

Boundedness of threefolds of Fano type with Mori fibration structures

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We show boundedness of 3-folds of ϵ -Fano type with Mori fibration structures. The proof is based on the birational boundedness result in our previous work [14] combining with arguments in Kawamata [17] and Kollár–Miyaoka–Mori–Takagi [24].

1. Introduction

Throughout this paper, we work over the field of complex numbers \mathbb{C} . See Subsection 2.1 for notation and conventions.

A normal projective variety X is of ϵ -Fano type if there exists an effective \mathbb{Q} -divisor B such that (X, B) is an ϵ -klt log Fano pair. 0-Fano type is also called Fano type for simplicity. The notion of Fano type was introduced by Prokhorov–Shokurov [31].

We are mainly interested in the boundedness of varieties of ϵ -Fano type. Our motivation is the following conjecture due to A. Borisov, L. Borisov, and V. Alexeev.

Conjecture 1.1 (BAB Conjecture). *Fix an integer $n > 0$, $0 < \epsilon < 1$. Then the set of all n -dimensional varieties of ϵ -Fano type is bounded.*

BAB Conjecture is one of the most important conjectures in birational geometry. It is related to the termination of flips (cf. [4, 7]) and has interesting application for the Jordan property of Cremona groups (cf. [32]). Besides, varieties of Fano type form a fundamental class in birational geometry according to Minimal Model Program and have many interesting properties (cf. [9, 12, 26]). Hence it is very interesting to understand the basic properties of this class, such as boundedness.

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BAB Conjecture in dimension two was proved by Alexeev [1] with a simplified argument by Alexeev–Mori [3]. In higher dimension, BAB Conjecture still remains open. There are only some partial boundedness results (cf. [2, 8, 17, 22, 24]).

We recall the following theorem proved in [13] by using Minimal Model Program.

Theorem 1.2 (cf. [13, Proof of Theorem 2.3]). *Fix an integer $n > 0$ and $0 < \epsilon < 1$. Every n -dimensional variety X of ϵ -Fano type is birational to an n -dimensional variety X' of ϵ -Fano type with a Mori fibration structure.*

We will recall the proof in Section 3. Here we emphasize that having a Mori fibration structure implies that having at most \mathbb{Q} -factorial terminal singularities by our definition (see Subsection 2.1). According to this theorem, it is important and interesting to investigate varieties of ϵ -Fano type with Mori fibration structures. In fact, in the proof of BAB Conjecture in dimension two (cf. [1, 3]), the first step is to classify (and bound) all surfaces of ϵ -Fano type with Mori fibration structures, which are just projective plane or Hirzebruch surfaces \mathbb{F}_n with $n < 2/\epsilon$. Therefore, we are interested in the boundedness of 3-folds of ϵ -Fano type with Mori fibration structures, as the first step towards BAB Conjecture in dimension three.

The following is our main theorem.

Theorem 1.3. *Fix $0 < \epsilon < 1$. The set of all 3-folds of ϵ -Fano type with Mori fibration structures is bounded.*

For interesting examples in dimension three, we refer to [28], where 3-folds of Fano type with conic bundle structures were constructed, which proves the birational unboundedness of 3-folds of 0-Fano type.

1.1. Sketch of the proof

Let X be a 3-fold of ϵ -Fano type with a Mori fibration $f : X \rightarrow Z$. If $\dim Z = 0$, then X is a \mathbb{Q} -factorial terminal Fano 3-fold of Picard number one, which is bounded by Kawamata [17]. So we only need to consider the case when $\dim Z > 0$.

We recall the following theorem from [14], by which we proved the birational boundedness of 3-folds of ϵ -Fano type.

Theorem 1.4 (cf. [14, Proof of Corollaries 1.5, 1.8]). *Fix $0 < \epsilon < 1$. Then there exist positive integers N_ϵ and V_ϵ depending only on ϵ , with the following property:*

If X is a 3-fold of ϵ -Fano type with a Mori fibration $f : X \rightarrow Z$.

- (1) *If $\dim Z = 1$ (i.e. $Z = \mathbb{P}^1$), take a general fiber F of f , then*
 - (1-1) *$-K_X + N_\epsilon F$ is ample;*
 - (1-2) *$(-K_X + N_\epsilon F)^3 \leq V_\epsilon$.*
- (2) *If $\dim Z = 2$, then there exists a very ample divisor H on Z such that*
 - (2-1) *$-K_X + N_\epsilon f^*H$ is ample;*
 - (2-2) *$(-K_X + N_\epsilon f^*H)^3 \leq V_\epsilon$.*

We will recall the proof in Section 4. By this theorem, to show the boundedness, it suffices to show the boundedness of Gorenstein indices due to Kollár's effective base point free theorem (see Subsection 5.3). We will explain the idea of bounding the Gorenstein indices.

For convenience, we define G and \mathcal{F}_X as following.

Definition 1.5. Let X be a 3-fold of ϵ -Fano type with a Mori fibration $f : X \rightarrow Z$ such that $\dim Z > 0$. Keep the notation in Theorem 1.4.

- 1) Define the projective smooth surface G to be a general fiber F (resp. a general element of $|f^*(H)|$) if $\dim Z = 1$ (resp. $\dim Z = 2$).
- 2) Define the torsion free sheaf $\mathcal{F}_X := \mathcal{T}_X \oplus \mathcal{O}_X(N_\epsilon G)$, where \mathcal{T}_X is the tangent sheaf on X .

We remark that G is a del Pezzo surface (resp. conic bundle over a general H) if $\dim Z = 1$ (resp. $\dim Z = 2$).

Following the idea of Kollár–Miyaoaka–Mori–Takagi [24], we can prove the pseudo-effectivity of $c_2(\mathcal{F}_X)$.

Theorem 1.6. *Fix $0 < \epsilon < 1$. Let X be a 3-fold of ϵ -Fano type with a Mori fibration $f : X \rightarrow Z$ such that $\dim Z > 0$. Keep the notation in Definition 1.5. Then $c_2(\mathcal{F}_X)$ is pseudo-effective.*

Following the idea of Kawamata [17], after bounding $(-K_X) \cdot c_2(X)$ from below, we can get an upper bound for Cartier index of K_X , which eventually implies the desired boundedness.

Theorem 1.7. *Fix $0 < \epsilon < 1$. Let X be a 3-fold of ϵ -Fano type with a Mori fibration $f : X \rightarrow Z$ such that $\dim Z > 0$, then*

$$1) (-K_X) \cdot c_2(X) \geq -M_\epsilon;$$

$$2) r_\epsilon K_X \text{ is Cartier,}$$

where $M_\epsilon = 5V_\epsilon + 12N_\epsilon$ and $r_\epsilon = (24 + M_\epsilon)!$, N_ϵ and V_ϵ are the numbers defined in Theorem 1.4.

This paper is organized as follows. For the reader's convenience, in Sections 3 and 4, we recall the proof of Theorems 1.2 and 1.4. In Section 5, we prove our main theorems, Theorems 1.6, 1.7, and 1.3.

2. Preliminaries

2.1. Notation and conventions

We adopt the standard notation and definitions in [18] and [25], and will freely use them.

A pair (X, B) consists of a normal projective variety X and an effective \mathbb{Q} -divisor B on X such that $K_X + B$ is \mathbb{Q} -Cartier.

The pair (X, B) is called a *log Fano pair* if $-(K_X + B)$ is ample.

Let $f : Y \rightarrow X$ be a log resolution of the pair (X, B) , write

$$K_Y = f^*(K_X + B) + \sum a_i F_i,$$

where $\{F_i\}$ are distinct prime divisors. For some $\epsilon \in [0, 1]$, the pair (X, B) is called

- (a) ϵ -kawamata log terminal (ϵ -klt, for short) if $a_i > -1 + \epsilon$ for all i ;
- (b) ϵ -log canonical (ϵ -lc, for short) if $a_i \geq -1 + \epsilon$ for all i ;
- (c) *terminal* if $a_i > 0$ for all f -exceptional divisors F_i and all f .

Usually we write X instead of $(X, 0)$ in the case $B = 0$. Note that 0-klt (resp. 0-lc) is just klt (resp. lc) in the usual sense.

F_i is called a *non-klt place* (resp. *non-lc place*) of (X, B) if $a_i \leq -1$ (resp. < -1). A subvariety $V \subset X$ is called a *non-klt center* (resp. *non-lc center*) of (X, B) if it is the image of a non-klt place (resp. non-lc place). The *non-klt locus* $\text{Nklt}(X, B)$ is the union of all non-klt centers of (X, B) . The *non-lc locus* $\text{Nlc}(X, B)$ is the union of all non-lc centers of (X, B) .

A normal projective variety X is of ϵ -Fano type if there exists an effective \mathbb{Q} -divisor B such that (X, B) is an ϵ -klt log Fano pair. 0-Fano type is also called Fano type for simplicity.

A projective morphism $f : X \rightarrow Z$ between normal projective varieties is called a *Mori fibration* (or *Mori fiber space*) if

- 1) X is \mathbb{Q} -factorial with terminal singularities;
- 2) f is a *contraction*, i.e., $f_*\mathcal{O}_X = \mathcal{O}_Z$;
- 3) $-K_X$ is ample over Z ;
- 4) $\rho(X/Z) = 1$;
- 5) $\dim X > \dim Z$.

We say that X is with a *Mori fibration structure* if there exists a Mori fibration $X \rightarrow Z$. In particular, in this situation, X has at most \mathbb{Q} -factorial terminal singularities by definition.

A collection of varieties $\{X_t\}_{t \in T}$ is said to be *bounded* (resp. *birationally bounded*) if there exists $h : \mathcal{X} \rightarrow S$ a projective morphism between schemes of finite type such that each X_t is isomorphic (resp. birational) to \mathcal{X}_s for some $s \in S$.

2.2. Volumes

Let X be an n -dimensional projective variety and D be a Cartier divisor on X . The *volume* of D is the real number

$$\mathrm{Vol}(X, D) = \limsup_{m \rightarrow \infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!}.$$

Note that the limsup is actually a limit. Moreover by the homogenous property of volumes, we can extend the definition to \mathbb{Q} -Cartier \mathbb{Q} -divisors. Note that if D is a nef \mathbb{Q} -divisor, then $\mathrm{Vol}(X, D) = D^n$.

For more background on volumes, see [27, 11.4.A]. It is easy to see the following inequality for volumes by comparing global sections by exact sequences.

Lemma 2.1 ([14, Lemma 2.5]). *Let X be a projective normal variety, D a \mathbb{Q} -Cartier \mathbb{Q} -divisor and S a semi-ample normal Cartier prime divisor. Then for any rational number $q > 0$,*

$$\mathrm{Vol}(X, D + qS) \leq \mathrm{Vol}(X, D) + q(\dim X)\mathrm{Vol}(S, D|_S + qS|_S).$$

2.3. Log canonical thresholds and α -invariants

Let (X, B) be an lc pair and $D \geq 0$ be a \mathbb{Q} -Cartier \mathbb{Q} -divisor. The *log canonical threshold* of D with respect to (X, B) is

$$\text{lct}(X, B; D) = \sup\{t \in \mathbb{Q} \mid (X, B + tD) \text{ is lc}\}.$$

For application, we need to consider the case when D is not effective. Let G be a \mathbb{Q} -Cartier \mathbb{Q} -divisor satisfying $G + B \geq 0$. The *generalized log canonical threshold* of G with respect to (X, B) is

$$\text{glct}(X, B; G) = \sup\{t \in [0, 1] \cap \mathbb{Q} \mid (X, B + tG) \text{ is lc}\}.$$

Note that the assumption $t \in [0, 1]$ guarantees that $B + tG \geq 0$.

If (X, B) is an lc log Fano pair, the (*generalized*) α -invariant of (X, B) is defined by

$$\alpha(X, B) = \inf\{\text{glct}(X, B; G) \mid G \sim_{\mathbb{Q}} -(K_X + B), G + B \geq 0\}.$$

2.4. Non-klt centers

The following lemma suggests a standard way to construct non-klt centers.

Lemma 2.2 (cf. [25, Lemma 2.29]). *Let (X, B) be a pair and $V \subset X$ a closed subvariety of codimension k such that V is not contained in the singular locus of X . If $\text{mult}_V B \geq k$, then V is a non-klt center of (X, B) .*

Recall that the *multiplicity* $\text{mult}_V F$ of a divisor F along a subvariety V is defined by the multiplicity $\text{mult}_x F$ of F at a general point $x \in V$.

Unfortunately, the converse of Lemma 2.2 is not true unless $k = 1$. Usually we do not have good estimates for the multiplicity along a non-klt center but the following lemma.

Lemma 2.3 (cf. [27, Theorem 9.5.13]). *Let (X, B) be a pair and $V \subset X$ a non-klt center of (X, B) such that V is not contained in the singular locus of X . Then $\text{mult}_V B \geq 1$.*

We have the following connectedness lemma of Kollár and Shokurov for non-klt locus (cf. Shokurov [34], Kollár [19, 17.4]).

Theorem 2.4 (Connectedness Lemma). *Let $f : X \rightarrow Z$ be a proper morphism of normal varieties with connected fibers and D a \mathbb{Q} -divisor such*

that $-(K_X + D)$ is \mathbb{Q} -Cartier, f -nef, and f -big. Write $D = D^+ - D^-$ where D^+ and D^- are effective with no common components. If D^- is f -exceptional (i.e. all of its components have image of codimension at least 2), then $\text{Nklt}(X, D) \cap f^{-1}(z)$ is connected for any $z \in Z$.

As an application, we have the following theorem on inversion of adjunction (cf. [25, Theorem 5.50]). Here we only use a weak version.

Theorem 2.5 (Inversion of adjunction). *Let (X, B) be a pair and $S \subset X$ a normal Cartier divisor not contained in the support of B . Then*

$$\text{Nklt}(X, B) \cap S \subset \text{Nklt}(S, B|_S).$$

In particular, if $\text{Nklt}(X, B) \cap S \neq \emptyset$, then $(S, B|_S)$ is not klt.

2.5. Length of extremal rays

Recall the result on length of extremal rays due to Kawamata.

Theorem 2.6 ([16]). *Let (X, B) be a klt pair. Then every $(K_X + B)$ -negative extremal ray R is generated by a rational curve C such that*

$$0 < -(K_X + B) \cdot C \leq 2 \dim X.$$

However, we need to deal with non-klt pairs in application. We have a slightly generalization of this theorem for non-klt pairs.

Theorem 2.7 ([10, Theorem 1.1(5)]). *Let (X, B) be a pair. Fix a $(K_X + B)$ -negative extremal ray R . Assume that*

$$R \cap \overline{NE}(X)_{\text{Nlc}(X, B)} = \{0\},$$

where

$$\overline{NE}(X)_{\text{Nlc}(X, B)} = \text{Im}(\overline{NE}(\text{Nlc}(X, B)) \rightarrow \overline{NE}(X)).$$

Then R is generated by a rational curve C such that

$$0 < -(K_X + B) \cdot C \leq 2 \dim X.$$

3. Proof of Theorem 1.2

In this section, for the reader's convenience, we recall the proof of Theorem 1.2 from [13]. We start with two lemmas. The first lemma is about equivalent definitions of ϵ -Fano type.

Lemma 3.1 (cf. [31, Lemma-Definition 2.6]). *Let Y be a projective normal variety, and $\epsilon \in [0, 1)$. The following are equivalent:*

- 1) Y is of ϵ -Fano type;
- 2) There exists an effective \mathbb{Q} -divisor Δ such that Δ is big, (Y, Δ) is ϵ -klt, and $K_Y + \Delta \equiv 0$.

Proof. First we assume that Y is of ϵ -Fano type, that is, there exists an effective \mathbb{Q} -divisor B on Y such that (Y, B) is ϵ -klt log Fano pair. Then take a general effective ample \mathbb{Q} -divisor A on Y such that $(Y, B + A)$ is ϵ -klt and

$$K_Y + B + A \sim_{\mathbb{Q}} 0.$$

We may take $\Delta = A + B$.

Then we assume that there exists an effective \mathbb{Q} -divisor Δ such that Δ is big, (Y, Δ) is ϵ -klt, and $K_Y + \Delta \equiv 0$. Since Δ is big, we may write $\Delta = A + G$ where A is an ample \mathbb{Q} -divisor and G is an effective \mathbb{Q} -divisor. We may take a sufficiently small $\delta > 0$ such that $(Y, \Delta + \delta G)$ is again ϵ -klt. Hence $(Y, (1 - \delta)\Delta + \delta G)$ is ϵ -klt, and

$$-(K_Y + (1 - \delta)\Delta + \delta G) \equiv \delta A$$

is ample. Hence Y is of ϵ -Fano type. □

Being of ϵ -Fano type is preserved by MMP according to the following lemma.

Lemma 3.2 (cf. [12, Lemma 3.1]). *Let Y be a projective normal variety and $f : Y \rightarrow Z$ be a projective birational contraction.*

- 1) *If Y is of ϵ -Fano type, so is Z ;*
- 2) *Assume that f is small, then Y is of ϵ -Fano type if and only if so is Z .*

In particular, minimal model program preserves ϵ -Fano type.

Proof. First we assume that Y is of ϵ -Fano type, that is, by Lemma 3.1, there exists an effective \mathbb{Q} -divisor Δ such that Δ is big, (Y, Δ) is ϵ -klt, and $K_Y + \Delta \equiv 0$. Pushing forward by f , by negativity lemma,

$$K_Y + \Delta = f^*(K_Z + f_*\Delta) \equiv 0.$$

Hence $f_*\Delta$ is big, $(Z, f_*\Delta)$ is ϵ -klt, and $K_Z + f_*\Delta \equiv 0$, that is, Z is of ϵ -Fano type.

Next we assume that f is small and Z is of ϵ -Fano type. Let Γ be an effective big \mathbb{Q} -divisor on Z such that (Z, Γ) is ϵ -klt and $K_Z + \Gamma \equiv 0$. Let Δ be the strict transform of Γ on Y . Then Δ is big since f is small. Again by f is small,

$$K_Y + \Delta = f^*(K_Z + \Gamma).$$

Hence (Y, Δ) is ϵ -klt and $K_Y + \Delta \equiv 0$. Hence Y is of ϵ -Fano type. □

Proof of Theorem 1.2. Fix $0 < \epsilon < 1$, an integer $n > 0$. Let X be a variety of ϵ -Fano type of dimension n , that is, by Lemma 3.1, there exists an effective \mathbb{Q} -divisor Δ such that Δ is big, (X, Δ) is ϵ -klt, and $K_X + \Delta \equiv 0$. By [6, Corollary 1.4.3], taking \mathbb{Q} -factorialization of (X, Δ) , we have a birational morphism $\phi : X_0 \rightarrow X$ where $K_{X_0} + \phi_*^{-1}\Delta = \phi^*(K_X + \Delta)$, X_0 is \mathbb{Q} -factorial, and ϕ is isomorphic in codimension one.

Again by [6, Corollary 1.4.3], taking terminalization of X_0 , we have a birational morphism $\pi : X_1 \rightarrow X_0$ where $K_{X_1} + \Delta_{X_1} = \pi^*(K_{X_0} + \phi_*^{-1}\Delta)$, $\Delta_{X_1} \geq \pi^*(\phi_*^{-1}\Delta)$ is an effective \mathbb{Q} -divisor, X_1 is \mathbb{Q} -factorial terminal. Here $K_{X_1} + \Delta_{X_1} \equiv 0$ and (X_1, Δ_{X_1}) is ϵ -klt. Since Δ is big and ϕ is small, $\Delta_{X_1} \geq \pi^*(\phi_*^{-1}\Delta)$ is big. Therefore, X_1 is \mathbb{Q} -factorial terminal and of ϵ -Fano type.

Running K -MMP on X_1 , we get a sequence of normal projective varieties:

$$X_1 \dashrightarrow X_2 \dashrightarrow X_3 \dashrightarrow \cdots \dashrightarrow X_r \rightarrow T.$$

Since $-K_{X_1}$ is big, this sequence ends up with a Mori fiber space $X_r \rightarrow T$ (cf. [6, Corollary 1.3.3]). Since we run K -MMP, X_r is again \mathbb{Q} -factorial terminal. By Lemma 3.2, for all i , X_i is of ϵ -Fano type. Now X_r is an n -dimensional variety of ϵ -Fano type with a Mori fiber structure by construction, which is birational to X . We complete the proof of Theorem 1.2. □

4. Proof of Theorem 1.4

In this section, for the reader's convenience, we recall the proof of Theorem 1.4 from [14]. The proof of Theorem 1.4 follows from Lemmas 4.3 and 4.4 below.

4.1. Setting

Fix $0 < \epsilon < 1$. Let X be a 3-fold of ϵ -Fano type with a Mori fibration $f : X \rightarrow Z$ and $\dim Z > 0$. Suppose (X, B) is an ϵ -klt log Fano pair. We will explain more about the surface G defined in Definition 1.5.

If $\dim Z = 1$, then $Z = \mathbb{P}^1$. In this case G is defined to be a general fiber of f , which is a smooth del Pezzo surface.

If $\dim Z = 2$, then we want to explain the choice of H first. We first claim that such Z forms a bounded family. Since X is of ϵ -Fano type, there exist effective \mathbb{Q} -divisors Δ and Δ' such that (Z, Δ) is klt, $-(K_Z + \Delta)$ is ample by [11, Corollary 3.3], and (Z, Δ') is δ -klt, $-(K_Z + \Delta') \sim_{\mathbb{Q}} 0$ by [5, Corollary 1.7]. Note that δ is a positive number depends only on ϵ . We may choose sufficiently small $t > 0$ such that $(Z, (1-t)\Delta' + t\Delta)$ is still δ -klt. In this case,

$$-(K_Z + (1-t)\Delta' + t\Delta) \sim -t(K_Z + \Delta)$$

is ample. Hence Z is of δ -Fano type. Hence by BAB Conjecture in dimension 2, such Z forms a bounded family. This means that there is a positive integer d_ϵ depending only on ϵ , and we may find a general very ample divisor H on Z such that $H^2 \leq d_\epsilon$. Now we take $G = f^*H$, which is a conic bundle over the curve H (i.e. $-K_G$ is ample over H). Note that H and G are smooth since H is general. Also $(G, B|_G)$ is ϵ -klt and $-(K_G + B|_G) + G|_G$ is ample by adjunction. Note that $G|_G = f^*(H|_H)$ is the sum of (H^2) fibers of f and $(H^2) \leq d_\epsilon$. Hence $-(K_G + B|_G) + d_\epsilon F$ is ample, where F is a general fiber of f .

4.2. Two boundedness theorems on surfaces

We recall two boundedness theorems on surfaces, the idea of proofs of them are from the proof of BAB Conjecture in dimension two by Alexeev–Mori [3].

Theorem 4.1 ([13, Theorem 2.8]). *Fix $0 < \epsilon < 1$. Then there exists a number $\mu(2, \epsilon) > 0$ depending only on ϵ with the following property:*

If (X, B) is an ϵ -klt log Fano pair and X is a smooth surface, then $\alpha(X, B) \geq \mu(2, \epsilon)$.

Theorem 4.2 ([14, Theorem 1.7]). *Fix $0 < \epsilon < 1$. Then there exists a number $\lambda(2, \epsilon) > 0$ depending only on ϵ , satisfying the following property:*

- 1) If (G, B) is an ϵ -klt log Fano pair and G is a smooth del Pezzo surface, then $(G, (1+t)B)$ is klt for $0 < t \leq \lambda(2, \epsilon)$;
- 2) If $f : G \rightarrow H$ is a conic bundle from a smooth surface G to a smooth curve, (G, B) is an ϵ -klt pair and $-(K_G + B) + d_\epsilon F$ is ample, then $(X, (1+t)B)$ is klt for $0 < t \leq \lambda(2, \epsilon)$. Here F is a general fiber of f and d_ϵ is the number depending only on ϵ defined in Subsection 4.1.

Note that in [14], such a conic bundle G in (2) is called a $(2, 1, d_\epsilon, \epsilon)$ -Fano fibration.

4.3. Boundedness of ampleness

Lemma 4.3 (cf. [14, Lemma 3.2]). *Keep the setting in Subsection 4.1, then there exists a positive integer N_ϵ depending only on ϵ such that $-K_X + kG$ is ample for all $k \geq N_\epsilon$.*

Proof. Let X be a 3-fold of ϵ -Fano type with a Mori fibration $f : X \rightarrow Z$ and $\dim Z > 0$. Suppose (X, B) is an ϵ -klt log Fano pair.

By our construction, in either case, $(G, B|_G)$ satisfies one of the two conditions in Theorem 4.2. Hence $(G, (1+\lambda)B|_G)$ is klt for $\lambda = \lambda(2, \epsilon)$.

Hence, in either case, every curve in $\text{Nklt}(X, (1+\lambda)B)$ is contracted by f by inversion of adjunction. In particular, every curve C_0 supported in $\text{Nklt}(X, (1+\lambda)B)$ satisfies that $G \cdot C_0 = 0$.

Now we consider an extremal ray R of $\overline{NE}(X)$. Since X is of Fano type, R is always generated by a rational curve by Cone Theorem.

If R is $(K_X + (1+\lambda)B)$ -non-negative, recall that $-(K_X + B)$ is ample, then

$$-K_X \cdot R = -\left(1 + \frac{1}{\lambda}\right)(K_X + B) \cdot R + \frac{1}{\lambda}(K_X + (1+\lambda)B) \cdot R > 0.$$

If R is $(K_X + (1+\lambda)B)$ -negative and $G \cdot R = 0$, then $-K_X \cdot R > 0$ since $-K_X$ is ample over Z and R is contracted by f .

If R is $(K_X + (1+\lambda)B)$ -negative and $G \cdot R > 0$, then every curve generating R is not supported in $\text{Nklt}(X, (1+\lambda)B)$. By Theorem 2.7, R is generated by a rational curve C such that

$$(K_X + (1+\lambda)B) \cdot C \geq -6.$$

On the other hand, $G \cdot C \geq 1$ since $G \cdot C > 0$ and G is Cartier. Hence

$$\begin{aligned} & \left(-K_X + \frac{6}{\lambda}G\right) \cdot C \\ &= -\left(1 + \frac{1}{\lambda}\right)(K_X + B) \cdot C + \frac{1}{\lambda}(K_X + (1 + \lambda)B) \cdot C + \frac{6}{\lambda}G \cdot C > 0. \end{aligned}$$

In summary,

$$(-K_X + kG) \cdot R > 0$$

holds for every extremal ray R and for all $k \geq \frac{6}{\lambda}$ (recall that G is nef). By Kleiman’s Ampleness Criterion, $-K_X + kG$ is ample for all $k \geq \frac{6}{\lambda}$. We may take

$$N_\epsilon = \frac{6}{\lambda(2, \epsilon)}$$

and complete the proof. □

4.4. Boundedness of volumes

Lemma 4.4 (cf. [14, Theorem 4.1]). *Keep the setting in Subsection 4.1, then there exists a positive number V_ϵ depending only on ϵ such that $(-K_X + N_\epsilon G)^3 \leq V_\epsilon$.*

Proof. Let X be a 3-fold of ϵ -Fano type with a Mori fibration $f : X \rightarrow Z$ and $\dim Z > 0$. Suppose (X, B) is an ϵ -klt log Fano pair.

If $\dim Z = 1$, G is a smooth del Pezzo surface and $(G, B|_G)$ is an ϵ -klt log Fano pair. Note that $\text{Vol}(G, -K_G) = K_G^2 \leq 9$. Assume that for some $w > 0$,

$$(-K_X + N_\epsilon G)^3 > 27(N_\epsilon + w).$$

It suffices to find an upper bound for w . We may assume that $w > 2$. By Lemma 2.1,

$$\text{Vol}(X, -K_X - wG) \geq \text{Vol}(X, -K_X + N_\epsilon G) - 3(N_\epsilon + w)\text{Vol}(G, -K_G) > 0.$$

Hence there exists an effective \mathbb{Q} -divisor $B' \sim_{\mathbb{Q}} -K_X - wG$. For two general fibers G_1 and G_2 , consider the pair $(X, (1 - \frac{2}{w})B + \frac{2}{w}B' + G_1 + G_2)$ where $2/w < 1$. Note that

$$-\left(K_X + \left(1 - \frac{2}{w}\right)B + \frac{2}{w}B' + G_1 + G_2\right) \sim_{\mathbb{Q}} -\left(1 - \frac{2}{w}\right)(K_X + B)$$

is ample. By Connectedness Lemma, $\text{Nklt}(X, (1 - \frac{2}{w})B + \frac{2}{w}B' + G_1 + G_2)$ is connected. On the other hand, it contains $G_1 \cup G_2$, hence contains a non-klt

center dominating Z . By inversion of adjunction, $(G, (1 - \frac{2}{w})B|_G + \frac{2}{w}B'|_G)$ is not klt for a general fiber G . On the other hand, $(G, B|_G)$ is an ϵ -klt log Fano pair of dimension 2, G is a del Pezzo surface, and $B'|_G - B|_G \sim_{\mathbb{Q}} -(K_G + B|_G)$. Hence by Theorem 4.1,

$$\frac{2}{w} \geq \text{glct}(G, B|_G; B'|_G - B|_G) \geq \mu(2, \epsilon).$$

Hence $w \leq \frac{2}{\mu(2, \epsilon)}$. In this case, we may take

$$V_\epsilon = 27\left(N_\epsilon + \frac{2}{\mu(2, \epsilon)}\right).$$

Now assume that $\dim Z = 2$. As constructed in Subsection 4.1, $G \rightarrow H$ is a conic bundle from a smooth surface to a smooth curve.

Claim 1. $\text{Vol}(G, -K_X|_G + N_\epsilon G|_G) \leq 8 + 4(N_\epsilon + 1)d_\epsilon$.

Proof of Claim 1. Note that $-K_X + N_\epsilon G$ is ample, so is $-K_X|_G + N_\epsilon G|_G$. Also note that $(H^2) \leq d_\epsilon$ and $G|_G = (H^2)F$ where $F \simeq \mathbb{P}^1$ is a general fiber of f . Hence

$$\begin{aligned} \text{Vol}(G, -K_X|_G + N_\epsilon G|_G) &= (-K_X|_G + N_\epsilon G|_G)^2 \\ &= (-K_G + (N_\epsilon + 1)G|_G)^2 \\ &= (K_G)^2 - 2(N_\epsilon + 1)K_G \cdot (H^2)F \\ &\leq 8 + 4(N_\epsilon + 1)d_\epsilon \end{aligned}$$

Here we use the fact that for the conic bundle G , $(K_G)^2 \leq 8$. □

Assume that for some $w > 0$,

$$(-K_X + N_\epsilon G)^3 > 3(N_\epsilon + w)(8 + 4(N_\epsilon + 1)d_\epsilon).$$

It suffices to find an upper bound for w . We may assume that $w > 3$. By Lemma 2.1 and Claim 1,

$$\begin{aligned} &\text{Vol}(X, -K_X - wG) \\ &\geq \text{Vol}(X, -K_X + N_\epsilon G) - 3(N_\epsilon + w)\text{Vol}(G, -K_X|_G + N_\epsilon G|_G) > 0. \end{aligned}$$

Hence there exists an effective \mathbb{Q} -divisor $B' \sim_{\mathbb{Q}} -K_X - wG$. For a general fiber F of X over $z \in Z$, there exists a number $\eta > 0$ (cf. [21, 4.8]) such that

for any general $H' \in |H|$ containing z ,

$$\text{Nklt}\left(X, \left(1 - \frac{3}{w}\right)B + \frac{3}{w}B'\right) = \text{Nklt}\left(X, \left(1 - \frac{3}{w}\right)B + \frac{3}{w}B' + \eta f^*(H')\right).$$

We may take general $H_j \in |H|$ containing z for $1 \leq j \leq J$ with $J > \frac{2}{\eta}$ and take $G_1 = \sum_{j=1}^J \frac{2}{j} f^*(H_j)$. Then $\text{mult}_F G_1 \geq 2$ and $G_1 \sim_{\mathbb{Q}} 2f^*(H) \sim_{\mathbb{Q}} 2G$. In particular, (X, G_1) is not klt at F and by construction, in a neighborhood of F ,

$$\begin{aligned} & \text{Nklt}\left(X, \left(1 - \frac{3}{w}\right)B + \frac{3}{w}B'\right) \cup F \\ &= \text{Nklt}\left(X, \left(1 - \frac{3}{w}\right)B + \frac{3}{w}B' + G_1\right). \end{aligned}$$

Take a general element $G_2 \in |f^*(H)|$ not containing F , consider the pair $(X, (1 - \frac{3}{w})B + \frac{3}{w}B' + G_1 + G_2)$ where $3/w < 1$. Then

$$-\left(K_X + \left(1 - \frac{3}{w}\right)B + \frac{3}{w}B' + G_1 + G_2\right) \sim_{\mathbb{Q}} -\left(1 - \frac{3}{w}\right)(K_X + B)$$

is ample. Since

$$F \cup G_2 \subset \text{Nklt}\left(X, \left(1 - \frac{3}{w}\right)B + \frac{3}{w}B' + G_1 + G_2\right),$$

by Connectedness Lemma, there is a curve C contained in $\text{Nklt}(X, (1 - \frac{3}{w})B + \frac{3}{w}B' + G_1 + G_2)$, intersecting F and not contracted by f . Hence C is contained in $\text{Nklt}(X, (1 - \frac{3}{w})B + \frac{3}{w}B')$ by the construction of G_1 and generality of G_2 . Since C intersects F , so does $\text{Nklt}(X, (1 - \frac{3}{w})B + \frac{3}{w}B')$. By inversion of adjunction, $(F, (1 - \frac{3}{w})B|_F + \frac{3}{w}B'|_F)$ is not klt for a general fiber F . On the other hand, $(F, B|_F)$ is ϵ -klt and $F \simeq \mathbb{P}^1$. Hence $\frac{3}{w} \geq \epsilon$ by comparing the coefficients of $\frac{3}{w}B'|_F$. Hence $w \leq \frac{3}{\epsilon}$. Hence we may take

$$V_{\epsilon} = 3\left(N_{\epsilon} + \frac{3}{\epsilon}\right)(8 + 4(N_{\epsilon} + 1)d_{\epsilon})$$

by definition of w . □

5. Proof of main theorems

5.1. Pseudo-effectivity of $c_2(\mathcal{F}_X)$

We recall a criterion of pseudo-effectivity of second Chern classes due to Miyaoka [30].

Definition 5.1 (cf. [30, Section 6]). Let X be an n -dimensional normal projective variety. A torsion free sheaf \mathcal{E} is called *generically semi-positive* (or *generically nef*) if one of the following equivalent conditions holds:

- 1) For every quotient torsion free sheaf $\mathcal{E} \rightarrow \mathcal{L}$ and any ample divisors H_i ,

$$c_1(\mathcal{L}) \cdot H_1 \cdot H_2 \cdots H_{n-1} \geq 0.$$

- 2) $\mathcal{E}|_C$ is nef for a general curve $C = D_1 \cap \cdots \cap D_{n-1}$ for general $D_i \in |m_i H_i|$ and $m_i \gg 0$ and any ample divisors H_i .

The equivalence of these two definitions follows from the Mehta–Ramanathan theorem [29] (cf. [30, Theorem 2.5]).

Theorem 5.2 ([30, Theorem 6.1]). *Let X be a normal projective variety which is smooth in codimension 2. Let \mathcal{E} be a torsion free sheaf on X such that*

- 1) $c_1(\mathcal{E})$ is a nef \mathbb{Q} -Cartier divisor, and
 2) \mathcal{E} is generically semi-positive.

Then $c_2(\mathcal{E})$ is pseudo-effective.

To check the generic semi-positivity of \mathcal{F}_X , it suffices to check that of \mathcal{T}_X , which is proved by the following theorem.

Theorem 5.3 ([24, Proof of 1.2 (1)]). *Let (X, B) be a \mathbb{Q} -factorial klt log Fano pair such that X is smooth in codimension 2. Then \mathcal{T}_X is generically semi-positive.*

This theorem is implicated by [24, Proof of 1.2 (1)], combining a structure theorem for the cone of nef curves (replacing [24, Theorem-Definition 2.2] by [6, Corollary 1.3.5]) and deformation theory of rational curves ([23, (1.3) Corollary]).

Proof of Theorem 1.6. Recall that X is of Fano type and with \mathbb{Q} -factorial terminal singularities (terminal singularities implies smooth in codimension 2). Since \mathcal{T}_X is generically semi-positive by Theorem 5.3 and G is nef, $\mathcal{F}_X = \mathcal{T}_X \oplus \mathcal{O}_X(N_\epsilon G)$ is again generically semi-positive. Since $c_1(\mathcal{F}_X) = -K_X + N_\epsilon G$ is ample, $c_2(\mathcal{F}_X)$ is pseudo-effective by Theorem 5.2. \square

5.2. Upper bound of Gorenstein indices

In this subsection, we prove Theorem 1.7. We start from the estimate of $(-K_X) \cdot c_2(X)$.

Proof of Theorem 1.7(1). Note that

$$c_2(\mathcal{F}_X) = c_2(\mathcal{T}_X \oplus \mathcal{O}_X(N_\epsilon G)) = c_2(X) - K_X \cdot N_\epsilon G.$$

By Theorem 1.6, $c_2(\mathcal{F}_X)$ is pseudo-effective, and thus

$$(-K_X + N_\epsilon G) \cdot (c_2(X) - K_X \cdot N_\epsilon G) \geq 0.$$

Hence

$$(-K_X) \cdot c_2(X) \geq -(-K_X + N_\epsilon G) \cdot (-K_X) \cdot N_\epsilon G - N_\epsilon G \cdot c_2(X).$$

It suffices to prove the following lemma.

Lemma 5.4. *The following inequalities hold:*

- 1) $(-K_X + N_\epsilon G) \cdot (-K_X) \cdot N_\epsilon G \leq V_\epsilon;$
- 2) $G \cdot c_2(X) \leq 12 + 4V_\epsilon/N_\epsilon.$

Proof. Recall that $-K_X$ is big, G is nef with $G^3 = 0$, and $-K_X + N_\epsilon G$ is ample with $(-K_X + N_\epsilon G)^3 \leq V_\epsilon$.

For statement (1),

$$\begin{aligned} & (-K_X + N_\epsilon G) \cdot (-K_X) \cdot N_\epsilon G \\ & \leq (-K_X + N_\epsilon G) \cdot (-K_X + N_\epsilon G) \cdot N_\epsilon G \\ & \leq (-K_X + N_\epsilon G)^3 \leq V_\epsilon. \end{aligned}$$

Now we prove statement (2).

If $\dim Z = 1$, then G is a del Pezzo surface and $G \cdot c_2(X) = c_2(G) \leq 11$.

If $\dim Z = 2$, by the exact sequence

$$0 \rightarrow \mathcal{T}_G \rightarrow \mathcal{T}_X|_G \rightarrow \mathcal{N}_{G/X} \rightarrow 0,$$

we have

$$\begin{aligned} G \cdot c_2(X) &= c_2(G) + c_1(G) \cdot c_1(\mathcal{N}_{G/X}) \\ &= 12\chi(\mathcal{O}_G) - K_G^2 - K_G \cdot G|_G. \end{aligned}$$

Note that G is a conic bundle over H , hence $\chi(\mathcal{O}_X) = 1 - g(H) \leq 1$. On the other hand,

$$\begin{aligned} -K_G^2 - K_G \cdot G|_G &= -(K_X + G)^2 \cdot G - (K_X + G) \cdot G^2 \\ &= -K_X \cdot (K_X + 3G) \cdot G \\ &\leq -K_X \cdot (N_\epsilon + 3)G \cdot G \\ &= (-K_X + N_\epsilon G) \cdot (N_\epsilon + 3)G^2 \\ &\leq (-K_X + N_\epsilon G) \cdot \frac{N_\epsilon + 3}{N_\epsilon^2} (-K_X + N_\epsilon G)^2 \\ &\leq \frac{N_\epsilon + 3}{N_\epsilon^2} V_\epsilon \leq \frac{4V_\epsilon}{N_\epsilon}. \end{aligned}$$

Hence $G \cdot c_2(X) \leq 12 + 4V_\epsilon/N_\epsilon$. □

By this lemma,

$$(-K_X) \cdot c_2(X) \geq -(5V_\epsilon + 12N_\epsilon).$$

Hence Theorem 1.7(1) is proved. □

By Reid’s Riemann–Roch formula, we can get the upper bound of Gorenstein indices. This method highly depends on the classification of 3-dimensional terminal singularities.

Proof of Theorem 1.7(2). Recall that X has at most terminal singularities. Recall that every terminal singularity in dimension 3 can be deformed to a collection of cyclic quotient terminal singularities. The *basket* of singularities of X is the set of all such cyclic quotient terminal singularities. We denote it by (b, r) if a cyclic quotient terminal singularity is of type $\frac{1}{r}(1, -1, b)$ for integers b and r . Hence a basket is a set (allowing multiplicities) of the form $\{(b_i, r_i) | i \in I\}$. Note that since r_i is the local index of singularities, $\text{l.c.m.}\{r_i\}K_X$ is a Cartier divisor. By Reid’s Riemann–Roch formula (cf. [15, Lemmas 2.2, 2.3] or [33, (10.3)]), we have

$$\chi(\mathcal{O}_X) = \frac{1}{24}(-K_X) \cdot c_2(X) + \frac{1}{24} \sum \left(r_i - \frac{1}{r_i} \right),$$

where r_i runs over the basket of singularities. Note that $\chi(\mathcal{O}_X) = 1$ by Kawamata–Viehweg vanishing theorem since X is of Fano type. Hence by

Theorem 1.7(1),

$$\sum \left(r_i - \frac{1}{r_i} \right) \leq 24 + M_\epsilon.$$

In particular, $r_i \leq 24 + M_\epsilon$. Hence $(24 + M_\epsilon)!$ is divisible by $\text{l.c.m.}\{r_i\}$, which implies that $(24 + M_\epsilon)!K_X$ is Cartier. \square

5.3. Proof of Theorem 1.3

Proof of Theorem 1.3. Let X be a 3-fold of ϵ -Fano type with a Mori fibration $f : X \rightarrow Z$. If $\dim Z = 0$, then X is a \mathbb{Q} -factorial terminal Fano 3-fold with Picard number one, which is bounded by Kawamata [17]. So we only need to consider the case when $\dim Z > 0$.

Keep the notation in Theorem 1.4. By Theorem 1.7, $L := r_\epsilon(-K_X + N_\epsilon G)$ is a Cartier ample divisor. Recall that X is of Fano type. By Kollár's effective base point free theorem (cf. [20, 1.1 Theorem, 1.2 Lemma]), $720L$ is base point free and $4321L$ is very ample. On the other hand, $L^3 \leq r_\epsilon^3 V_\epsilon$. Hence X is a subvariety of projective spaces with bounded degree. Such X forms a bounded family by the boundedness of Chow variety. \square

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