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Mean curvature flows of closed hypersurfaces in warped product manifolds

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We investigate the mean curvature flows in a class of warped products manifolds with closed hypersurfaces fibering over $\mathbb{R}$. In particular, we prove that under some natural conditions on the warping function and Ricci curvature bound for the ambient space, there exists a large class of closed initial hypersurfaces, as geodesic graphs over the totally geodesic hypersurface N, such that the mean curvature flow starting from S_0 exists for all time and converges to N.

1. Introduction

1.1. Motivation and Main Theorem

The study of the mean curvature flow equation has attracted major attentions in geometric analysis over the past decades. The flow has the following formulation:

$$\begin{cases} \dfrac{\partial}{\partial t} F(x,t) = -H(x,t)\boldsymbol{\nu}(x,t) \ , \\ F(\cdot,0) = F_0 \ , \end{cases} \tag{1.1}$$

where $H(x,t)$ and $\boldsymbol{\nu}(x,t)$ are the mean curvature and unit outward normal vector respectively at $F(x,t)$ of the evolving surface $S(t)$, and our convention of the mean curvature is the sum of the principal curvatures. All other terms will be made transparent later.

A fundamental theorem of Huisken ([10]) in the theory of mean curvature flow states that any mean curvature flow of a closed and strictly convex initial hypersurface $N^{n-1} \subset \mathbb{R}^n$ (with $n \geq 3$) stays strictly convex, and develops singularity in finite time. This was generalised to several classes of Riemannian manifolds ([11]). Since then there are extensive studies in the

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field to understand the singularity formulation in various settings and for various mean curvature flows, see for instance [5, 6, 12, 13, 15] and many others.

We consider a topic similar to that in [10], namely, under what (natural) conditions, a mean curvature flow of closed hypersurfaces in some Riemannian manifolds may converge to some canonical objects. A prototype is a class of hyperbolic three-manifolds ([9]) where negative curvature and special topology (preventing large balls to appear) help to keep a graphical mean curvature flow staying graphical and converging to the totally geodesic surface.

In this paper, we consider a much wider class of warped product manifolds as the ambient space. To describe our setting more precisely, let's first fix notations. Throughout this paper, we use N to denote a closed Riemannian manifold of dimension $n-1$, where $n \geq 3$. We always assume N satisfies the following condition:

(C0) $Ric_N \geq (n-1)\rho g_N$,

for some constant ρ. Note that here ρ is different from the one in [4] for simplicity in calculations. The ambient manifold in the current work is $M^n = N^{n-1} \times (-\bar{r}, \bar{r})$ for $\bar{r} \in (0, \infty]$, and the warped product metric is

$$g = g_M = dr \otimes dr + h^2(r) g_N, \tag{1.2}$$

with the warping function $h(r) : (-\bar{r}, \bar{r}) \to (0, \infty)$. Geometrically M has the structure of closed hypersurfaces fibering over the real line. The warping function $h(r)$ is assumed to satisfy the following conditions:

(C1) $h(0) = 1$ and $h'(0) = 0$;

(C2) $h'(r) > 0$ for all $r \in (0, \bar{r})$ and $h'(r) < 0$ for all $r \in (-\bar{r}, 0)$;

(C3) for $c = \max\{0, \rho\}$ and any $r \in (-\bar{r}, \bar{r})$,

$$h(r)h''(r) - h'(r)^2 + \rho \geq c. \tag{1.3}$$

The above conditions might seem artificial at the first look, and so let's motivate them below.

Conditions (C1–2) are natural geometrical conditions. As a consequence, N is totally geodesic in M, and the unique such hypersurface which is also fixed by the mean curvature flow. The case of h being even provides a natural class of examples in practice. The choice of $h(0) = 1$ is just for convenience and certainly not essential.

Condition (C3) allows negative Ricci curvature on the hypersurface N for suitable warping function. Moreover, Condition (C3) ensures the function $|h'(r)/h(r)|$ is non-decreasing in r, which is important in the proof of Theorem 3.7. For $r = 0$, by (C1), (1.3) becomes $h''(0) + \rho \geq c$, and so it is easy to generate examples with a proper $\bar{r}$ value satisfying all the conditions.

Notice that M is not required to be complete as the mean curvature flow in our study stays local in M by the barrier argument as discussed in Lemma 3.3. Of course, there are still plenty of examples of complete M, for instance, the one studied in [9]. Simply speaking, if we choose N to be a closed hyperbolic surface of constant curvature -1, then $n = 3$, $h(r) = \cosh(r)$, and $c = \frac{1}{2}$, then we have M is the Fuchsian manifold, a complete hyperbolic three-manifold as a warped product. All the conditions described above are satisfied. The authors in [9] used the special structure of the Fuchsian manifold and initiated the study on how graphical mean curvature flows behave in that setting.

Furthermore, these conditions are comparable with but different from Brendle's in his work on constant mean curvature hypersurfaces ([4]), where such interval like $[0, \bar{r})$ is considered from general relativity perspective. Indeed, our argument can be applied without any change to the Brendle's setting of $M = N \times [0, \bar{r})$ and Conditions (C1–3) will simply be restricted to $[0, \bar{r}) \subset (-\bar{r}, \bar{r})$. Hence just as in [4], our main result Theorem 1.2 can be applied to the de Sitter-Schwarzschild manifold which is of great interest in general relativity.

Let's now fix some notions for the purpose of this paper.

Definition 1.1. A hypersurface S is called a **graph** over N or **starshaped**, if the angle function $\Theta = \langle \mathbf{n}, \boldsymbol{\nu} \rangle > 0$, where $\mathbf{n} = \frac{\partial}{\partial r}$, and $\boldsymbol{\nu}$ is the unit normal to S. Clearly by definition we have $\Theta \in [0, 1]$, and in particular if $\Theta \equiv 1$, we call S is **parallel** (or **equidistant**) to N.

In this work, we generalize this phenomenon of converging mean curvature flow of closed hypersurfaces in [9] to a much more general class of Riemannian manifolds, namely, those satisfying Conditions (C0–3). More precisely, the following result is proved.

Theorem 1.2. *Let M^n be a warped product manifold satisfying Conditions (C0–3) and S_0 a smooth closed hypersurface which is a geodesic graph over the unique totally geodesic hypersurface N in M^n. Then for any $a_0 > 0$, if*

S_0 lies in distance no more than a_0 to N, and the initial angle satisfies

$$\min_{p\in S_0}\Theta(p) \geq \sqrt{1-\frac{1}{h^2(a_0)}}, \tag{1.4}$$

the mean curvature flow (1.1) *with the initial hypersurface S_0 exists for all time, remains as geodesic graph over N and converges continuously to N. Moreover, the convergence is smooth if the above inequality is strict.*

Note that the warped product structure in our work is hypersurface fibering over the line. One may also define a type of warped product manifolds as line bundles fibering over hypersurfaces, and some remarkable results of the behaviors of the mean curvature flows in such manifolds were obtained in [3]. In that setting, graphs are equidistant graphs, not geodesic graphs. In general we do not expect a geodesic graph stays graphical along the mean curvature flow. One can also define the mean curvature flow of higher codimensions, as well as warped product manifolds where the base manifold is of higher dimensions, and we will not pursue these generalisations here. There are many other interesting papers on various flows in several classes of warped product manifolds (see for example [8, 17, 18] and others).

1.2. Outline of the paper

In Section 2, we discuss important equations and estimates used in the proof of the main result, including the evolution equations for the height and angle functions along the mean curvature flow with a general warped Riemannian manifold as the ambient space. The main result is then proved in Section 3.

2. Preliminaries

In this section, we fix the notations and introduce some preliminary facts that will be used later in this paper.

2.1. Mean curvature flow

For completeness, we start by collecting and deriving a number of evolution equations for geometric quantities on $S(t)$, $t \in [0,T)$, which are involved in our calculations. Let S_0 be an embedded closed hypersurface in M^n, and F_0

be the local diffeomorphism representing the embedding:

$$F_0 : U \subset \mathbb{R}^{n-1} \to F_0(U) \subset S_0 \subset M.$$

We consider the mean curvature flow, i.e. a family of maps $F(\cdot,t)$ satisfying (1.1):

$$\begin{cases} \dfrac{\partial}{\partial t} F(x,t) = -H(x,t)\boldsymbol{\nu}(x,t) \ , \\ \quad F(\cdot,0) = F_0 \, . \end{cases}$$

We denote $S(t)$ as the evolving hypersurface which is the image of the map $F(\cdot,t)$. Here $H(\cdot,t)$ is the mean curvature function of $S(t)$ and $\boldsymbol{\nu}$ is the correspondingly chosen unit normal of $S(t)$. The short time existence of the mean curvature flow was established, and it was further shown that the flow can be extended as long as the norm of the second fundamental form is controlled ([10, 11]).

As we are primarily working with the graphical case, let's set two functions on $S(t)$: *height function* $u(x,t)$, which records the distance, with respect to the metric g_M, between any point $x \in S(t)$ and the fixed reference hypersurface N, and the *angle function* (or the gradient function) $\Theta(x,t)$ as defined in Definition 1.1. Of course, we always have $\Theta(x,t) \in [0,1]$, and it is clear that the hypersurface $S(t)$ is a geodesic graph over N if $\Theta > 0$ everywhere on $S(t)$ with properly chosen $\boldsymbol{\nu}$. The general evolution equations for these two functions are as follows.

Theorem 2.1. *([1, 7, 9]) The evolution equations of u and Θ have the following form:*

$$\frac{\partial}{\partial t} u = - H\Theta, \tag{2.1}$$

$$\left(\frac{\partial}{\partial t} - \Delta\right)\Theta = (|A|^2 + Ric_M(\boldsymbol{\nu},\boldsymbol{\nu}))\Theta + \mathbf{n}(H_{\mathbf{n}}) - H\left\langle \overline{\nabla}_{\boldsymbol{\nu}}\mathbf{n} \, , \boldsymbol{\nu}\right\rangle \tag{2.2}$$

where $\mathbf{n}(H_{\mathbf{n}})$ is the variation of mean curvature function of $S(t)$ under the deformation vector field $\mathbf{n}$.

In practice the equation for Θ is difficult to work with, especially the term $\mathbf{n}(H_{\mathbf{n}})$. In the case of Fuchsian manifolds, a much simplified and explicit equation was derived ([9]), using special hyperbolic geometry of a Fuchsian manifold. In the current situation, we no longer assume the ambient warped product manifold M has constant curvature, and so the evolution equation for Θ will be more involved. We start with curvature properties in both M and N.

2.2. Ricci curvature in warped product manifolds

We denote the covariant derivatives of $S(t)$ (induced metric) and M by ∇ and $\overline{\nabla}$, respectively. The relationship between the Ricci curvatures on M and N is given by the following well-known formula:

Lemma 2.2. *[4] Let $h = h(r)$ be the warping function, then*

$$Ric_M = Ric_N - (hh'' + (n-2)h'^2)g_N - (n-1)\frac{h''}{h}dr \otimes dr. \tag{2.3}$$

Proof. This follows from the calculations in [2]. Let $\{e_1, \ldots, e_{n-1}\}$ be a local orthonormal frame on N such that $g_N(e_i, e_j) = \delta_{ij}$. Then one gets

$$\begin{aligned} Ric_M(e_i, e_j) &= Ric_N(e_i, e_j) - (hh'' + (n-2)h'^2)\delta_{ij} \\ Ric_M(e_i, \mathbf{n}) = 0, &\quad Ric_M(\mathbf{n}, \mathbf{n}) = -(n-1)\tfrac{h''}{h}. \end{aligned} \tag{2.4}$$

∎

Using vectors $\mathbf{n} = \frac{\partial}{\partial r}$ and $\boldsymbol{\nu}$, it is standard to decompose vector fields into tangential and normal components, with respect to either $\mathbf{n}$ or $\boldsymbol{\nu}$, as follows.

Definition 2.3. For any vector field X in M, we define

$$\begin{aligned} X_\nu &= X - \langle X, \boldsymbol{\nu}\rangle \boldsymbol{\nu}, \\ X_n &= X - \langle X, \mathbf{n}\rangle \mathbf{n}. \end{aligned}$$

With these notations, we have the following decompositions:

$$\boldsymbol{\nu}_n = \boldsymbol{\nu} - \Theta\mathbf{n}, \tag{2.5}$$

$$\mathbf{n}_\nu = \mathbf{n} - \Theta\boldsymbol{\nu} = \sum_{k=1}^{n-1} \langle \mathbf{n}, e_k\rangle e_k. \tag{2.6}$$

Clearly, $\boldsymbol{\nu}_n$ is perpendicular to $\mathbf{n}$ and $\mathbf{n}_\nu$ is perpendicular to $\boldsymbol{\nu}$. When $\Theta = 1$, we have $\mathbf{n} = \boldsymbol{\nu}$ and $\boldsymbol{\nu}_n = 0$. We further normalize $\boldsymbol{\nu}_n$ by the metric g_N to set

$$\vec{v} = \begin{cases} \dfrac{\boldsymbol{\nu}_n}{|\boldsymbol{\nu}_n|_{g_N}} & \text{if} \quad \boldsymbol{\nu}_n \neq 0 \\ 0 & \text{if} \quad \boldsymbol{\nu}_n = 0. \end{cases} \tag{2.7}$$

Now we derive the following technical lemma.

Lemma 2.4. *We have*

$$Ric_M(\boldsymbol{\nu}, \mathbf{n}_\nu) = -\frac{\Theta(1-\Theta^2)}{h^2}\{(n-2)(hh''-(h')^2) + Ric_N(\vec{v},\vec{v})\}. \tag{2.8}$$

Proof. Let us assume $\boldsymbol{\nu}_n \neq 0$ or the assertion is trivial. We will omit M in Ric_M for simplicity of notation. By (2.5) and (2.6), we apply (2.4) to have

$$\begin{aligned} Ric(\boldsymbol{\nu}, \mathbf{n}_\nu) &= Ric(\Theta\mathbf{n} + \boldsymbol{\nu}_n, (1-\Theta^2)\mathbf{n} - \Theta\boldsymbol{\nu}_n) \\ &= \Theta(1-\Theta^2)Ric(\mathbf{n},\mathbf{n}) - \Theta Ric(\boldsymbol{\nu}_n, \boldsymbol{\nu}_n) \\ &= -(n-1)\Theta(1-\Theta^2)\frac{h''}{h} - \Theta Ric(\boldsymbol{\nu}_n, \boldsymbol{\nu}_n). \end{aligned} \tag{2.9}$$

Since $\boldsymbol{\nu}_n$ is tangential to N, we find

$$\begin{aligned} Ric(\boldsymbol{\nu}_n, \boldsymbol{\nu}_n) &= Ric_N(\boldsymbol{\nu}_n, \boldsymbol{\nu}_n) - (hh'' + (n-2)h'^2)g_N(\boldsymbol{\nu}_n, \boldsymbol{\nu}_n) \\ &= (1-\Theta^2)\frac{Ric_N(\vec{v},\vec{v})}{h^2} - \frac{(1-\Theta^2)}{h^2}(hh'' + (n-2)h'^2). \end{aligned} \tag{2.10}$$

Now the conclusion follows by putting everything above together. □

Another useful fact for our warped product manifold M is that all slices are umbilic. Namely, let $N(a)$ be the equidistant hypersurface (with signed constant distance a) to N, and then $N(a)$ is umbilic with constant principal curvature $\frac{h'(a)}{h(a)}$.

2.3. Elliptic equations for height and angle

One of the most beautiful geometric properties for warped product manifolds is the existence of the following special vector field which we denote by V,

$$V = h(r)\frac{\partial}{\partial r} = h(r)\mathbf{n}. \tag{2.11}$$

For any tangential vector field X in M, we have ([16]):

$$\overline{\nabla}_X V = h'(r)X. \tag{2.12}$$

As an application, we immediately have

$$\overline{\nabla}_X \mathbf{n} = \frac{h'(r)}{h(r)}(X - \langle X\, , \mathbf{n}\rangle\, \mathbf{n}). \tag{2.13}$$

for any tangential vector field X in M. We also calculate the Laplace of the height function r restricted to the evolving hypersurface $S(t)$ which is denoted by u.

Proposition 2.5. *Let Δ be the Laplace operator on the hypersurface $S(t)$. Then we have:*

$$\Delta u = \frac{h'(u)}{h(u)}(n - 2 + \Theta^2) - H\Theta. \tag{2.14}$$

Proof. For any point $x \in S = S(t)$, we choose $\{e_1, \ldots, e_{n-1}\}$ (with $e_n = \boldsymbol{\nu}$) to be a local normal frame of S at x. Without loss of generality, we assume that $u(x) \geq 0$. Then at x, we have

$$\begin{aligned}
\Delta u &= \sum_{i=1}^{n-1} \nabla_{e_i} \nabla_{e_i} u \\
&= \sum_{i=1}^{n-1} \nabla_{e_i} \langle \mathbf{n}, e_i\rangle \\
&= \sum_{i=1}^{n-1} e_i\, \langle \mathbf{n}\, , e_i\rangle \\
&= \sum_{i=1}^{n-1} \langle \frac{h'(u)}{h(u)}(e_i - \langle \mathbf{n}, e_i\rangle \mathbf{n}), e_i\rangle + \sum_{i=1}^{n-1} \langle \mathbf{n}, \bar{\nabla}_{e_i} e_i\rangle \\
&= (n-1)\frac{h'(u)}{h(u)} - \frac{h'(u)}{h(u)}(1 - \Theta^2) - H\Theta \\
&= \frac{h'(u)}{h(u)}(n - 2 + \Theta^2) - H\Theta, \qquad (2.15)
\end{aligned}$$

where we used (2.13) for the fourth equality. □

An immediate consequence is:

Corollary 2.6. *Using* (2.1)*, we have the evolution equation for the height function u of $S(t)$ along the mean curvature flow :*

$$u_t - \Delta u = -\frac{h'}{h}(n - 2 + \Theta^2). \tag{2.16}$$

We now derive the main technical tool in this work, the calculation for the Laplacian of the angle function Θ.

Theorem 2.7. *We have*

$$\Delta\Theta = \langle \nabla H\,, \mathbf{n}\rangle - |A|^2\Theta + \frac{h'(r)}{h(r)}\{H(\Theta^2+1) - 2\,\langle \mathbf{n}\,, \nabla\Theta\rangle\} - (n-1)\frac{h'(r)^2}{h^2(r)}\Theta$$
$$- \frac{\Theta(1-\Theta^2)}{h^2(r)}[(n-1)(h(r)h''(r) - h'(r)^2) + Ric_N(\vec{v},\vec{v})],$$

where $\vec{v}$ is defined in (2.7)*.*

Proof. We first work with the auxilliary function:

$$\eta = \langle V\,, \boldsymbol{\nu}\rangle = h(r)\Theta. \tag{2.17}$$

We still use the local normal frame $\{e_1, \ldots, e_{n-1}\}$ (with $e_n = \boldsymbol{\nu}$) at any point $x \in S = S(t)$ such that at this point x, we have

$$\nabla_{e_i} e_k(x) = 0 (i \neq k), \quad \overline{\nabla}_{e_i} e_i(x) = -a_{ii}\boldsymbol{\nu}.$$

where $A = (a_{ij})$ is the second fundamental form of the hypersurface S.

We then have the following computation at x:

$$\begin{aligned}
\Delta\eta &= \sum_{i=1}^{n-1} \nabla_{e_i}\nabla_{e_i}\langle V\,, \boldsymbol{\nu}\rangle \\
&= \sum_{i=1}^{n-1} \left\langle \overline{\nabla}_{e_i}\overline{\nabla}_{e_i} V\,, \boldsymbol{\nu}\right\rangle + 2\left\langle \overline{\nabla}_{e_i} V\,, \overline{\nabla}_{e_i}\boldsymbol{\nu}\right\rangle + \left\langle V\,, \overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\boldsymbol{\nu}\right\rangle \\
&= \sum_{i=1}^{n-1} \left\langle \overline{\nabla}_{e_i}(h'(u)e_i)\,, \boldsymbol{\nu}\right\rangle + 2\sum_{i,k=1}^{n-1} h'(u)\,\langle e_i\,, a_{ik}e_k\rangle + \sum_{i,k=1}^{n-1}\left\langle V\,, \overline{\nabla}_{e_i}(a_{ik}e_k)\right\rangle \\
&= -h'(u)H + 2h'(u)H + \sum_{i,k=1}^{n-1} a_{ik}\left\langle V\,, \overline{\nabla}_{e_i}e_k\right\rangle + a_{ik,i}\,\langle V\,, e_k\rangle \\
&= h'(u)H - |A|^2\eta + \sum_{i,k=1}^{n-1} a_{ik,i}\,\langle V\,, e_k\rangle
\end{aligned}$$

where we have made use of the properties of the normal frame at the point x under investigation. Let's examine the last summation more closely. First we use (2.6) to get

$$\sum_{k=1}^{n-1} \langle V\,, e_k\rangle\, e_k = h(r)\sum_{k=1}^{n-1} \langle \mathbf{n}\,, e_k\rangle\, e_k = h(r)\mathbf{n}_\nu.$$

Then recall the Codazzi equation for $S \subset M$:

$$e_i(a_{ik}) - e_k(a_{ii}) = \bar{R}(\boldsymbol{\nu}, e_i, e_k, e_i), \tag{2.18}$$

where $\bar{R}$ is the curvature tensor on M. Thus, we have

$$\begin{aligned}\sum_{i,k=1}^{n-1} a_{ik,i}\langle V\,, e_k\rangle &= \sum_{i,k=1}^{n-1} e_k(a_{ii})\langle V\,, e_k\rangle + \bar{R}(\boldsymbol{\nu}, e_i, e_k, e_i)\langle V\,, e_k\rangle \\ &= \langle V\,, \nabla H\rangle + \sum_{i,k=1}^{n-1} \bar{R}(\boldsymbol{\nu}, e_i, \langle V\,, e_k\rangle\, e_k, e_i) \\ &= \langle V\,, \nabla H\rangle + h(r)\sum_{i=1}^{n-1} \bar{R}(\boldsymbol{\nu}, e_i, \mathbf{n}_\nu, e_i) \\ &= \langle V\,, \nabla H\rangle + h(r) Ric_M(\boldsymbol{\nu}, \mathbf{n}_\nu)\,.\end{aligned}$$

Now we are in position to apply Lemma 2.4 and arrive at:

$$\begin{aligned}\Delta\eta &= h'(r)H - |A|^2\eta + \langle V\,, \nabla H\rangle \\ &\quad - \frac{\Theta(1-\Theta^2)}{h(r)}[(n-2)(h(r)h''(r) - (h'(r))^2) + Ric_N(\vec{v}, \vec{v})].\end{aligned} \tag{2.19}$$

Since $\eta = h(r)\Theta$, and applying Lemma 2.5, we have

$$\begin{aligned}\Delta\eta &= h\Delta\Theta + 2h'\langle \mathbf{n}\,, \nabla\Theta\rangle + \Theta(h'\Delta r + h''|\nabla r|^2) \\ &= h\Delta\Theta + 2h'\langle \mathbf{n}\,, \nabla\Theta\rangle + \Theta(h'\Delta r + h''(1-\Theta^2)) \\ &= h\Delta\Theta + 2h'\langle \mathbf{n}\,, \nabla\Theta\rangle + \Theta\left[\frac{h'^2}{h}(n-2+\Theta^2) - h'H\Theta + h''(1-\Theta^2)\right].\end{aligned}$$

Now the assertion follows by isolating $\Delta\Theta$ on one side of the equation. □

3. Proof of Main Theorem

The key is to obtain a positive lower bound for the angle function Θ.

3.1. Evolution equation for the angle function squared

In Theorem 2.7, we have derived the Laplacian of Θ. Now let us derive the evolution equation.

Theorem 3.1. *The angle function* $\Theta(\cdot, t)$ *satisfies the following evolution equation along the mean curvature flow* (1.1)*:*

$$\Theta_t - \Delta\Theta = |A|^2\Theta + \frac{2h'(r)}{h(r)}\{\langle \mathbf{n}, \nabla\Theta\rangle - H\} + (n-1)\frac{h'(r)^2}{h^2(r)}\Theta$$

$$+ \frac{\Theta(1-\Theta^2)}{h^2(r)}\{(n-1)(h(r)h''(r) - h'(r)^2) + Ric_N(\vec{v}, \vec{v})\}, \tag{3.1}$$

where $\vec{v}$ *is defined in* (2.7)*.*

Proof. For the mean curvature flow we have ([11]) $\frac{\partial}{\partial t}\boldsymbol{\nu} = \nabla H$. Then we have

$$\begin{aligned}\frac{\partial\Theta}{\partial t} &= \frac{\partial}{\partial t}\langle \mathbf{n}, \boldsymbol{\nu}\rangle = \left\langle \frac{\partial}{\partial t}\boldsymbol{\nu}, \mathbf{n}\right\rangle + \langle \boldsymbol{\nu}, \overline{\nabla}_{-H\boldsymbol{\nu}}\mathbf{n}\rangle \\ &= \langle \nabla H, \mathbf{n}\rangle - H\langle \boldsymbol{\nu}, \overline{\nabla}_{\boldsymbol{\nu}}\mathbf{n}\rangle.\end{aligned}$$

Using (2.13), we have

$$\overline{\nabla}_{\boldsymbol{\nu}}\mathbf{n} = \frac{h'}{h}(\boldsymbol{\nu} - \Theta\mathbf{n}) = \frac{h'}{h}\boldsymbol{\nu}_n,$$

and so that

$$\frac{\partial\Theta}{\partial t} = \langle \nabla H, \mathbf{n}\rangle - \frac{Hh'}{h}(1-\Theta^2). \tag{3.2}$$

Now the assertion follows from combining this with Theorem 2.7. □

It's actually easier to work with Θ^2. Let's denote $f(\cdot, t) = \Theta^2$ which satisfies the following differential inequality.

Corollary 3.2. *As long as* $f(\cdot, t) = \Theta^2 > 0$ *and the inequality* (1.3) *holds, we have*

$$f_t - \Delta f \geq \left\langle \frac{2h'(r)}{h(r)}\mathbf{n} - \frac{\nabla f}{2f}, \nabla f\right\rangle + G(f, r), \tag{3.3}$$

where

$$G(f, r) = \frac{2(n-1)(1-f)}{h^2(r)}\{cf - h'(r)^2\}, \tag{3.4}$$

and c is the constant from condition (C3).

Proof. A direct calculation from Theorem 3.1 shows that f satisfies the following evolution equation:

$$\begin{aligned} f_t - \Delta f = {} & 2|A|^2 f + \left\langle \frac{2h'(r)}{h(r)}\mathbf{n} - \frac{\nabla f}{2f}, \nabla f \right\rangle \\ & - \frac{4h'(r)\sqrt{f}}{h(r)} H + 2(n-1)\frac{h'(r)^2}{h^2(r)} f \\ & + \frac{2f(1-f)}{h^2(r)}\{(n-1)(h(r)h''(r) - h'(r)^2) + Ric_N(\vec{v}, \vec{v})\}. \end{aligned} \tag{3.5}$$

By (1.3), we have

$$\begin{aligned} f_t - \Delta f \geq {} & 2|A|^2 f + \left\langle \frac{2h'(r)}{h(r)}\mathbf{n} - \frac{\nabla f}{2f}, \nabla f \right\rangle - \frac{4h'(r)\sqrt{f}}{h(r)} H \\ & + 2(n-1)\frac{h'(r)^2}{h^2(r)} f + \frac{2f(1-f)}{h^2(r)}\{(n-1)c\}, \end{aligned} \tag{3.6}$$

where the special case of $\vec{v} = 0$ is taken care of by the factor $1 - f$ in the last term on the right hand side of (3.5) because $f = \Theta^2 = 1$ in this case.

Now the corollary follows from using the fact that $|H| \leq \sqrt{n-1}|A|$ and completing the square. □

3.2. Barriers

In this subsection, we treat the model case, namely, the mean curvature flow of initial hypersurface parallel (i.e equidistant) to N. By the well-known avoidance principle in mean curvature flows of closed hypersurfaces, this special mean curvature flow will serve as barriers to control the behavior of our mean curvature flow for a more general class of initial hypersurfaces.

Lemma 3.3. *Let N be a closed $(n-1)$-dimensional Riemannian manifold and $M = N \times \mathbb{R}$ or $N \times [0, \infty)$ the warped product manifold with the metric given by* (1.2) *satisfying Conditions (C1–2). Then any mean curvature flow* (1.1) *in M of initial hypersurface $N(a)$, where $N(a)$ is a hypersurface of constant (signed) distance $a \in \mathbb{R}$ to N, exists for all time, stays umbilic and converges smoothly to N as $t \to \infty$.*

Proof. Since $N(a)$ is parallel to N, the initial angle function $\Theta(0) \equiv 1$, moreover, $N(a)$ is umbilic with principal curvature $\frac{h'(a)}{h(a)}$. By uniqueness of the mean curvature flow, we have $\Theta \equiv 1$ and the evolving surface stays parallel to N.

It's trivial when $a = 0$. Let's assume $a > 0$. Let $R(t)$ be the height function of the evolving hypersurface at time t. Note that it is a function of t only, since $N(R)$ is parallel to N. Then by either (2.1) or (2.16), we have

$$\begin{cases} \dfrac{dR(t)}{dt} = -(n-1)\dfrac{h'(R(t))}{h(R(t))} \\ R(0) = a > 0\ . \end{cases} \tag{3.7}$$

Since h is a positive function over $\mathbb{R}$, the solution $R(t)$ exists forever. In light of the direction field of this ODE, mostly just the sign of derivative, we know that $R(t)$ decreases and stays positive. Assuming $R(t) \to A \geq 0$ as $t \to \infty$, one can easily rule out the case of $A > 0$ since $\frac{h'(A)}{h(A)} > 0$, so $A = 0$ and $N(R)$ converges smoothly to N at time infinity. The case of $a < 0$ can be treated in the same way. In the case of $M = N \times [0, \infty)$, $N = N \times \{0\}$ itself serves as the other barrier. This completes the proof. □

As a corollary, we obtain the convergence part of the main theorem.

Corollary 3.4. *Let* (1.1) *be any mean curvature flow in the above warped product manifold M with a closed initial hypersurface $S(0)$. If it exists for all time, then it converges continuously to N.*

Proof. Since the initial hypersurface is closed, there is a constant $a > 0$ such that $S(0)$ is enclosed in the region between parallel hypersurfaces $N(-a)$ and $N(a)$. The corollary follows from the fact that mean curvature flows of initial hypersurfaces $N(-a)$ and $N(a)$ both converge to N and the avoidance principle. □

We conclude this subsection by the following lemma which will be used later to obtain the key estimate.

Lemma 3.5. *Let $R(t) > 0$ be the solution for the initial value problem* (3.7)*, and $\bar{f}(t)$ be the solution to the initial value problem:*

$$\begin{cases} \dfrac{d\bar{f}(t)}{dt} = -2(n-1)(1-\bar{f})\dfrac{h'(R(t))^2}{h^2(R(t))} \\ \bar{f}(0) = \bar{f}_0 \in [0,1]\ , \end{cases} \tag{3.8}$$

Then we have the following identity:

$$(1 - \bar{f}(t))h^2(R(t)) = (1 - \bar{f}_0)h^2(a), \tag{3.9}$$

for all $t \geq 0$.

Proof. Let's assume $\bar{f}_0 < 1$, otherwise the equation (3.8) forces $\bar{f}(t) = 1$ and we are done. Since we have both $h(r) > 0$ and $h'(r) > 0$ for all $r > 0$, then

$$\frac{d\bar{f}(t)}{dt} = -2(n-1)(1-\bar{f})\frac{h'^2(R(t))}{h^2(R(t))} < 0.$$

So the solution to (3.8) exists for all time. To prove the lemma, we set the function $\Lambda(t) = (1-\bar{f}(t))h^2(R(t))$ and prove that it is actually independent of t. We justify this by a direct calculation using both equations (3.7) and (3.8):

$$\begin{aligned}\frac{d\Lambda(t)}{dt} &= -h^2(R(t))\frac{d\bar{f}(t)}{dt} + (1-\bar{f}(t))(2h(R(t))h'(R(t))\frac{dR(t)}{dt}) \\ &= 2(n-1)(1-\bar{f})h'^2(R(t)) + 2(1-\bar{f})hh'\left(-(n-1)\frac{h'(R(t))}{h(R(t))}\right) \\ &= 0. \end{aligned} \tag{3.10}$$

□

Remark 3.6. As an immediate consequence, we have the limit

$$\lim_{t\to\infty} \bar{f}(t) = 1 - (1-\bar{f}_0)h^2(a). \tag{3.11}$$

This is the only place that $h(0) = 1$ is used which can of course be easily adjusted using any positive constant instead.

3.3. Gradient estimate

Now we apply the barriers established in the previous subsection and comparison equations to control the lower bound for Θ and establish the gradient estimate. In particular, we prove:

Theorem 3.7. *Let M^n be a warped product manifold satisfying Conditions (C0–3) and S_0 a smooth closed hypersurface which is a geodesic graph over the unique totally geodesic hypersurface Σ in M^n, and suppose there is a constant $a_0 > 0$ such that S_0 lies between $\Sigma(\pm a_0)$. Then if S_0 satisfies*

$$\min_{p\in S_0} \Theta(p) \geq \sqrt{1 - \frac{1}{h^2(a_0)}},$$

the mean curvature flow (1.1) *with initial hypersurface S_0 remains graphical over Σ, namely, $\Theta(\cdot, t) > 0$ as long as the flow exists.*

Proof. Let us first recall the evolution inequality satisfied by $f = \Theta^2$ as in Corollary 3.2:

$$f_t - \Delta f \geq \left\langle \frac{2h'(r)}{h(r)}\mathbf{n} - \frac{\nabla f}{2f}, \nabla f \right\rangle + G(f, r),$$

where

$$G(f, r) = \frac{2(n-1)(1-f)}{h^2(r)}\{cf - h'(r)^2\},$$

and c is the constant from condition (C3). Let $\phi(t)$ be the spatial minimum of $f(\cdot, t)$ on the evolving hypersurface $S(t)$, namely,

$$\phi(t) = \min_{S(t)} f.$$

We only have to establish $\phi \in (0, 1]$ for a priori estimate. At the spatial minimum of f, we have $\nabla f = 0$ and $\Delta f \geq 0$, and so for $t > 0$ (using Hamilton's trick), we find:

$$\begin{aligned} \frac{d\phi}{dt} &\geq \frac{\partial f}{\partial t} - \Delta f \\ &\geq \frac{2(n-1)(1-f)}{h^2(r)}\{cf - h'(r)^2\} \\ &\geq -\frac{2(n-1)(1-f)}{h^2(r)}\{h'(r)^2\} \\ &= -2(n-1)(1-\phi)\left\{\frac{h'(r)}{h(r)}\right\}^2. \end{aligned}$$

By our conditions on the warping function $h(r)$, in particular (1.3), we have $\left|\frac{h'(r)}{h(r)}\right|$ is nondecreasing in $|r|$. Let $R(t) > 0$ be the solution to the initial value problem (3.7), namely, the evolving distance of the mean curvature flow with initial hypersurface $N(a_0)$ at any time $t \geq 0$. Since our evolving hypersurfaces $S(t)$ in the mean curvature flow (1.1) of initial hypersurface S_0 is squeezed by the barriers $N(R(t))$, by Lemma 3.3, we have $|r(t)| \leq R(t)$, and therefore

$$\begin{cases} \dfrac{1}{1-\phi}\dfrac{d\phi}{dt} \geq -2(n-1)\dfrac{h'^2(R(t))}{h^2(R(t))} \\ \phi(0) = \phi_0 \in (0,1) \; . \end{cases} \tag{3.12}$$

Recall that our comparison initial value problem (3.8) is equivalent to:

$$\begin{cases} \dfrac{1}{1-\bar{f}}\dfrac{d\bar{f}(t)}{dt} = -2(n-1)\dfrac{h'^2(R(t))}{h^2(R(t))} \\ \bar{f}(0) = \bar{f}_0 \in [0,1], \end{cases}$$

Choosing $\phi_0 = \bar{f}_0$, we find, for $t > 0$,

$$\frac{d}{dt}\left(\log\frac{1-\bar{f}(t)}{1-\phi(t)}\right) \geq 0,$$

which implies $\phi(t) \geq \bar{f}(t)$. Finally, by Lemma 3.5 and Remark 3.6, setting $a = a_0$, we have

$$\phi(t) \geq \bar{f}(t) \geq \lim_{t\to\infty} \bar{f}(t) = 1 - (1-\bar{f}_0)h^2(a_0).$$

where the second $\geq$ is strict unless the initial surface is N, for which there is nothing to proof.

Now we apply the initial angle condition $\min_{p\in S_0}\Theta(p) \geq \sqrt{1-\frac{1}{h^2(a_0)}}$, and choose $\bar{f}_0$ to be $(\min_{p\in S_0}\Theta(p))^2$, we arrive at $\phi(t) > 0$ as long as the flows exists. This completes the proof. □

3.4. Completing the proof of the main result

Now we can assemble the ingredients and complete the proof of Theorem 1.2.

Proof. (of Theorem 1.2) We have shown in Theorem 3.7 that the mean curvature flow (1.1) stays graphical as long as it exists. This provides the gradient estimate for the mean curvature flow for any finite time interval. By the classical theory of parabolic equations in divergent form (for instance [14]), the higher regularity and a priori estimates of the solution follow in the standard way. This yields the long time existence of the flow by Huisken ([11]). Thus, by Lemma 3.3 and the avoidance principle, the continuous convergence of the flow also follows.

When the inequality in the assumption of the theorem is strict, the proof of Theorem 3.7 gives a uniform positive lower bound of the angle for all time, and so the higher order estimates are uniform for all time, providing the smooth convergence. Hence, the proof of Theorem 1.2 is completed. □

3.5. Applications and remarks

We begin by showing that the main result can be applied for the de Sitter-Schwarzschild manifold from general relativity. The discussion is adjusted from the same consideration in [4]. Usually in literature, such manifold is described in the following setting: $M = S^{n-1} \times (\underline{s}, \overline{s})$ with the warped metric

$$g = \frac{1}{\omega(s)} ds \otimes ds + s^2 g_{S^{n-1}}$$

where S^{n-1} is the unit $(n-1)$-sphere, $\omega(s) > 0$ in $(\underline{s}, \overline{s}) \subset (0, \infty)$ and smooth up to $\underline{s}$ with $\omega(\underline{s}) = 0$. $Ric_{S^{n-1}} = (n-2) g_{S^{n-1}}$ gives Condition (C0) with $\rho = \frac{n-2}{n-1}$. For the de Sitter-Schwarzschild manifold,

$$\omega(s) = 1 - m s^{2-n} - \kappa s^2$$

where the mass constant $m > 0$, the cosmological constant κ can be either non-positive in which case $\overline{s} = \infty$ or satisfy

$$n^n m^2 \kappa^{n-2} < 4(n-2)^{n-2}$$

so that $\omega(\overline{s}) = 0$ for $0 < \underline{s} < \overline{s} < \infty$. We then use the change of variable in [4], $r = F(s)$ with

$$\frac{dr}{ds} = F'(s) = \frac{1}{\sqrt{\omega(s)}}, \quad F(\underline{s}) = 0.$$

The assumption on $\omega(s)$ makes sure that this is a legitimate change of variable from $s \in [\underline{s}, \overline{s})$ to $r \in [0, F(\overline{s}))$. This provides a natural way to extend the manifold to $s = \underline{s}$ or $r = 0$.

Using the new warping variable r, we have

$$g = dr \otimes dr + h^2(r) g_{S^{n-1}},$$

where $h(r) = s = F^{-1}(r)$. So we have

$$h'(r) = \frac{ds}{dr} = \sqrt{\omega(s)}$$

which clearly satisfies Conditions (C1–2) except that $h(0) = \underline{s}$ which is not a problem as described before. Moreover, we have

$$h''(r) = \frac{1}{2\sqrt{\omega(s)}} \omega'(s) \frac{ds}{dr} = \frac{\omega'(s)}{2}.$$

Regarding Condition (C3), $c = \rho > 0$, we need

$$h(r)h''(r) - h'^2(r) = \frac{1}{2}s\ \omega'(s) - \omega(s) \geq 0$$

and so at $r = 0$, i.e. $s = \underline{s}$,

$$h(0)h''(0) - h'^2(0) = \frac{1}{2}\underline{s}\ \omega'(\underline{s}).$$

For the de Sitter-Schwarzschild manifold,

$$\omega'(s) = -m(2-n)s^{1-n} - 2\kappa s,$$

and so we have

$$h(r)h''(r) - h'^2(r) = \frac{1}{2}s\ \omega'(s) - \omega(s) = \frac{1}{2}mns^{2-n} - 1.$$

Hence we need $\frac{1}{2}mns^{2-n} - 1 \geq 0$, and so

$$s \leq \left(\frac{mn}{2}\right)^{\frac{1}{n-2}}.$$

Claim: $\left(\frac{mn}{2}\right)^{\frac{1}{n-2}} \in (\underline{s}, \overline{s})$.

Proof. There are two cases to deal with.

If $\kappa \leq 0$, since $N \geq 3$, it's obvious that

$$h(0)h''(0) - h'^2(0) = \frac{1}{2}\underline{s}\ \omega'(\underline{s}) > 0.$$

So $\left(\frac{mn}{2}\right)^{\frac{1}{n-2}} \in (\underline{s}, \infty) = (\underline{s}, \overline{s})$.

If $\kappa > 0$ and $n^n m^2 \kappa^{n-2} < 4(n-2)^{n-2}$, as s increases from 0 to ∞, it's clear that $\omega'(s) = -m(2-n)s^{1-n} - 2\kappa s$ strictly decreases from ∞ to $-\infty$, and so we know that $\omega(s)$ strictly increases (passing $\omega(\underline{s}) = 0$) from $-\infty$ to its positive maximum which is achieves at some $S \in (\underline{s}, \overline{s})$, and then strictly decreases (passing $\omega(\overline{s}) = 0$) to $-\infty$. So we clearly have $\omega'(\underline{s}) > 0$, and

$$h(0)h''(0) - h'^2(0) = \frac{1}{2}\underline{s}\ \omega'(\underline{s}) > 0.$$

So $\left(\frac{mn}{2}\right)^{\frac{1}{n-2}} > \underline{s}$. Meanwhile, at $s = S$ where the maximum of $\omega(s)$ is achieved,

$$\frac{1}{2}S\ \omega'(S) - \omega(S) = -\omega(S) < 0,$$

and so $\overline{s} > S > \left(\frac{mn}{2}\right)^{\frac{1}{n-2}}$.

So the claim is justified. □

Now we can conclude that Condition (C3) is satisfied in $S^{n-1} \times [\underline{s}, \left(\frac{mn}{2}\right)^{\frac{1}{n-2}}]$ as part of the de Sitter-Schwarzschild manifold, where we can apply the main result of Theorem 1.2.

Finally, we would like to point out that there are examples where graphical complete hypersurfaces of warped product manifolds fail to stay graphical along the mean curvature flow after some finite time, for example, as discussed in Appendix A of [19]. A priori it's not clear whether this will indicate the development of geometric singularities along the flow. We hope to address this problem in future works.

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