

On degrees of birational mappings

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We prove that the degrees of the iterates $\deg(f^n)$ of a birational map satisfy $\liminf(\deg(f^n)) < +\infty$ if and only if the sequence $\deg(f^n)$ is bounded, and that the growth of $\deg(f^n)$ cannot be arbitrarily slow, unless $\deg(f^n)$ is bounded.

1. Degree sequences

Let \mathbf{k} be a field. Consider a projective variety X , a polarization H of X (given by hyperplane sections of X in some embedding $X \subset \mathbb{P}^N$), and a birational transformation f of X , all defined over the field \mathbf{k} . Let k be the dimension of X . The **degree** of f with respect to the polarization H is the integer

$$(1.1) \quad \deg_H(f) = (f^*H) \cdot H^{k-1}$$

where f^*H is the total transform of H , and $(f^*H) \cdot H^{k-1}$ is the intersection product of f^*H with $k - 1$ copies of H . The degree is a positive integer, which we shall simply denote by $\deg(f)$, even if it depends on H . When f is a birational transformation of the projective space \mathbb{P}^k and the polarization is given by $\mathcal{O}_{\mathbb{P}^k}(1)$ (i.e. by hyperplanes $H \subset \mathbb{P}^k$), then $\deg(f)$ is the degree of the homogeneous polynomial formulas defining f in homogeneous coordinates.

The degrees are submultiplicative, in the following sense:

$$(1.2) \quad \deg(f \circ g) \leq c_{X,H} \deg(f) \deg(g)$$

for some positive constant $c_{X,H}$ and for every pair of birational transformations. Also, if the polarization H is changed into another polarization H' , there is a positive constant c which depends on X , H and H' but not on f , such that

$$(1.3) \quad \deg_H(f) \leq c \deg_{H'}(f)$$

We refer to [11, 16, 18] for these fundamental properties.

The **degree sequence** of f is the sequence $(\deg(f^n))_{n \geq 0}$; it plays an important role in the study of the dynamics and the geometry of f . There are infinitely, but only countably many degree sequences (see [4, 19]); unfortunately, not much is known on these sequences when $\dim(X) \geq 3$ (see [3, 10] for $\dim(X) = 2$). In this article, we obtain the following basic results.

- The sequence $(\deg(f^n))_{n \geq 0}$ is bounded if and only if it is bounded along an infinite subsequence (see Theorems A and B in § 2 and § 3).
- If the sequence $(\deg(f^n))_{n \geq 0}$ is unbounded, then its growth can not be arbitrarily slow; for instance, $\max_{0 \leq j \leq n} \deg(f^j)$ is asymptotically bounded from below by the inverse of the diagonal Ackermann function when $X = \mathbb{P}_{\mathbf{k}}^k$ (see Theorem C in § 4 for a better result).

We focus on birational transformations because a rational dominant transformation which is not birational has a topological degree $\delta > 1$, and this forces an exponential growth of the degrees: $1 < \delta^{1/k} \leq \lim_n (\deg(f^n)^{1/n})$ where $k = \dim(X)$ (see [11] and [6], pages 120–126).

2. Automorphisms of the affine space

We start with the simpler case of automorphisms of the affine space; the goal of this section is to introduce a p -adic method to study degree sequences.

Theorem A (Urech).— *Let f be an automorphism of the affine space $\mathbb{A}_{\mathbf{k}}^k$. If $\deg(f^n)$ is bounded along an infinite subsequence, then it is bounded.*

2.1. Urech's proof

In [19], Urech proves a stronger result. Writing his proof in an intrinsic way, we extend it to affine varieties:

Theorem 2.1. *Let $X = \text{Spec } A$ be an irreducible affine variety of dimension k over the field \mathbf{k} . Let $f : X \rightarrow X$ be an automorphism. If $(\deg(f^n))$ is unbounded there exists $\alpha > 0$ such that $\#\{n \geq 0 \mid \deg(f^n) \leq d\} \leq \alpha d^k$; in particular, $\max_{0 \leq j \leq n} \deg(f^j)$ is bounded from below by $(n/\alpha)^{1/k}$.*

Here, the degree of f^n , depends on the choice of a projective compactification Y of X and an ample line bundle L on Y . However, by Equation (1.3), the statement of Theorem 2.1 does not depend on the choice of (Y, L) . Since

automorphisms of X always lift to its normalization, we may assume that X is normal. To prove this theorem, we shall introduce another equivalent notion of degree.

2.1.1. Degrees on affine varieties. Consider X as a subvariety $X \subseteq \mathbb{A}^N \subseteq \mathbb{P}^N$. Let \bar{X} be the Zariski closure of X in \mathbb{P}^N and $H_1 := \mathbb{P}^N \setminus \mathbb{A}^N$ be the hyperplane at infinity. Let $\pi : Y \rightarrow \bar{X}$ be its normalization: Y is a normal projective compactification of X . Since $\pi : Y \rightarrow \bar{X}$ is finite, there exists $m \geq 1$ such (i) $H := \pi^*(mH_1|_{\bar{X}})$ is very ample on Y and (ii) H is projectively normal on Y i.e. for every $n \geq 0$, the morphism $(H^0(Y, H))^{\otimes n} \rightarrow H^0(Y, nH)$ is surjective.

If $P \in A$ is a regular function on X , we extend it as a rational function on Y , we denote by $(P) = (P)_0 - (P)_\infty$ the divisor defined by P on Y , and we define

$$(2.1) \quad \Delta(P) = \min\{d \geq 0 \mid (P) + dH \geq 0 \text{ on } Y\},$$

$$(2.2) \quad A_d = \{P \in A \mid \Delta(P) \leq d\}, \quad (\forall d \geq 0).$$

Then $A = \cup_{d \geq 0} A_d$. Since $Y \setminus X$ is the support of H , we get an isomorphism $i_n : H^0(Y, nH) \rightarrow A_n \subseteq A$ for every $n \geq 0$. Thus, A_1 generates A and the morphism $A_1^{\otimes n} \rightarrow A_n$ is surjective. Now we define

$$(2.3) \quad \deg^H(f) = \min\{m \geq 0 \mid \Delta(f^*P) \leq m \text{ for every } P \in A_1\}.$$

For every $P \in A_n$, we can write $P = \sum_{i=1}^l g_{1,i} \cdots g_{1,n}$ for some $g_{i,j} \in A_1$. We get $f^*P = \sum_{i=1}^l f^*g_{1,i} \cdots f^*g_{1,n} \in A_{\deg^H(f)n}$ and

$$(2.4) \quad \Delta(f^*P) \leq \deg^H(f)\Delta(P).$$

Since A is generated by A_1 , we get an embedding

$$(2.5) \quad \text{End}(A) \subseteq \text{Hom}_{\mathbf{k}}(A_1, A) = \cup_{d \geq 1} \text{Hom}_{\mathbf{k}}(A_1, A_d).$$

Set $\text{End}(A)_d = \text{End}(A) \cap \text{Hom}_{\mathbf{k}}(A_1, A_d)$. For any automorphism $f : X \rightarrow X$, $\deg^H(f) \leq d$ if and only if $f \in \text{End}(A)_d$. By Riemann-Roch theorem, there exists $\gamma > 0$ such that $\dim A_n \leq \gamma n^k$, and this gives the upper bound

$$(2.6) \quad \dim \text{End}(A)_d \leq \text{Hom}_{\mathbf{k}}(A_1, A_d) \leq (\gamma d^k) \dim A_1.$$

The following proposition, proved in the Appendix, shows that this new degree $\deg^H(f)$ is equivalent to the degree $\deg_H(f)$ introduced in Section 1.

Proposition 2.2. *For every automorphism $f \in \text{Aut}(X)$ we have*

$$\text{deg}^H(f) \leq \text{deg}_H(f) \leq (H^k) \text{deg}^H(f).$$

2.1.2. Proof of Theorem 2.1. By Proposition 2.2, the initial notion of degree can be replaced by deg^H . Let γ be as in Equation (2.6). Set $\ell = (\gamma d^k) \dim A_1 + 1$, and assume that $\text{deg}^H(f^{n_i}) \leq d$ for some sequence of positive integers $n_1 < n_2 < \dots < n_\ell$. Each $(f^*)^{n_i}$ is in $\text{End}(A)_d$ and, because $\ell > \dim \text{End}(A)_d$, there is a non-trivial linear relation between the $(f^*)^{n_i}$ in the vector space $\text{End}(A)_d$:

$$(2.7) \quad (f^*)^n = \sum_{m=1}^{n-1} a_m (f^*)^m$$

for some integer $n \leq n_\ell$ and some coefficients $a_m \in \mathbf{k}$. Then, the subalgebra $\mathbf{k}[f^*] \subseteq \text{End}(A)$ is of finite dimension and $\mathbf{k}[f^*] \subseteq E_B$ for some $B \geq 0$. This shows that the sequence $(\text{deg}^H(f^N))_{N \geq 0}$ is bounded.

Thus, if we set $\alpha = \gamma \dim A_1$, and if the sequence $(\text{deg}^H(f^n))$ is not bounded, we obtain $\#\{n \geq 0 \mid \text{deg}^H(f^n) \leq d\} \leq \alpha d^k$. This proves the first assertion of the theorem; the second follows easily.

2.2. The p -adic argument

Let us give another proof of Theorem A when $\text{char}(\mathbf{k}) = 0$, which will be generalized in § 3 for birational transformations.

2.2.1. Tate diffeomorphisms. Let p be a prime number. Let K be a field of characteristic 0 which is complete with respect to an absolute value $|\cdot|$ satisfying $|p| = 1/p$; such an absolute value is automatically ultrametric (see [13], Ex. 2 and 3, Chap. I.2). Let $R = \{x \in K; |x| \leq 1\}$ be the valuation ring of K ; in the vector space K^k , the unit **polydisk** is the subset $U = R^k$.

Fix a positive integer k , and consider the ring $R[\mathbf{x}] = R[\mathbf{x}_1, \dots, \mathbf{x}_k]$ of polynomial functions in k variables with coefficients in R . For f in $R[\mathbf{x}]$, define the norm $\|f\|$ to be the supremum of the absolute values of the coefficients of f :

$$(2.8) \quad \|f\| = \sup_I |a_I|$$

where $f = \sum_{I=(i_1, \dots, i_k)} a_I \mathbf{x}^I$. By definition, the **Tate algebra** $R\langle \mathbf{x} \rangle$ is the completion of $R[\mathbf{x}]$ with respect to this norm. It coincides with the set of

formal power series $f = \sum_I a_I \mathbf{x}^I$ converging (absolutely) on the closed poly-disk R^k . Moreover, the absolute convergence is equivalent to $|a_I| \rightarrow 0$ as $\text{length}(I) \rightarrow \infty$. Every element g in $R\langle \mathbf{x} \rangle^k$ determines a **Tate analytic** map $g: \mathbf{U} \rightarrow \mathbf{U}$.

For f and g in $R\langle \mathbf{x} \rangle$ and c in \mathbf{R}_+ , the notation $f \in p^c R\langle \mathbf{x} \rangle$ means $\|f\| \leq |p|^c$ and the notation $f \equiv g \pmod{p^c}$ means $\|f - g\| \leq |p|^c$; we then extend such notations component-wise to $(R\langle \mathbf{x} \rangle)^m$ for all $m \geq 1$.

For indeterminates $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_k)$ and $\mathbf{y} = (\mathbf{y}_1, \dots, \mathbf{y}_m)$, the composition $R\langle \mathbf{y} \rangle \times R\langle \mathbf{x} \rangle^m \rightarrow R\langle \mathbf{x} \rangle$ is well defined, and coordinatewise we obtain

$$(2.9) \quad R\langle \mathbf{y} \rangle^n \times R\langle \mathbf{x} \rangle^m \rightarrow R\langle \mathbf{x} \rangle^n.$$

When $m = n = k$, we get a semigroup $R\langle \mathbf{x} \rangle^k$. The group of (Tate) **analytic diffeomorphisms** of \mathbf{U} is the group of invertible elements in this semigroup; we denote it by $\text{Diff}^{an}(\mathbf{U})$. Elements of $\text{Diff}^{an}(\mathbf{U})$ are bijective transformations $f: \mathbf{U} \rightarrow \mathbf{U}$ given by $f(\mathbf{x}) = (f_1, \dots, f_k)(\mathbf{x})$ where each f_i is in $R\langle \mathbf{x} \rangle$ with an inverse $f^{-1}: \mathbf{U} \rightarrow \mathbf{U}$ that is also defined by power series in the Tate algebra.

The following result is due to Jason Bell and Bjorn Poonen (see [1, 17]).

Theorem 2.3. *Let f be an element of $R\langle \mathbf{x} \rangle^k$ with $f \equiv \text{id} \pmod{p^c}$ for some real number $c > 1/(p - 1)$. Then f is a Tate diffeomorphism of $\mathbf{U} = R^k$ and there exists a unique Tate analytic map $\Phi: R \times \mathbf{U} \rightarrow \mathbf{U}$ such that*

- 1) $\Phi(n, \mathbf{x}) = f^n(\mathbf{x})$ for all $n \in \mathbf{Z}$;
- 2) $\Phi(s + t, \mathbf{x}) = \Phi(s, \Phi(t, \mathbf{x}))$ for all t, s in R .

2.2.2. Second proof of Theorem A. Denote by S the finite set of all the coefficients that appear in the polynomial formulas defining f and f^{-1} . Let $R_S \subset \mathbf{k}$ be the ring generated by S over \mathbf{Z} , and let K_S be its fraction field:

$$(2.10) \quad \mathbf{Z} \subset R_S \subset K_S \subset \mathbf{k}.$$

Since $\text{char}(\mathbf{k}) = 0$, there exists a prime $p > 2$ such that R_S embeds into \mathbf{Z}_p (see [15], §4 and 5, and [1], Lemma 3.1). We apply this embedding to the coefficients of f and get an automorphism of $\mathbb{A}_{\mathbf{Q}_p}^k$ which is defined by polynomial formulas in $\mathbf{Z}_p[\mathbf{x}_1, \dots, \mathbf{x}_k]$; for simplicity, we keep the same notation f for this automorphism (embedding R_S in \mathbf{Z}_p does not change the value of the degrees $\text{deg}(f^n)$). Since f and f^{-1} are polynomial automorphisms

with coefficients in \mathbf{Z}_p , they determine elements of $\text{Diff}^{an}(\mathbf{U})$, the group of analytic diffeomorphisms of the polydisk $\mathbf{U} = \mathbf{Z}_p^k$.

Reducing the coefficients of f and f^{-1} modulo $p^2\mathbf{Z}_p$, one gets two permutations of the finite set $\mathbb{A}^k(\mathbf{Z}_p/p^2\mathbf{Z})$ (equivalently, f and f^{-1} permute the balls of $\mathbf{U} = \mathbf{Z}_p^k$ of radius p^{-2} , and these balls are parametrized by $\mathbb{A}^k(\mathbf{Z}_p/p^2\mathbf{Z})$; see [7]). Thus, there exists a positive integer m such that $f^m(0) \equiv 0 \pmod{p^2}$. Taking some further iterate, we may also assume that the differential Df_0^m satisfies $Df_0^m \equiv \text{Id} \pmod{p}$. We fix such an integer m and replace f by f^m . The following lemma follows from the submultiplicativity of degrees (see Equation (1.2) in Section 1). It shows that replacing f by f^m is harmless if one wants to bound the degrees of the iterates of f .

Lemma 2.4. *If the sequence $\text{deg}(f^{mn})$ is bounded for some $m > 0$, then the sequence $\text{deg}(f^n)$ is bounded too.*

Denote by $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_k)$ the coordinate system of \mathbb{A}^k , and by m_p the multiplication by p : $m_p(\mathbf{x}) = p\mathbf{x}$. Change f into $g := m_p^{-1} \circ f \circ m_p$; then $g \equiv \text{Id} \pmod{p}$ in the sense of Section 2.2.1. Since $p \geq 3$, Theorem 2.3 gives a Tate analytic flow $\Phi: \mathbf{Z}_p \times \mathbb{A}^k(\mathbf{Z}_p) \rightarrow \mathbb{A}^k(\mathbf{Z}_p)$ which extends the action of g : $\Phi(n, \mathbf{x}) = g^n(\mathbf{x})$ for every integer $n \in \mathbf{Z}$. Since Φ is analytic, one can write

$$(2.11) \quad \Phi(\mathbf{t}, \mathbf{x}) = \sum_J A_J(\mathbf{t})\mathbf{x}^J$$

where J runs over all multi-indices $(j_1, \dots, j_k) \in (\mathbf{Z}_{\geq 0})^k$ and each A_J defines a p -adic analytic curve $\mathbf{Z}_p \rightarrow \mathbb{A}^k(\mathbf{Q}_p)$. By submultiplicativity of the degrees, there is a constant $C > 0$ such that $\text{deg}(g^{n_i}) \leq CB^m$. Thus, we obtain $A_J(n_i) = 0$ for all indices i and all multi-indices J of length $|J| > CB^m$. The A_J being analytic functions of $t \in \mathbf{Z}_p$, the principle of isolated zeros implies that

$$(2.12) \quad A_J = 0 \text{ in } \mathbf{Z}_p\langle t \rangle, \quad \forall J \text{ with } |J| > CB^m.$$

Thus, $\Phi(t, \mathbf{x})$ is a polynomial automorphism of degree $\leq CB^m$ for all $t \in \mathbf{Z}_p$, and $g^n(\mathbf{x}) = \Phi(n, \mathbf{x})$ has degree at most CB^m for all n . By Lemma 2.4, this proves that $\text{deg}(f^n)$ is a bounded sequence.

3. Birational transformations

Theorem B.— *Let \mathbf{k} be a field of characteristic 0. Let X be a projective variety and $f: X \dashrightarrow X$ be a birational transformation of X , both defined*

over \mathbf{k} . If the sequence $(\deg(f^n))_{n \geq 0}$ is not bounded, then it goes to $+\infty$ with n :

$$\liminf_{n \rightarrow +\infty} \deg(f^n) = +\infty.$$

This extends Theorem A to birational transformations. With a theorem of Weil, we get: *if f is a birational transformation of the projective variety X , over an algebraically closed field of characteristic 0, and if the degrees of its iterates are bounded along an infinite subsequence f^{n_i} , then there exist a birational map $\psi: Y \dashrightarrow X$ and an integer $m > 0$ such that $f_Y := \psi^{-1} \circ f \circ \psi$ is in $\text{Aut}(Y)$, and f_Y^m is in the connected component $\text{Aut}(Y)^0$ (see [5] and references therein).*

Urech’s argument does not apply to this context; the basic obstruction is that rational transformations of $\mathbb{A}_{\mathbf{k}}^k$ of degree $\leq B$ generate an infinite dimensional \mathbf{k} -vector space for every $B \geq 1$ (the maps $z \in \mathbb{A}_{\mathbf{k}}^1 \mapsto (z - a)^{-1}$ with $a \in \mathbf{k}$ are linearly independent); looking back at the proof in Section 2.1.2, the problem is that the field of rational functions on an affine variety X is not finitely generated as a \mathbf{k} -algebra. We shall adapt the p -adic method described in Section 2.2.2. In what follows, f and X are as in Theorem B; we assume, without loss of generality, that $\mathbf{k} = \mathbf{C}$ and X is smooth. We suppose that there is an infinite sequence of integers $n_1 < \dots < n_j < \dots$ and a number B such that $\deg(f^{n_j}) \leq B$ for all j . We fix a finite subset $S \subset \mathbf{C}$ such that X , f and f^{-1} are defined by equations and formulas with coefficients in S , and we embed the ring $R_S \subset \mathbf{C}$ generated by S in some \mathbf{Z}_p , for some prime number $p > 2$. According to [7, Section 3], we may assume that X and f have good reduction modulo p .

3.1. The Hrushovski’s theorem and p -adic polydisks

According to a theorem of Hrushovski (see [12]), there is a periodic point z_0 of f in $X(\mathbf{F})$ for some finite field extension \mathbf{F} of the residue field \mathbf{F}_p , the orbit of which does not intersect the indeterminacy points of f and f^{-1} . If ℓ is the period of z_0 , then $f^\ell(z_0) = z_0$ and $Df_{z_0}^\ell$ is an element of the finite group $\text{GL}((TX_{\mathbf{F}_q})_{z_0}) \simeq \text{GL}(k, \mathbf{F}_q)$. Thus, there is an integer $m > 0$ such that $f^m(z_0) = z_0$ and $Df_{z_0}^m = \text{Id}$.

Replace f by its iterate $g = f^m$. Then, g fixes z_0 in $X(\mathbf{F})$, g is an isomorphism in a neighborhood of z_0 , and $Dg_{z_0} = \text{Id}$. According to [2] and [7, Section 3], this implies that there is

- a finite extension K of \mathbf{Q}_p , with valuation ring $R \subset K$;

- a point z in $X(K)$ and a polydisk $V_z \simeq R^k \subset X(K)$ which is g -invariant and such that $g|_{V_z} \equiv \text{Id} \pmod{(p)}$ (in the coordinate system $(\mathbf{x}_1, \dots, \mathbf{x}_k)$ of the polydisk).

When the point z_0 is in $X(\mathbf{F}_p)$ and is the reduction of a point $z \in X(\mathbf{Z}_p)$, the polydisk V_z is the set of points $w \in X(\mathbf{Z}_p)$ with $|z - w| < 1$; one identifies this polydisk to $\mathbf{U} = (\mathbf{Z}_p)^k$ via some p -adic analytic diffeomorphism $\varphi: \mathbf{U} \rightarrow V_z$; changing φ into $\varphi \circ m_p$ if necessary, we obtain $g_{V_z} \equiv \text{Id} \pmod{(p)}$ (see Section 2.2.2 and [7], Section 3.2.1). In full generality, a finite extension K of \mathbf{Q}_p is needed because z_0 is a point in $X(\mathbf{F})$ for some extension \mathbf{F} of \mathbf{F}_p .

3.2. Controlling the degrees

As in Section 2.2.1, denote by \mathbf{U} the polydisk $R^k \simeq V_z$; thus, \mathbf{U} is viewed as the polydisk R^k and also as a subset of $X(K)$. Applying Theorem 2.3 to g , we obtain a p -adic analytic flow

$$(3.1) \quad \Phi: R \times \mathbf{U} \rightarrow \mathbf{U}, \quad (t, \mathbf{x}) \mapsto \Phi(t, \mathbf{x})$$

such that $\Phi(n, \mathbf{x}) = g^n(\mathbf{x})$ for every integer n . In other words, the action of g on \mathbf{U} extends to an analytic action of the additive compact group $(R, +)$.

Let $\pi_1: X \times X \rightarrow X$ denote the projection onto the first factor. Denote by $\text{Bir}_D(X)$ the set of birational transformations of X of degree D ; once birational transformations are identified to their graphs, this set becomes naturally a finite union of irreducible, locally closed algebraic subsets in the Hilbert scheme of $X \times X$ (see [5], Section 2.2, and references therein). Taking a subsequence, there is a positive integer D , an irreducible component B_D of $\text{Bir}_D(X)$, and a strictly increasing, infinite sequence of integers (n_j) such that

$$(3.2) \quad g^{n_j} \in B_D$$

for all j . Denote by $\overline{B_D}$ the Zariski closure of B_D in the Hilbert scheme of $X \times X$. To every element $h \in \overline{B_D}$ corresponds a unique algebraic subset \mathcal{G}_h of $X \times X$ (the graph of h , when h is in B_D). Our goal is to show that, for every $t \in R$, the graph of $\Phi(t, \cdot)$ is the intersection $\mathcal{G}_{h_t} \cap \mathbf{U}^2$ for some element $h_t \in \overline{B_D}$; this will conclude the proof because $g^n(\mathbf{x}) = \Phi(n, \mathbf{x})$ for all $n \geq 0$.

We start with a simple remark, which we encapsulate in the following lemma.

Lemma 3.1. *There is a finite subset $E \subset U \subset X(K)$ with the following property. Given any subset \tilde{E} of $U \times U$ with $\pi_1(\tilde{E}) = E$, there is at most one element $h \in \overline{B_D}$ such that $\tilde{E} \subset \mathcal{G}_h$.*

Fix such a set E , and order it to get a finite list $E = (x_1, \dots, x_{\ell_0})$ of elements of U . Let $E' = (x_1, \dots, x_{\ell_0}, x_{\ell_0+1}, \dots, x_\ell)$ be any list of elements of U which extends E . For every element h in $\overline{B_D}$, the variety \mathcal{G}_h determines a correspondance $\mathcal{G}_h \subset X \times X$. The subset of elements $(h, (x_i, y_i)_{1 \leq i \leq \ell})$ in $\overline{B_D} \times (X \times X)^\ell$ defined by the incidence relation

$$(3.3) \quad (x_i, y_i) \in \mathcal{G}_h$$

for every $1 \leq i \leq \ell$ is an algebraic subset of $\overline{B_D} \times (X \times X)^\ell$. Add one constraint, namely that the first projection $(x_i)_{1 \leq i \leq \ell}$ coincides with E' , and project the resulting subset on $(X \times X)^\ell$: we get a subset $G(E')$ of $(X \times X)^\ell$. Then, define a p -adic analytic curve $\Lambda: R \rightarrow (X \times X)^\ell$ by

$$(3.4) \quad \Lambda(t) = (x_i, \Phi(t, x_i))_{1 \leq i \leq \ell}.$$

If $t = n_j$, g^{n_j} is an element of B_D and $\Lambda(n_j)$ is contained in the graph of g^{n_j} ; hence, $\Lambda(n_j)$ is an element of $G(E')$. By the principle of isolated zeros, the analytic curve $t \mapsto \Lambda(t) \subset (X \times X)^\ell$ is contained in $G(E')$ for all $t \in R$. Thus, for every t there is an element $h_t \in \overline{B_D}$ such that $\Lambda(t)$ is contained in the subset $\mathcal{G}_{h_t}^\ell$ of $(X \times X)^\ell$. From the choice of E and the inclusion $E \subset E'$, we know that h_t does not depend on E' . Thus, the graph of $\Phi(t, \cdot)$ coincides with the intersection of \mathcal{G}_{h_t} with $U \times U$. This implies that the graph of $g^n(\cdot) = \Phi(n, \cdot)$ coincides with \mathcal{G}_{h_n} , and that the degree of g^n is at most D for all values of n .

4. Lower bounds on degree growth

We now prove that the growth of $(\deg(f^n))$ can not be arbitrarily slow unless $(\deg(f^n))$ is bounded. For simplicity, we focus on birational transformations of the projective space; there is no restriction on the characteristic of \mathbf{k} .

4.1. A family of integer sequences

Fix two positive integers k and d ; k will be the dimension of $\mathbb{P}_{\mathbf{k}}^k$, and d will be the degree of $f: \mathbb{P}^k \dashrightarrow \mathbb{P}^k$. Set

$$(4.1) \quad m = (d - 1)(k + 1).$$

Then, consider an auxiliary integer $D \geq 1$, which will play the role of the degree of an effective divisor in the next paragraphs, and define

$$(4.2) \quad q = (dD + 1)^m.$$

Thus, q depends on k, d and D because m depends on k and d . Then, set

$$(4.3) \quad a_0 = \binom{k + D}{k} - 1, \quad b_0 = 1, \quad c_0 = D + 1.$$

Starting from the triple (a_0, b_0, c_0) , we define a sequence $((a_j, b_j, c_j))_{j \geq 0}$ inductively by

$$(4.4) \quad (a_{j+1}, b_{j+1}, c_{j+1}) = (a_j, b_j - 1, qc_j^2)$$

if $b_j \geq 2$, and by

$$(4.5) \quad (a_{j+1}, b_{j+1}, c_{j+1}) = (a_j - 1, qc_j^2, qc_j^2) = (a_j - 1, c_{j+1}, c_{j+1})$$

if $b_j = 1$. By construction, $(a_1, b_1, c_1) = (a_0 - 1, qc_0^2, qc_0^2)$.

Define $\Phi: \mathbf{Z}^+ \rightarrow \mathbf{Z}^+$ by

$$(4.6) \quad \Phi(c) = qc^2.$$

Lemma 4.1. *Define the sequence of integers $(F_i)_{i \geq 1}$ recursively by $F_1 = q(D + 1)^2$ and $F_{i+1} = \Phi^{F_i}(F_i)$ for $i \geq 1$ (where Φ^{F_i} is the F_i -iterate of Φ). Then*

$$(a_{1+F_1+\dots+F_i}, b_{1+F_1+\dots+F_i}, c_{1+F_1+\dots+F_i}) = (a_0 - i - 1, F_{i+1}, F_{i+1}).$$

The proof is straightforward. Now, define $S: \mathbf{Z}^+ \rightarrow \mathbf{Z}^+$ as the sum

$$(4.7) \quad S(j) = 1 + F_1 + F_2 + \dots + F_j$$

for all $j \geq 1$; it is increasing and goes to $+\infty$ extremely fast with j . Then, set

$$(4.8) \quad \chi_{d,k}(n) = \max \left\{ D \geq 0 \mid S\left(\binom{k + D}{k} - 2\right) < n \right\}.$$

Lemma 4.2. *The function $\chi_{d,k}: \mathbf{Z}^+ \rightarrow \mathbf{Z}^+$ is non-decreasing and goes to $+\infty$ with n .*

Remark 4.3. *The function S is primitive recursive* (see [9], Chapters 3 and 13). In other words, S is obtained from the basic functions (the zero function, the successor $s(x) = x + 1$, and the projections $(x_i)_{1 \leq i \leq m} \rightarrow x_i$) by a finite sequence of compositions and recursions. Equivalently, there is a program computing S , all of whose instructions are limited to (1) the zero initialization $V \leftarrow 0$, (2) the increment $V \leftarrow V + 1$, (3) the assignment $V \leftarrow V'$, and (4) loops of definite length. Writing such a program is an easy exercise. Now, consider the diagonal Ackermann function $A(n)$ (see [9], Section 13.3). It grows asymptotically faster than any primitive recursive function; hence, the inverse of the Ackermann diagonal function $\alpha(n) = \max\{D \geq 0 \mid \text{Ack}(D) \leq n\}$ is, asymptotically, a lower bound for $\chi_{d,k}(n)$. Showing that $\chi_{d,k}$ is in the \mathcal{L}_6 hierarchy of [9], Chapter 13, one gets an asymptotic lower bound by the inverse of the function f_7 of [9], independent of the values of d and k .

4.2. Statement of the lower bound

We can now state the result that will be proved in the next paragraphs.

Theorem C.— *Let f be a birational transformation of the complex projective space $\mathbb{P}_{\mathbf{k}}^k$ of degree d . If the sequence $(\max_{0 \leq j \leq n}(\deg(f^j)))_{n \geq 0}$ is unbounded, then it is bounded from below by the sequence of integers $(\chi_{d,k}(n))_{n \geq 0}$.*

Remark 4.4. There are infinitely, but only countably many sequences of degrees $(\deg(f^n))_{n \geq 0}$ (see [4, 19]). Consider the countably many sequences

$$(4.9) \quad \left(\max_{0 \leq j \leq n} (\deg(f^j)) \right)_{n \geq 0}$$

restricted to the family of birational maps for which $(\deg(f^n))$ is unbounded. We get a countable family of *non-decreasing, unbounded sequences of integers*. Let $(u_i)_{i \in \mathbf{Z}_{\geq 0}}$ be any countable family of such sequences of integers $(u_i(n))$. Define $w(n)$ as follows. First, set $v_j = \min\{u_0, u_1, \dots, u_j\}$; this defines a new family of sequences, with the same limit $+\infty$, but now $v_j(n) \geq v_{j+1}(n)$ for every pair (j, n) . Then, set $m_0 = 0$, and define m_{n+1} recursively to be the first positive integer such that $v_{n+1}(m_{n+1}) \geq v_n(m_n) + 1$. We have $m_{n+1} \geq m_n + 1$ for all $n \in \mathbf{Z}_{\geq 0}$. Set $w(n) := v_{r_n}(m_{r_n})$ where r_n is the unique non-negative integer satisfying $m_{r_n} \leq n \leq m_{r_n+1} - 1$. By construction, $w(n)$ goes to $+\infty$ with n and $u_i(n)$ is asymptotically bounded from below by $w(n)$.

In Theorem C, the result is more explicit. Firstly, the lower bound is explicitly given by the sequence $(\chi_{d,k}(n))_{n \geq 0}$. Secondly, the lower bound is not asymptotic: it works for every value of n . In particular, if $\deg(f^j) < \chi_{d,k}(n)$ for $0 \leq j \leq n$ and $\deg(f) = d$, then the sequence $(\deg(f^n))$ is bounded.

4.3. Divisors and strict transforms

To prove Theorem C, we consider the action of f by strict transform on effective divisors. As above, $d = \deg(f)$ and $m = (d-1)(k+1)$ (see Section 4.1).

4.3.1. Exceptional locus. Let X be a smooth projective variety and π_1 and $\pi_2: X \rightarrow \mathbb{P}^k$ be two birational morphisms such that $f = \pi_2 \circ \pi_1^{-1}$; then, consider the exceptional locus $\text{Exc}(\pi_2) \subset X$, project it by π_1 into \mathbb{P}^k , and list its irreducible components of codimension 1: we obtain a finite number

$$(4.10) \quad E_1, \dots, E_{m(f)}$$

of irreducible hypersurfaces, contained in the zero locus of the jacobian determinant of f . Since this critical locus has degree m , we obtain:

$$(4.11) \quad m(f) \leq m, \quad \text{and} \quad \deg(E_i) \leq m \quad (\forall i \geq 1).$$

4.3.2. Effective divisors. Denote by M the semigroup of effective divisors of $\mathbb{P}^k_{\mathbf{k}}$. There is a partial ordering \leq on M , which is defined by $E \leq E'$ if and only if the divisor $E' - E$ is effective.

We denote by $\deg: M \rightarrow \mathbf{Z}_{\geq 0}$ the degree function. For every degree $D \geq 0$, we denote by M_D the set $\mathbb{P}(H^0(\mathbb{P}^k_{\mathbf{k}}, \mathcal{O}_{\mathbb{P}^k_{\mathbf{k}}}(D)))$ of effective divisors of degree D ; thus, M is the disjoint union of all the M_D , and each of these components will be endowed with the Zariski topology of $\mathbb{P}(H^0(\mathbb{P}^k_{\mathbf{k}}, \mathcal{O}_{\mathbb{P}^k_{\mathbf{k}}}(D)))$. The dimension of M_D is equal to the integer $a_0 = a_0(D, k)$ from Section 4.1:

$$(4.12) \quad \dim(M_D) = \binom{k+D}{k} - 1.$$

Let $G \subset M$ be the semigroup generated by the E_i :

$$(4.13) \quad G = \bigoplus_{i=1}^{m(f)} \mathbf{Z}_{\geq 0} E_i.$$

The elements of G are the effective divisors which are supported by the exceptional locus of f . For every $E \in G$, there is a translation operator

$T_E: M \rightarrow M$, defined by $T_E: E' \mapsto E + E'$; it restricts to a linear projective embedding of the projective space M_D into the projective space $M_{D+\deg(E)}$. We define

$$(4.14) \quad M_D^\circ = M_D \setminus \bigcup_{E \in G \setminus \{0\}, \deg(E) \leq D} T_E(M_{D-\deg(E)}).$$

Thus, M_D° is the complement in M_D of finitely many proper linear projective subspaces. Also, $M_0^\circ = M_0$ is a point and M_1° is obtained from $M_1 = (\mathbb{P}_k^k)^\vee$ by removing finitely many points, corresponding to the E_i of degree 1 (the hyperplanes contracted by f). Set $M^\circ = \bigcup_{D \geq 0} M_D^\circ$. This is the set of effective divisors without any component in the exceptional locus of f . The inclusion of M° in M will be denoted by $\iota: M^\circ \rightarrow M$. There is a natural projection $\pi_G: M \rightarrow G$; namely, $\pi_G(E)$ is the maximal element such that $E - \pi_G(E)$ is effective. We denote by $\pi_\circ: M \rightarrow M^\circ$ the projection $\pi_\circ = \text{Id} - \pi_G$; this homomorphism removes the part of an effective divisor E which is supported on the exceptional locus of f .

Remark 4.5. The restriction of the map π_\circ to the projective space M_D is piecewise linear, in the following sense. Consider the subsets $U_{E,D}$ of M_D which are defined for every $E \in G$ with $\deg(E) \leq D$ by

$$U_{E,D} = T_E(M_{D-\deg(E)}) \setminus \bigcup_{E' > E, E' \in G, \deg(E') \leq D} T_{E'}(M_{D-\deg(E')}).$$

They define a stratification of M_D by (open subsets of) linear subspaces, and π_\circ coincides with the linear map inverse of T_E on each $U_{E,D}$. Moreover, $\pi_\circ(Z)$ is closed for any closed subset $Z \subseteq M_D$.

We say that a scheme theoretic point $x \in M$ (resp. M°) is **irreducible** if the divisor of \mathbb{P}^k corresponding to x is irreducible. In other words, x is irreducible, if a general closed point $y \in \overline{\{x\}} \subseteq M$ is irreducible.

4.3.3. Strict transform. First, we consider the total transform $f^*: M \rightarrow M$, which is defined by $f^*(E) = (\pi_1)_* \pi_2^*(E)$ for every divisor $E \in M$. This is a homomorphism of semigroups; it is injective on non-closed irreducible points. Let $[x_0, \dots, x_k]$ be homogeneous coordinates on \mathbb{P}^k . If E is defined by the homogeneous equation $P = 0$, then $f^*(E)$ is defined by $P \circ f = 0$; thus, f^* induces a linear projective embedding of M_D into M_{dD} for every D .

Then, we denote by $f^\circ: M^\circ \rightarrow M^\circ$ the strict transform. It is defined by

$$(4.15) \quad f^\circ(E) = (\pi_\circ \circ f^* \circ \iota)(E).$$

This is a homomorphism of semigroups. If $x \in M$ is an irreducible point, its total transform $f^*(x)$ is not necessarily irreducible, but $f^\circ(x)$ is irreducible.

In general, $(f^\circ)^n \neq (f^n)^\circ$, but for non-closed irreducible point $x \in M$, we have $(f^\circ)^n(x) = (f^n)^\circ(x)$ for $n \geq 0$. Indeed, a non-closed irreducible point $x \in M$ can be viewed as an irreducible hypersurface on X which is defined over some transcendental extension of \mathbf{k} , but not over \mathbf{k} . Then $f^\circ(x)$ is the unique irreducible component E of $f^*(x)$, on which $f|_E$ is birational to its image. (Note that when \mathbf{k} is uncountable, one can also work with very general points of M_D for every $D \geq 1$, instead of irreducible, non-closed points).

4.4. Proof of Theorem C

Let η be the generic point of M_1° (η corresponds to a generic hyperplane of $\mathbb{P}_{\mathbf{k}}^k$). Note that η is non-closed and irreducible. The degree of $f^*(\eta)$ is equal to the degree of f , and since η is generic, $f^*(\eta)$ coincides with $f^\circ(\eta)$. Thus, $\deg(f) = \deg(f^\circ(\eta))$ and more generally

$$(4.16) \quad \deg(f^n) = \deg((f^\circ)^n \eta) \quad (\forall n \geq 1).$$

Fix an integer $D \geq 0$. Write $M_{\leq D}^\circ$ for the disjoint union of the $M_{D'}^\circ$, with $D' \leq D$, and define recursively $Z_D(0) = M_{\leq D}^\circ$ and

$$(4.17) \quad Z_D(i+1) = \{E \in Z_D(i) \mid f^\circ(E) \in Z_D(i)\}$$

for $i \geq 0$. A divisor $E \in M_{\leq D}^\circ$ is in $Z_D(i)$ if its strict transform $f^\circ(E)$ is of degree $\leq D$, and $f^\circ(f^\circ(E))$ is also of degree $\leq D$, up to $(f^\circ)^i(E)$ which is also of degree at most D .

Let us describe $Z_D(i+1)$ more precisely. For each i , and each $E \in G$ of degree $\deg(E) \leq dD$ consider the subset $T_E(\overline{\iota(Z_D(i))}) \cap M_{dD}$; this is a subset of M_{dD} which is made of divisors W such that $\pi_\circ(W)$ is contained in $Z_D(i)$, and the union of all these subsets when E varies is exactly the set of points W in M_{dD} with a projection $\pi_\circ(W)$ in $Z_D(i)$. Thus, we consider

$$(4.18) \quad (f^*)^{-1}(T_E(\overline{\iota(Z_D(i))})) = \{V \in M_{\leq D} \mid f^*(V) \in T_E(\overline{\iota(Z_D(i))})\}.$$

These sets are closed subsets of $M_{\leq D}$, and

$$(4.19) \quad Z_D(i + 1) = Z_D(i) \cap \bigcup_{E \in G, \deg(E) \leq dD} \pi_\circ \left((f^*)^{-1}(T_E(\overline{Z_D(i)})) \right).$$

Since $Z_D(0)$ is closed in $M_{\leq D}^\circ$ and π_\circ is closed on $M_{\leq D}$, by induction, $Z_D(i)$ is closed for all $i \geq 0$. The subsets $Z_D(i)$ form a decreasing sequence of Zariski closed subsets (in the disjoint union $M_{\leq D}^\circ$ of the $M_{D'}^\circ$, $D' \leq D$). The strict transform f° maps $Z_D(i + 1)$ into $Z_D(i)$. By Noetherianity, there exists a minimal integer $\ell(D) \geq 0$ such that

$$(4.20) \quad Z_D(\ell(D)) = \bigcap_{i \geq 0} Z_D(i);$$

we denote this subset by $Z_D(\infty) = Z_D(\ell(D))$. By construction, $Z_D(\infty)$ is stable under the operator f° ; more precisely,

$$f^\circ(Z_D(\infty)) = Z_D(\infty) = (f^\circ)^{-1}(Z_D(\infty)).$$

Let $\tau: \mathbf{Z}_{\geq 0} \rightarrow \mathbf{Z}_{\geq 0}$ be a lower bound for the inverse function of ℓ :

$$(4.21) \quad \ell(\tau(n)) \leq n \quad (\forall n \geq 0).$$

Assume that $\max\{\deg(f^m) \mid 0 \leq m \leq n_0\} \leq \tau(n_0)$ for some $n_0 \geq 1$. Then $\deg((f^\circ)^i(\eta)) \leq \tau(n_0)$ for every integer i between 0 and n_0 ; this implies that η is in the set $Z_{\tau(n_0)}(\ell(\tau(n_0))) = Z_{\tau(n_0)}(\infty)$, so that the degree of $(f^\circ)^m(\eta)$ is bounded from above by $\tau(n_0)$ for all $m \geq 0$. From Equation (4.16) we deduce that the sequence $(\deg(f^m))_{m \geq 0}$ is bounded. This proves the following lemma.

Lemma 4.6. *Let τ be a lower bound for the inverse function of ℓ . If*

$$\max\{\deg(f^m) \mid 0 \leq m \leq n_0\} \leq \tau(n_0)$$

for some $n_0 \geq 1$, then the sequence $(\deg(f^n))_{n \geq 0}$ is bounded by $\tau(n_0)$.

So, to conclude, we need to compare $\ell: \mathbf{Z}_{\geq 0} \rightarrow \mathbf{Z}^+$ to the function $S: \mathbf{Z}_{\geq 0} \rightarrow \mathbf{Z}^+$ of paragraph 4.1 (recall that S depends on the parameters $k = \dim(\mathbb{P}_{\mathbf{k}}^k)$ and $d = \deg(f)$ and that ℓ depends on f). Now, write $Z'_D(i) = Z_D(i) \setminus Z_D(\infty)$, and note that it is a strictly decreasing sequence of open subsets of $Z_D(i)$ with $Z'_D(j) = \emptyset$ for all $j \geq \ell(D)$. We shall say that a closed subset of $M_{\leq D}^\circ \setminus Z_D(\infty)$ for the Zariski topology is **piecewise linear** if

all its irreducible components are equal to the intersection of $M_{\leq D}^\circ \setminus Z_D(\infty)$ with a linear projective subspace of some $M_{D'}$, $D' \leq D$. We note that the intersection of two irreducible linear projective subspaces is still an irreducible linear projective subspace.

Let $\text{Lin}(a, b, c)$ be the family of closed piecewise linear subsets of $M_{\leq D}^\circ \setminus Z_D(\infty)$ of dimension a , with at most c irreducible components, and at most b irreducible components of maximal dimension a . Then,

1)

$$Z'_D(i + 1) = \{F \in Z'_D(i) \mid f^\circ(F) \in Z'_D(i)\} = \pi_\circ(f^*Z'_D(i) \cap \bigcup_E T_E(Z'_D(i))),$$

where E runs over the elements of G of degree $\deg(E) \leq dD$;

2) in this union, each irreducible component of $T_E(Z'_D(i))$ is piecewise linear.

Recall that $q = (dD + 1)^m$ (see Section 4.1). If Z is any closed piecewise linear subset of $M_{\leq D}^\circ \setminus Z_D(\infty)$ that contains exactly c irreducible components, the set

$$\begin{aligned} \pi_\circ(f^*Z \cap \bigcup_{E \in G, \deg(E) \leq dD} T_E(Z)) &= \bigcup_{E \in G, \deg(E) \leq dD} \pi_\circ(f^*Z \cap T_E(Z)) \\ &= \bigcup_{E \in G, \deg(E) \leq dD} T_E^{-1}|_{T_E(Z)}(f^*Z \cap T_E(Z)) \end{aligned}$$

has at most $qc^2 = (dD + 1)^m c^2$ irreducible components (this is a crude estimate: $f^*Z \cap T_E(Z)$ has at most c^2 irreducible components, $T_E^{-1}|_{T_E(Z)}$ is injective and the factor $(dD + 1)^m$ comes from the fact that G contains at most $(dD + 1)^m$ elements of degree $\leq dD$). Let us now use that the sequence $Z'_D(i)$ decreases strictly as i varies from 0 to $\ell(D)$, with $Z'_D(\ell(D)) = \emptyset$. If $0 \leq i \leq \ell(D) - 1$, and if $Z'_D(i)$ is contained in $\text{Lin}(a, b, c)$, we obtain

- 1) if $b \geq 2$, then $Z'_D(i + 1)$ is contained in $\text{Lin}(a, b - 1, qc^2)$;
- 2) if $b = 1$, then $Z'_D(i + 1)$ is contained in $\text{Lin}(a - 1, qc^2, qc^2)$.

This shows that

$$(4.22) \quad \ell(D) \leq S \left(\binom{k + D}{k} - 2 \right) + 1$$

where S is the function introduced in the Equation (4.7) of Section 4.1. Since $\chi_{d,k}$ satisfies $\ell(\chi_{d,k}(n)) \leq n$ for every $n \geq 1$, the conclusion follows.

5. Appendix: Proof of Proposition 2.2

We keep the notation from Section 2.1.1. Let f be an automorphism of X . There exist a normal projective irreducible variety Z and two birational morphisms $\pi_1: Z \rightarrow Y$ and $\pi_2: Z \rightarrow Y$ such that π_1 and π_2 are isomorphisms over X , and $f = \pi_2 \circ \pi_1^{-1}$.

Lemma 5.1. *We have $\Delta(f^*P) \leq \Delta(P) \deg_H(f)$ for every $P \in A$.*

Proof of Lemma 5.1. Let E_1, \dots, E_s be the irreducible components of the divisor $Y \setminus X$, and set $D = \sum_{i=1}^s E_i$; by construction $H \geq D$. By definition $(P) + \Delta(P)H \geq 0$, thus

$$(5.1) \quad (f^*P) + \Delta(P)f^*(H) \geq 0.$$

Since f is an automorphism of X , $f^*H = \sum_{i=1}^r a_i E_i$ for some $a_i \geq 0$. We have $\deg_H(f) = (f^*H \cdot H^{k-1}) \geq a_i(E_i \cdot H^{k-1}) \geq a_i$ for every index i . Thus,

$$(5.2) \quad f^*(H) \leq \deg_H(f)D \leq \deg_H(f)H.$$

From Equations (5.1) and (5.2) we infer $(f^*P) + \Delta(P) \deg_H(f)H \geq 0$; thus $\Delta(f^*P) \leq \deg_H(f)\Delta(P)$ by definition of $\Delta(f)$. □

Lemma 5.1 shows that $\deg^H(f) \leq \deg_H(f)$. We now prove the reverse direction: $\deg_H(f) \leq (H^k) \deg^H(f)$.

Since H is very ample, Bertini's theorem gives an irreducible divisor $D \in |H|$ such that $\pi_2(E) \not\subseteq D$ for every prime divisor E of Z in $Z \setminus \pi_2^*(X)$; hence, π_2^*D is equal to the strict transform $\pi_2^\circ D$. By definition, $D = (P) + H$ for some $P \in A_1$. Thus, $(\pi_1)_*\pi_2^*H$ is linearly equivalent to $(\pi_1)_*\pi_2^*D = (\pi_1)_*\pi_2^\circ D$, and this irreducible divisor $(\pi_1)_*\pi_2^\circ D$ is the closure D_{f^*P} of $\{f^*P = 0\} \subseteq X$ in Y . Writing $(f^*P) = D_{f^*P} - F$ where F is supported on $Y \setminus X$ we also get that $(\pi_1)_*\pi_2^*H$ is linearly equivalent to F . Since $\Delta(f^*P) \leq \deg^H(f)\Delta(P) = \deg^H(f)$, the definition of Δ gives

$$(5.3) \quad D_{f^*P} - F + \deg^H(f)H = (f^*P) + \deg^H(f)H \geq 0.$$

Thus, $F \leq \deg^H(f)H$ because D_{f^*P} is irreducible and is not supported on $Y \setminus X$. Altogether, this gives $\deg_H(f) = ((\pi_1)_*\pi_2^*H \cdot H^{k-1}) = (F \cdot H^{k-1}) \leq \deg^H(f)(H^k)$.

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RECEIVED AUGUST 31, 2018

ACCEPTED JANUARY 6, 2019

