

A remark on a 3-fold constructed by Colliot-Thélène and Voisin

FUMIAKI SUZUKI

A classical question asks whether the Abel-Jacobi map is universal among all regular homomorphisms. In this paper, we prove that we can construct a 4-fold which gives the negative answer in codimension 3 if the generalized Bloch conjecture holds for a 3-fold constructed by Colliot-Thélène and Voisin in the context of the study of the defect of the integral Hodge conjecture in degree 4.

1. Introduction

For a smooth complex projective variety X , we denote by $CH^p(X)$ the p -th Chow group of X and by $A^p(X)$ (resp. $CH^p(X)_{\text{hom}}$) the subgroup of cycle classes algebraically equivalent to zero (resp. homologous to zero). A homomorphism $\phi: A^p(X) \rightarrow A$ to an Abelian variety A is called *regular* if for any smooth connected projective variety S with a base point s_0 and for any codimension p cycle Γ on $S \times X$, the composition

$$S \rightarrow A^p(X) \rightarrow A, s \mapsto \phi(\Gamma_*(s - s_0))$$

is a morphism of algebraic varieties. An important example of such homomorphisms is given as follows. We consider the Abel-Jacobi map

$$CH^p(X)_{\text{hom}} \rightarrow J^p(X),$$

where $J^p(X) = H^{2p-1}(X, \mathbb{C}) / (H^{2p-1}(X, \mathbb{Z}(p)) + F^p H^{2p-1}(X, \mathbb{C}))$ is the p -th Griffiths intermediate Jacobian. Then the image $J_a^p(X) \subset J^p(X)$ of the restriction of the Abel-Jacobi map to $A^p(X)$ is an Abelian variety, and the induced map

$$\psi^p: A^p(X) \rightarrow J_a^p(X),$$

which we also call Abel-Jacobi, is regular [9][11]. A classical question [16, Section 7] asks whether the Abel-Jacobi map $\psi^p: A^p(X) \rightarrow J_a^p(X)$ is *universal* among all regular homomorphisms $\phi: A^p(X) \rightarrow A$, that is, whether every

such ϕ factors through ψ^p . It is true for $p = 1$ by the theory of Picard variety, and for $p = \dim X$ by the theory of Albanese variety. It is also true for $p = 2$, which is proved by Murre [14][15] using the Merkurjev-Suslin theorem [13]. To the author’s knowledge, this question is open for $3 \leq p \leq \dim X - 1$.

Some progress has been made. As an application of the theory of the Lawson homology and the morphic cohomology, the following theorem is proved by Walker [26]: the Abel-Jacobi map ψ^p factors as

$$\begin{array}{ccc}
 & J(N^{p-1}H^{2p-1}(X, \mathbb{Z}(p))) , & \\
 & \nearrow \tilde{\psi}^p & \downarrow \pi^p \\
 A^p(X) & \xrightarrow{\psi^p} & J_a^p(X)
 \end{array}$$

where $J(N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)))$ is the intermediate Jacobian for the mixed Hodge structure given by the coniveau filtration $N^{p-1}H^{2p-1}(X, \mathbb{Z}(p))$, π^p is the natural map, and $\tilde{\psi}^p$ is a surjective regular homomorphism. If the Abel-Jacobi map ψ^p is universal, then the kernel

$$\begin{aligned}
 \text{Ker}(\pi^p) &= \text{Coker} (H^{2p-1}(X, \mathbb{Z}(p))_{\text{tors}} \\
 &\rightarrow (H^{2p-1}(X, \mathbb{Z}(p))/N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)))_{\text{tors}})
 \end{aligned}$$

is trivial. In other words, the sublattice

$$N^{p-1}H^{2p-1}(X, \mathbb{Z}(p))/\text{tors} \subset H^{2p-1}(X, \mathbb{Z}(p))/\text{tors}$$

is primitive.

We prove that the converse also holds for any $p \in \{3, \dim X - 1\}$ if the Chow group $CH_0(X)$ of 0-cycles is supported on a 3-dimensional closed subset. It is a consequence of an analogue of the Roitman theorem [18, Theorem 3.1]:

Theorem 1.1. *Let Y be a smooth projective variety such that $CH_0(Y)$ is supported on a 3-dimensional closed subset. Let $p \in \{3, \dim Y - 1\}$. Then the restriction*

$$\tilde{\psi}^p|_{\text{tors}} : A^p(Y)_{\text{tors}} \rightarrow J(N^{p-1}H^{2p-1}(Y, \mathbb{Z}(p)))_{\text{tors}}$$

is an isomorphism. Moreover the Walker map $\tilde{\psi}^p$ is universal.

Remark 1.2. There is a 4-fold Y such that $A^3(Y)$ has infinite l -torsion for some prime l [20]. Therefore the assumption on $CH_0(Y)$ is essential.

Then we prove that we can construct a 4-fold which does not meet the primitiveness condition for $p = 3$ and gives the negative answer to the universality question in codimension 3 if the generalized Bloch conjecture holds for a 3-fold constructed by Colliot-Thélène and Voisin in the context of the study of the defect

$$Z^4(X) = Hdg^4(X, \mathbb{Z})/H_{\text{alg}}^4(X, \mathbb{Z}(2))$$

of the integral Hodge conjecture in degree 4. The 3-fold is constructed as an example which conjecturally gives the negative answer to their question [8, Section 6] asking whether $Z^4(X)$ is trivial for a 3-fold X such that $CH_0(X)$ is supported on a proper closed subset. It is constructed as follows [8, Subsection 5.7]. Let $G = \mathbb{Z}/5$. We fix a generator $g \in G$, then G acts on $\mathbb{P}^1 = \text{Proj } \mathbb{C}[X, Y]$ and $\mathbb{P}^3 = \text{Proj } \mathbb{C}[X_0, \dots, X_3]$ by

$$g^*X = X, g^*Y = \zeta Y, g^*X_i = \zeta^i X_i \ (i = 0, \dots, 3),$$

where ζ is a primitive 5-th root of unity. Let $H \subset \mathbb{P}^1 \times \mathbb{P}^3$ be the hypersurface defined by a very general G -invariant polynomial of type $(3, 4)$, and X be a resolution of H/G . Then X satisfies the following two properties [8, Proposition 5.7]:

- (i) $H^p(X, \mathcal{O}_X) = 0$ for all $p > 0$;
- (ii) there is a class $\alpha \in H^4(X, \mathbb{Z}(2)) = Hdg^4(X, \mathbb{Z})$ such that $\alpha \cdot F = 5$, but $C \cdot F$ is even for any curves $C \subset X$, where F is the class of fibers of the morphism $X \rightarrow \mathbb{P}^1$ induced by the first projection $H \rightarrow \mathbb{P}^1$.

The property (i) implies $CH_0(X) = \mathbb{Z}$ if the generalized Bloch conjecture [23, Conjecture 11.23] holds for X . The property (ii) implies more than $Z^4(X) \neq 0$. We have $\text{Coker}(H^4(X, \mathbb{Z}(2))_{\text{tors}} \rightarrow Z^4(X)) \neq 0$, where a non-zero element is given by the image of α . Equivalently, the sublattice

$$H_{\text{alg}}^4(X, \mathbb{Z}(2))/\text{tors} \subset Hdg^4(X, \mathbb{Z})/\text{tors}$$

is not primitive. Our main theorem is:

Theorem 1.3. *Let X be a smooth projective variety such that $CH_0(X)$ is supported on a 2-dimensional closed subset and the sublattice*

$$H_{\text{alg}}^4(X, \mathbb{Z}(2))/\text{tors} \subset Hdg^4(X, \mathbb{Z})/\text{tors}$$

is not primitive. Then there exists a smooth elliptic curve E such that the sublattice

$$N^2 H^5(X \times E, \mathbb{Z}(3))/\text{tors} \subset H^5(X \times E, \mathbb{Z}(3))/\text{tors}$$

is not primitive and the Abel-Jacobi map $\psi^3: A^3(X \times E) \rightarrow J_a^3(X \times E)$ is not universal.

Corollary 1.4. *Let X be the 3-fold constructed by Colliot-Thélène and Voisin. Assume that the generalized Bloch conjecture holds for X . Then there exists a smooth elliptic curve E such that the sublattice*

$$N^2 H^5(X \times E, \mathbb{Z}(3))/\text{tors} \subset H^5(X \times E, \mathbb{Z}(3))/\text{tors}$$

is not primitive and the Abel-Jacobi map $\psi^3: A^3(X \times E) \rightarrow J_a^3(X \times E)$ is not universal.

This paper is organized as follows. In Section 2, we study regular homomorphisms on the torsion subgroup $A^p(X)_{\text{tors}}$. Then we prove Theorem 1.1 and its corollary. In Section 3, we prove a proposition on non-algebraic integral Hodge classes of Kollár type and torsion elements in the Abel-Jacobi kernel. Then we conclude Theorem 1.3 and its “homology counterpart”. In Section 4, we discuss stable birational invariants related to our problem. In Appendix A, we prove the Roitman theorem for the Walker maps by using the formalism of decomposition of the diagonal.

Throughout this paper, the base field is the field of complex numbers \mathbb{C} .

Notation. For a smooth projective variety X , we denote by $N^i H^j(X, \mathbb{Z}(k))$ the coniveau filtration on $H^j(X, \mathbb{Z}(k))$. Recall that it is defined as

$$N^i H^j(X, \mathbb{Z}(k)) = \text{Ker} \left(H^j(X, \mathbb{Z}(k)) \rightarrow \varinjlim_{Z \subset X} H^j(X - Z, \mathbb{Z}(k)) \right),$$

where $Z \subset X$ runs through all codimension $\geq i$ closed subsets of X . For an Abelian group G and a prime number l , we denote by $G_{l\text{-tors}}$ the subgroup of l -primary torsion elements of G .

2. Regular homomorphisms on the torsion subgroup $A^p(X)_{\text{tors}}$

Lemma 2.1. *Let X be a smooth projective variety and $\phi: A^p(X) \rightarrow A$ be a surjective regular homomorphism. Assume that the restriction $\phi|_{\text{tors}}: A^p(X)_{\text{tors}} \rightarrow A_{\text{tors}}$ is an isomorphism. Then ϕ is universal.*

Proof. First we prove the existence of a universal regular homomorphism $\phi_0: A^p(X) \rightarrow A_0$. By Saito’s criterion [19, Theorem 2.2] (see also [15, Proposition 2.1]), it is enough to prove $\dim B \leq \dim A$ for any surjective regular homomorphism $\psi: A^p(X) \rightarrow B$. Such a homomorphism ψ restricts to a surjection $\psi|_{\text{tors}}: A^p(X)_{\text{tors}} \rightarrow B_{\text{tors}}$. Indeed, by [19, Proposition 1.2] (see also [15, Lemma 1.6.2]), there exists an Abelian variety C and $\Gamma \in CH^p(C \times X)$ such that the composition

$$C \rightarrow A^p(X) \rightarrow B, s \mapsto \psi(\Gamma_*(s - s_0))$$

is an isogeny, and the restriction $C_{\text{tors}} \rightarrow B_{\text{tors}}$ is a surjection. By assumption, we have $A^p(X)_{\text{tors}} \cong A_{\text{tors}}$. Then we have

$$\dim B = \frac{1}{2} \text{co-rank } B_{\text{tors}} \leq \frac{1}{2} \text{co-rank } A_{\text{tors}} = \dim A.$$

The existence follows.

The map ϕ_0 is surjective since the image of a regular homomorphism is an Abelian variety [15, Lemma 1.6.2]. Thus ϕ_0 restricts to a surjection $\phi_0|_{\text{tors}}: A^p(X)_{\text{tors}} \rightarrow (A_0)_{\text{tors}}$. The induced map $A_0 \rightarrow A$ is surjective and restricts to an isomorphism $(A_0)_{\text{tors}} \cong A_{\text{tors}}$, therefore it is an isomorphism. The proof is done. □

We review the Bloch-Ogus theory on the coniveau spectral sequence [4]. For a smooth projective variety X , we define $\mathcal{H}^q(\mathbb{Z}(r))$ to be the Zariski sheaf on X associated to the presheaf $U \mapsto H^q(U, \mathbb{Z}(r))$. Then the E_2 term of the coniveau spectral sequence is given by

$$E_2^{p,q} = H^p(X, \mathcal{H}^q(\mathbb{Z}(r))) \Rightarrow N^\bullet H^{p+q}(X, \mathbb{Z}(r)),$$

and we have $E_2^{p,q} = 0$ if $p > q$ [4, Corollary 6.3]. We also have

$$E_2^{p,q} = 0 \text{ if } (p, q) \notin [0, \dim X] \times [0, \dim X].$$

Let $f^p: H^{p-1}(X, \mathcal{H}^p(\mathbb{Z}(p))) \rightarrow H^{2p-1}(X, \mathbb{Z}(p))$ be the edge homomorphism.

Lemma 2.2. *There is a short exact sequence:*

$$\begin{aligned}
 0 &\rightarrow H^{p-1}(X, \mathcal{K}_p) \otimes \mathbb{Q}_l/\mathbb{Z}_l \\
 &\rightarrow \text{Ker} (f^p \otimes \mathbb{Q}_l/\mathbb{Z}_l: H^{p-1}(X, \mathcal{H}^p(\mathbb{Z}(p))) \otimes \mathbb{Q}_l/\mathbb{Z}_l \\
 &\quad \rightarrow N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)) \otimes \mathbb{Q}_l/\mathbb{Z}_l) \\
 &\rightarrow \text{Ker} \left(\tilde{\psi}^p|_{l\text{-tors}}: A^p(X)_{l\text{-tors}} \rightarrow J(N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)))_{l\text{-tors}} \right) \rightarrow 0
 \end{aligned}$$

for any smooth projective variety X and any prime number l , where \mathcal{K}_p is the Zariski sheaf on X associated to the Quillen K -theory.

Proof. We use the Bloch map $\lambda_l^p: CH^p(X)_{l\text{-tors}} \rightarrow H^{2p-1}(X, \mathbb{Q}_l/\mathbb{Z}_l(p))$ [2] (see also [6]). By the construction of the Bloch map and [12, Theorem 5.1], we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^{p-1}(X, \mathcal{K}_p) \otimes \mathbb{Q}_l/\mathbb{Z}_l & \longrightarrow & H^{p-1}(X, \mathcal{H}^p(\mathbb{Z}(p))) \otimes \mathbb{Q}_l/\mathbb{Z}_l & \longrightarrow & A^p(X)_{l\text{-tors}} \longrightarrow 0 \\
 & & \downarrow & & \downarrow f^p \otimes \mathbb{Q}_l/\mathbb{Z}_l & & \downarrow -\lambda_l^p \\
 & & 0 & \longrightarrow & H^{2p-1}(X, \mathbb{Z}(p)) \otimes \mathbb{Q}_l/\mathbb{Z}_l & \longrightarrow & H^{2p-1}(X, \mathbb{Q}_l/\mathbb{Z}_l(p))
 \end{array}$$

We prove that it induces another commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^{p-1}(X, \mathcal{K}_p) \otimes \mathbb{Q}_l/\mathbb{Z}_l & \longrightarrow & H^{p-1}(X, \mathcal{H}^p(\mathbb{Z}(p))) \otimes \mathbb{Q}_l/\mathbb{Z}_l & \longrightarrow & A^p(X)_{l\text{-tors}} \longrightarrow 0 \\
 & & \downarrow & & \downarrow f^p \otimes \mathbb{Q}_l/\mathbb{Z}_l & & \downarrow -\tilde{\lambda}_l^p \\
 & & 0 & \longrightarrow & N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)) \otimes \mathbb{Q}_l/\mathbb{Z}_l & \xlongequal{\quad} & N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)) \otimes \mathbb{Q}_l/\mathbb{Z}_l \longrightarrow 0
 \end{array}$$

It is enough to prove the image of

$$H^{p-1}(X, \mathcal{K}_p) \otimes \mathbb{Q}_l/\mathbb{Z}_l \quad \text{in} \quad N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)) \otimes \mathbb{Q}_l/\mathbb{Z}_l$$

is zero. This follows by observing that $H^{p-1}(X, \mathcal{K}_p) \otimes \mathbb{Q}_l/\mathbb{Z}_l$ is divisible and

$$\begin{aligned}
 &\text{Ker} (N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)) \otimes \mathbb{Q}_l/\mathbb{Z}_l \rightarrow H^{2p-1}(X, \mathbb{Z}(p)) \otimes \mathbb{Q}_l/\mathbb{Z}_l) \\
 &= \text{Coker} (H^{2p-1}(X, \mathbb{Z}(p))_{l\text{-tors}} \\
 &\quad \rightarrow (H^{2p-1}(X, \mathbb{Z}(p))/N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)))_{l\text{-tors}})
 \end{aligned}$$

is finite. We prove that $\tilde{\lambda}^p$ coincides with the restriction $\tilde{\psi}^p|_{l\text{-tors}}$. In commutative triangles

$$\begin{array}{ccc}
 & N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)) \otimes \mathbb{Q}_l/\mathbb{Z}_l, & \\
 \tilde{\lambda}_l^p \text{ (resp. } \tilde{\psi}^p|_{l\text{-tors}}) \nearrow & & \downarrow \\
 A^p(X)_{l\text{-tors}} \xrightarrow{\lambda_l^p \text{ (resp. } \psi^p|_{l\text{-tors}})} & H^{2p-1}(X, \mathbb{Z}(p)) \otimes \mathbb{Q}_l/\mathbb{Z}_l &
 \end{array}$$

$\tilde{\lambda}_l^p$ (resp. $\tilde{\psi}^p|_{l\text{-tors}}$) is the unique lift of λ_l^p (resp. $\psi^p|_{l\text{-tors}}$) since $A^p(X)_{l\text{-tors}}$ is divisible [4, Lemma 7.10]. Therefore it is enough to prove that λ_l^p coincides with $\psi^p|_{l\text{-tors}}$. This follows from [2, Proposition 3.7]. The proof is done by the snake lemma. \square

Proof of Theorem 1.1. The second statement follows from the first one by Lemma 2.1.

We prove that $\tilde{\psi}^3|_{\text{tors}}$ is an isomorphism. By Lemma 2.2, it is enough to prove that

$$\text{Ker}(f^3) = \text{Im} (H^0(Y, \mathcal{H}^4(\mathbb{Z}(3))) \rightarrow H^2(Y, \mathcal{H}^3(\mathbb{Z}(3))))$$

is torsion. The group $H^0(Y, \mathcal{H}^4(\mathbb{Z}(3)))$ is torsion by [8, Proposition 3.3 (i)] (it is actually zero as a consequence of the Bloch-Kato conjecture, see [8, Theorem 3.1]), so the result follows.

Let $d = \dim Y$. We prove that $\tilde{\psi}^{d-1}|_{\text{tors}}$ is an isomorphism. By Lemma 2.2, it is enough to prove that

$$\text{Ker}(f^{d-1}) = \text{Im} \left(H^{d-4}(Y, \mathcal{H}^d(\mathbb{Z}(d-1))) \rightarrow H^{d-2}(Y, \mathcal{H}^{d-1}(\mathbb{Z}(d-1))) \right)$$

is torsion. The group $H^{d-4}(Y, \mathcal{H}^d(\mathbb{Z}(d-1)))$ is torsion by [8, Proposition 3.3 (ii)], so the result follows. \square

Corollary 2.3. *Under the assumptions of Theorem 1.1, the following are equivalent:*

- (i) *the Abel-Jacobi map ψ^p is universal;*
- (ii) *the sublattice $N^{p-1}H^{2p-1}(Y, \mathbb{Z}(p))/\text{tors} \subset H^{2p-1}(Y, \mathbb{Z}(p))/\text{tors}$ is primitive;*
- (iii) *the restriction $\psi^p|_{\text{tors}}: A^p(Y)_{\text{tors}} \rightarrow J_a^p(Y)_{\text{tors}}$ is an isomorphism.*

Proof. It is enough to prove that (ii) and (iii) are equivalent. By Theorem 1.1, we have an isomorphism

$$\text{Ker}(\pi^p) \cong \text{Ker}(\psi^p|_{\text{tors}}: A^p(Y)_{\text{tors}} \rightarrow J_a^p(Y)_{\text{tors}}).$$

The result follows. □

3. Non-algebraic integral Hodge classes of Kollár type and torsion elements in the Abel-Jacobi kernel

Inspired by the work of Soulé and Voisin [21], we prove:

Proposition 3.1. *Let X be a smooth projective variety such that the sublattice*

$$H_{\text{alg}}^{2p}(X, \mathbb{Z}(p))/\text{tors} \subset \text{Hdg}^{2p}(X, \mathbb{Z})/\text{tors}$$

is not primitive. Then there exists a smooth elliptic curve E such that the restriction

$$\psi^{p+1}|_{\text{tors}}: A^{p+1}(X \times E)_{\text{tors}} \rightarrow J_a^{p+1}(X \times E)_{\text{tors}}$$

is not an isomorphism.

Remark 3.2. The assumption of Proposition 3.1 for $p = 2$ is satisfied by Kollár’s example [1, p.134, Lemma] (see also [21, Section 2]). It is a very general hypersurface in \mathbb{P}^4 of degree l^3 for a prime number $l \geq 5$. When it contains a certain smooth degree l curve, the same conclusion follows from [21, Theorem 4]. The details are given in [21, Section 4].

Proof of Proposition 3.1. We define

$$\overline{Z}^{2p}(X) = \text{Coker} \left(H^{2p}(X, \mathbb{Z}(p))_{\text{tors}} \rightarrow \text{Hdg}^{2p}(X, \mathbb{Z})/H_{\text{alg}}^{2p}(X, \mathbb{Z}(p)) \right).$$

Then we have the following exact sequence:

$$0 \rightarrow \overline{Z}^{2p}(X)_{\text{tors}} \rightarrow H_{\text{alg}}^{2p}(X, \mathbb{Z}(p)) \otimes \mathbb{Q}/\mathbb{Z} \rightarrow \text{Hdg}^{2p}(X, \mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z}.$$

We have $\overline{Z}^{2p}(X)_{\text{tors}} \neq 0$ by the assumption. Let $\alpha \in \overline{Z}^{2p}(X)_{\text{tors}}$ be a non-trivial element; we use the same notation for its image in $H_{\text{alg}}^{2p}(X, \mathbb{Z}(p)) \otimes$

\mathbb{Q}/\mathbb{Z} . Let $\tilde{\alpha} \in CH^p(X) \otimes \mathbb{Q}/\mathbb{Z}$ be an element which maps to α via the surjection

$$cl^p \otimes \mathbb{Q}/\mathbb{Z}: CH^p(X) \otimes \mathbb{Q}/\mathbb{Z} \rightarrow H_{\text{alg}}^{2p}(X, \mathbb{Z}(p)) \otimes \mathbb{Q}/\mathbb{Z}.$$

Let $k \subset \mathbb{C}$ be an algebraically closed field such that $\text{tr. deg}_{\mathbb{Q}} k < \infty$ and both X and $\tilde{\alpha}$ are defined over k . Let E be a smooth elliptic curve such that $j(E) \notin k$. We fix one component \mathbb{Q}/\mathbb{Z} of $CH^1(E)_{\text{tors}} = (\mathbb{Q}/\mathbb{Z})^2$, and we identify $\tilde{\alpha}$ with an element in $CH^p(X) \otimes CH^1(E)_{\text{tors}}$. By the Schoen theorem [20, Theorem 0.2], the image β of $\tilde{\alpha}$ by the exterior product map

$$CH^p(X) \otimes CH^1(E)_{\text{tors}} \rightarrow CH^{p+1}(X \times E)$$

is non-zero. Then $\beta \in A^{p+1}(X \times E)_{\text{tors}}$. We prove

$$\beta \in \text{Ker}(\psi^{p+1}: A^{p+1}(X \times E) \rightarrow J_a^{p+1}(X \times E)).$$

It is enough to prove that β is in the kernel of the cycle class map of the Deligne cohomology:

$$cl_{\mathcal{D}}^{p+1}: CH^{p+1}(X \times E) \rightarrow H_{\mathcal{D}}^{2p+2}(X \times E, \mathbb{Z}(p+1)).$$

The composition of $cl_{\mathcal{D}}^{p+1}$ with the exterior product map factors through

$$cl_{\mathcal{D}}^p \otimes cl_{\mathcal{D}}^1: CH^p(X) \otimes CH^1(E)_{\text{tors}} \rightarrow H_{\mathcal{D}}^{2p}(X, \mathbb{Z}(p)) \otimes H_{\mathcal{D}}^2(E, \mathbb{Z}(1))_{\text{tors}}.$$

Now it is enough to prove that $\tilde{\alpha}$ is in the kernel of this map. Since we have an isomorphism

$$H_{\mathcal{D}}^{2p}(X, \mathbb{Z}(p)) \otimes H_{\mathcal{D}}^2(E, \mathbb{Z}(1))_{\text{tors}} \cong \text{Hdg}^{2p}(X, \mathbb{Z}) \otimes H_{\mathcal{D}}^2(E, \mathbb{Z}(1))_{\text{tors}},$$

the proof is done by the choice of $\tilde{\alpha}$. □

Proof of Theorem 1.3. For any smooth projective curve E , the group $CH_0(X \times E)$ is supported on a 3-dimensional closed subset. The proof is done by applying Corollary 2.3 for $p = 3$ to $Y = X \times E$ and Proposition 3.1 for $p = 2$. □

The same arguments yield a “homology counterpart” of Theorem 1.3:

Theorem 3.3. *Let X be a smooth projective variety such that $CH_0(X)$ is supported on a 2-dimensional closed subset and the sublattice*

$$H_{2,\text{alg}}(X, \mathbb{Z}(1))/\text{tors} \subset Hdg_2(X, \mathbb{Z})/\text{tors}$$

is not primitive. Then there exists a smooth elliptic curve E such that the sublattice

$$N_2H_3(X \times E, \mathbb{Z}(1))/\text{tors} \subset H_3(X \times E, \mathbb{Z}(1))/\text{tors}$$

is not primitive and the Abel-Jacobi map $\psi_1: A_1(X \times E) \rightarrow J_{1,a}(X \times E)$ is not universal.

4. Stable birational invariants

Lemma 4.1. *The groups $\text{Ker}(\psi^3|_{\text{tors}})$, $\text{Ker}(\psi^{d-1}|_{\text{tors}})$, $\text{Ker}(\pi^3)$, and $\text{Ker}(\pi^{d-1})$, where $d = \dim X$, are stable birational invariants of smooth projective varieties X .*

Remark 4.2. A related result is proved by Voisin [24, Lemma 2.2].

Proof of Lemma 4.1. For each group, it is enough to check

- (i) the invariance under the blow-up along a smooth subvariety;
- (ii) the invariance under taking the product with \mathbb{P}^n .

By the formulas under these operations for the Chow groups and the Deligne cohomology groups (resp. the coniveau spectral sequence and the integral cohomology groups) and by their compatibility with the cycle class maps (resp. the differentials and the edge homomorphisms), (i) and (ii) are reduced to the triviality of the groups $\text{Ker}(\psi^i|_{\text{tors}})$ and $\text{Ker}(\pi^i)$ for $i \leq 2$ and $i = \dim Y$ on a smooth projective variety Y . The triviality of $\text{Ker}(\psi^2|_{\text{tors}})$ (resp. $\text{Ker}(\psi^{\dim Y}|_{\text{tors}})$) follows from the Roitman theorem for codimension 2-cycles due to Murre [15, Theorem 10.3] (resp. the Roitman theorem [18, Theorem 3.1]). The triviality of $\text{Ker}(\pi^2)$ (resp. $\text{Ker}(\pi^{\dim Y})$) follows from the universality of ψ^2 (resp. $\psi^{\dim Y}$). The rest is clear. The proof is done. \square

Corollary 4.3. *Let X be a smooth projective stably rational variety. Let $p \in \{3, \dim X - 1\}$. Then $\text{Ker}(\psi^p|_{\text{tors}}) = \text{Ker}(\pi^p) = 0$. Therefore the Abel-Jacobi map ψ^p is universal and the sublattice*

$$N^{p-1}H^{2p-1}(X, \mathbb{Z}(p))/\text{tors} \subset H^{2p-1}(X, \mathbb{Z}(p))/\text{tors}$$

is primitive.

For a smooth projective variety X , let

$$Z^{2p}(X) = \text{Hdg}^{2p}(X, \mathbb{Z}) / H_{\text{alg}}^{2p}(X, \mathbb{Z}(p))$$

be the defect of the integral Hodge conjecture in degree $2p$. We define

$$\overline{Z}^{2p}(X) = \text{Coker} (H^{2p}(X, \mathbb{Z}(p))_{\text{tors}} \rightarrow Z^{2p}(X)).$$

Then $\overline{Z}^{2p}(X)_{\text{tors}} = 0$ if and only if the sublattice

$$H_{\text{alg}}^{2p}(X, \mathbb{Z}(p)) / \text{tors} \subset \text{Hdg}^{2p}(X, \mathbb{Z}) / \text{tors}$$

is primitive.

Lemma 4.4. *The groups $\overline{Z}^4(X)$ and $\overline{Z}^{2d-2}(X)$, where $d = \dim X$, are stable birational invariants of smooth projective varieties X .*

Remark 4.5. The groups $Z^4(X)$ and $Z^{2d-2}(X)$, where $d = \dim X$, are stable birational invariants of smooth projective varieties X [22][25] and related to the unramified cohomology groups [8].

Proof of Lemma 4.4. The proof is reduced to the triviality of the groups $\overline{Z}^2(Y)$ and $\overline{Z}^{2\dim Y}(Y)$ on a smooth projective variety Y . The triviality of $\overline{Z}^2(Y)$ follows from the Lefschetz (1, 1)-theorem. The triviality of $\overline{Z}^{2\dim Y}(Y)$ is clear. The proof is done. \square

We recall the following question (see [8, Subsection 5.6]):

Question 4.6. *Let X be a smooth projective rationally connected variety. Then is the group $\overline{Z}^4(X)$ trivial? Equivalently, is the sublattice $H_{\text{alg}}^4(X, \mathbb{Z}(2)) / \text{tors} \subset \text{Hdg}^4(X, \mathbb{Z}) / \text{tors}$ primitive?*

The negative answer to this question would provide us with another example to which we can apply Theorem 1.3. We note that there is a unirational 6-fold X constructed by Colliot-Thélène and Ojanguren [7] with $Z^4(X)$ later proved to be non-zero [8, Theorem 1.3]. The 6-fold X is a smooth model of a quadric bundle Y over \mathbb{P}^3 . It is hard to analyze $\overline{Z}^4(X)$ while the construction of Y is explicit.

Appendix A. The Roitman theorem for the Walker maps and decomposition of the diagonal

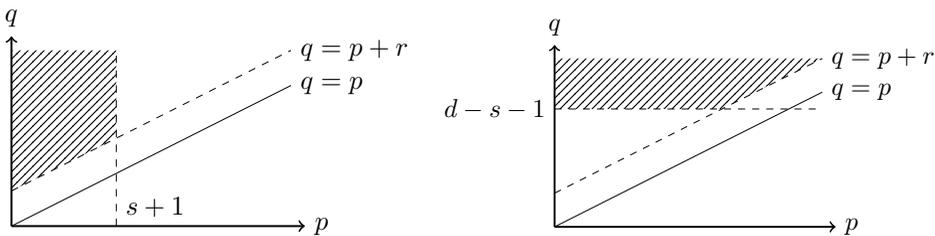
A smooth projective variety with $CH_0(X)$ supported on a proper closed subset admits a decomposition of the diagonal due to Bloch [3] and Bloch-Srinivas [5]. This result is generalized by Paranjape [17] and Laterveer [10]. We follow Laterveer’s formulation here. Let X be a smooth projective variety of dimension d . For non-negative integers r and s , we consider the following condition: $CH_i(X)_{\mathbb{Q}}$ is supported on an $(i + r)$ -dimensional closed subset for any $0 \leq i \leq s$. We call this condition $L_{r,s}$. Assume that $L_{r,s}$ holds for X . Then X admits a generalized decomposition of the diagonal [10, Theorem 1.7] (see also [17, Proposition 6.1]): there exist closed subsets V_0, \dots, V_s and W_0, \dots, W_{s+1} of X with $\dim V_j \leq j + r$ ($j = 0, \dots, s$) and $\dim W_j \leq d - j$ ($j = 0, \dots, s + 1$) such that we have a decomposition

$$\Delta_X = \Delta_0 + \dots + \Delta_s + \Delta^{s+1}$$

in $CH^d(X \times X)_{\mathbb{Q}}$, where Δ_j is supported on $V_j \times W_j$ ($j = 0, \dots, s$) and Δ^{s+1} is supported on $X \times W_{s+1}$.

For a smooth projective variety Y , let $E_2^{p,q}(Y) = H^p(Y, \mathcal{H}^q(\mathbb{Z}))$. For the action of correspondences on the coniveau spectral sequence, we refer the reader to [8, Appendice A].

Lemma A.1. *Let X be a smooth projective variety of dimension d such that $L_{r,s}$ holds for X . Then $E_2^{p,q}(X)$ is torsion if $p + r < q$ and $p < s + 1$, or if $p + r < q$ and $q > d - s - 1$.*



Remark A.2. The case $s = 0$ is [8, Proposition 3.3 (i)(ii)].

Proof of Lemma A.1. We may assume that the inequalities about the dimensions of V_j, W_j are equal. Let N be a positive integer such that

$$N\Delta_X = N\Delta_0 + \dots + N\Delta_s + N\Delta^{s+1} \in CH^d(X \times X).$$

Let $\widetilde{V}_j (j = 0, \dots, s)$ and $\widetilde{W}_j (j = 0, \dots, s + 1)$ be resolutions of V_j and W_j , and $\widetilde{\Delta}_j$ be d -cycles on $\widetilde{V}_j \times \widetilde{W}_j$ pushed forward to $c_j \Delta_j$ for some positive integer c_j . We may assume that $c_0 = \dots = c_{s+1}$. Let $N' = N \cdot c_0$. We prove $N' \cdot E_2^{p,q}(X) = 0$ if $p + r < q$ and $p < s + 1$, or if $p + r < q$ and $q > d - s - 1$.

For $0 \leq j \leq s$, we prove that

- (i) $(N' \Delta_j)_* = 0$ if $(p, q) \notin [j, j + r] \times [j, j + r]$;
- (ii) $(N' \Delta_j)^* = 0$ if $(p, q) \notin [d - j - r, d - j] \times [d - j - r, d - j]$.

We have a commutative diagram

$$\begin{array}{ccc}
 E_2^{p,q}(X) & \xrightarrow{(N' \Delta_j)_*} & E_2^{p,q}(X) & . \\
 \downarrow & & \uparrow & \\
 E_2^{p,q}(\widetilde{V}_j) & & E_2^{p-j,q-j}(\widetilde{W}_j) & \\
 \downarrow & & \uparrow & \\
 E_2^{p,q}(\widetilde{V}_j \times \widetilde{W}_j) & \xrightarrow{\cup(N' \widetilde{\Delta}_j)} & E_2^{p+r,q+r}(\widetilde{V}_j \times \widetilde{W}_j) &
 \end{array}$$

To prove (i), it is enough to observe that $E_2^{p,q}(\widetilde{V}_j) = 0$ if $p > j + r$ or $q > j + r$, and $E_2^{p-j,q-j}(\widetilde{W}_j) = 0$ if $p < j$ or $q < j$. Similarly, we have a commutative diagram

$$\begin{array}{ccc}
 E_2^{p,q}(X) & \xrightarrow{(N' \Delta_j)^*} & E_2^{p,q}(X) & . \\
 \downarrow & & \uparrow & \\
 E_2^{p,q}(\widetilde{W}_j) & & E_2^{p+r-d+j,q+r-d+j}(\widetilde{V}_j) & \\
 \downarrow & & \uparrow & \\
 E_2^{p,q}(\widetilde{V}_j \times \widetilde{W}_j) & \xrightarrow{\cup(N' \widetilde{\Delta}_j)} & E_2^{p+r,q+r}(\widetilde{V}_j \times \widetilde{W}_j) &
 \end{array}$$

To prove (ii), it is enough to observe that $E_2^{p,q}(\widetilde{W}_j) = 0$ if $p > d - j$ or $q > d - j$, and $E_2^{p+r-d+j,q+r-d+j}(\widetilde{V}_j) = 0$ if $p < d - r - j$ or $q < d - r - j$.

For Δ^{s+1} , we prove that

- (iii) $(N' \Delta^{s+1})_* = 0$ if $(p, q) \notin [s + 1, d] \times [s + 1, d]$;
- (iv) $(N' \Delta^{s+1})^* = 0$ if $(p, q) \notin [0, d - s - 1] \times [0, d - s - 1]$.

We have a commutative diagram

$$\begin{array}{ccc}
 E_2^{p,q}(X) & \xrightarrow{(N'\Delta^{s+1})^*} & E_2^{p,q}(X) & . \\
 \downarrow & & \uparrow & \\
 & & E_2^{p-s-1,q-s-1}(\widetilde{W}_{s+1}) & \\
 & & \uparrow & \\
 E_2^{p,q}(X \times \widetilde{W}_{s+1}) & \xrightarrow{\cup(N\widetilde{\Delta}^{s+1})} & E_2^{p+d-s-1,q+d-s-1}(X \times \widetilde{W}_{s+1}) &
 \end{array}$$

To prove (iii), it is enough to observe that $E_2^{p-s-1,q-s-1}(\widetilde{W}_{s+1}) = 0$ if $p < s + 1$ or $q < s + 1$. Similarly, we have a commutative diagram

$$\begin{array}{ccc}
 E_2^{p,q}(X) & \xrightarrow{(N'\Delta^{s+1})^*} & E_2^{p,q}(X) & . \\
 \downarrow & & \uparrow & \\
 E_2^{p,q}(\widetilde{W}_{s+1}) & & & \\
 \downarrow & & & \\
 E_2^{p,q}(X \times \widetilde{W}_{s+1}) & \xrightarrow{\cup(N\widetilde{\Delta}^{s+1})} & E_2^{p+d-s-1,q+d-s-1}(X \times \widetilde{W}_{s+1}) &
 \end{array}$$

To prove (iv), it is enough to observe that $E^{p,q}(\widetilde{W}_{s+1}) = 0$ if $p > d - s - 1$ or $q > d - s - 1$.

The proof is done by (i), (ii), (iii) and (iv). □

Theorem A.3. *Let X be a smooth projective variety of dimension d such that $L_{3,s}$ holds for X . Let $p \in [3, s + 3] \cup [d - s - 1, d - 1]$. Then the restriction*

$$\widetilde{\psi}^p|_{\text{tors}} : A^p(X)_{\text{tors}} \rightarrow J(N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)))_{\text{tors}}$$

is an isomorphism. Moreover the Walker map $\widetilde{\psi}^p$ is universal.

Remark A.4. The case $s = 0$ is Theorem 1.1.

Proof of Theorem A.3. The second statement follows from the first one by Lemma 2.1.

We prove that the restriction $\tilde{\psi}^p|_{\text{tors}}$ is an isomorphism. By Lemma 2.2, it is enough to prove that $\text{Ker}(f^p)$ is torsion. By Lemma A.1, the groups

$$E_2^{p-3,p+1}(X), \dots, E_2^{0,2p-2}(X)$$

are torsion, so the result follows. \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT CHICAGO
851 S. MORGAN ST., CHICAGO, IL 60607, USA
E-mail address: fsuzuk2@uic.edu

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