On Fujita's conjecture for pseudo-effective thresholds

Jingjun Han and Zhan Li

We show Fujita's spectrum conjecture for ϵ-log canonical pairs and Fujita's log spectrum conjecture for log canonical pairs. Then, we generalize the pseudo-effective threshold of a single divisor to multiple divisors and establish the analogous finiteness and the DCC properties.

1. Introduction

Let (X, Δ) be a log pair with X a normal projective variety over $\mathbb{C}$. Suppose D is a big $\mathbb{R}$-Cartier divisor, then the *pseudo-effective threshold* of D with respect to (X, Δ) is defined by

$$(1.1) \qquad \tau(X, \Delta; D) := \inf\{t \in \mathbb{R}_{\geq 0} \mid K_X + \Delta + tD \text{ is pseudo-effective}\}.$$

Recall that an $\mathbb{R}$-divisor is called *pseudo-effective* if it is a limit of effective divisors in $N^1(X)_{\mathbb{R}}$.

It is one of Shokurov's philosophy that the pseudo-effective threshold is a natural generalization of Fano index when $K_X + \Delta$ is an $\mathbb{R}$-Cartier divisor and D is an ample divisor. He introduced the theory of complements when he studied the ACC for Fano index. The theory of complements is in a central place of birational geometry, and it plays a crucial role in the proof of BAB (Borisov-Alexeev-Borisov) conjecture [2]. It is also known that the theory of complements is related to flips, the minimal log discrepancy, the index conjecture and log canonical thresholds. Thus, it is interesting to study the pseudo-effective threshold from this point of view.

In order to study the adjunction theory for polarized varieties, Fujita made the following two conjectures [8, 9].

Theorem 1.1 (Fujita's spectrum conjecture, [7] Theorem 1.1). *Let n be a natural number, S_n be the set of pseudo-effective thresholds $\tau(X, H) :=*

$\tau(X, \emptyset; H)$ of an ample Cartier divisor H with respect to a smooth projective variety X of dimension n. Then $S_n \cap [\epsilon, +\infty)$ is a finite set for any $\epsilon > 0$.

Theorem 1.2 (Fujita's log spectrum conjecture, [6] Theorem 1.2). *Let S_n^{ls} be the set of pseudo-effective thresholds $\tau(X, \Delta; H)$, where X is a smooth projective variety of dimension n, Δ is a reduced divisor with simple normal crossing support, and H is an ample Cartier divisor on X. Then S_n^{ls} satisfies the ACC.*

Fujita showed that his spectrum conjecture is a consequence of the minimal model program and the BAB (Borisov-Alexeev-Borisov) conjecture, [9]. Recently, Di Cerbo studied these two problems. He proved Fujita's spectrum conjecture by using the special BAB conjecture, [7], and proved Fujita's log spectrum conjecture by using the ACC for the lc thresholds, [6].

Recall that for a partially ordered set $(\mathcal{S}, \succeq)$, it is said to satisfy the DCC (*descending chain condition*) if any non-increasing sequence $a_1 \succeq a_2 \succeq \cdots \succeq a_k \succeq \cdots$ in S stabilizes. It is said to satisfy the ACC (*ascending chain condition*) if any non-decreasing sequence in S stabilizes. When S is a set of real numbers, we consider the usual relation "$\leq$".

In this paper, we study Fujita's spectrum conjecture and Fujita's log spectrum conjecture in a more general setting, namely we allow the pair (X, Δ) to have singularities, and the coefficients of Δ and H to vary in some fixed set.

Fix a positive integer n, a positive real number ϵ, and a subset $\mathcal{I} \subset [0, 1]$. We will consider the following set,

$$\mathcal{T}_{n,\epsilon}(\mathcal{I}) := \{\tau(X, \Delta; H) \mid \dim X = n, (X, \Delta) \text{ is } \epsilon\text{-lc}, \Delta \in \mathcal{I},$$
$$H \text{ is a big and nef } \mathbb{Q}\text{-Cartier Weil divisor}\}.$$

Here "$\Delta \in \mathcal{I}$" means the coefficients of Δ are in the set $\mathcal{I}$, and we emphasize that H is an (integral) Weil divisor.

We can now state one of the main results in this paper.

Theorem 1.3. *Let n be a natural number, and ϵ be a positive real number.*

1) *If $\mathcal{I} \subset [0, 1]$ is a finite set, then $\mathcal{T}_{n,\epsilon}(\mathcal{I}) \cap [\delta, +\infty)$ is a finite set for any $\delta > 0$.*

2) *If $\mathcal{I} \subset [0, 1]$ is a DCC set, then $\mathcal{T}_{n,\epsilon}(\mathcal{I})$ satisfies the ACC.*

Roughly speaking, we generalize Di Cerbo's result from a smooth variety with an ample divisor H (Theorem 1.2 and Theorem 1.1) to an ϵ-lc pair with a big and nef divisor. Theorem 1.3 (1) was a question asked by Di Cerbo (cf. [7] P.244).

Our argument of the above result relies on the minimal model program and the recent progress on the BAB conjecture [2, 3]. In some sense, the ϵ-lc condition is the weakest possible condition for Theorem 1.3 to hold. If we relax the singularities from ϵ-lc to klt, Theorem 1.3 is no longer true, see Example 3.1 for more details. However, if H is assumed to be an ample Cartier divisor in Theorem 1.3 (2), then the condition "(X, Δ) is lc" is enough.

Theorem 1.4. *Let n be a natural number and $\mathcal{I}$ a DCC set of nonnegative real numbers. Let $\mathcal{CT}_n(\mathcal{I})$ be the set of pseudo-effective threshold $\tau(X, \Delta; H)$ which satisfies the following conditions.*

1) *X is a normal projective variety of dimension n,*

2) *(X, Δ) is lc and the coefficients of Δ are in $\mathcal{I}$, and*

3) *$H = \sum \mu_i H_i$, where H_i is a nef Cartier divisor for each i, $\mu_i \in \mathcal{I}$, and H is big.*

Then $\mathcal{CT}_n(\mathcal{I})$ satisfies the ACC.

The version where (X, Δ) is log smooth, Δ is reduced, and H itself is an ample divisor (Theorem 1.2) was proven by Di Cerbo by the ACC for the log canonical thresholds and Global ACC for the lc pairs. Roughly speaking, the main difficulty in our setting comes from the fact that $(X, \Delta + \tau(X, \Delta; H)H)$ may not be log canonical, and we can not apply Global ACC directly. Instead, we use Global ACC for generalized polarized pairs, [4], which generalizes Global ACC for the lc pairs, [13], to the generalized lc pairs. The notation and the theory of generalized polarized pairs were introduced and developed in [2, 4]. Furthermore, we can show a slightly stronger version of Theorem 1.4 by using an effective birationality result for generalized polarized pairs of general type established in [4], see Theorem 3.2.

It is natural to generalize the pseudo-effective threshold of a single divisor to multiple divisors. Let (X, Δ) be a log pair and $D_1, \ldots, D_m$ be big $\mathbb{R}$-Cartier divisors on X. The *pseudo-effective polytope* (PE-polytope) of

$D_1, \ldots, D_m$ with respect to X is defined as

$$(1.2) \quad P(X, \Delta; D_1, \ldots, D_m) := \{(t_1, \ldots, t_m) \in \mathbb{R}^m_{\geq 0} \mid$$
$$(K_X + \Delta + t_1 D_1 + \cdots + t_m D_m) \text{ is pseudo-effective}\}.$$

We will show in Proposition 4.2 that $P(X, \Delta; D_1, \ldots, D_m)$ is indeed an unbounded polytope if (X, Δ) has klt singularities and D_i is a big and nef $\mathbb{Q}$-Weil divisor for each i. For convenience, we include all $(t_1, \ldots, t_m) \in \mathbb{R}^m_{\geq 0}$ such that $K_X + \Delta + \sum_{i=1}^m t_i D_i$ is pseudo-effective in $P(X, \Delta; D_1, \ldots, D_m)$, though the *thresholds* only happen on the boundary. In particular, by comparing with (1.1), we see that $P(X, \Delta; D)$ is the interval $[\tau(X, \Delta; D), +\infty)$.

Fix two positive integers n, m, a subset $\mathcal{I} \subset [0, 1]$, a positive real number ϵ, and a nonnegative real number δ, we consider the following set of truncated PE-polytopes.

$$\mathcal{P}_{n,m,\mathcal{I},\epsilon,\delta} := \{P(X, \Delta; H_1, \ldots, H_m) \cap [\delta, +\infty)^m \mid \dim X = n, (X, \Delta) \text{ is } \epsilon\text{-lc},$$
$$\Delta \in \mathcal{I}, H_i \text{ is a big and nef } \mathbb{Q}\text{-Cartier Weil divisor for each } i\}.$$

For simplicity, if $\delta = 0$, we denote $\mathcal{P}_{n,m,\mathcal{I},\epsilon} := \mathcal{P}_{n,m,\mathcal{I},\epsilon,\delta}$. We will show the following results for PE-polytopes.

Theorem 1.5. *Let n, m be two natural numbers and ϵ, δ be two positive real numbers, and $\mathcal{I} \subseteq [0, 1]$ be a finite set. Then the set of truncated PE-polytope $\mathcal{P}_{n,m,\mathcal{I},\epsilon,\delta}$ is finite.*

Letting $m = 1$ in Theorem 1.5, we get Theorem 1.3 (1).

Theorem 1.6. *Let n, m be two natural numbers ,ϵ be a positive real number, and $\mathcal{I} \subseteq [0, 1]$ a DCC set.*
Then the set of PE-polytopes $\mathcal{P}_{n,m,\mathcal{I},\epsilon}$ is a DCC set under the inclusion relation "$\supseteq$".

Letting $m = 1$ in Theorem 1.6, we get Theorem 1.3 (2).

We note that we can not apply Theorem 1.3 to prove Theorem 1.5 and Theorem 1.6 directly. For example, suppose we consider two testing divisors, and thus $P := P(X, \Delta; H_1, H_2)$ is a two dimensional polytope (suppose it is non-degenerate). It is possible to construct a sequence of strictly decreasing sequence of convex polytopes $\{P_i\}_{i \in \mathbb{N}}$, such that $\{P_i\}_{i \in \mathbb{N}}$ stabilizes along any vertical line, any horizontal line, and any line passing through the origin.

For example, in Figure 1, the sequence of polytopes $\{P_i\}_{i \in \mathbb{N}}$ is not stable near the point τ. It is our hope that Theorem 1.5 and Theorem 1.6 would give more information on the testing divisors.
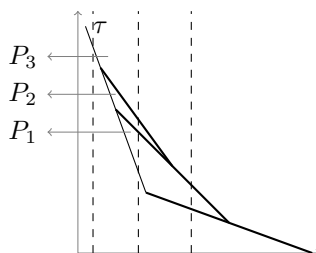

Figure 1.

As a corollary of Theorem 1.3 (1), by the same argument of [11], we can improve their main result for big and nef divisors (rather than big and semiample). This corollary was firstly proven by [17] assuming a weak version of the BAB conjecture.

Recall that for smooth variety X, and a big $\mathbb{R}$-Cartier divisor L on X, the a-constant is defined by $a(X, L) := \tau(X, L)$. For a singular projective variety X, the a-constant is defined by $a(X, L) := a(Y, \pi^* L)$, where $\pi : Y \to X$ is any log resolution of (X, L).

Corollary 1.7 ([17] Theorem 1.3). *Let X be a smooth uniruled projective variety and L a big and nef $\mathbb{Q}$-divisor on X. Then there exists a proper closed subset $W \subset X$ such that every subvariety Y satisfying $a(Y, L) > a(X, L)$ is contained in W.*

The above corollary was used to study the Manin's Conjecture [17], [15], [16].

Finally, it is reasonable to propose the following conjecture for PE-polytopes, which was proven in Theorem 1.4 for a single divisor.

DCC for PE-polytopes. Let n, m be two natural numbers, $\mathcal{I}$ be a DCC set of nonnegative real numbers. Then the set of PE-polytopes

$$\mathcal{P}_{n,m,\mathcal{I}} := \{P(X, \Delta; H_1, \ldots, H_m) \mid \dim X = n, (X, \Delta) \text{ is lc},$$
$$\Delta \in \mathcal{I}, H_i \text{ is a big and nef Cartier divisor for each } i\}$$

satisfies the DCC under the inclusion of polytopes.

2. Preliminaries

2.1. Singularities

For basic definitions of log discrepancies and log canonical (lc), divisorially log terminal (dlt), kawamata log terminal (klt) singularities, we refer to [14]. Recall that a log pair (X, Δ) is ϵ-lc for some $\epsilon \geq 0$, if its minimal log discrepancy is greater or equal to ϵ.

For reader's convenience, we state the following lemma which is known as dlt modifications.

Lemma 2.1 (dlt modifications, c.f. [13] Proposition 3.3.1). *Let (X, Δ) be a lc pair. There is a proper birational morphism $f : X' \to X$ with reduced exceptional divisors E_i, such that*

1) $(X', \Delta') := (X', f_*^{-1}\Delta + \sum E_i)$ *is dlt,*

2) X' *is $\mathbb{Q}$-factorial,*

3) $K_{X'} + \Delta' = f^*(K_X + \Delta)$.

In particular, if (X, Δ) is klt, then f is small.

We can apply dlt modifications to reduce the study of PE-polytopes from lc (resp. klt) pairs to $\mathbb{Q}$-factorial dlt (resp. klt) pairs.

Lemma 2.2. *Let (X, Δ) be a lc pair, and H_i be a big $\mathbb{R}$-Cartier divisor for any $1 \leq i \leq m$. If $f : (X', \Delta') \to (X, \Delta)$ is a dlt modification of (X, Δ), then*

$$P(X, \Delta; H_1, \ldots, H_m) = P(X', \Delta'; H_1', \ldots, H_m'),$$

where $H_i' = f^*H_i$.

Proof. On one hand, let $(t_1, \ldots, t_m) \in P(X, \Delta; H_1, \ldots, H_m)$, then $K_X + \Delta + \sum_{i=1}^{m} t_i H_i$ is pseudo-effective. We have $K_{X'} + \Delta' + \sum_{i=1}^{m} t_i H_i' = f^*(K_X + \Delta + \sum_{i=1}^{m} t_i H_i)$ is also pseudo-effective.

On the other hand, let $(t_1, \ldots, t_m) \in P(X', \Delta'; H_1', \ldots, H_m')$, $K_{X'} + \Delta' + \sum_{i=1}^{m} t_i H_i'$ is pseudo-effective. We have $K_X + \Delta + \sum_{i=1}^{m} t_i H_i = f_*(K_{X'} + \Delta' + \sum_{i=1}^{m} t_i H_i')$ is pseudo-effective. $\square$

2.2. Boundedness of Fano varieties

A set of varieties $\mathcal{X}$ is said to form a *bounded family* if there is a projective morphism of schemes $g : W \to T$, with T of finite type, such that for every $X \in \mathcal{X}$, there is a closed point $t \in T$ and an isomorphism $W_t \simeq X$, where W_t is the fibre of g at t. A variety X is called *Fano* if it is lc and $-K_X$ is ample. The following result is a variant of the conjecture of Borisov-Alexeev-Borisov.

Theorem 2.3 ([3] Corollary 1.2). *Let n be a natural number and ϵ a positive real number. Then the projective varieties X such that*

1) *(X, Δ) is ϵ-lc of dimension n for some boundary Δ, and*

2) *$K_X + \Delta \equiv_\mathbb{R} 0$ and Δ is big,*

form a bounded family.

2.3. Generalized polarized pairs

The theory of generalized pair was developed in [4].

Definition 2.4 (generalized polarized pairs). A generalized polarized pair consists of a normal variety X equipped with projective morphisms

$$W \xrightarrow{f} X \to Z$$

where f is birational, W is normal, an $\mathbb{R}$-boundary $\Delta \geq 0$, and an $\mathbb{R}$-Cartier divisor H_W on W which is nef over Z such that $K_X + \Delta + H$ is $\mathbb{R}$-Cartier with $H = f_* H_W$. We call Δ the boundary part and H_W the nef part. We usually refer to the pair by saying $(X, \Delta + H)$ is a generalized pair with data $W \xrightarrow{f} X \dashrightarrow Z$ and H_W.

In this paper, we only need to use the case that Z is a point. Thus, we will drop Z, and say the pair is projective. Note that if $W' \to W$ is a projective birational morphism from a normal variety, then there is no harm in replacing W with W' and replacing H_W with its pullback to W'.

Definition 2.5 (generalized lc). Let $(X, \Delta + H)$ be a generalized polarized pair, which comes with the data $W \xrightarrow{f} X \dashrightarrow Z$, and replacing W, we

may assume that f is a log resolution of (X, Δ). We can write

$$K_W + \Delta_W + H_W = f^*(K_X + \Delta + H)$$

for some uniquely determined Δ_W.

We say $(X, \Delta + H)$ is *generalized lc* if every coefficient of Δ_W is less than or equal to 1.

Remark 2.6 ([4] Remark 4.2(6)). Let $(X, \Delta + H)$ be a generalized projective pair with data $W \xrightarrow{f} X$ and H_W. We may assume that f is a log resolution of (X, Δ). Assume that there is a contraction $X \to Y$. Let F be a general fibre of $W \to Y$, T the corresponding fibre of $X \to Y$, and $g : F \to T$ the induced morphism. Let

$$\Delta_F = \Delta_W|_F, H_F = H_W|_F, \Delta_T = g_*(\Delta_F), H_T = g_* H_F.$$

Then $(T, \Delta_T + H_T)$ is a generalized polarized projective pair with the data $F \xrightarrow{g} T$ and H_F. Moreover,

$$K_T + \Delta_T + H_T = (K_X + \Delta + H)|_T.$$

In addition, $\Delta_T = \Delta|_T$ and $H_T = H|_T$.

For more properties of generalized polarized pairs, we refer to [2, 4]. We will need the following result to prove Theorem 1.4.

Theorem 2.7 (global ACC for generalized pairs, [4] Theorem 1.6). *Let $\mathcal{I}$ be a DCC set of nonnegative real numbers and n a natural number. Then there is a finite subset $\mathcal{I}_0 \subseteq \mathcal{I}$ depending only on $\mathcal{I}, n$ such that*

1) *X is a normal projective variety of dimension n,*

2) *$(X, \Delta + H)$ is generalized lc with data $W \xrightarrow{f} X$ and H_W,*

3) *$H_W = \sum \mu_j H_{j,W}$, where $H_{j,W}$ are nef Cartier divisors and $\mu_j \in \mathcal{I}$,*

4) *$\mu_j = 0$ if $H_{j,W} \equiv 0$,*

5) *the coefficients of Δ belong to $\mathcal{I}$, and*

6) *$K_X + \Delta + H \equiv_{\mathbb{R}} 0$,*

then the coefficients of Δ and μ_j belong to $\mathcal{I}_0$.

3. Fujita's spectrum conjecture and Fujita's log spectrum conjecture

Proof of Theorem 1.3. By Lemma 2.2, we can assume that X is $\mathbb{Q}$-factorial. Let $\tau = \tau(X, \Delta; H)$, then $K_X + \Delta + \tau H$ is pseudo-effective but not big. We can assume that K_X is not pseudo-effective.

Since H is big and nef, there is an effective divisor E, and ample $\mathbb{Q}$-divisors A_k such that $H \sim_{\mathbb{Q}} A_k + \frac{E}{k}$, for any $k \gg 1$. Let N be a natural number such that $(X, \Delta + \tau \frac{E}{N})$ is $\frac{\epsilon}{2}$-lc, and N' a natural numbers such that $N' A_N$ is a very ample divisor. Let $A' \in |N' A_N|$ be a general very ample divisor, and $H' := \frac{1}{N'} A' + \frac{E}{N} \sim_{\mathbb{Q}} H$, then

$$(X, \Gamma) := (X, \Delta + \tau H')$$

is still $\frac{\epsilon}{2}$-lc.

According to [1], we may run a $(K_X + \Gamma)$-MMP, $\phi : X \dashrightarrow Y$, such that ϕ is a birational map, $K_Y + \phi_* \Gamma$ is semiample and $(Y, \phi_* \Gamma)$ is still $\frac{\epsilon}{2}$-lc. As ϕ is $(K_X + \Gamma)$-negative, $K_Y + \phi_* \Gamma$ is not big as well.

Now $K_Y + \phi_* \Gamma$ defines a contraction $f : Y \to Z$. Let F be a general fiber of f, we have $\dim(F) > 0$. Restricting to F, we get

$$(3.1) \qquad K_F + \phi_* \Gamma|_F = K_F + \phi_* \Delta|_F + \tau H_Y|_F \equiv_{\mathbb{R}} 0,$$

where H_Y is the strict transform of H' on Y. We note that H_Y is big, and $H_Y|_F$ is also big since it is the restriction of a big divisor on a general fiber.

Since $K_F + \phi_* \Gamma|_F$ is $\frac{\epsilon}{2}$-lc, according to Theorem 2.3, F belongs to a bounded family. We may find a very ample Cartier divisor M_F on F, so that $-K_F \cdot M_F^{\dim F - 1}$ is bounded. Besides, as $H' \sim_{\mathbb{Q}} H$, we have $\phi_*(H') \equiv_{\mathbb{Q}} \phi_*(H)$. Recall that H is a Weil divisor, hence $\phi_*(H)$ is again a Weil divisor, and the intersection number $d := H_Y|_F \cdot M_F^{\dim F - 1} = \phi_*(H)|_F \cdot M_F^{\dim F - 1}$ is a positive integer. Let $\phi_* \Delta|_F = \sum_j a_j \Delta_{F,j}$, where $a_j \in \mathcal{I}$, and $\Delta_{F,j}$ is a Weil divisor. By intersecting (3.1) with $M_F^{\dim F - 1}$, we obtain an equation for τ,

$$(3.2) \qquad \tau d + \sum_j a_j b_j = c,$$

where $c = -K_F \cdot M_F^{\dim F - 1}$ is a nonnegative integral with only finite possibilities, $b_j = \Delta_{F,j} \cdot M_F^{\dim F - 1}$ are nonnegative integers.

First, we prove Theorem 1.3 (1). By assumption $\tau \geq \delta > 0$ and $\mathcal{I}$ is finite, d, b_j are bounded above and thus only have finite possibilities. Hence there are only finite possibilities for the τ in equation (3.2).

Next, we prove Theorem 1.3 (2). Recall that $a_j \in \mathcal{I}$ which is a DCC set, and $b_j \in \mathbb{N}$, then the finite summations $\sum_j a_j b_j \in \sum \mathcal{I}$ still form a DCC set. Now

$$(3.3) \qquad \tau = \frac{1}{d}\left(c - \sum_j a_j b_j\right).$$

The right hand side of the equation (3.3) belongs to an ACC set, and $\mathcal{T}_{n,\epsilon}(\mathcal{I})$ satisfies the ACC. $\qquad \square$

Proof of Corollary 1.7. Replacing Theorem 1.1 by Theorem 1.3 in the proof of [11], we get Corollary 1.7. $\qquad \square$

In the following example, we see that Theorem 1.3 (1) is no longer true even for an ample Cartier divisor if we relax the singularities from ϵ-lc to klt, and Theorem 1.3 (2) is no longer true even for an ample $\mathbb{Q}$-Cartier Weil divisor if we relax the singularities from ϵ-lc to klt. We thank Chen Jiang for providing us this example.

Example 3.1. Let n be a natural number and $\mathbb{P}(1,1,n)$ be the weighted projective space. It is a toric variety with the lattice N spanned by $\{(1,0), (\frac{1}{n}, \frac{1}{n})\}$. Let $v_1 = (1,0), v_2 = (0,1)$ and $v_3 = (-\frac{1}{n}, -\frac{1}{n})$, then the fan is generated by maximal cones $\sigma_1 = \langle v_1, v_2 \rangle, \sigma_2 = \langle v_2, v_3 \rangle$ and $\sigma_3 = \langle v_1, v_3 \rangle$ (c.f. [10] P.35). Let D_i be the toric invariant divisor corresponding to v_i. By choosing $(n,0)$ and $(0,n)$ in the dual lattice $N^\vee$, we see that $nD_1 \sim D_3$ and $nD_2 \sim D_3$. Thus

$$-K_{\mathbb{P}(1,1,n)} \sim_{\mathbb{Q}} D_1 + D_2 + D_3 \sim_{\mathbb{Q}} \left(1 + \frac{2}{n}\right) D_3.$$

Moreover D_3 is an ample Cartier divisor as it is associated to the lattice $(0,0) \in N^\vee$ for σ_1, $(n,0) \in N^\vee$ for σ_2 and $(0,n) \in N^\vee$ for σ_3. As $\mathbb{P}(1,1,n)$ has Picard number one, $(1 + \frac{2}{n})$ is the pseudo-effective threshold of D_3. This gives a set of varieties whose pseudo-effective spectrum is infinite away from 0. Notice that the minimal log discrepancy of $\mathbb{P}(1,1,n)$ is $\frac{2}{n}$ (c.f. [5]). Hence, it is a counterexample for Theorem 1.3 (1) if we replace ϵ-lc by klt in the statement.

Besides, D_1 is an integral divisor and

$$-K_{\mathbb{P}(1,1,n)} \sim_{\mathbb{Q}} D_1 + D_2 + D_3 \sim_{\mathbb{Q}} (n+2)D_1.$$

Hence the pseudo-effective threshold of D_1 is $n+2$. Thus we get a family whose pseudo-effective thresholds (with respect to D_1) are strictly increasing. This gives a counterexample if we replace ϵ-lc by klt in Theorem 1.3 (2).

Proof of Theorem 1.4. Let $\tau = \tau(X, \Delta, H)$. We first prove the theorem for the case that (X, Δ) is $\mathbb{Q}$-factorial klt.

By assumption, H is big and nef, as in the proof of Theorem 1.3, we can run a $(K_X + \Delta + \tau H)$-MMP, $\phi : X \dashrightarrow Y$, and reach a minimal model $(Y, \phi_*(\Delta + \tau H))$, on which $K_Y + \phi_*(\Delta + \tau H)$ is semiample defining a contraction $Y \to Z$.

Taking a common log resolution $p : W \to (X, \Delta)$, $q : W \to (Y, \phi_*\Delta)$, let $H_W = p^*H$. Then $(X, \Delta + \tau H)$ is generalized lc, as (X, Δ) is lc. Since $p^*(K_X + \Delta + \tau H) \geq q^*(K_Y + \phi_*(\Delta + \tau H))$, $(Y, \phi_*(\Delta + \tau H))$ is also generalized lc. Let F be a general fiber of $W \to Z$, and T be the corresponding fiber of $Y \to Z$. Again, we have $\dim(T) > 0$. By restricting to the general fiber T, $(T, \phi_*(\Delta + \tau H)|_T)$ is generalized lc, and $K_T + \phi_*(\Delta + \tau H)|_T \equiv 0$. Since H_W is big, $H_W|_F$ is not numerically trivial, and there exists some component $q^*(H_j)|_F$ of $H_W|_F$ which is not numerically trivial. If $\{\tau\}$ forms a strictly increasing sequence, then $\{\mu_j \tau\}$ belongs to a DCC set. According to Theorem 2.7, $\{\mu_j \tau\}$ belongs to a finite set, a contradiction. Therefore, $\{\tau\}$ belongs to an ACC set.

For the general case, according to Lemma 2.2, we may assume that (X, Δ) is $\mathbb{Q}$-factorial dlt. If the statement were not true, then there exists a sequence of lc pairs $(X^{(i)}, \Delta^{(i)})$, and a big $\mathbb{R}$-Cartier $H^{(i)}$ on $X^{(i)}$ satisfying the assumption of Theorem 1.4, but $\tau_i := \tau(X^{(i)}, \Delta^{(i)}, H^{(i)})$ is strictly increasing. For any $1 \geq \epsilon \geq 0$, let $\tau_{i,\epsilon} := \tau(X^{(i)}, (1 - \epsilon)\Delta^{(i)}; H^{(i)})$. It is clear that $\tau_{i,\epsilon} \geq \tau_i$, and there exists a decreasing sequence, $\epsilon_i \to 0$, such that $\tau_{i+1} > \tau_{i,\epsilon_i} \geq \tau_i$. Let $\mathcal{J} = \{1 - \epsilon_i\}$. Now $(X^{(i)}, (1 - \epsilon_i)\Delta^{(i)})$ is $\mathbb{Q}$-factorial klt, the coefficients of $(1 - \epsilon_i)\Delta^{(i)}$ belong to $\mathcal{IJ}$, which is a DCC set. However, the sequence $\{\tau_{i,\epsilon_i}\}_{i \in \mathbb{N}}$ is strictly increasing. This contradicts to the above $\mathbb{Q}$-factorial case. $\square$

Inspired by a private communication with Di Cerbo, we can prove a slightly stronger version of Theorem 1.4 by using an effective birationality result for generalized polarized pairs of general type established in [4].

Theorem 3.2. *Let n be a natural number and $\mathcal{I}$ a DCC set of nonnegative real numbers. Let $\mathcal{D}_n(\mathcal{I})$ be the set of pseudo-effective threshold $\tau(X, \Delta; M)$ which satisfies the following conditions.*

1) *X is a normal projective variety of dimension n,*

2) *(X, Δ) is lc and the coefficients of Δ are in $\mathcal{I}$,*

3) *$M = \sum \mu_i M_i$, where M_i is a nef Cartier divisor for each i, $\mu_i \in \mathcal{I}$, and*

4) *$K_X + \Delta + M$ is big.*

Then $\mathcal{D}_n(\mathcal{I})$ satisfies the ACC.

Proof. Otherwise, there exists a sequence $\{(X^{(i)}, \Delta^{(i)}, M^{(i)})\}$, such that $\alpha_i = \tau(X^{(i)}, \Delta^{(i)}; M^{(i)})$ is strictly increasing and $\lim_{i \to +\infty} \alpha_i = \alpha \le 1$. Then,

$$\tau(X^{(i)}, \Delta^{(i)}; \alpha M^{(i)}) = \frac{\alpha_i}{\alpha} \to 1,$$

and $K_X^{(i)} + \Delta^{(i)} + \alpha M^{(i)}$ is big. Since the coefficients of αM^i belong to the DCC set $\mathcal{I} \cup \alpha \mathcal{I}$, by Theorem 8.2 in [4], there exists a natural number m depending only on n and $\mathcal{I} \cup \alpha \mathcal{I}$, such that the linear system $\lfloor\lfloor m(K_X + \Delta)\rfloor\rfloor + \sum \lfloor m\alpha\mu_j \rfloor M_j |$ defines a birational map. By Lemma 2.3.4 in [12], $K_X + \Delta + (2n+1)(m(K_X + \Delta + \alpha M))$ is big. Since

$$K_X + \Delta + (2n+1)(m(K_X + \Delta + \alpha M))$$
$$\sim_{\mathbb{R}} ((2n+1)m + 1) \left(K_X + \Delta + \frac{(2n+1)m\alpha}{(2n+1)m+1} M \right),$$

we have $\alpha_i = \tau(X^{(i)}, \Delta^{(i)}; M^{(i)}) \le \frac{(2n+1)m}{(2n+1)m+1}\alpha$. This implies that

$$\lim_{i \to +\infty} \alpha_i \le \frac{(2n+1)m}{(2n+1)m+1}\alpha < \alpha,$$

a contradiction. $\square$

Remark 3.3. In Theorem 3.2, we do not require that M to be big, and Theorem 3.2 implies Theorem 1.4 . In fact, let $\mathcal{I}$ be a DCC set, (X, Δ) be a projective $\mathbb{Q}$-factorial dlt pair of dimension n, $H = \sum \mu_i H_i$ be a big and nef $\mathbb{R}$-Cartier divisor. Set $\delta = \min\{\mathcal{I}^{>0}\}$, where $\Delta, \mu_j \in \mathcal{I}$. One can show that $K_X + \Delta + \frac{(2n+1)}{\delta} H$ is also big (see, for example, [2] Lemma 2.30). Let $M = \frac{(2n+1)}{\delta} H$, Theorem 1.4 follows from Lemma 2.2 and Theorem 3.2.

4. The Finiteness and the DCC property for pseudo-effective polytopes

In this section, we prove Theorem 1.5 and Theorem 1.6, which generalize Theorem 1.3 from a single divisor to multiple divisors. The proofs are similar.

4.1. Pseudo-effective polytopes

Let X be a normal projective variety, and V be a finite dimensional affine subspace of the real vector space $\mathrm{WDiv}_{\mathbb{R}}(X)$ of Weil divisors on X. Fix an $\mathbb{R}$-divisor $A \geq 0$ and define (see [1] Definition 1.1.4)

$$\mathcal{E}_A(V) = \{\Delta = A + B \mid B \in V, B \geq 0, K_X + \Delta \text{ is lc and pseudo-effective}\}.$$

Notice that $\mathcal{E}_A(V)$ is a compact set by the lc requirement. Under this notation, we have the following result.

Theorem 4.1 ([1] Corollary 1.1.5). *Let X be a normal projective variety, V be a finite dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$ which is defined over rationals. Suppose there is a divisor $\Delta_0 \in V$ such that $K_X + \Delta_0$ is klt. Let A be a general ample $\mathbb{Q}$-divisor, which has no components in common with any element of V. Then $\mathcal{E}_A(V)$ is a rational polytope.*

We can deduce that PE-polytopes are indeed polytopes under suitable assumptions from Theorem 4.1. Notice that there are no lc restrictions for PE-polytopes. See Figure 2 for a PE-polytope of two divisors.
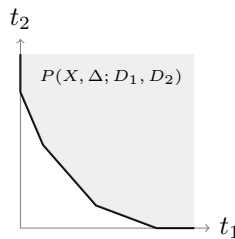

Figure 2: A PE-polytope of two divisors.

Proposition 4.2. *Let (X, Δ) be a klt pair and $H_1, \ldots, H_m$ be big and nef $\mathbb{Q}$-Weil divisors. Then $P(X, \Delta; H_1, \ldots, H_m)$ is an unbounded polytope.*

Proof. By definition (see (1.2)), $P(X, \Delta; H_1, \ldots, H_m)$ is convex. If K_X is pseudo-effective, then $P(X, \Delta; H_1, \ldots, H_m) = \mathbb{R}^m_{\geq 0}$, hence we can assume that K_X is not pseudo-effective. Then there exists a rational number $a > 0$ such that $K_X + a(H_1 + \cdots + H_m)$ is also not pseudo-effective.

Because H_i are big divisors, there exists constant $\tau > 0$, such that $K_X + \Delta + \sum_{i=1}^m t_i H_i$ is pseudo-effective when $t_i \geq \tau$ for some i. In other words, $\mathbb{R}^m_{\geq 0} \setminus [0, \tau]^m$ is contained in $P(X, \Delta; H_1, \ldots, H_m)$. Thus, it suffices to show that $P(X, \Delta; H_1, \ldots, H_m) \cap [0, \tau]^m$ is a polytope.

Recall that a divisor is pseudo-effective if and only if the divisor which is numerical to it is also pseudo-effective, thus,

$$(4.1) \qquad P(X, \Delta; H_1, \ldots, H_m) = P(X, \Delta; H_1', \ldots, H_m')$$

for any $H_i \equiv_{\mathbb{Q}} H_i'$. Since H_i is big and nef, there is an effective divisors E_i, and ample $\mathbb{Q}$-divisor $A_{i,k}$ such that $H_i \equiv A_{i,k} + \frac{E_i}{k}$, for any $k \gg 1$. Let N be a natural number such that $(X, \Delta + \tau \sum_{i=1}^m \frac{E_i}{N})$ is klt, and $N_i > \tau$ be a natural number such that $N_i A_{i_N}$ is a very ample divisor. Let $A_i' \in |N_i A_{i_N}|$ be a general very ample divisor, and $H_i' = \frac{1}{N_i} A_i' + \frac{E_i}{N}$, $(X, \Delta + \tau \sum_{i=1}^m H_i')$ is also klt. Now, we only need to show that $P(X, \Delta; H_1', \ldots, H_m') \cap [0, \tau]^m$ is a polytope.

For each i, let $M_i' \in |A_i'|$ be another general very ample divisor, and let $M_i = \frac{a}{N_i} M_i'$. Let V_i be the affine space $\Delta + \frac{a}{N} E_i + \sum_{i=1}^m \mathbb{R} H_i'$. By Theorem 4.1, $\mathcal{E}_{M_i}(V_i)$ is a rational polytope.

Let

$$\mathcal{E}_i := \left\{ (t_1, \ldots, t_{i-1}, t_i + a, t_{i+1}, \ldots, t_m) \mid \right.$$
$$\left. \Delta + M_i + \frac{a}{N} E_i + \sum_{i=1}^m t_i H_i' \in \mathcal{E}_{M_i}(V_i) \right\}.$$

It is clear that $\mathcal{E}_i \cap [0, \tau]^m \subseteq P(X, \Delta; H_1', \ldots, H_m') \cap [0, \tau]^m$ for each i. Now if $(t_1, \ldots, t_m) \in P(X, \Delta; H_1', \ldots, H_m') \cap [0, \tau]^m$, there exists at least one k such that $t_k \geq a$ (recall that we chose a such that $K_X + a(H_1 + \cdots + H_m)$ is not pseudo-effective). This implies that $\Delta + M_k + \frac{a}{N} E_k + \sum_{i=1}^m t_i H_i' - a H_k' \in \mathcal{E}_{M_k}(V_k)$, and $(t_1, \ldots, t_m) \in \mathcal{E}_k$. Thus,

$$\bigcup_{i=1}^m \mathcal{E}_i \cap [0, \tau]^m = P(X, \Delta; H_1', \ldots, H_m') \cap [0, \tau]^m.$$

In particular, $P(X, \Delta; H_1', \ldots, H_m') \cap [0, \tau]^m$ is a polytope by convexity. $\square$
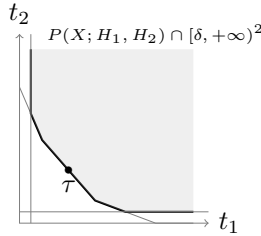
4.2. Proofs of Theorem 1.5 and Theorem 1.6

Figure 3: A truncated PE-polytope.

Proof of Theorem 1.5. By Lemma 2.2, we can assume that X is $\mathbb{Q}$-factorial. Choose arbitrary $\tau = (\tau_1, \ldots, \tau_m)$ on the boundary of the truncated PE-polytope $P(X, \Delta; H_1, \ldots, H_m) \cap [\delta, +\infty)^m$ and in the interior of $[\delta, +\infty)^m$, then $K_X + \Delta + \tau_1 H_1 + \cdots \tau_m H_m$ is pseudo-effective but not big. If K_X is pseudo-effective, then $P(X, \Delta; H_1, \ldots, H_m) \cap [\delta, +\infty)^m = [\delta, +\infty)^m$, we can assume that K_X is not pseudo-effective.

Since H_i is big and nef, by the same argument in the proof of Theorem 1.3, we can find $\mathbb{Q}$-divisors $H_i' \sim_{\mathbb{Q}} H_i$, such that

$$(X, \Gamma) := \left(X, \Delta + \sum_{i=1}^{m} \tau_i H_i' \right)$$

is $\frac{\epsilon}{2}$-lc, and we may run a $(K_X + \Gamma)$-MMP, $\phi : X \dashrightarrow Y$, such that ϕ is a birational map, $K_Y + \phi_* \Gamma$ is semiample and $(Y, \phi_* \Gamma)$ is still $\frac{\epsilon}{2}$-lc.

Now $K_Y + \phi_* \Gamma$ defines a contraction $f : Y \to Z$. Let F be a general fiber of f, we have $\dim(F) > 0$. By restricting to F, we get

$$(4.2) \qquad K_F + \phi_* \Gamma|_F = K_F + \phi_* \Delta_F + \sum_{i=1}^{m} \tau_i H_{Y,i}|_F \equiv 0,$$

where $H_{Y,i}$ is the strict transform of H_i' on Y for each i.

Since $K_F + \phi_* \Delta + \sum_{i=1}^{m} \tau_i H_{Y,i}|_F$ is $\frac{\epsilon}{2}$-lc, according to Theorem 2.3, F belongs to a bounded family. Hence, we may find a very ample Cartier divisor M on F, so that $-K_F \cdot M^{\dim F - 1}$ is bounded. Besides, as $H_i' \sim_{\mathbb{Q}} H_i$, we have $\phi_*(H_i') \equiv_{\mathbb{Q}} \phi_*(H_i)$, and the intersection numbers $d_i := H_{Y,i}|_F \cdot M^{\dim F - 1} =$

$\phi_*(H_i)|_F \cdot M^{\dim F-1}$ is a positive integer. Let $\phi_*\Delta|_F = \sum_j a_j \Delta_{F,j}$, where $a_j \in \mathcal{I}$, and $\Delta_{F,j}$ are Weil divisors. By intersecting (4.2) with $M^{\dim F-1}$, we obtain an equation for τ_i,

$$(4.3) \qquad \sum_j a_j b_j + \sum_{i=1}^{m} \tau_i d_i = c,$$

where $c = -K_F \cdot M^{\dim F-1}$ is a nonnegative integer with only finite possibilities, $b_j = \Delta_{F,j} \cdot M^{\dim F-1}$ is a positive integer. Since $\tau_i \geq \delta$ for all i, and $\mathcal{I}$ is finite, d_i, b_i are bounded above and thus only have finite possibilities. Hence there are only finite possibilities for the equations (4.3). In other words, $(\tau_1, \ldots, \tau_m)$ can only lie on finitely many hyperplanes $L_k \subseteq \mathbb{R}^m, k \in I$. By Proposition 4.2, $P(X, \Delta; H_1, \ldots, H_m) \cap [\epsilon, +\infty)^m$ is a polytope. If Θ is a facet of $P(X; H_1, \ldots, H_m) \cap [\epsilon, +\infty)^m$ (i.e. an $(m-1)$-dimensional face), then

$$\Theta \subseteq (\cup_{k \in I} L_k) \cup (\cup_{i=1}^m \{t_i = \delta\}).$$

By irreducibility, $\Theta \subseteq L_k$ or $\Theta \subseteq \{t_i = \delta\}$, and hence finite possibilities. This shows that the set $\mathcal{P}_{n,m,\mathcal{I},\epsilon,\delta}$ is finite. $\qquad\square$
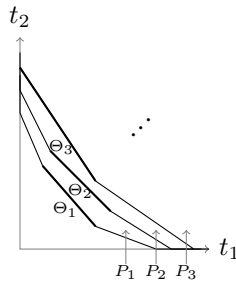

Figure 4: A decreasing sequence of PE-polytopes.

Proof of Theorem 1.6. By Lemma 2.2, we can assume that X is $\mathbb{Q}$-factorial. Without loss of generality, we can assume that K_X is not pseudo-effective.

Choose arbitrary $\tau = (\tau_1, \ldots, \tau_m)$ on the boundary of the PE-polytope $P(X, \Delta; H_1, \ldots, H_m)$ which is in the interior of $\mathbb{R}^m_{\geq 0}$. We then proceed the same way as in the proof of Theorem 1.3. We can find $\mathbb{Q}$-divisors $H'_i \sim_{\mathbb{Q}} H_i$,

such that

$$(X, \Gamma) := \left(X, \Delta + \sum_{i=1}^{m} \tau_i H_i' \right)$$

is $\frac{\epsilon}{2}$-lc, and we may run a $(K_X + \Gamma)$-MMP, $\phi : X \dashrightarrow Y$, such that $K_Y + \phi_* \Gamma$ is semiample. $K_Y + \phi_* \Gamma$ defines a contraction $f : Y \to Z$. By restricting to a general fiber F, we get a similar equation as (4.2)

$$(4.4) \qquad K_F + \phi_* \Gamma|_F = K_F + \sum_j a_j \Delta_{F,j} + \sum_{i=1}^{m} \tau_i H_{Y,i}|_F \equiv 0.$$

Here we write $\phi_* \Delta|_F = \sum_j a_j \Delta_{F,j}$, where $\Delta_{F,j}$ is a Weil divisor. As $(F, \phi_* \Gamma|_F)$ is $\frac{\epsilon}{2}$-lc, it belongs to a bounded family by Theorem 2.3. By taking a very ample divisor M, and intersecting $M^{\dim F - 1}$ with (4.4), we get

$$(4.5) \qquad \sum_j a_j b_j + \sum_{i=1}^{m} \tau_i d_i = c,$$

where $c = -K_F \cdot M^{\dim F - 1}$ is a nonnegative integer with only finite possibilities, $b_j = \Delta_{Y,j} \cdot M^{\dim F - 1}$, $d_i = H_{Y,i}|_F \cdot M^{\dim F - 1}$ are nonnegative integers.

Now, all $a_j b_j$ form a DCC set, and the finite summations $\sum_j a_j b_j \in \sum \mathcal{I}$ also form a DCC set.

Set $P^{(k)} = P(X^{(k)}, \Delta^{(k)}; H_1^{(k)}, \ldots, H_m^{(k)})$. Suppose $P^{(1)} \supsetneq P^{(2)} \supsetneq \cdots \supsetneq P^{(k)} \supsetneq \cdots$ is a sequence of decreasing polytopes. Then there are *countably* many linear equations $\{L^{(s)}\}_{s \in \mathbb{N}}$, where

$$(4.6) \qquad L^{(s)} := \sum_{i=1}^{m} t_i d_i^{(s)} + \sum_j a_j^{(s)} b_j^{(s)} - c^{(s)},$$

where $d_i^{(s)}, b_j^{(s)} \in \mathbb{N}$, $a_j^{(s)} \in \mathcal{I}$, and $c^{(s)}$ belongs to a finite set. If $\Theta^{(k)}$ is a facet of $P^{(k)}$, then $\Theta^{(k)}$ must lie on the *countably* union of hyperplanes

$$(\cup_{s \in \mathbb{N}} \{L^{(s)} = 0\}) \cup (\cup_{i=1}^{m} \{t_i = 0\}),$$

where $\{t_i = 0\}$ is the i-th coordinate hyperplane. If $\Theta^{(k)}$ is not contained in any of $\{L^{(s)} = 0\}$ nor $\{t_i = 0\}$, then their intersections on $\Theta^{(k)}$ are measure zero sets. Thus, their countable unions is still of measure zero. This is impossible and hence $\Theta^{(k)}$ must be contained in one of $\{L^{(s)} = 0\}$ or $\{t_i = 0\}$.

Since $\{P^{(k)}\}_{k\in\mathbb{N}}$ is a strictly decreasing sequence, there must exist a sequence $\{\Theta^{(k)}\}_{k\in\mathbb{N}}$ such that $\Theta^{(k)} \subseteq P^{(k)}$ is a facet, $\Theta^{(k)} \subseteq \{t\in\mathbb{R}^m_{\geq 0} \mid L^{(k)}(t) = 0\}$, and

$$\{t \in \mathbb{R}^m_{\geq 0} \mid L^{(k)}(t) = 0\}_{k\in\mathbb{N}}$$

are different sets (c.f. Figure 4).

Using the DCC property, by passing to a subsequence, we can assume that, in (4.6), $c^{(k)} = c$ is fixed for all k, the sequence $\{\sum_j a_j^{(k)} b_j^{(k)}\}_{k\in\mathbb{N}}$ is non-decreasing, and the sequence $\{n_i^{(k)}\}_{k\in\mathbb{N}}$ is non-decreasing for each i.

Now, the set

(4.7) $$\{t \in \mathbb{R}^m_{\geq 0} \mid L^{(k)}(t) \geq 0\}_{k\in\mathbb{N}}$$

is strictly increasing. However, by assumption, we have

$$\{t \in \mathbb{R}^m_{\geq 0} \mid L^{(k)}(t) \geq 0\} \supseteq P^{(k)} \supsetneq P^{(k+1)} \supseteq \Theta^{(k+1)}.$$

This is impossible as there exists $\theta \in \Theta^{(k+1)}$ lies on $L^{(k+1)} = 0$ which is not contained in $\{t \in \mathbb{R}^m_{\geq 0} \mid L^{(k)}(t) \geq 0\}$ by (4.7). $\square$

Acknowledgements

References

[1] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468.

[2] Caucher Birkar, *Anti–pluricanonical systems on Fano varieties*, to appear in Ann. of Math. `arXiv:math/1603.05765`, (2016).

[3] Caucher Birkar, *Singularities of linear systems and boundedness of Fano varieties*, `arXiv:math/1609.05543`, (2016).

[4] Caucher Birkar and De-Qi Zhang, *Effectivity of Iitaka fibrations and pluricanonical systems of polarized pairs*, Publ. Math. Inst. Hautes Études Sci. **123** (2016), 283–331.

[5] Alexandr Borisov, *Minimal discrepancies of toric singularities*, Manuscripta Math. **92** (1997), no. 1, 33–45.

[6] Gabriele Di Cerbo, *On Fujita's log spectrum conjecture*, Math. Ann. **366** (2016), no. 1-2, 447–457.

[7] Gabriele Di Cerbo, *On Fujita's spectrum conjecture*, Adv. Math. **311** (2017), 238–248.

[8] Takao Fujita, *On Kodaira energy and adjoint reduction of polarized manifolds*, Manuscripta Math. **76** (1992), no. 1, 59–84.

[9] Takao Fujita, *On Kodaira energy of polarized log varieties*, J. Math. Soc. Japan **48** (1996), no. 1, 1–12.

[10] William Fulton, Introduction to Toric Varieties, Vol. 131 of Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, (1993). The William H. Roever Lectures in Geometry.

[11] Christopher D. Hacon and Chen Jiang, *On Fujita invariants of subvarieties of a uniruled variety*, Algebraic Geomerty **4** (2017), no. 3, 304–310.

[12] Christopher D. Hacon, James McKernan, and Chenyang Xu, *On the birational automorphisms of varieties of general type*, Ann. of Math. **177** (2013), no. 3, 1077–1111.

[13] Christopher D. Hacon, James McKernan, and Chenyang Xu, *ACC for log canonical thresholds*, Ann. of Math. **180** (2014), no. 2, 523–571.

[14] János Kollár and Shigefumi Mori, Birational Geometry of Algebraic Varieties, Vol. 134 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, (1998). With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[15] B. Lehmann, S. Tanimoto, *On the geometry of thin exceptional sets in Manin's Conjecture*, Duke Math. J. **166** (2017), no. 15, 2815–2869.

[16] B. Lehmann, S. Tanimoto, *Geometric Manin's Conjecture and rational curves*, Compos. Math. **155** (2019), no. 5, 833–862.

[17] B. Lehmann, S. Tanimoto, and Y. Tschinkel, *Balanced line bundles on Fano varieties*, J. Reine Angew. Math. **743** (2018), 91–131.

Beijing International Center for Mathematical Research
Peking University, Beijing 100871, China
Current address:
Department of Mathematics, Johns Hopkins University
Baltimore, MD 21218, USA
E-mail address: `jhan@math.jhu.edu`

Department of Mathematics
Southern University of Science and Technology
1088 Xueyuan Rd, Shenzhen 518055, China
E-mail address: `lizhan@sustech.edu.cn`