it is the limit of the average number of nodal domains for a random spherical harmonic divided by the degree of this spherical harmonic. The factor 4π can be interpreted as the area of the unit sphere. Bogomolny and Schmit gave in [5] a highly heuristical argument, based on percolation, which yielded the

¹We refer the reader to [1] for the definition and properties of random fields.

value

$$\nu_{BS} \simeq 0.0624.$$

However, numerical simulations carried out by Nastasescu (in [9]), Konrad (in [7]) and Beliaev and Kereta (in [3]) showed that $\nu_{BS} \simeq 0.0589$, contradicting Bogomolny and Schmit's prediction by a few percents. Experimental measurements of ν_{BS} were also realised by Kuhl, Höhmann, Stöckmann and Gnutzmann ([8]), using Berry's conjecture ([4]) that high frequency eigenmodes in a chaotic cavity should locally behave like the monochromatic random wave f . From the mathematical point of view, though, very little is known regarding the value of ν_{BS} . The best rigorous lower bound so far was the one given in [9], which is of the order of 10^{-319} . The aim of this note is obtain a much better lower bound by elementary means:

Theorem 1.1.

$$\nu_{BS} \geq 1.39 \times 10^{-4}.$$

This bound is much smaller than the expected value of ν_{BS} . However, our method does not take into account all nodal domains, but only those which are included in circles of radius 3.8 (the first minimum of the Bessel function J_0). After visual inspection of computer simulations we expect that these are not very common. Aside from this, in our use of Lemma 2.2, we ignore the fact that small, isolated nodal domains should be somewhat rare. We hope that our methods can be used to count more general nodal domains, and to obtain sharper lower bounds on ν_{BS} .

Before proving Theorem 1.1 let us recall two basic facts about random monochromatic plane waves.

- The function f almost surely satisfies $\Delta f + f = 0$ where Δ is the Laplace operator. This follows by applying the Laplace operator with respect to x to the expression $\mathbb{E}[f(x)f(y)] = J_0(|x - y|)$ because J_0 is also an eigenfunction of the Laplace operator.
- In polar coordinates, f can be given the simple expression

$$(2) \quad f(r, \theta) = X_0 J_0(r) - \sqrt{2} \sum_{n \geq 1} J_n(r) (X_n \cos(n\theta) + Y_n \sin(n\theta)) .$$

where for each $n \in \mathbb{Z}$ we denote by J_n the n th Bessel function of the first kind, that is $J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{in\theta - x \sin(\theta)} d\theta$, and where $(X_k)_{k \geq 0}$ and $(Y_k)_{k \geq 1}$ are two families of centred Gaussian random variables

with unit variance and independent as a whole. An explanation for this fact can be found for instance in Section 4.2 of [11].

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2. Proof of Theorem 1.1

We now prove Theorem 1.1. The idea is to consider points such that f does not vanish on a certain circle around this point. We start by producing a lower bound for the probability that 0 is such a point.

Lemma 2.1. *Fix $r > 0$ and write $\Psi : t \mapsto \int_t^{+\infty} e^{-t^2/2} \frac{dt}{\sqrt{2\pi}}$. Let \mathcal{E}_r be the event that the zero set of f does not intersect the circle of radius r . Then, for each $T \in \mathbb{R}$,*

$$\mathbb{P}[\mathcal{E}_r] \geq 2\Psi(T) - \sqrt{2}r\Psi\left(\frac{T}{\sqrt{1 - J_0(r)^2}}\right).$$

Proof. In this proof we use the expression (2) for f . Fix $r > 0$ and for each $\theta \in [0, 2\pi]$, let $u(\theta) = X_0 J_0(r) - f((r \cos(\theta), r \sin(\theta)))$. We first fix $x_0 > 0$ and try to estimate the probability that $u(\theta)$ crosses the level $x_0 J_0(r)$ when θ varies on the unit circle. Also, throughout our calculations, we will use the following two Bessel function identities:

$$(3) \quad J_{-n}(x) = (-1)^n J_n(x); \quad \sum_{n \in \mathbb{Z}} J_n(x)^2 = 1; \quad \sum_{n \in \mathbb{Z}} n^2 J_n(x)^2 = \frac{x^2}{2}.$$

These identities follow from the following classical formula (cf. [13, Chapter 2])

$$\sum_{n \in \mathbb{Z}} t^n J_n(x) = e^{\frac{x}{2}(t-t^{-1})}$$

by setting $t = e^{i\theta}$ and applying Parseval's formula.

Now, observe that, since f is stationary, isotropic and $(f, \nabla f)$ is non-degenerate, u is a stationary Gaussian field on the unit circle and the field (u, u') is non-degenerate so that u crosses the level $x_0 J_0(r)$ almost surely an even number of times. In particular, by Markov's inequality,

$$\mathbb{P}[\exists \theta; u(\theta) = x_0 J_0(r)] \leq \frac{1}{2} \mathbb{E}[\text{Card}\{\theta \in [0, 2\pi]; u(\theta) = x_0 J_0(r)\}] .$$

To compute the right-hand side of this inequality, we apply the Kac-Rice formula (see Theorem 6.2 from [2]). Define $\alpha(r) > 0$ by $\alpha(r)^2 = \text{Var}(u(\theta)) = 2 \sum_{n \geq 1} J_n(r)^2$. Then, by the first and second identity in (3), $\alpha(r)^2 = 1 - J_0(r)^2$. By the Kac-Rice formula,

$$\begin{aligned} & \mathbb{E}[\text{Card}\{\theta \in [0, 2\pi]; u(\theta) = x_0 J_0(r)\}] \\ &= \int_0^{2\pi} \mathbb{E}[|u'(\theta)| \mid u(\theta) = x_0 J_0(r)] \frac{e^{-\frac{x_0^2 J_0(r)^2}{2\alpha(r)^2}}}{\alpha(r)\sqrt{2\pi}} d\theta . \end{aligned}$$

The fact that u is stationary, implies first that the integrand is independent of θ and second, that $u'(\theta)$ is independent of $u(\theta)$. Observe that $\text{Var}(u'(\theta)) = 2 \sum_{n \geq 1} n^2 J_n(r)^2$. By the first and third identities in (3), $\text{Var}(u'(\theta)) = \frac{r^2}{2}$. Moreover, if ξ is a centred Gaussian of variance one, $\mathbb{E}[|\xi|] = \sqrt{\frac{2}{\pi}}$. Therefore,

$$\mathbb{E}[\text{Card}\{\theta \in [0, 2\pi]; u(\theta) = x_0 J_0(r)\}] = \frac{\sqrt{2}r}{\alpha(r)} \exp\left(-\frac{x_0^2 J_0(r)^2}{2\alpha(r)^2}\right) .$$

On the other hand, we have

$$\begin{aligned} \mathbb{P}[-\mathcal{E}_r; X_0 = x_0] &= \mathbb{P}[\exists \theta \in [0, 2\pi]; u(\theta) = x_0 J_0(r)] \\ &\leq \mathbb{E}[\text{Card}\{\theta \in [0, 2\pi]; u(\theta) = x_0 J_0(r)\}] \end{aligned}$$

In particular, for each $T \in \mathbb{R}$, the probability that f has a zero on the circle of radius r centred at 0 satisfies

$$(4) \quad \mathbb{P}[\mathcal{E}_r] \geq \mathbb{E}\left[\left(1 - \frac{r}{\sqrt{2}\alpha(r)} \exp\left(-\frac{X_0^2 J_0(r)^2}{2\alpha(r)^2}\right)\right) \mathbb{1}_{[|X_0| \geq T]}\right] .$$

Let $\Psi(t) = \mathbb{P}[X_0 \geq t] = \int_t^{+\infty} e^{-t^2/2} \frac{t}{\sqrt{2\pi}}$. Then, the right-hand side is

$$2\Psi(T) - 2\frac{r}{\sqrt{2}\alpha(r)} \mathbb{E}\left[\exp\left(-\frac{X_0^2 J_0(r)^2}{2\alpha(r)^2}\right) \mathbb{1}_{[X_0 \geq T]}\right] .$$

But for each $a > 0$, a simple calculation shows that $\mathbb{E} [e^{-aX_0^2} \mathbf{1}_{[X_0 \geq T]}] = \frac{\Psi(\sqrt{1+2a}T)}{\sqrt{1+2a}}$ so

$$\mathbb{P} [\mathcal{E}_r] \geq 2\Psi(T) - \frac{\sqrt{2}r}{\sqrt{\alpha(r)^2 + J_0(r)^2}} \Psi \left(\sqrt{1 + J_0(r)^2/\alpha(r)^2} T \right)$$

Replacing $\alpha(r)$ by its expression, we get, for any $T > 0$ and any $r > 0$,

$$(5) \quad \mathbb{P} [\mathcal{E}_r] \geq 2\Psi(T) - \sqrt{2}r \Psi \left(\frac{T}{\sqrt{1 - J_0(r)^2}} \right). \quad \square$$

Let $G_r = G_r(f) \subset \mathbb{R}^2$ be the (random) set of points $x \in \mathbb{R}^2$ for which f does not vanish on the circle of radius r centred at x . If $x, y \in \mathbb{R}^2$, we will write $x \sim_r y$ if $x, y \in G_r$ and x, y belong to the same connected component of $\mathbb{R}^2 \setminus f^{-1}(\{0\})$. The next step of the proof is to show that the connected components of G_r are not too large.

Lemma 2.2. *Let r_1 and r_2 denote the first and second zeros of J_0 and let $r \in]r_1, r_2[$. Then, for each $x \in G_r$, the connected component of x in $\mathbb{R}^2 \setminus f^{-1}(\{0\})$ is included in $B(x, r)$. In particular, if $x, y \in G_r$ are such that $x \sim_r y$, then $|x - y| \leq r$.*

Remark 2.3. The result of this lemma may be optimal, deterministically speaking. But, we expect equivalence classes of G_r of diameter close to r to be very rare. We probably lose a large factor in this step. However, this intuition seems difficult to quantify.

Proof of Lemma 2.2. In this proof, we use the fact that f satisfies $\Delta f + f = 0$ almost surely. Without loss of generality, we may suppose that $f(x) > 0$. We claim that $f(z) < 0$ for all z such that $|x - z| = r$. We already know that $f(z)$ has constant sign for all z such that $|x - z| = r$. Consider the function

$$g(z) := \frac{1}{2\pi} \int_0^{2\pi} f(R_\theta z) d\theta,$$

where R_θ is the rotation of centre x and of angle θ . The function g satisfies $(\Delta + 1)g = 0$ (because $f(R_\theta \cdot)$ does for each $\theta \in [0, 2\pi]$), and it is radially

symmetric around x . Therefore, we must have

$$g(z) = \lambda J_0(|z - x|)$$

for some $\lambda \neq 0$. Since $f(x) > 0$, we must have $\lambda > 0$, and hence, $f(z) < 0$ for all z such that $|x - z| = r$. Therefore, the connected component of $\mathbb{R}^2 \setminus f^{-1}(\{0\})$ containing x is included in $B(x, r)$ and the Lemma follows. \square

We now finish off the proof of Theorem 1.1 by estimating the expected size of G_r using Lemma 2.1. Let $0 < r_1 < r_2$ be the first two zeros of the Bessel function J_0 and fix $r \in]r_1, r_2[$. Then, by Lemma 2.2, for each $x \in G_r$, the equivalence class of x has diameter at most r . By the isodiametric inequality (see paragraph 10 of [6]) its area is no greater than $\frac{\pi}{4}r^2$. Also, two different equivalence classes are included in different connected components of $\mathbb{R}^2 \setminus f^{-1}(\{0\})$. Finally, if $R > r$ and $x \in B(0, R - r)$, then the connected component of $\mathbb{R}^2 \setminus f^{-1}(\{0\})$ containing x is included in $B(0, R)$. Thus, for each $R > r$,

$$\text{Vol}(G_r \cap B(0, R - r)) \leq \sum_c \text{Vol}(c) \leq \frac{\pi}{4}r^2 \times N(R, f)$$

where the sum runs over the equivalence classes of G_r intersecting $B(0, R - r)$. Taking expectations, by stationarity, we get

$$\text{Vol}(B(R - r)) \mathbb{P}[0 \in G_r] \leq \frac{\pi}{4}r^2 \times \mathbb{E}[N(R, f)].$$

Dividing by $\text{Vol}(B(0, R)) = \pi R^2$ and letting $R \rightarrow +\infty$, we get $\mathbb{P}[0 \in G_r] \leq \frac{1}{16}r^2 \nu_{BS}$ which yields the following lower bound for the Bogomolny-Schmit constant:

$$\nu_{BS} \geq \frac{16\mathbb{P}[0 \in G_r]}{r^2}.$$

By Lemma 2.1, we have, for each $T > 0$ and each $r > r_0$,

$$\nu_{BS} \geq \frac{32}{r^2} \left[\Psi(T) - \frac{r}{\sqrt{2}} \Psi \left(\frac{T}{\sqrt{1 - J_0(r)^2}} \right) \right].$$

Taking $r = 3.8$ (the first minimum of J_0) and $T = 3.35$ (the smallest T for which in (4), the function whose expectation we compute is always positive),

we get

$$\frac{32}{r^2} \geq 2.216; \quad \frac{T}{\sqrt{1 - J_0(r)^2}} \geq 3.659; \quad \frac{r}{\sqrt{2}} \leq 2.69.$$

We therefore obtain the announced lower bound

$$\nu_{BS} \geq 1.39 \times 10^{-4}.$$

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