Khovanov homology detects the Hopf links

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We prove that any link in S^3 whose Khovanov homology is the same as that of a Hopf link must be isotopic to that Hopf link. This holds for both reduced and unreduced Khovanov homology, and with coefficients in either $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$.

Khovanov homology [4] associates to each link $L \subset S^3$ a bigraded group $Kh^{*,*}(L)$, whose graded Euler characteristic recovers the Jones polynomial $V_L(q)$, as well as a reduced variant $Kh_{\mathrm{red}}^{*,*}(L)$ [5]. In their landmark paper [8], Kronheimer and Mrowka proved that Khovanov homology detects the unknot. Since then, Khovanov homology has been shown to detect both the n-component unlink for all n [2] and the trefoils [1]. In this article, we prove the same for the Hopf links $H_\pm$, which are oriented so that the two components have linking number ± 1. This is the first Khovanov homology detection result for any nontrivial link with more than one component.

Recall that the Khovanov homology of $H_\pm$ is given by

$$Kh(H_+; \mathbb{Z}) \cong \mathbb{Z}_{(0,0)} \oplus \mathbb{Z}_{(0,2)} \oplus \mathbb{Z}_{(2,4)} \oplus \mathbb{Z}_{(2,6)}$$
$$Kh(H_-; \mathbb{Z}) \cong \mathbb{Z}_{(0,0)} \oplus \mathbb{Z}_{(0,-2)} \oplus \mathbb{Z}_{(-2,-4)} \oplus \mathbb{Z}_{(-2,-6)}$$

where $\mathbb{Z}_{(h,q)}$ denotes a copy of $\mathbb{Z}$ in bigrading (h, q). The reduced Khovanov homology of a link depends in general on a choice of distinguished component, which we typically suppress from the notation, but we have that

$$Kh_{\mathrm{red}}(H_+; \mathbb{Z}) \cong \mathbb{Z}_{(0,1)} \oplus \mathbb{Z}_{(2,5)}$$
$$Kh_{\mathrm{red}}(H_-; \mathbb{Z}) \cong \mathbb{Z}_{(0,-1)} \oplus \mathbb{Z}_{(-2,-5)}$$

regardless of this choice. Our main result is the following.

Theorem 1. *Let L be a link in S^3 such that either $Kh(L) \cong Kh(H_\pm)$ or $Kh_{\mathrm{red}}(L) \cong Kh_{\mathrm{red}}(H_\pm)$ as bigraded groups, with coefficients in either $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$. Then $L = H_\pm$.*

Our proof makes use of a handful of spectral sequences involving Khovanov homology. First, we combine Batson and Seed's link splitting spectral sequence [2] with Kronheimer and Mrowka's spectral sequence converging to singular instanton knot homology [8] to conclude that a link L satisfying the hypotheses of Theorem 1 must have exactly two unknotted components with linking number ± 1. The hypotheses on L also determine the module structure on $Kh_{\mathrm{red}}(L)$. Given this module structure, we apply a refinement by the third author [13] of Kronheimer and Mrowka's spectral sequence to conclude that the instanton knot homology $KHI(L)$ of [7] has rank at most 4. Finally, we use all of the above to determine the Alexander grading on $KHI(L)$, which tells us that L has Seifert genus zero, and we conclude that L must be a Hopf link.

If L is a link with r components, then $Kh^{*,*}(L)$ is invariant as a module over the ring

$$R_r = \mathbb{Z}[x_1, \ldots, x_r]/\langle x_1^2, \ldots, x_r^2 \rangle$$

[3, 5], in which each x_i preserves the h ("homological") grading while lowering the q ("quantum") grading. If we define the reduced Khovanov homology of L using the component which corresponds to x_r, then $Kh_{\mathrm{red}}^{*,*}(L)$ is a module over R_{r-1}. We begin by determining this module structure.

Proposition 2. *Let L be a link in S^3, and let H be either H_+ or H_-. Let $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. Suppose that any one of the following is true as an isomorphism of bigraded groups:*

1) $Kh(L; \mathbb{Z}) \cong Kh(H; \mathbb{Z})$.

2) $Kh(L; \mathbb{F}) \cong Kh(H; \mathbb{F})$.

3) $Kh_{\mathrm{red}}(L; \mathbb{Z}) \cong Kh_{\mathrm{red}}(H; \mathbb{Z})$ for some choice of component of L.

4) $Kh_{\mathrm{red}}(L; \mathbb{F}) \cong Kh_{\mathrm{red}}(H; \mathbb{F})$ for some choice of component of L.

Then L has exactly two components, which are both unknots, and if $H = H_\pm$ then their linking number is ± 1. Moreover, we have $\mathrm{rank}\, Kh_{\mathrm{red}}(L; \mathbb{Z}) = 2$, with

$$Kh_{\mathrm{red}}(L; \mathbb{Q}) \cong \begin{cases} \mathbb{Q}_{(0,1)} \oplus \mathbb{Q}_{(2,5)}, & H = H_+ \\ \mathbb{Q}_{(0,-1)} \oplus \mathbb{Q}_{(-2,-5)}, & H = H_-, \end{cases}$$

and the R_1-action on $Kh_{\mathrm{red}}(L; \mathbb{Q})$ is trivial.

Proof. Suppose that $H = H_+$; the case of H_- is identical. We first claim that each of conditions (1), (2), and (3) implies condition (4), and then use the latter to prove the rest of the proposition. Certainly (1) implies (2) and (3) implies (4) by the universal coefficient theorem. Moreover, (2) and (4) are equivalent by the identity

$$Kh^{h,q}(K; \mathbb{F}) \cong Kh_{\mathrm{red}}^{h,q-1}(K; \mathbb{F}) \oplus Kh_{\mathrm{red}}^{h,q+1}(K; \mathbb{F})$$

where K is an arbitrary link [11, Corollary 3.2.C], so this proves the claim.

We suppose from now on that condition (4) holds; since this is equivalent to (2), we have

$$Kh(L; \mathbb{F}) \cong \mathbb{F}_{(0,0)} \oplus \mathbb{F}_{(0,2)} \oplus \mathbb{F}_{(2,4)} \oplus \mathbb{F}_{(2,6)}$$
$$Kh_{\mathrm{red}}(L; \mathbb{F}) \cong \mathbb{F}_{(0,1)} \oplus \mathbb{F}_{(2,5)}.$$

If L were a knot, then the rank of $Kh_{\mathrm{red}}(L; \mathbb{F})$ would be congruent mod 2 to $V_L(-1) = \pm \det(L) \equiv 1 \pmod 2$, so L must be a link, say $L = K_1 \cup K_2 \cup \cdots \cup K_r$ with $r \geq 2$.

We apply the rank inequality [2, Corollary 1.6] derived from Batson and Seed's link splitting spectral sequence, namely that if $\mathbb{K}$ is any field then

$$(1) \qquad \mathrm{rank}\, Kh(L; \mathbb{K}) \geq \mathrm{rank} \bigotimes_{i=1}^{r} Kh(K_i; \mathbb{K}).$$

For $\mathbb{K} = \mathbb{F}$, the left side is 4 while the right side is at least 2^r with equality if and only if K_i is unknotted [8], so $r = 2$ and K_1 and K_2 are both unknotted. (See [2, Proposition 7.1].) In fact, this inequality respects the grading $\ell = h - q$: Define

$$\mathrm{rank}^\ell Kh(K; \mathbb{F}) = \mathrm{rank} \left(\bigoplus_{h-q=\ell} Kh^{h,q}(K; \mathbb{F}) \right).$$

If we let $t = 2\,\mathrm{lk}(K_1, K_2)$, then [2, Corollary 4.4] says that

$$\mathrm{rank}^\ell Kh(L; \mathbb{F}) \geq \mathrm{rank}^{\ell+t} \left(\bigotimes_i Kh(K_i; \mathbb{F}) \right) = \mathrm{rank}^{\ell+t} \left(\mathbb{F}_{(0,-1)} \oplus \mathbb{F}_{(0,1)} \right)^{\otimes 2}.$$

The left side has ranks $1, 2, 1$ in ℓ-gradings $0, -2, -4$ respectively, and the right side has rank 2 when $\ell + t = 0$, so we must have $-2 + t = 0$, or $\mathrm{lk}(K_1, K_2) = \frac{1}{2} t = 1$.

Finally, applying the inequality (1) with $\mathbb{K} = \mathbb{Q}$ says that rank $Kh(L; \mathbb{Q}) \geq 4$, and there is an exact triangle

$$\cdots \to Kh(L; \mathbb{Q}) \to Kh_{\mathrm{red}}(L; \mathbb{Q}) \to Kh_{\mathrm{red}}(L; \mathbb{Q}) \to \cdots,$$

so we must have rank $Kh_{\mathrm{red}}(L; \mathbb{Q}) \geq 2$. Applying the universal coefficient theorem repeatedly, we determine first that rank $Kh_{\mathrm{red}}(L; \mathbb{Z}) \geq 2$, and that if this inequality is strict then rank $Kh_{\mathrm{red}}(L; \mathbb{F}) > 2$ as well, which is false. Thus rank $Kh_{\mathrm{red}}(L; \mathbb{Z}) = 2 = \mathrm{rank}\, Kh_{\mathrm{red}}(L; \mathbb{F})$, and so the $\mathbb{Z}$ summands of the former have the same bigrading as the $\mathbb{F}$ summands of the latter. We deduce from this that

$$Kh_{\mathrm{red}}(L; \mathbb{Q}) \cong \mathbb{Q}_{(0,1)} \oplus \mathbb{Q}_{(2,5)}.$$

Now if we pick a distinguished component of L so that $Kh_{\mathrm{red}}(L; \mathbb{Q})$ is a module over $R_1 = \mathbb{Z}[x_1]/\langle x_1^2 \rangle$, then x_1 acts with square zero on each $Kh_{\mathrm{red}}^{h,*}(L; \mathbb{Q})$, which is either 0 or $\mathbb{Q}$, and so it must act trivially as claimed. $\qquad \square$

We will now make use of Kronheimer and Mrowka's *instanton knot homology KHI*, defined in [7] and extended to links in [6]. They showed in [8] that for any knot K, the complex vector space $KHI(K)$ is isomorphic to the reduced singular instanton knot homology $I^\natural(K; \mathbb{C})$, and using a spectral sequence $Kh(\overline{K}) \Rightarrow I^\natural(K)$ they deduced a rank inequality

$$\mathrm{rank}\, Kh_{\mathrm{red}}(K) \geq \mathrm{rank}\, KHI(K).$$

Based on results in [7], this proved that Kh_{red} has rank 1 if and only if K is unknotted. The third author [13] extended this from knots to pointed links and incorporated the module structure on Khovanov homology to prove the following.

Theorem 3 ([13, Theorem 5.4]). *Let L be a link of r components in S^3, and fix a base point on the rth component, equipping $Kh_{\mathrm{red}}(L; \mathbb{Z})$ with an R_{r-1}-module structure. Let $X' = (x_1, \ldots, x_{r-1}) \in R_{r-1}^{\oplus r-1}$. Then*

$$\mathrm{rank}\, KHI(L) \leq \mathrm{rank}\, H^*(Kh_{\mathrm{red}}(L; \mathbb{Z}) \otimes_{R_{r-1}} K(X')),$$

where $K(X')$ is the Koszul complex

$$0 \to R_{r-1} \xrightarrow{\wedge X'} \Lambda^1 R_{r-1}^{\oplus r-1} \xrightarrow{\wedge X'} \Lambda^2 R_{r-1}^{\oplus r-1} \xrightarrow{\wedge X'} \cdots \xrightarrow{\wedge X'} \Lambda^{r-1} R_{r-1}^{\oplus r-1} \to 0.$$

Proposition 4. *Let $L \subset S^3$ be a link satisfying any of the hypotheses of Proposition 2. Then* $\mathrm{rank}\, KHI(L) \leq 4$.

Proof. Proposition 2 tells us that L is a 2-component link, and that $Kh_{\mathrm{red}}(L; \mathbb{Q}) \cong \mathbb{Q}^2$ has a trivial action of $R_1 = \mathbb{Z}[x_1]/\langle x_1^2 \rangle$. In this case the Koszul complex $K(X')$ is

$$0 \to R_1 \xrightarrow{\cdot x_1} R_1 \to 0,$$

and $Kh_{\mathrm{red}}(L; \mathbb{Q}) \cong (R_1/\langle x_1 \rangle \otimes_{\mathbb{Z}} \mathbb{Q})^{\oplus 2}$ as R_1-modules. The complex

$$(R_1/\langle x_1 \rangle \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{R_1} K(X') = \left(0 \to \mathbb{Q} \xrightarrow{0} \mathbb{Q} \to 0 \right)$$

has cohomology $\mathbb{Q}^2$, so we conclude that

$$\mathrm{rank}\, H^*(Kh_{\mathrm{red}}(L; \mathbb{Z}) \otimes_{R_1} K(X')) = 4$$

and the proposition now follows from Theorem 3. $\qquad\square$

For links in S^3 (and more generally for null-homologous links with a choice of Seifert surface), instanton knot homology is equipped with an Alexander grading

$$KHI(L) = \bigoplus_{j=-g+1-r}^{g-1+r} KHI(L, j),$$

where L has r components and Seifert genus g. This grading is symmetric in the sense that $KHI(L, j) \cong KHI(L, -j)$ for all j. Each $KHI(L, j)$ is canonically $\mathbb{Z}/2\mathbb{Z}$-graded, and the Euler characteristics of these summands determine the Alexander polynomial of L by

$$(2) \qquad \sum_j \chi(KHI(L, j)) t^j = -(t^{1/2} - t^{-1/2})^{r-1} \Delta_L(t).$$

See [6, Theorem 3.6] or [9].

Let $\tilde{\Delta}_L(t_1, t_2)$ denote the multivariable Alexander polynomial of a 2-component link $L = K_1 \cup K_2$. This is related to the single-variable Alexander polynomial $\Delta_L(t)$ by

$$\Delta_L(t) = \pm(t^{1/2} - t^{-1/2})\tilde{\Delta}_L(t, t),$$

see e.g. [10, Lemma 10.1]. It follows from (2) that

$$\sum_j \chi(KHI(L, j))t^j = \pm(t^{1/2} - t^{-1/2})^2\tilde{\Delta}_L(t, t),$$

and hence that the right side of

(3) $$\tilde{\Delta}_L(t, t) = \pm\left(\frac{\sum_j \chi(KHI(L, j))t^j}{t - 2 + t^{-1}}\right)$$

must be a Laurent polynomial, i.e. that $t - 2 + t^{-1}$ divides the numerator. Torres [12] proved that $\tilde{\Delta}_L(t_1, t_2)$ also satisfies $\tilde{\Delta}_L(1, 1) = \pm \mathrm{lk}(K_1, K_2)$.

Proposition 5. *Let $L = K_1 \cup K_2$ be a two-component link with $\mathrm{lk}(K_1, K_2) = \pm 1$. If $\mathrm{rank}\, KHI(L) \leq 4$, then*

$$KHI(L) = \mathbb{C}_1 \oplus (\mathbb{C}^{\oplus 2})_0 \oplus \mathbb{C}_{-1},$$

where the subscripts denote the Alexander grading, and $\Delta_L(t) = \pm(t^{1/2} - t^{-1/2})$.

Proof. We first claim that in fact $KHI(L)$ is nonzero in at least three different Alexander gradings. If not, then (3) has the form

$$\tilde{\Delta}_L(t, t) = \pm\left(\frac{c_1 t^{e_1} + c_2 t^{e_2}}{t - 2 + t^{-1}}\right)$$

for some integers c_1, c_2 which are not both zero and some integers $e_1 \neq e_2$. (The numerator cannot be identically zero because then $\tilde{\Delta}_L(t, t) = 0$, contradicting $\tilde{\Delta}_L(1, 1) = \pm 1$.) The denominator has a double root at $t = 1$, so it cannot divide the numerator, which is either a monomial (and is thus nonzero when $t = 1$) or a binomial with only simple roots, and we have a contradiction.

Next, we show that $\mathrm{rank}\, KHI(L) = 4$. It is already at least three, since it has positive rank in each of at least three Alexander gradings. But the

rank must be even, since it has the same parity as $\sum_j \chi(KHI(L,j))(-1)^j$ and this sum is equal to $\pm 4\tilde{\Delta}_L(-1,-1) \in 4\mathbb{Z}$ by setting $t = -1$ in (3). Thus the total rank is 4 as claimed.

We let $m > k$ be the two largest Alexander gradings in which $KHI(L)$ is nonzero. By symmetry $-m$ is the lowest such grading, so $-m < k < m$; then $KHI(L,-k) \neq 0$ as well, so we must have had $k \geq 0$. If $k > 0$ then $KHI(L,j)$ has positive rank for each of $j = \pm m, \pm k$, and its total rank is 4, so it must have rank exactly one in each of these gradings. Otherwise $k = 0$, and by symmetry we have $\mathrm{rank}(KHI(L)) \equiv \mathrm{rank}(KHI(L,0)) \pmod 2$; the former is even, so the latter is as well, and $KHI(K,j)$ now has rank at least $1, 2, 1$ for each of $j = m, 0, -m$ and total rank 4. In either case we conclude that

$$KHI(L) \cong \mathbb{C}_m \oplus \mathbb{C}_k \oplus \mathbb{C}_{-k} \oplus \mathbb{C}_{-m}$$

where $m > k \geq 0$.

We do not know the $\mathbb{Z}/2\mathbb{Z}$ grading of each summand, but from (3) there must be signs $\epsilon_m, \epsilon_k, \epsilon_{-k}, \epsilon_{-m} \in \{\pm 1\}$ such that

$$\tilde{\Delta}_L(t,t) = \pm \left(\frac{\epsilon_m t^m + \epsilon_k t^k + \epsilon_{-k} t^{-k} + \epsilon_{-m} t^{-m}}{t - 2 + t^{-1}} \right).$$

The numerator must be a multiple of $(t-1)^2$, so both it and its derivative are zero at $t = 1$, giving us the conditions

$$\epsilon_m + \epsilon_k + \epsilon_{-k} + \epsilon_{-m} = 0, \qquad m(\epsilon_m - \epsilon_{-m}) + k(\epsilon_k - \epsilon_{-k}) = 0.$$

From the second equation we have $\epsilon_m = \epsilon_{-m}$, or else the first term would have magnitude $2m$ and thus be strictly greater than $|k(\epsilon_k - \epsilon_{-k})| \leq 2k$; and from $\sum \epsilon_j = 0$ we now have $\epsilon_k + \epsilon_{-k} = -2\epsilon_m$, so that $\epsilon_k = \epsilon_{-k} = -\epsilon_m$. We conclude that

$$\tilde{\Delta}_L(t,t) = \pm \left(\frac{t^m - t^k - t^{-k} + t^{-m}}{t - 2 + t^{-1}} \right).$$

Finally, we use the fact that $\tilde{\Delta}_L(1,1) = \pm 1$ to determine m and k. Given an equation involving Laurent polynomials of the form $p(t) = t^c(t-1)^2 q(t)$ we have $q(1) = \frac{1}{2}p''(1)$, and so taking $p(t) = \pm(t^m - t^k - t^{-k} + t^{-m})$ and $q(t) = \tilde{\Delta}_L(t,t)$ gives us

$$\tilde{\Delta}_L(1,1) = \pm \frac{1}{2} \frac{d^2}{dt^2}(t^m - t^k - t^{-k} + t^{-m})\Big|_{t=1} = \pm(m^2 - k^2).$$

Since $m > k \geq 0$, this can only be equal to ± 1 if $m = 1$ and $k = 0$. Thus $KHI(L)$ is exactly as claimed, while $\tilde{\Delta}_L(t,t) = \pm 1$ and $\Delta_L(t) = \pm(t^{1/2} - t^{-1/2})\tilde{\Delta}_L(t,t) = \pm(t^{1/2} - t^{-1/2})$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now know enough about any link L with the same Khovanov homology as a Hopf link $H_\pm$ to determine its link type.

Proof of Theorem 1. We have $L = K_1 \cup K_2$, with both K_i unknotted and $\mathrm{lk}(K_1, K_2) = \pm 1$, by Proposition 2. Proposition 4 gives us the bound rank $KHI(L) \leq 4$, so then

$$KHI(L) \cong \mathbb{C}_1 \oplus (\mathbb{C}^{\oplus 2})_0 \oplus \mathbb{C}_{-1}$$

by Proposition 5, where again the subscripts denote the Alexander grading.

We now claim that the Seifert genus of L is 0, as a consequence of the more general

$$g(L) + r - 1 = \max\{j \mid KHI(L,j) \neq 0\}$$

for r-component links in S^3 with irreducible complement. This generalizes the case $r = 1$ of [7, Proposition 7.16], and the proof is essentially the same. If Σ is a genus-g Seifert surface for L, then we identify the appropriate Alexander grading with a sutured instanton homology group,

$$KHI(L, g + r - 1) \cong SHI(S^3(\Sigma)),$$

where $S^3(\Sigma)$ is obtained by cutting open the complement of L (with a pair of meridional sutures on each component) along Σ. (The reason for the shift by $r - 1$ is that the jth Alexander grading is the generalized $2j$-eigenspace of an operator $\mu(\bar{\Sigma})$ on the instanton homology of some closed manifold, and the maximal real eigenvalue of $\mu(\bar{\Sigma})$ is $2g(\bar{\Sigma}) - 2 = 2(g + r - 1)$, as in [6, §2.5].) If Σ is genus-minimizing then $S^3(\Sigma)$ is a taut sutured manifold, so $SHI(S^3(\Sigma)) \neq 0$ by [7, Theorem 7.12] and the claim follows.

Thus L bounds an annulus A, and the core of A is isotopic to either of the unknots K_i on its boundary. The boundary is in particular a cable of the unknot, and the only such 2-component links with linking number ± 1 are the positive and negative Hopf links. $\qquad\qquad\qquad\qquad\square$

Acknowledgements

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