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Improved bounds for the bilinear spherical maximal operators

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In this paper we study the bilinear multiplier operator of the form

$$H^t(f,g)(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} m(t\xi, t\eta)\, \mathrm{e}^{2\pi \mathrm{i}t|(\xi,\eta)|}\, \widehat{f}(\xi)\, \widehat{g}(\eta)\, \mathrm{e}^{2\pi \mathrm{i}x(\xi+\eta)}\, d\xi d\eta, \quad 1 \le t \le 2$$

where m satisfies the Marcinkiewicz-Mikhlin-Hörmander's derivative conditions. And by obtaining some estimates for H^t, we establish the $L^{p_1}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ estimates for the bi(sub)-linear spherical maximal operators

$$\mathcal{M}(f,g)(x) = \sup_{t>0} \left| \int_{\mathbb{S}^{2d-1}} f(x-ty)\, g(x-tz)\, d\sigma_{2d}(y,z) \right|$$

which was considered by Barrionevo et al in [1], here σ_{2d} denotes the surface measure on the unit sphere $\mathbb{S}^{2d-1}$. In order to investigate $\mathcal{M}$ we use the asymptotic expansion of the Fourier transform of the surface measure σ_{2d} and study the related bilinear multiplier operator $H^t(f,g)$. To treat the bad behavior of the term $\mathrm{e}^{2\pi \mathrm{i}t|(\xi,\eta)|}$ in H^t, we rewrite $\mathrm{e}^{2\pi \mathrm{i}t|(\xi,\eta)|}$ as the summation of $\mathrm{e}^{2\pi \mathrm{i}t\sqrt{N^2+|\eta|^2}} a_N(t\xi, t\eta)$'s where N's are positive integers, $a_N(\xi,\eta)$ satisfies the Marcinkiewicz-Mikhlin-Hörmander condition in η, and $\mathrm{supp}(a_N(\cdot,\eta)) \subset \{\xi : N \le |\xi| < N+1\}$. By using these decompositions, we significantly improve the results of Barrionevo et al in [1].

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1. Introduction and statement of results

For each positive integer $n \geq 2$, let σ_n be the surface measure on the unit sphere $\mathbb{S}^{n-1}$. Then it is well-known that the spherical maximal function

$$\sup_{t>0} \left| \int_{\mathbb{S}^{n-1}} f(x-ty)\, d\sigma_n(y) \right|$$

is bounded on $L^p(\mathbb{R}^n)$ if and only if $p > \frac{n}{n-1}$. This result was first proved by Stein [16] for $n \geq 3$ and by Bourgain [2] for $n = 2$. See also [3, 6, 14, 15]. In this paper we study the bilinear spherical maximal function

$$\mathcal{M}(f,g)(x) = \sup_{t>0} \left| \int_{\mathbb{S}^{2d-1}} f(x-ty)\, g(x-tz)\, d\sigma_{2d}(y,z) \right| \tag{1.1}$$

initially defined for Schwartz functions f, g on $\mathbb{R}^d$. These operators were studied by J. A. Barrionevo, L. Grafakos, D. He, P. Honzík, and L. Oliveira in [1] and they obtained the following results.

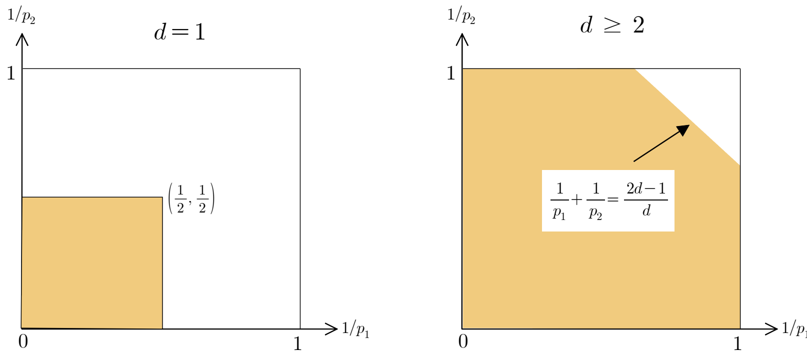
Theorem A (The results in [1]). *Let $d \geq 8$. Then the bilinear spherical maximal operator $\mathcal{M}$, when restricted to Schwartz functions, is bounded from $L^{p_1}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ with $1/p = 1/p_1 + 1/p_2$ for all indices $(1/p_1, 1/p_2)$ in the open rhombus with vertices the points*

$$\overrightarrow{P_0} = (0,0), \quad \overrightarrow{P_1} = (1,0), \quad \overrightarrow{P_2} = (0,1), \quad \overrightarrow{P_3} = \left(\frac{2d-10}{2d-5}, \frac{2d-10}{2d-5} \right).$$

See Figure 2 for the results.

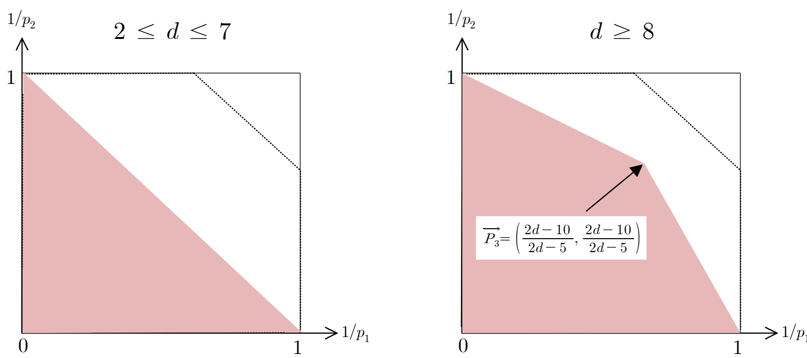
We also refer a recent paper of L. Grafakos, D. He, and P. Honzík [8] that presents an alternative way to study the bilinear spherical maximal operator $\mathcal{M}$ on $L^2 \times L^2 \to L^1$ and improves some results(see [1]) in case $d \geq 4$.

Notation. Throughout this paper, for two quantities A and B, we shall write $A \lesssim B$ if $A \leq CB$ for some positive constant C, depending on the



(a) The case $d = 1$ (b) The case $d \geq 2$

Figure 1: Necessary conditions.



(a) The case $2 \leq d \leq 7$ (b) The case $d \geq 8$

Figure 2: The results in [1].

dimension and possibly other parameters apparent form the context. We denote the Fourier transform and inverse Fourier transform of f by $\widehat{f}$ and $(f)^{\vee}$, respectively. For a measurable set E, $|E|$ denotes the measure of E. M denote the Hardy-Littlewood maximal function

$$\mathrm{M}f(x) = \sup_{r>0} \frac{1}{|B(0,r)|} \int_{|y|<r} |f(x-y)|\, dy.$$

Necessary conditions

From homogeneity it is necessary that $1/p = 1/p_1 + 1/p_2$. For $d \geq 1$, it is proved in [1] that the bilinear spherical maximal operator $\mathcal{M}$ is unbounded from $L^{p_1}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ when $1 \leq p_1, p_2 \leq \infty$, $1/p = 1/p_1 + 1/p_2$, and $p \leq \frac{d}{2d-1}$. However, when $d = 1$, for a sufficiently small $\delta > 0$ if we set

$$f(y) = \chi_{[-\delta,\delta]}(y), \quad g(z) = \chi_{[-10,10]}(z)$$

where χ_I denotes the characteristic function of the interval I, then we easily see that

$$\mathcal{M}(f,g)(x) \geq C\delta^{1/2}, \qquad 1 \leq |x| \leq 2.$$

Thus from the observation

$$\delta^{1/2} \lesssim \|\mathcal{M}(f,g)\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} \lesssim \delta^{1/p_1},$$

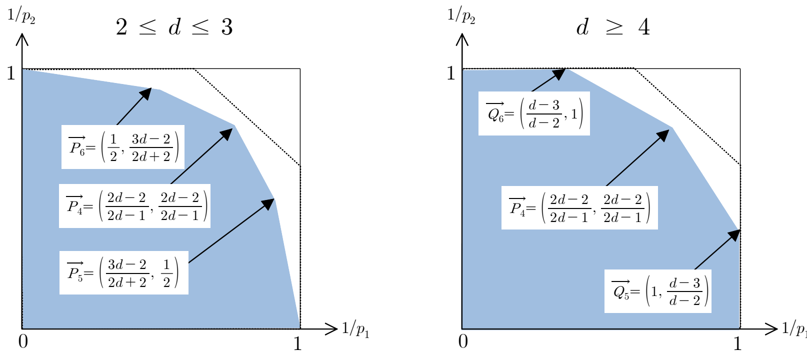
we obtain the necessary condition $p_1 \geq 2$ for $d = 1$. Similarly we obtain the necessary condition $p_2 \geq 2$ for $d = 1$. See Figure 1 for the necessary conditions.

In this paper we improve the results for the bilinear spherical maximal operator $\mathcal{M}$ treated in [1] via the Fourier transform of σ_{2d}

$$\widehat{\sigma_{2d}}(\xi,\eta) = 2\pi|(\xi,\eta)|^{(2-2d)/2} J_{(2d-2)/2}\big(2\pi|(\xi,\eta)|\big)$$

and the asymptotic expansion of the Bessel function $J_{(2d-2)/2}$. Our main results are as follows.

Theorem 1 (Improved ranges). *If $2 \leq d \leq 3$, then the bilinear spherical maximal operator $\mathcal{M}$, when restricted to Schwartz functions, is bounded from $L^{p_1}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ with $1/p = 1/p_1 + 1/p_2$ for all indices $(1/p_1, 1/p_2)$ in the open hexagon with vertices the points $\overrightarrow{P_0} = \big(0,0\big)$,*



(a) The case $2 \le d \le 3$ (b) The case $d \ge 4$

Figure 3: Improved ranges.

$\overrightarrow{P_1} = (1,0)$, $\overrightarrow{P_2} = (0,1)$,

$$\overrightarrow{P_4} = \left(\frac{2d-2}{2d-1}, \frac{2d-2}{2d-1}\right), \quad \overrightarrow{P_5} = \left(\frac{3d-2}{2d+2}, \frac{1}{2}\right), \quad \overrightarrow{P_6} = \left(\frac{1}{2}, \frac{3d-2}{2d+2}\right).$$

If $d \ge 4$, then the bilinear spherical maximal operator $\mathcal{M}$, when restricted to Schwartz functions, is bounded from $L^{p_1}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ with $1/p = 1/p_1 + 1/p_2$ for all indices $(1/p_1, 1/p_2)$ in the open hexagon with vertices the points $\overrightarrow{P_0} = (0,0)$, $\overrightarrow{P_1} = (1,0)$, $\overrightarrow{P_2} = (0,1)$,

$$\overrightarrow{P_4} = \left(\frac{2d-2}{2d-1}, \frac{2d-2}{2d-1}\right), \quad \overrightarrow{Q_5} = \left(1, \frac{d-3}{d-2}\right), \quad \overrightarrow{Q_6} = \left(\frac{d-3}{d-2}, 1\right).$$

See Figure 3 for the results.

Some crucial estimates in this paper

Note that the bilinear maximal operator $\mathcal{M}$ in (1.1) can be written as

$$\mathcal{M}(f,g)(x) = \sup_{t>0} \left| \int_{\mathbb{R}^{2d}} \widehat{\sigma_{2d}}(t\xi, t\eta)\, \widehat{f}(\xi)\, \widehat{g}(\eta)\, \mathrm{e}^{2\pi \mathrm{i} x\cdot(\xi+\eta)}\, d\xi\, d\eta \right|.$$

Let Ψ be a smooth function on $\mathbb{R}^{2d}$ so that $\widehat{\Psi}$ is supported on the annulus $|(\xi,\eta)| \sim 1$. For each positive integer j, let

$$\mathcal{M}_j(f,g)(x) = \sup_{t>0} \left| \int_{\mathbb{R}^{2d}} \widehat{\sigma_{2d}}(t\xi,t\eta)\widehat{\Psi}\big(2^{-j}(t\xi,t\eta)\big)\, \widehat{f}(\xi)\, \widehat{g}(\eta)\, \mathrm{e}^{2\pi \mathrm{i}x\cdot(\xi+\eta)}\, d\xi\, d\eta \right|.$$

The crucial estimate in [1] is

$$\|\mathcal{M}_j(f,g)\|_{L^1} \lesssim j2^{(\frac{3}{2}-\frac{d}{5})j}\|f\|_{L^2}\|g\|_{L^2}, \tag{1.2}$$

which holds for $d \geq 8$, whose proof is based on Corollary 8 in [9]. On the other hand, let ψ be a smooth function on $\mathbb{R}^d$ so that $\widehat{\psi}$ is supported on the annulus $|\xi| \sim 1$. For each positive integers i and j, let

$$\mathcal{M}^0_{ij}(f,g)(x) = \sup_{1\leq t\leq 2} \left| \int_{\mathbb{R}^{2d}} \widehat{f}(\xi)\, \widehat{g}(\eta)\, \widehat{\sigma_{2d}}(t\xi,t\eta)\widehat{\psi}(2^{-i}\xi)\widehat{\psi}(2^{-j}\eta)\, \mathrm{e}^{2\pi \mathrm{i}x\cdot(\xi+\eta)}\, d\xi\, d\eta \right|.$$

In this paper, on the contrary to the estimate (1.2) in [1], we obtain the following three estimates

$$\|\mathcal{M}^0_{ij}(f,g)\|_{L^{\frac{2}{3}}} \leq C\big(\max(2^i,2^j)\big)^{\frac{-d+4}{2}} \|f*\psi_i\|_{L^{p_1}}\|g*\psi_j\|_{L^{p_2}} \tag{1.3}$$

which holds for all $1 \leq p_1,\, p_2 \leq 2$ with $3/2 = 1/p_1 + 1/p_2$,

$$\|\mathcal{M}^0_{ij}(f,g)\|_{L^1} \leq C\big(\max(2^i,2^j)\big)^{\frac{-2d+3}{2}} \|f*\psi_i\|_{L^2}\|g*\psi_j\|_{L^2}, \tag{1.4}$$

and

$$\mathcal{M}^0_{ij}(f,g)(x) \lesssim \max(2^i,2^j)\, \mathrm{M}(f)(x)\, \mathrm{M}(g)(x). \tag{1.5}$$

Then by using the estimates (1.3), (1.4), and (1.5) together with interpolation, scaling, and Littlewood-Paley Theory, we significantly improve the results of Barrionevo et al in [1] for $\mathcal{M}$ (see Figure 3 for the results).

2. Reductions and some main lemmas

Let $n \geq 2$ and σ_n be the surface measure on the unit sphere $\mathbb{S}^{n-1}$. Then

$$\widehat{\sigma_n}(\xi) = 2\pi|\xi|^{(2-n)/2}J_{(n-2)/2}(2\pi|\xi|) \tag{2.1}$$

where $J_{(n-2)/2}$ denotes the Bessel function. We use the complete asymptotic expansion,

$$J_{\frac{n-2}{2}}(r) \sim r^{-\frac{1}{2}} \mathrm{e}^{\mathrm{i}r} \sum_{j=0}^{\infty} a_j r^{-j} + r^{-\frac{1}{2}} \mathrm{e}^{-\mathrm{i}r} \sum_{j=0}^{\infty} b_j r^{-j} \quad \text{as } r \to \infty,$$

for some coefficients a_j and b_j, in the sense that, for all nonnegative integer N and α

(2.2)
$$\left(\frac{d}{dr}\right)^{\alpha} \left[J_{\frac{n-2}{2}}(r) - r^{-\frac{1}{2}} \mathrm{e}^{\mathrm{i}r} \sum_{j=0}^{N} a_j r^{-j} - r^{-\frac{1}{2}} \mathrm{e}^{-\mathrm{i}r} \sum_{j=0}^{N} b_j r^{-j} \right] = O\big(r^{-(N+\frac{1}{2})}\big)$$

as $r \to \infty$. For this asymptotic expansion we refer Stein's book [17], Chapter VIII, §5.2.

The main idea of this paper is to take advantage of the Fourier transform (2.1) and the asymptotic expansion (2.2). We now describe some main ingredients of this paper. Let i, j be positive integers. In view of (2.1) and (2.2) we define the bilinear operator H^t_{ij} by

(2.3)
$$H^t_{ij}(f,g)(x) := \int_{\mathbb{R}^{2d}} m(t\xi, t\eta)\, \mathrm{e}^{2\pi \mathrm{i}\, t|(\xi,\eta)|}\, \widehat{f}(\xi)\, \widehat{g}(\eta)\, \widehat{\psi}(2^{-i}\xi)\, \widehat{\psi}(2^{-j}\eta)\; \mathrm{e}^{2\pi \mathrm{i} x\cdot(\xi+\eta)}\, d\xi\, d\eta,$$

where $\widehat{\psi}$ is a smooth function supported in $|\xi| \sim 1$, and the function m satisfies the following derivative conditions

$$|\partial^{\beta} m(\xi,\eta)| \le C_{\beta}(|(\xi,\eta)|)^{-|\beta|} \tag{2.4}$$

for all multi-indices β. Actually the study of the bilinear maximal operator $\mathcal{M}$ is closely related to that of the bilinear operator H^t_{ij}. The classical multiplier results of Coifman-Meyer [4] read that if the multiplier m satisfies the derivative conditions (2.4), then the bilinear operator

$$\int_{\mathbb{R}^{2d}} m(\xi,\eta)\, \widehat{f}(\xi)\, \widehat{g}(\eta)\, \mathrm{e}^{2\pi \mathrm{i} x\cdot(\xi+\eta)}\, d\xi d\eta,$$

maps $L^p \times L^q$ into L^r as long as $1 < p, q \le \infty$, $1/p + 1/q = 1/r$ and $0 < r < \infty$. See also [5, 11–13] for the classical Coifman-Meyer theorem on multilinear singular integrals. In studying the bilinear operator H^t_{ij}, the term $\mathrm{e}^{2\pi \mathrm{i}|(\xi,\eta)|}$

in (2.3) behaves badly in obtaining the derivative conditions (2.4). In order to conquer this obstacle, we restrict the size of ξ so that $N \le |\xi| < N+1$ for some positive integer $N \sim 2^i$. Then we rewrite $\mathrm{e}^{2\pi \mathrm{i}\, t|(\xi,\eta)|}$ as

$$\mathrm{e}^{2\pi \mathrm{i}\, t|(\xi,\eta)|} = \mathrm{e}^{2\pi \mathrm{i}\, t\, \Psi_N(\xi,\eta)}\, \mathrm{e}^{2\pi \mathrm{i}\, t\left(\sqrt{N^2+|\eta|^2}\right)},$$

where

$$\Psi_N(\xi,\eta) := \sqrt{|\xi|^2+|\eta|^2} - \sqrt{N^2+|\eta|^2} = \frac{(|\xi|+N)(|\xi|-N)}{\sqrt{|\xi|^2+|\eta|^2}+\sqrt{N^2+|\eta|^2}}.$$

Note that the function $\Psi_N(\xi,\eta)$ satisfies the derivative conditions

$$|\partial_\eta^\beta \Psi_N(\xi,\eta)| \le C_\beta |\eta|^{-|\beta|}$$

for all multi-indices β uniformly when $N \le |\xi| < N+1$. For each positive integer N, let χ_N denote the characteristic function of the interval $[N, N+1)$. Then we have

$$\begin{aligned} H_{ij}^t(f,g)(x) &= \sum_{N=2^{i-1}}^{2^{i+1}} \int_{\mathbb{R}^{2d}} \left[m_{ij}^t(\xi,\eta)\, \mathrm{e}^{2\pi \mathrm{i}\, t \Psi_N(\xi,\eta)}\right] \\ &\qquad \times \left[\widehat{\mathrm{F}_N(f)}(\xi)\right] \left[\widehat{\mathrm{U}_j^{t,N}(g)}(\eta)\right] \mathrm{e}^{2\pi \mathrm{i} x\cdot(\xi+\eta)}\, d\xi\, d\eta \\ &:= \sum_{N=2^{i-1}}^{2^{i+1}} H_{ij}^{t,N}(f,g)(x), \end{aligned}$$

where

$$\begin{aligned} m_{ij}^t(\xi,\eta) &:= m(t\xi, t\eta)\, \widehat{\psi}(2^{-i}\xi)\, \widehat{\psi}(2^{-j}\eta), \\ \mathrm{F}_N(f)(x) &:= \int_{\mathbb{R}^d} \widehat{f}(\xi)\chi_N(|\xi|)\, \mathrm{e}^{2\pi \mathrm{i} x\cdot\xi}\, d\xi, \end{aligned}$$

and

$$\mathrm{U}_j^{t,N}(g)(y) := \int_{\mathbb{R}^d} \widehat{g}(\eta)\widehat{\psi}(2^{-j}\eta)\mathrm{e}^{2\pi \mathrm{i}\, t\left(\sqrt{N^2+|\eta|^2}\right)}\, \mathrm{e}^{2\pi \mathrm{i} y\cdot\eta}\, d\eta.$$

In Lemma 2.2 we obtain the following estimate

$$\left\| \left(\int_1^2 |\mathrm{U}_j^{t,N}(g)(y)|^2\, dt \right)^{\frac{1}{2}} \right\|_{L_y^1} \lesssim 2^{\frac{d}{2}j} \|g\|_{L^1}. \tag{2.5}$$

In Lemma 2.3 by using the derivative conditions

$$\left|\partial_\eta^\alpha \left(m(\xi,\eta)\mathrm{e}^{2\pi\mathrm{i}\,\Psi_N(\xi,\eta)}\right)\right| \le C_\alpha\,|\eta|^{-|\alpha|} \quad \text{for all } \alpha,$$

together with the estimate (2.5), we obtain some estimates for the operator $H_{ij}^{t,N}(f,g)$. Then by summing the results for $H_{ij}^{t,N}(f,g)$ in $N \sim 2^i$ we get

$$\left\| \left(\int_1^2 \left| H_{ij}^t(f,g)(x) \right|^2 dt \right)^{\frac{1}{2}} \right\|_{L_x^{\frac{2}{3}}} \le C\,2^{\frac{d+2}{2}\max(i,j)}\,\|f\|_{L^{p_1}(\mathbb{R}^d)}\,\|g\|_{L^{p_2}(\mathbb{R}^d)} \tag{2.6}$$

for any $1 \le p_1, p_2 \le 2$ with $3/2 = 1/p_1 + 1/p_2$, and

$$\left\| \left(\int_1^2 \left| H_{ij}^t(f,g)(x) \right|^2 dt \right)^{\frac{1}{2}} \right\|_{L_x^1} \le C\,2^{\frac{1}{2}\max(i,j)}\,\|f\|_{L^2(\mathbb{R}^d)}\,\|g\|_{L^2(\mathbb{R}^d)}. \tag{2.7}$$

In Section 3, by using the estimates (2.6) and (2.7), we prove that the maximal operator

$$\mathcal{M}^0(f,g)(x) = \sup_{1\le t\le 2}\left| \int_{\mathbb{R}^{2d}} \widehat{\sigma_{2d}}(t\xi,t\eta)\,\widehat{f}(\xi)\,\widehat{g}(\eta)\,\mathrm{e}^{2\pi\mathrm{i}x\cdot(\xi+\eta)}\,d\xi\,d\eta \right|$$

is bounded from $L^{p_1}(\mathbb{R}^d)\times L^{p_2}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ if $(1/p_1, 1/p_2, 1/p)$ lies in the open hexagon stated in Theorem 1. In Section 4, via scaling, interpolation, and Littlewood-Paley theorem, we extend the results for $\mathcal{M}^0$ to those for the maximal operator $\mathcal{M}$. Extension of the results from $\mathcal{M}^0$ to $\mathcal{M}$ is similar to that in [7]. The following Lemmas 2.2 and 2.3 are our main estimates.

Choose a radial function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\widehat{\varphi}(\xi) = \begin{cases} 1, & \text{if } |\xi| \le 1; \\ 0, & \text{if } |\xi| \ge 2. \end{cases}$$

Let $\widehat{\psi}(\xi) = \widehat{\varphi}(\xi) - \widehat{\varphi}(2\xi)$, then $\widehat{\psi}$ is supported in $\{1/2 < |\xi| < 2\}$ and

$$\widehat{\varphi}(\xi) + \sum_{j=1}^{\infty} \widehat{\psi}(2^{-j}\xi) = 1 \quad \text{for all } \xi.$$

Lemma 2.1. *Let i and j be positive integers. Let $m(\xi,\eta)$ be a $C^{2d+4}\big(\mathbb{R}^d\setminus\{0\}\times\mathbb{R}^d\setminus\{0\}\big)$ function that satisfies*

$$\big|\partial_\xi^\alpha\partial_\eta^\beta m(\xi,\eta)\big| \le C_{\alpha,\beta}\,|\xi|^{-|\alpha|}|\eta|^{-|\beta|} \tag{2.8}$$

for all multi-indices α and β such that $|\alpha|+|\beta|\le 2d+4$. Then the bilinear operator

$$S_{ij}(f,g)(x)=\int_{\mathbb{R}^{2d}}\widehat{f}(\xi)\,\widehat{g}(\eta)\,m(\xi,\eta)\,\widehat{\psi}(2^{-i}\xi)\,\widehat{\psi}(2^{-j}\eta)\,\mathrm{e}^{2\pi\mathrm{i}x\cdot(\xi+\eta)}\,d\xi\,d\eta$$

satisfies

$$\big|S_{ij}(f,g)(x)\big|\le C\,\mathrm{M}f(x)\,\mathrm{M}g(x)$$

where the constant C is proportional to $\sup_{|\alpha|+|\beta|\le 2d+4}C_{\alpha,\beta}$.

Proof of Lemma 2.1. By using $\widehat{f}(\xi)=\int_{\mathbb{R}^d}f(y)\mathrm{e}^{-2\pi\mathrm{i}y\cdot\xi}dy$, we write

$$\begin{aligned}S_{ij}(f,g)(x)=&\int_{\mathbb{R}^{2d}}f(x-y)g(x-z)\\&\times\left(\int_{\mathbb{R}^{2d}}m(\xi,\eta)\,\widehat{\psi}(2^{-i}\xi)\,\widehat{\psi}(2^{-j}\eta)\,\mathrm{e}^{2\pi\mathrm{i}(y\cdot\xi+z\cdot\eta)}d\xi\,d\eta\right)\,dydz.\end{aligned}\tag{2.9}$$

Let $\lceil x\rceil$ denote the integer part of the real number x. Then if we integrate by parts $\lceil d/2\rceil+1$-times via

$$\big(1-2^{2i}\triangle_\xi\big)\left(\mathrm{e}^{2\pi\mathrm{i}y\cdot\xi}\right)=\big(1+4\pi^2|2^iy|^2\big)\,\mathrm{e}^{2\pi\mathrm{i}y\cdot\xi},$$

we obtain that

$$\begin{aligned}&\big(1+4\pi^2|2^iy|^2\big)^{\lceil d/2\rceil+1}\int_{\mathbb{R}^{2d}}\mathrm{e}^{2\pi\mathrm{i}[y\cdot\xi+z\cdot\eta]}\,m(\xi,\eta)\,\widehat{\psi}(2^{-i}\xi)\,\widehat{\psi}(2^{-j}\eta)\,d\xi\,d\eta\\=&\int_{\mathbb{R}^{2d}}\mathrm{e}^{2\pi\mathrm{i}[y\cdot\xi+z\cdot\eta]}\,\big(1-2^{2i}\triangle_\xi\big)^{\lceil d/2\rceil+1}\Big(m(\xi,\eta)\,\widehat{\psi}(2^{-i}\xi)\,\widehat{\psi}(2^{-j}\eta)\Big)\,d\xi\,d\eta.\end{aligned}$$

Next if we integrate by parts $\lceil d/2\rceil+1$-times by using

$$\big(1-2^{2j}\triangle_\eta\big)\left(\mathrm{e}^{2\pi\mathrm{i}z\cdot\eta}\right)=\big(1+4\pi^2|2^jz|^2\big)\,\mathrm{e}^{2\pi\mathrm{i}z\cdot\eta},$$

we obtain that

$$\begin{aligned}&\big(1+4\pi^2|2^iy|^2\big)^{\lceil d/2\rceil+1}\big(1+4\pi^2|2^jz|^2\big)^{\lceil d/2\rceil+1}\\&\times\int_{\mathbb{R}^{2d}}\mathrm{e}^{2\pi\mathrm{i}[y\cdot\xi+z\cdot\eta]}m(\xi,\eta)\widehat{\psi}(2^{-i}\xi)\widehat{\psi}(2^{-j}\eta)d\xi d\eta\\=&\int_{\mathbb{R}^{2d}}\mathrm{e}^{2\pi\mathrm{i}[y\cdot\xi+z\cdot\eta]}\big(1-2^{2j}\triangle_\eta\big)^{\lceil d/2\rceil+1}\big(1-2^{2i}\triangle_\xi\big)^{\lceil d/2\rceil+1}\\&\times\big(m(\xi,\eta)\widehat{\psi}(2^{-i}\xi)\widehat{\psi}(2^{-j}\eta)\big)d\xi d\eta.\end{aligned}$$

By (2.8) we have

$$\Big|\big(1-2^{2j}\triangle_\eta\big)^{\lceil d/2\rceil+1}\big(1-2^{2i}\triangle_\xi\big)^{\lceil d/2\rceil+1}\Big(m(\xi,\eta)\,\widehat{\psi}(2^{-i}\xi)\,\widehat{\psi}(2^{-j}\eta)\Big)\Big|\lesssim 1$$

and so

$$\begin{aligned}&\Big|\int_{\mathbb{R}^{2d}}\mathrm{e}^{2\pi\mathrm{i}[y\cdot\xi+z\cdot\eta]}\,m(\xi,\eta)\,\widehat{\psi}(2^{-i}\xi)\,\widehat{\psi}(2^{-j}\eta)\,d\xi\,d\eta\Big|\\&\lesssim\frac{2^{id}}{(1+|2^iy|^2)^{\lceil d/2\rceil+1}}\frac{2^{jd}}{(1+|2^jz|^2)^{\lceil d/2\rceil+1}}.\end{aligned}\tag{2.10}$$

By (2.9) and (2.10), we obtain that $\big|S_{ij}(f,g)(x)\big|\leq C\,\mathrm{M}f(x)\,\mathrm{M}g(x)$. □

The following two lemmas are the main estimates of this paper.

Lemma 2.2. *Let $N>0$ and*

$$\mathrm{U}_j^{t,N}(g)(x):=\int_{\mathbb{R}^d}\widehat{g}(\eta)\,\widehat{\psi}(2^{-j}\eta)\,\mathrm{e}^{2\pi\mathrm{i}t\sqrt{N^2+|\eta|^2}}\,\mathrm{e}^{2\pi\mathrm{i}x\cdot\eta}d\eta.$$

Then

$$\Big\|\Big(\int_1^2|\mathrm{U}_j^{t,N}(g)(x)|^2\,dt\Big)^{\frac12}\Big\|_{L^1_x}\lesssim 2^{\frac d2 j}\|g\|_{L^1}.$$

Proof of Lemma 2.2. We write

$$\mathrm{U}_j^{t,N}(g)(x)=\int_{\mathbb{R}^d}g(x-y)\,\kappa_j^N(y,t)\,dy,$$

where

$$\kappa_j^N(y,t):=\int_{\mathbb{R}^d}\widehat{\psi}(2^{-j}\eta)\,\mathrm{e}^{2\pi\mathrm{i}t\sqrt{N^2+|\eta|^2}}\,\mathrm{e}^{2\pi\mathrm{i}y\cdot\eta}d\eta.$$

We split $\mathrm{U}_j^{t,N}(g) = \mathrm{U}_{j,1}^{t,N}(g) + \mathrm{U}_{j,2}^{t,N}(g)$ where

$$\mathrm{U}_{j,1}^{t,N}(g)(x) := \int_{2^j|y|\lesssim 1} g(x-y)\,\kappa_j^N(y,t)\,dy,$$

and

$$\mathrm{U}_{j,2}^{t,N}(g)(x) := \int_{2^j|y|\gtrsim 1} g(x-y)\,\kappa_j^N(y,t)\,dy.$$

For the term $\mathrm{U}_{j,1}^{t,N}(g)$, since $|\kappa_j^N(y,t)| \lesssim 2^{jd}$, we have

$$\left\| \left(\int_1^2 |\mathrm{U}_{j,1}^{t,N}(g)(x)|^2\,dt \right)^{\frac{1}{2}} \right\|_{L_x^1} \lesssim 2^{jd} \int_{\mathbb{R}^d} \left(\int_{2^j|y|\lesssim 1} |g(x-y)|\,dy \right) dx \lesssim \|g\|_{L^1}.$$

Therefore it suffices to show that

$$\left\| \left(\int_1^2 |\mathrm{U}_{j,2}^{t,N}(g)(x)|^2\,dt \right)^{\frac{1}{2}} \right\|_{L_x^1} \lesssim 2^{\frac{d}{2}j}\|g\|_{L^1}.$$

Let $L := \frac{2^j}{\sqrt{N^2+2^{2j}}} \lesssim 1$. We consider the term $\mathrm{U}_{j,2}^{t,N}(g)$ in two parts

$$\begin{aligned}\mathrm{U}_{j,2}^{t,N}(g)(x) &= \int_{\substack{|y|<\frac{1}{8}L,\,|y|>8L \\ 2^j|y|\gtrsim 1}} g(x-y)\,\kappa_j^N(y,t)\,dy \\ &\quad + \int_{\substack{\frac{1}{8}L\le|y|\le 8L \\ 2^j|y|\gtrsim 1}} g(x-y)\,\kappa_j^N(y,t)\,dy \\ &:= \mathrm{U}_{j,3}^{t,N}(g)(x) + \mathrm{U}_{j,4}^{t,N}(g)(x).\end{aligned}$$

Estimates for the term $\mathbf{U}_{j,3}^{t,N}(g)$

By Minkowski's inequality we have

$$\left(\int_1^2 \left|\mathrm{U}_{j,3}^{t,N}(g)(x)\right|^2 dt \right)^{\frac{1}{2}} \le \int_{\substack{|y|<\frac{1}{8}L,\,|y|>8L \\ 2^j|y|\gtrsim 1}} |g(x-y)| \left(\int_1^2 \left|\kappa_j^N(y,t)\right|^2 dt \right)^{\frac{1}{2}} dy,$$

and so

$$\left\| \left(\int_1^2 \left|\mathrm{U}_{j,3}^{t,N}(g)(x)\right|^2 dt \right)^{\frac{1}{2}} \right\|_{L_x^1} \le \|g\|_{L^1} \int_{\substack{|y|<\frac{1}{8}L,\,|y|>8L \\ 2^j|y|\gtrsim 1}} \left(\int_1^2 \left|\kappa_j^N(y,t)\right|^2 dt \right)^{\frac{1}{2}} dy.$$

Thus it suffices to prove that

$$\int_{\substack{|y|<\frac{1}{8}L,\,|y|>8L\\ 2^j|y|\gtrsim 1}} \left(\int_1^2 \left|\kappa_j^N(y,t)\right|^2 dt\right)^{\frac{1}{2}} dy \lesssim 1. \tag{2.11}$$

By using spherical coordinates we have

$$\kappa_j^N(y,t) = 2^{jd}\int_0^\infty \left(\int_{\mathbb{S}^{d-1}} \mathrm{e}^{2\pi \mathrm{i} 2^j y\cdot r\theta} d\theta\right) \mathrm{e}^{2\pi \mathrm{i} t\sqrt{N^2+2^{2j}r^2}}\, r^{d-1}\widehat{\psi}(r)\, dr. \tag{2.12}$$

Since $2^j|y| \gtrsim 1$, by using (2.1) and (2.2) we see that the main contribution of $\int_{\mathbb{S}^{d-1}} \mathrm{e}^{2\pi \mathrm{i} 2^j y\cdot r\theta} d\theta$ comes from

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} \mathrm{e}^{2\pi \mathrm{i} 2^j y\cdot r\theta} d\theta \sim a_0|2^j yr|^{(-d+1)/2}\mathrm{e}^{2\pi \mathrm{i} 2^j|y|r}&\\ + b_0|2^j yr|^{(-d+1)/2}\mathrm{e}^{-2\pi \mathrm{i} 2^j|y|r}&.\end{aligned} \tag{2.13}$$

Let $t\sim 1$, $|y| < \frac{1}{8}L$ or $|y| > 8L$. Then we claim that for any positive integer M

$$\left|\int_0^\infty \mathrm{e}^{2\pi \mathrm{i}\left(t\sqrt{N^2+2^{2j}r^2}\pm 2^j|y|r\right)}\, \widetilde{\psi}(r)\, dr\right| \le C_M \left(1+\max\left(2^jL, 2^j|y|\right)\right)^{-M} \tag{2.14}$$

where $\widetilde{\psi}(r)$ is a smooth function supported in $r\sim 1$. Then by (2.12), (2.13), (2.14), and Minkowski's inequality we have

$$\begin{aligned}&\int_{\substack{|y|<\frac{1}{8}L,\,|y|>8L\\ 2^j|y|\gtrsim 1}} \left(\int_1^2 \left|\kappa_j^N(y,t)\right|^2 dt\right)^{\frac{1}{2}} dy\\ &\lesssim \int_{\substack{|y|<\frac{1}{8}L,\,|y|>8L\\ 2^j|y|\gtrsim 1}} \frac{2^{jd}}{(2^j|y|)^{\frac{d-1}{2}}\left(1+\max\left(2^jL, 2^j|y|\right)\right)^M} dy.\end{aligned}$$

If we take $M > (d+1)/2$, then

$$\int_{|y|<\frac{1}{8}L} \frac{2^{jd}}{(2^j|y|)^{\frac{d-1}{2}}\left(1+\max\left(2^jL, 2^j|y|\right)\right)^M} dy \lesssim (2^jL)^{\frac{d+1}{2}}(1+2^jL)^{-M} \lesssim 1,$$

and

$$\int_{\substack{|y|>8L\\ 2^j|y|\gtrsim 1}} \frac{2^{jd}}{(2^j|y|)^{\frac{d-1}{2}}\left(1+\max\left(2^jL, 2^j|y|\right)\right)^M} dy \lesssim \int \frac{2^{jd}}{(1+2^j|y|)^{M+\frac{d-1}{2}}} dy \lesssim 1.$$

Therefore we have (2.11). Now it remains to prove (2.14). To see this, define

$$\phi(r) := 2\pi(t\sqrt{N^2 + 2^{2j}r^2} \pm 2^j|y|r).$$

Then since $t \sim 1$, $|y| < \frac{1}{8}L$ or $|y| > 8L$, if $r \sim 1$, then

$$|\phi'(r)| = 2\pi \left| t\frac{2^{2j}r}{\sqrt{N^2 + 2^{2j}r^2}} \pm 2^j|y| \right| \gtrsim \max\left(2^jL, 2^j|y|\right). \tag{2.15}$$

Recall $L := \frac{2^j}{\sqrt{N^2+2^{2j}}}$, and so for any positive integer $\alpha \geq 2$ we have

$$|\phi^{(\alpha)}(r)| \lesssim 2^jL \quad \text{if } r \sim 1 \text{ and } t \sim 1. \tag{2.16}$$

Thus by (2.15) and (2.16), for any positive integer α we have

$$\frac{|\phi^{(\alpha)}(r)|}{|\phi'(r)|} \lesssim 1 \quad \text{if } r \sim 1 \text{ and } t \sim 1. \tag{2.17}$$

Now if we integrate by parts M times via

$$\frac{1}{\mathrm{i}\phi'(r)}\frac{d}{dr}\left(\mathrm{e}^{\mathrm{i}\phi(r)}\right) = \mathrm{e}^{\mathrm{i}\phi(r)},$$

we obtain that

$$\begin{aligned}
\int \widetilde{\psi}(r)\,\mathrm{e}^{\mathrm{i}\phi(r)}\,dr &= \int \frac{\widetilde{\psi}(r)}{\mathrm{i}\phi'(r)}\frac{d}{dr}\left(\mathrm{e}^{\mathrm{i}\phi(r)}\right)\,dr \\
&= \int -\frac{d}{dr}\left(\frac{\widetilde{\psi}(r)}{\mathrm{i}\phi'(r)}\right)\mathrm{e}^{\mathrm{i}\phi(r)}\,dr \\
&\;\;\vdots \\
&= \left(\frac{-1}{\mathrm{i}}\right)^M \int \underbrace{\frac{d}{dr}\left(\frac{1}{\phi'(r)}\cdots\frac{d}{dr}\left(\frac{1}{\phi'(r)}\right.\right.}_{(M-1)-times}\left.\left.\frac{d}{dr}\left(\frac{\widetilde{\psi}(r)}{\phi'(r)}\right)\right)\right)\mathrm{e}^{\mathrm{i}\phi(r)}\,dr.
\end{aligned} \tag{2.18}$$

By (2.17), for any positive integer α we have

$$\left|\frac{d^\alpha}{dr^\alpha}\left(\frac{1}{\phi'(r)}\right)\right| \lesssim \frac{1}{|\phi'(r)|}, \quad \text{and} \quad \left|\frac{d^\alpha}{dr^\alpha}\left(\frac{\widetilde{\psi}(r)}{\phi'(r)}\right)\right| \lesssim \frac{1}{|\phi'(r)|}. \tag{2.19}$$

And so by (2.19) for any positive integer M

$$\left|\underbrace{\frac{d}{dr}\left(\frac{1}{\phi'(r)}\cdots\frac{d}{dr}\left(\frac{1}{\phi'(r)}\right.\right.}_{(M-1)-times}\frac{d}{dr}\left(\frac{\widetilde{\psi}(r)}{\phi'(r)}\right)\bigg)\bigg)\right| \lesssim \frac{1}{|\phi'(r)|^M}. \tag{2.20}$$

By applying (2.20) to (2.18) we obtain (2.14).

Estimates for the term $\mathrm{U}_{j,4}^{t,N}(g)$

Note that

$$\mathrm{U}_{j,4}^{t,N}(g)(x) = \int_{\mathbb{R}^d}\int_{\substack{\frac{1}{8}L\le|y|\le 8L\\ 2^j|y|\gtrsim 1}} g(x-y)\,\psi(2^{-j}\eta)\,\mathrm{e}^{2\pi \mathrm{i}t\sqrt{N^2+|\eta|^2}}\,\mathrm{e}^{2\pi\mathrm{i}y\cdot\eta}\,dy\,d\eta.$$

Let $\phi\in\mathcal{S}(\mathbb{R})$ be a smooth function supported in $\{1/2<t<3\}$ and $\phi(t)=1$ on $1\le t\le 2$. Then we have

$$\begin{aligned}\int_1^2 \left|\mathrm{U}_{j,4}^{t,N}(g)(x)\right|^2 dt &\le \int \phi(t)\left|\mathrm{U}_{j,4}^{t,N}(g)(x)\right|^2 dt\\ &= \int_{\substack{\frac{1}{8}L\le|y|\le 8L\\ 2^j|y|\gtrsim 1}}\int_{\substack{\frac{1}{8}L\le|y'|\le 8L\\ 2^j|y'|\gtrsim 1}} g(x-y)\overline{g(x-y')}\,K_j^N(y,y')\,dydy',\end{aligned} \tag{2.21}$$

where

$$\begin{aligned}K_j^N(y,y') &:= \int_{\mathbb{R}^{2d}}\widehat{\psi}(2^{-j}\eta)\overline{\widehat{\psi}(2^{-j}\eta')}\\ &\quad\times\left(\int_{\mathbb{R}}\phi(t)\mathrm{e}^{2\pi\mathrm{i}t\left(\sqrt{N^2+|\eta|^2}-\sqrt{N^2+|\eta'|^2}\right)}dt\right)\mathrm{e}^{2\pi\mathrm{i}(y\cdot\eta-y'\cdot\eta')}\,d\eta\,d\eta'\\ &= 2^{2jd}\int_{\mathbb{R}^2} r^{d-1}\widehat{\psi}(r)r'^{d-1}\overline{\widehat{\psi}(r')}\\ &\quad\times\widehat{\phi}\big(\sqrt{N^2+2^{2j}r^2}-\sqrt{N^2+2^{2j}r'^2}\,\big)\widehat{\sigma_d}(2^jry)\widehat{\sigma_d}(2^jr'y')\,drdr',\end{aligned}$$

by using the spherical coordinates. Thus by using the Fourier decay estimates

$$\left|\widehat{\sigma_d}(2^jry)\right|\lesssim(2^j|y|)^{\frac{-d+1}{2}},\quad \left|\widehat{\sigma_d}(2^jr'y')\right|\lesssim(2^j|y'|)^{\frac{-d+1}{2}},$$

we have

$$\begin{aligned}|K_j^N(y,y')| &\lesssim 2^{2dj}(2^j|y|)^{\frac{-d+1}{2}}(2^j|y'|)^{\frac{-d+1}{2}} \\ &\quad\times \int_1^2\int_1^2 \left|\widehat{\phi}\left(\sqrt{N^2+2^{2j}r^2}-\sqrt{N^2+2^{2j}r'^2}\right)\right| dr\, dr' \\ &= 2^{2dj}(2^j|y|)^{\frac{-d+1}{2}}(2^j|y'|)^{\frac{-d+1}{2}} \\ &\quad\times \int_1^2\int_1^2 \left|\widehat{\phi}\left(\frac{2^{2j}|r-r'||r+r'|}{\sqrt{N^2+2^{2j}r^2}+\sqrt{N^2+2^{2j}r'^2}}\right)\right| dr\, dr'.\end{aligned}$$

For each $r\sim 1$ and $r'\sim 1$, we have

$$\frac{2^j|r+r'|}{\sqrt{N^2+2^{2j}r^2}+\sqrt{N^2+2^{2j}r'^2}} \sim L.$$

Hence we estimate

$$\begin{aligned}&\int_1^2\int_1^2 \left|\widehat{\phi}\left(\frac{2^{2j}|r-r'||r+r'|}{\sqrt{N^2+2^{2j}r^2}+\sqrt{N^2+2^{2j}r'^2}}\right)\right| dr\, dr' \\ \lesssim &\int_1^2\int_1^2 \left(1+2^jL|r-r'|\right)^{-100} drdr' \lesssim (1+2^jL)^{-1},\end{aligned}$$

and

$$\left|K_j^N(y,y')\right| \lesssim 2^{2dj}(2^j|y|)^{\frac{-d+1}{2}}(2^j|y'|)^{\frac{-d+1}{2}}(1+2^jL)^{-1}. \tag{2.22}$$

By (2.21) and (2.22) we obtain

$$\begin{aligned}\int_1^2 \left|\mathrm{U}_{j,4}^{t,N}(g)(x)\right|^2 dt &\le \int_{|y|\sim L}\int_{|y'|\sim L} |g(x-y)|\,|g(x-y')| \\ &\quad\times \left(2^{2dj}(2^j|y|)^{\frac{-d+1}{2}}(2^j|y'|)^{\frac{-d+1}{2}}(1+2^jL)^{-1}\right) dy\, dy' \\ &\lesssim 2^{dj}L^{-d}\left(\int_{|y|\sim L}|g(x-y)|\,dy\right)\left(\int_{|y'|\sim L}|g(x-y')|\,dy'\right),\end{aligned}$$

and by Hölder's inequality and the condition $L := \frac{2^j}{\sqrt{N^2+2^{2j}}} \lesssim 1$

$$\left\|\left(\int_1^2 \left|\mathrm{U}_{j,4}^{t,N}(g)(x)\right|^2 dt\right)^{\frac{1}{2}}\right\|_{L_x^1} \lesssim \left[2^{dj}L^{-d}\right]^{\frac{1}{2}}(L^d)\,\|g\|_{L^1} \lesssim 2^{\frac{d}{2}j}\|g\|_{L^1}.$$

□

Lemma 2.3. *Let i and j be positive integers. Let $m(\xi,\eta)$ be a $C^{d+2}\big(\mathbb{R}^d \setminus \{0\} \times \mathbb{R}^d \setminus \{0\}\big)$ function that satisfies*

$$\left|\partial_\xi^\alpha \partial_\eta^\beta m(\xi,\eta)\right| \le C_{\alpha,\beta}\,|\xi|^{-|\alpha|}|\eta|^{-|\beta|} \tag{2.23}$$

for all multi-indices α and β such that $|\alpha|+|\beta|\le d+2$. Let $H_{ij}^t(f,g)$ be as in (2.3). Then for any $1\le p_1,p_2\le 2$ with $3/2 = 1/p_1+1/p_2$ we have

$$\begin{aligned}\left\|\left(\int_1^2 \left|H_{ij}^t(f,g)(x)\right|^2 dt\right)^{\frac12}\right\|_{L_x^{\frac23}} \\ \le C\big(\max(2^i,2^j)\big)^{\frac{d+2}{2}} \|f\|_{L^{p_1}(\mathbb{R}^d)}\,\|g\|_{L^{p_2}(\mathbb{R}^d)},\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}\left\|\left(\int_1^2 \left|H_{ij}^t(f,g)(x)\right|^2 dt\right)^{\frac12}\right\|_{L_x^{1}} \\ \le C\big(\max(2^i,2^j)\big)^{\frac12} \|f\|_{L^{2}(\mathbb{R}^d)}\,\|g\|_{L^{2}(\mathbb{R}^d)},\end{aligned} \tag{2.25}$$

where the constant C is proportional to $\sup_{|\alpha|+|\beta|\le 2d+4} C_{\alpha,\beta}$.

Proof of Lemma 2.3. For $1\le t\le 2$, let $m_{ij}^t(\xi,\eta) = m(t\xi,t\eta)\,\widehat{\psi}(2^{-i}\xi)\,\widehat{\psi}(2^{-j}\eta)$. For each positive integer N, let χ_N denote the characteristic function of the interval $[N,N+1)$. Then

$$\begin{aligned}H_{ij}^t(f,g)(x) &= \sum_{N=2^{i-1}}^{2^{i+1}} \int_{\mathbb{R}^{2d}} \widehat{f}(\xi)\,\chi_N(|\xi|)\,\widehat{g}(\eta)\,m_{ij}^t(\xi,\eta) \\ &\qquad\qquad \times e^{2\pi i\, t|(\xi,\eta)|}\, e^{2\pi i x\cdot(\xi+\eta)}\, d\xi\, d\eta \\ &:= \sum_{N=2^{i-1}}^{2^{i+1}} H_{ij}^{t,N}(f,g)(x).\end{aligned}$$

We decompose $e^{2\pi i\, t|(\xi,\eta)|}$ as $e^{2\pi i\, t|(\xi,\eta)|} = e^{2\pi i\, t\,\Psi_N(\xi,\eta)}\, e^{2\pi i\, t\left(\sqrt{N^2+|\eta|^2}\right)}$ where

$$\Psi_N(\xi,\eta) := \sqrt{|\xi|^2+|\eta|^2} - \sqrt{N^2+|\eta|^2} = \frac{(|\xi|+N)(|\xi|-N)}{\sqrt{|\xi|^2+|\eta|^2}+\sqrt{N^2+|\eta|^2}}.$$

Then we have

$$H_{ij}^{t,N}(f,g)(x) = \int_{\mathbb{R}^{2d}} \left[m_{ij}^t(\xi,\eta)\, \mathrm{e}^{2\pi \mathrm{i}\, t \Psi_N(\xi,\eta)} \right] \times \left[\widehat{\mathrm{F}_N(f)}(\xi) \right] \left[\widehat{\mathrm{U}_j^{t,N}(g)}(\eta) \right] \mathrm{e}^{2\pi \mathrm{i} x\cdot(\xi+\eta)}\, d\xi\, d\eta$$

where

$$\mathrm{F}_N(f)(x) := \int_{\mathbb{R}^d} \widehat{f}(\xi) \chi_N(|\xi|)\, \mathrm{e}^{2\pi \mathrm{i} x\cdot\xi}\, d\xi,$$
$$\mathrm{U}_j^{t,N}(g)(y) := \int_{\mathbb{R}^d} \widehat{g}(\eta)\widehat{\psi}(2^{-j}\eta) \mathrm{e}^{2\pi \mathrm{i}\, t\left(\sqrt{N^2+|\eta|^2}\right)}\, \mathrm{e}^{2\pi \mathrm{i} y\cdot\eta}\, d\eta.$$

Since $N \sim 2^i$, $N \le |\xi| < N+1$, and $|\eta| \sim 2^j$ we have

$$\left|\partial_\eta^\alpha \Psi_N(\xi,\eta)\right| \le C_\alpha\, 2^{-|\alpha| j} \quad \text{for all } \alpha. \tag{2.26}$$

Then by (2.23) and (2.26) we obtain that

$$\left| \partial_\eta^\alpha \left(m_{ij}^t(\xi,\eta) \mathrm{e}^{2\pi \mathrm{i}\, t \Psi_N(\xi,\eta)} \right) \right| \le C_\alpha\, 2^{-|\alpha| j} \quad \text{for all } |\alpha| \le d+2, \text{ and } t \sim 1.$$

By using $\widehat{\mathrm{U}_j^{t,N}(g)}(\eta) = \int_{\mathbb{R}^d} \mathrm{U}_j^{t,N}(g)(y)\, \mathrm{e}^{-2\pi \mathrm{i} y\cdot\eta}\, dy$ we have

$$H_{ij}^{t,N}(f,g)(x) = \int_{\mathbb{R}^{2d}} \widehat{\mathrm{F}_N(f)}(\xi)\, \mathrm{U}_j^{t,N}(g)(y) \times \left(\int_{\mathbb{R}^d} \mathrm{e}^{2\pi \mathrm{i}(x-y)\cdot\eta} m_{ij}^t(\xi,\eta) \mathrm{e}^{2\pi \mathrm{i}\, t \Psi_N(\xi,\eta)} d\eta \right) \mathrm{e}^{2\pi \mathrm{i} x\cdot\xi}\, d\xi dy.$$

Note that

$$\left(1 - 2^{2j} \triangle_\eta\right) \left(\mathrm{e}^{2\pi \mathrm{i}(x-y)\cdot\eta} \right) = \left(1 + 4\pi^2 |2^j(x-y)|^2\right) \mathrm{e}^{2\pi \mathrm{i}(x-y)\cdot\eta}. \tag{2.27}$$

Thus if we integrate by parts by using (2.27), we obtain

$$\begin{aligned} &\int_{\mathbb{R}^d} \mathrm{e}^{2\pi \mathrm{i}(x-y)\cdot\eta}\, m_{ij}^t(\xi,\eta)\, \mathrm{e}^{2\pi \mathrm{i}\, t \Psi_N(\xi,\eta)}\, d\eta \\ &= \frac{1}{\left(1 + 4\pi^2 |2^j(x-y)|^2\right)} \\ &\quad \times \int_{\mathbb{R}^d} \mathrm{e}^{2\pi \mathrm{i}(x-y)\cdot\eta} \left(1 - 2^{2j} \triangle_\eta\right) \left(m_{ij}^t(\xi,\eta)\, \mathrm{e}^{2\pi \mathrm{i}\, t \Psi_N(\xi,\eta)} \right) d\eta. \end{aligned} \tag{2.28}$$

So by applying the estimate (2.28) $\lceil d/2\rceil+1$-times we get

$$H_{ij}^{t,N}(f,g)(x)=\int_{\mathbb{R}^d}\mathrm{A}_j\big(\mathrm{U}_j^{t,N}(g)\big)(x,\eta)\,\mathrm{B}_{ij}^{t,N}(\mathrm{F}_N(f))(x,\eta)\,d\eta,$$

where

$$\mathrm{A}_j\big(\mathrm{U}_j^{t,N}(g)\big)(x,\eta):=\int_{\mathbb{R}^d}\frac{\mathrm{U}_j^{t,N}(g)(y)\,\mathrm{e}^{2\pi\mathrm{i}(x-y)\cdot\eta}}{\big(1+4\pi^2|2^j(x-y)|^2\big)^{\lceil d/2\rceil+1}}\,dy,$$

and

$$\begin{aligned}\mathrm{B}_{ij}^{t,N}\big(\mathrm{F}_N(f)\big)(x,\eta):=\int_{\mathbb{R}^d}&\widehat{\mathrm{F}_N(f)}(\xi)\,\big(1-2^{2j}\triangle_\eta\big)^{\lceil d/2\rceil+1}\\&\times\Big(m_{ij}^t(\xi,\eta)\,\mathrm{e}^{2\pi\mathrm{i}\,t\Psi_N(\xi,\eta)}\Big)\,\mathrm{e}^{2\pi\mathrm{i}x\cdot\xi}\,d\xi.\end{aligned}\tag{2.29}$$

Note that

$$\begin{aligned}\big|\mathrm{A}_j\big(\mathrm{U}_j^{t,N}(g)\big)(x,\eta)\big|\le\int_{\mathbb{R}^d}&\frac{\big|\mathrm{U}_j^{t,N}(g)(y)\big|}{\big(1+4\pi^2|2^j(x-y)|^2\big)^{\lceil d/2\rceil+1}}\,dy\\&:=\mathrm{C}_j\big(\mathrm{U}_j^{t,N}(g)\big)(x).\end{aligned}\tag{2.30}$$

Thus for $1\le t\le 2$ we have

$$\big|H_{ij}^{t,N}(f,g)(x)\big|\le\mathrm{C}_j\big(\mathrm{U}_j^{t,N}(g)\big)(x)\int_{\mathbb{R}^d}\sup_{1\le t\le 2}\big|\mathrm{B}_{ij}^{t,N}\big(\mathrm{F}_N(f)\big)(x,\eta)\big|\,d\eta,$$

and so

$$\begin{aligned}\left(\int_1^2\big|H_{ij}^{t,N}(f,g)(x)\big|^2\,dt\right)^{\frac12}\le&\left(\int_{\mathbb{R}^d}\sup_{1\le t\le 2}\big|\mathrm{B}_{ij}^{t,N}\big(\mathrm{F}_N(f)\big)(x,\eta)\big|\,d\eta\right)\\&\times\left(\int_1^2\big|\mathrm{C}_j\big(\mathrm{U}_j^{t,N}(g)\big)(x)\big|^2\,dt\right)^{\frac12}.\end{aligned}\tag{2.31}$$

Then by Hölder's inequality and Minkowski's integral inequality we have

$$\begin{aligned}(2.32)\qquad &\left\|\left(\int_1^2 |H_{ij}^{t,N}(f,g)(x)|^2\,dt\right)^{\frac{1}{2}}\right\|_{L_x^{\frac{2}{3}}}\\ &\le \left\|\int_{\mathbb{R}^d}\sup_{1\le t\le 2}|\mathrm{B}_{ij}^{t,N}\big(\mathrm{F}_N(f)\big)(x,\eta)|\,d\eta\right\|_{L_x^2}\\ &\quad\times\left\|\left(\int_1^2 \left|\mathrm{C}_j\big(\mathrm{U}_j^{t,N}(g)\big)(x)\right|^2 dt\right)^{\frac{1}{2}}\right\|_{L_x^1}\\ &\le \left(\int_{\mathbb{R}^d}\left\|\sup_{1\le t\le 2}|\mathrm{B}_{ij}^{t,N}\big(\mathrm{F}_N(f)\big)(x,\eta)|\right\|_{L_x^2} d\eta\right)\\ &\quad\times\left\|\left(\int_1^2 \left|\mathrm{C}_j\big(\mathrm{U}_j^{t,N}(g)\big)(x)\right|^2 dt\right)^{\frac{1}{2}}\right\|_{L_x^1}.\end{aligned}$$

Now we claim that

$$(2.33)\qquad \left\|\left(\int_1^2 \left|\mathrm{C}_j\big(\mathrm{U}_j^{t,N}(g)\big)(x)\right|^2 dt\right)^{\frac{1}{2}}\right\|_{L_x^1} \lesssim 2^{-\frac{d}{2}j}\,\|g\|_{L^1},$$

$$(2.34)\qquad \left\|\left(\int_1^2 \left|\mathrm{C}_j\big(\mathrm{U}_j^{t,N}(g)\big)(x)\right|^2 dt\right)^{\frac{1}{2}}\right\|_{L_x^2} \lesssim 2^{-dj}\,\|g\|_{L^2},$$

and

$$(2.35)\qquad \int_{\mathbb{R}^d}\left\|\sup_{1\le t\le 2}\left|\mathrm{B}_{ij}^{t,N}\big(\mathrm{F}_N(f)\big)(x,\eta)\right|\right\|_{L_x^2} d\eta \lesssim 2^{dj}\left\|\widehat{f}(\xi)\chi_N(|\xi|)\right\|_{L^2}.$$

Then by applying the estimates (2.33) and (2.35) to (2.32) we have

$$\left\|\left(\int_1^2 \left|H_{ij}^{t,N}(f,g)(x)\right|^2 dt\right)^{\frac{1}{2}}\right\|_{L_x^{\frac{2}{3}}} \le C\,2^{\frac{d}{2}j}\left\|\widehat{f}(\xi)\chi_N(|\xi|)\right\|_{L^2}\|g\|_{L^1},$$

and so

$$\left\|\left(\int_1^2 \left|H_{ij}^t(f,g)(x)\right|^2 dt\right)^{\frac{1}{2}}\right\|_{L_x^{\frac{2}{3}}} \tag{2.36}$$
$$\leq C\, 2^{\frac{d}{2}j}\left(\sum_{N=2^{i-1}}^{2^{i+1}} \left\|\widehat{f}(\xi)\chi_N(|\xi|)\right\|_{L^2}^{\frac{2}{3}} \|g\|_{L^1}^{\frac{2}{3}}\right)^{\frac{3}{2}}$$
$$\leq C\, 2^{\frac{d}{2}j}2^i\|f\|_{L^2}\,\|g\|_{L^1}.$$

Similarly by changing the role of ξ and η we have

$$\left\|\left(\int_1^2 \left|H_{ij}^t(f,g)(x)\right|^2 dt\right)^{\frac{1}{2}}\right\|_{L_x^{\frac{2}{3}}} \leq C2^{\frac{d}{2}i}2^j\|f\|_{L^1}\|g\|_{L^2}. \tag{2.37}$$

Then (2.24) follows by interpolating (2.36) and (2.37). For (2.25), by (2.31), (2.34) and (2.35) we have

$$\left\|\left(\int_1^2 \left|H_{ij}^{t,N}(f,g)(x)\right|^2 dt\right)^{\frac{1}{2}}\right\|_{L_x^1}$$
$$\leq \left\|\int_{\mathbb{R}^d} \sup_{1\leq t\leq 2} \left|\mathrm{B}_{ij}^{t,N}\big(\mathrm{F}_N(f)\big)(x,\eta)\right| d\eta\right\|_{L_x^2} \left\|\left(\int_1^2 \left|\mathrm{C}_j\big(\mathrm{U}_j^{t,N}(g)\big)(x)\right|^2 dt\right)^{\frac{1}{2}}\right\|_{L_x^2}$$
$$\leq C\left(\int_{\mathbb{R}^d} \left\|\sup_{1\leq t\leq 2} \left|\mathrm{B}_{ij}^{t,N}\big(\mathrm{F}_N(f)\big)(x,\eta)\right|\right\|_{L_x^2} d\eta\right)\left(2^{-jd}\|g\|_{L^2}\right)$$
$$\leq C\left\|\widehat{f}(\xi)\chi_N(|\xi|)\right\|_{L^2}\|g\|_{L^2},$$

and so

$$\left\|\left(\int_1^2 \left|H_{ij}^t(f,g)(x)\right|^2 dt\right)^{\frac{1}{2}}\right\|_{L_x^1} \leq C\sum_{N=2^{i-1}}^{2^{i+1}} \left\|\widehat{f}(\xi)\chi_N(|\xi|)\right\|_{L^2}\|g\|_{L^2}$$
$$\leq C2^{\frac{i}{2}}\|f\|_{L^2}\|g\|_{L^2}.$$

Now it remains to prove the estimates (2.33), (2.34), and (2.35). For the estimate (2.33), by (2.30) and Minkowski's inequality we have

$$\left(\int_1^2 \left|\mathrm{C}_j\big(\mathrm{U}_j^{t,N}(g)\big)(x)\right|^2 dt\right)^{\frac{1}{2}} \le \int_{\mathbb{R}^d}\left(\int_1^2 \left|\mathrm{U}_j^{t,N}(g)(x-y)\right|^2 dt\right)^{\frac{1}{2}} \times \frac{1}{\big(1+4\pi^2|2^j y|^2\big)^{\lceil d/2\rceil+1}}\,dy$$

and so by Lemma 2.2

$$\begin{aligned}\int_{\mathbb{R}^d}\left(\int_1^2 \left|\mathrm{C}_j\big(\mathrm{U}_j^{t,N}(g)\big)(x)\right|^2 dt\right)^{\frac{1}{2}} dx &\lesssim 2^{-jd}\int_{\mathbb{R}^d}\left(\int_1^2 \left|\mathrm{U}_j^{t,N}(g)(x)\right|^2 dt\right)^{\frac{1}{2}} dx\\ &\lesssim 2^{-jd}\left(2^{\frac{d}{2}j}\|g\|_{L^1}\right).\end{aligned}$$

We now estimate (2.34). We use (2.30), Minkowski's inequality, Plancherel's identity and $t \sim 1$ to obtain

$$\begin{aligned}\left\|\mathrm{C}_j\big(\mathrm{U}_j^{t,N}(g)\big)(x)\right\|_{L_x^2} &\le \left(\int_{\mathbb{R}^d}\frac{1}{\big(1+4\pi^2|2^j y|^2\big)^{\lceil d/2\rceil+1}}\,dy\right)\left\|\mathrm{U}_j^{t,N}(g)(x)\right\|_{L_x^2}\\ &\lesssim 2^{-dj}\,\|g\|_{L^2}.\end{aligned}$$

For the estimate (2.35), let $\phi \in \mathcal{S}(\mathbb{R})$ be a smooth function supported in $\{1/2 < t < 3\}$ and $\phi(t) = 1$ on $1 \le t \le 2$. Then for $1 \le t \le 2$, we have

$$\Big(\mathrm{B}_{ij}^{t,N}\big(\mathrm{F}_N(f)\big)(x,\eta)\Big)^2 = \int_{\frac{1}{2}}^{t}\frac{d}{ds}\Big(\phi(s)\mathrm{B}_{ij}^{s,N}\big(\mathrm{F}_N(f)\big)(x,\eta)\Big)^2\,ds.$$

Hence for $1 \le t \le 2$, we dominate $\left|\mathrm{B}_{ij}^{t,N}\big(\mathrm{F}_N(f)\big)(x,\eta)\right|^2$ by

$$2\left(\int_{\frac{1}{2}}^{2}\left|\phi(s)\mathrm{B}_{ij}^{s,N}\big(\mathrm{F}_N(f)\big)(x,\eta)\right|^2 ds\right)^{\frac{1}{2}} \times\left(\int_{\frac{1}{2}}^{2}\left|\frac{d}{ds}\Big(\phi(s)\mathrm{B}_{ij}^{s,N}\big(\mathrm{F}_N(f)\big)(x,\eta)\Big)\right|^2 ds\right)^{\frac{1}{2}}.$$

It follows that

$$\left\|\sup_{1\le t\le 2}|\mathrm{B}_{ij}^{t,N}\big(\mathrm{F}_N(f)\big)(x,\eta)|\right\|_{L^2_x}^2 \tag{2.38}$$
$$\le 2\left(\int_{\frac12}^2 \left\|\phi(s)\mathrm{B}_{ij}^{s,N}\big(\mathrm{F}_N(f)\big)(x,\eta)\right\|_{L^2_x}^2 ds\right)^{\frac12}$$
$$\times\left(\int_{\frac12}^2 \left\|\frac{d}{ds}\Big(\phi(s)\mathrm{B}_{ij}^{s,N}\big(\mathrm{F}_N(f)\big)(x,\eta)\Big)\right\|_{L^2_x}^2 ds\right)^{\frac12}.$$

Recall the definition (2.29) of $\mathrm{B}_{ij}^{t,N}\big(\mathrm{F}_N(f)\big)$. By using the conditions (2.23) and (2.26) we have

$$\left|(1-2^{2j}\triangle_\eta)^{\lceil d/2\rceil+1}\Big(m_{ij}^t(\xi,\eta)\,\mathrm{e}^{2\pi\mathrm{i}\,t\Psi_N(\xi,\eta)}\Big)\right| \lesssim \chi_{(2^{j-1}\le|\eta|\le 2^{j+1})}(\eta),$$
$$\left|\frac{d}{dt}\Big[(1-2^{2j}\triangle_\eta)^{\lceil d/2\rceil+1}\Big(m_{ij}^t(\xi,\eta)\,\mathrm{e}^{2\pi\mathrm{i}\,t\Psi_N(\xi,\eta)}\Big)\Big]\right| \lesssim \chi_{(2^{j-1}\le|\eta|\le 2^{j+1})}(\eta),$$

and so by Plancherel's identity and (2.29) for any $t\sim 1$

$$\left\|\phi(t)\mathrm{B}_{ij}^{t,N}\big(\mathrm{F}_N(f)\big)(x,\eta)\right\|_{L^2_x} \le C\,\|\widehat{\mathrm{F}_N(f)}\|_{L^2}\left[\chi_{(2^{j-1}\le|\eta|\le 2^{j+1})}(\eta)\right],$$
$$\left\|\frac{d}{dt}\Big(\phi(t)\mathrm{B}_{ij}^{t,N}\big(\mathrm{F}_N(f)\big)(x,\eta)\Big)\right\|_{L^2_x} \le C\,\|\widehat{\mathrm{F}_N(f)}\|_{L^2}\left[\chi_{(2^{j-1}\le|\eta|\le 2^{j+1})}(\eta)\right].$$

Hence by (2.38) we have

$$\left\|\sup_{1\le t\le 2}|\mathrm{B}_{ij}^{t,N}\big(\mathrm{F}_N(f)\big)(x,\eta)|\right\|_{L^2_x} \le C\,\|\widehat{\mathrm{F}_N(f)}\|_{L^2}\left[\chi_{(2^{j-1}\le|\eta|\le 2^{j+1})}(\eta)\right].$$

Therefore (2.35) is proved. □

3. The case $t\in[1,2]$

First we consider the case $t\in[1,2]$ and the corresponding maximal operator

$$\mathcal{M}^0(f,g)(x)=\sup_{1\le t\le 2}\left|\int_{\mathbb{R}^{2d}}\widehat{f}(\xi)\,\widehat{g}(\eta)\,\widehat{\sigma_{2d}}(t\xi,t\eta)\,\mathrm{e}^{2\pi\mathrm{i}x\cdot(\xi+\eta)}\,d\xi\,d\eta\right|.$$

We prove that $\mathcal{M}^0$ is bounded from $L^{p_1}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ if $(1/p_1, 1/p_2)$ with $1/p = 1/p_1 + 1/p_2$ lies in the open region stated in Theorem 1. Note that

$$\widehat{\varphi}(\xi) \sum_{j=1}^{\infty} \widehat{\psi}(2^{-j}\eta) = \widehat{\varphi}(\xi)\,(1-\widehat{\varphi}(\eta))\,, \quad \widehat{\varphi}(\eta) \sum_{i=1}^{\infty} \widehat{\psi}(2^{-i}\xi) = \widehat{\varphi}(\eta)\,(1-\widehat{\varphi}(\xi))\,,$$

and so

$$\begin{aligned} 1 &= \left(\widehat{\varphi}(\xi) + \sum_{i=1}^{\infty} \widehat{\psi}(2^{-i}\xi)\right)\left(\widehat{\varphi}(\eta) + \sum_{j=1}^{\infty} \widehat{\psi}(2^{-j}\eta)\right) \\ &= \widehat{\varphi}(\xi)\widehat{\varphi}(\eta) + \widehat{\varphi}(\xi)\sum_{j=1}^{\infty} \widehat{\psi}(2^{-j}\eta) + \widehat{\varphi}(\eta)\sum_{i=1}^{\infty} \widehat{\psi}(2^{-i}\xi) + \sum_{i,\,j\geq 1} \widehat{\psi}(2^{-i}\xi)\,\widehat{\psi}(2^{-j}\eta) \\ &= -\widehat{\varphi}(\xi)\widehat{\varphi}(\eta) + \widehat{\varphi}(\xi) + \widehat{\varphi}(\eta) + \sum_{i,\,j\geq 1} \widehat{\psi}(2^{-i}\xi)\,\widehat{\psi}(2^{-j}\eta) \end{aligned}$$

for all $(\xi, \eta) \in \mathbb{R}^d \times \mathbb{R}^d$. Thus we have

$$\begin{aligned} \mathcal{M}^0(f,g)(x) \leq \mathcal{M}^0_{00}(f,g)(x) + \mathcal{M}^0_{0\infty}(f,g)(x) + \mathcal{M}^0_{\infty 0}(f,g)(x) \\ + \sum_{i,j\geq 1} \mathcal{M}^0_{ij}(f,g)(x) \end{aligned}$$

where

$$\begin{aligned} \mathcal{M}^0_{00}(f,g)(x) &= \sup_{1\leq t\leq 2} \left| \int_{\mathbb{R}^{2d}} \widehat{f}(\xi)\,\widehat{g}(\eta)\,\widehat{\sigma_{2d}}(t\xi, t\eta)\widehat{\varphi}(\xi)\widehat{\varphi}(\eta)\,\mathrm{e}^{2\pi \mathrm{i} x\cdot(\xi+\eta)}\,d\xi\,d\eta \right|, \\ \mathcal{M}^0_{0\infty}(f,g)(x) &= \sup_{1\leq t\leq 2} \left| \int_{\mathbb{R}^{2d}} \widehat{f}(\xi)\,\widehat{g}(\eta)\,\widehat{\sigma_{2d}}(t\xi, t\eta)\widehat{\varphi}(\xi)\,\mathrm{e}^{2\pi \mathrm{i} x\cdot(\xi+\eta)}\,d\xi\,d\eta \right|, \\ \mathcal{M}^0_{\infty 0}(f,g)(x) &= \sup_{1\leq t\leq 2} \left| \int_{\mathbb{R}^{2d}} \widehat{f}(\xi)\,\widehat{g}(\eta)\,\widehat{\sigma_{2d}}(t\xi, t\eta)\widehat{\varphi}(\eta)\,\mathrm{e}^{2\pi \mathrm{i} x\cdot(\xi+\eta)}\,d\xi\,d\eta \right|, \\ \mathcal{M}^0_{ij}(f,g)(x) &= \sup_{1\leq t\leq 2} \left| \int_{\mathbb{R}^{2d}} \widehat{f}(\xi)\,\widehat{g}(\eta)\,\widehat{\sigma_{2d}}(t\xi, t\eta)\widehat{\psi}(2^{-i}\xi)\widehat{\psi}(2^{-j}\eta)\,\mathrm{e}^{2\pi \mathrm{i} x\cdot(\xi+\eta)}\,d\xi\,d\eta \right|. \end{aligned}$$

Lemma 3.1. *We have*

1) $\mathcal{M}^0_{00}(f,g)(x) \lesssim \mathrm{M}f(x)\mathrm{M}g(x)$,
2) $\mathcal{M}^0_{0\infty}(f,g)(x) \lesssim \mathrm{M}f(x)\left(\sup_{1\leq t\leq 2} \int_{\mathbb{S}^{2d-1}} |g(x-tz)|\,d\sigma_{2d}(y,z)\right)$,
3) $\mathcal{M}^0_{\infty 0}(f,g)(x) \lesssim \left(\sup_{1\leq t\leq 2} \int_{\mathbb{S}^{2d-1}} |f(x-ty)|\,d\sigma_{2d}(y,z)\right)\mathrm{M}g(x)$.

Proof of Lemma 3.1. Note that if $|y| \lesssim 1$, $|z| \lesssim 1$, and $|t| \lesssim 1$, then

$$|f * \varphi(x - ty)| \lesssim \mathrm{M}(f)(x), \quad |g * \varphi(x - tz)| \lesssim \mathrm{M}(g)(x).$$

And the proof follows from

$$\begin{aligned}
\mathcal{M}^0_{00}(f,g)(x) &= \sup_{1\le t\le 2} \left| \int_{\mathbb{S}^{2d-1}} f * \varphi(x-ty)\, g * \varphi(x-tz)\, d\sigma_{2d}(y,z) \right|, \\
\mathcal{M}^0_{0\infty}(f,g)(x) &= \sup_{1\le t\le 2} \left| \int_{\mathbb{S}^{2d-1}} f * \varphi(x-ty)\, g(x-tz)\, d\sigma_{2d}(y,z) \right|, \\
\mathcal{M}^0_{\infty 0}(f,g)(x) &= \sup_{1\le t\le 2} \left| \int_{\mathbb{S}^{2d-1}} f(x-ty)\, g * \varphi(x-tz)\, d\sigma_{2d}(y,z) \right|.
\end{aligned}$$

□

Corollary 3.1. *Let $d \ge 2$. Then the operators $\mathcal{M}^0_{00}(f,g)$, $\mathcal{M}^0_{0\infty}(f,g)$, and $\mathcal{M}^0_{\infty 0}(f,g)$ map $L^{p_1}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ when $1/p = 1/p_1 + 1/p_2$, $1 < p_1, p_2 \le \infty$.*

Proof of Corollary 3.1. If $d \ge 2$, then the maximal operators

$$\begin{aligned}
&\sup_{t>0} \left(\int_{\mathbb{S}^{2d-1}} |f(x-ty)|\, d\sigma_{2d}(y,z) \right), \\
\text{and} \quad &\sup_{t>0} \left(\int_{\mathbb{S}^{2d-1}} |g(x-tz)|\, d\sigma_{2d}(y,z) \right)
\end{aligned}$$

are bounded on $L^p(\mathbb{R}^d)$ when $1 < p \le \infty$ (see [1, 15]). And the results follow immediately from Lemma 3.1. □

Lemma 3.2. *For $1 \le p_1, p_2 \le 2$ with $3/2 = 1/p_1 + 1/p_2$ we have*

$$\|\mathcal{M}^0_{ij}(f,g)\|_{L^{\frac{2}{3}}} \le C\big(\max(2^i, 2^j)\big)^{\frac{-d+4}{2}} \|f * \psi_i\|_{L^{p_1}} \|g * \psi_j\|_{L^{p_2}}. \tag{3.1}$$

And

$$\|\mathcal{M}^0_{ij}(f,g)\|_{L^1} \le C\big(\max(2^i, 2^j)\big)^{\frac{-2d+3}{2}} \|f * \psi_i\|_{L^2} \|g * \psi_j\|_{L^2}. \tag{3.2}$$

Lemma 3.3. *We have*

$$\mathcal{M}^0_{ij}(f,g)(x) = \sup_{1\le t\le 2} |T^t_{ij}(f,g)(x)| \lesssim \max(2^i,2^j)\,\mathrm{M}f(x)\,\mathrm{M}g(x).$$

Thus we have

$$\|\mathcal{M}^0_{ij}(f,g)\|_{L^p} \le C\max(2^i,2^j)\|f\|_{L^{p_1}}\|g\|_{L^{p_2}} \tag{3.3}$$

for any $1/p = 1/p_1 + 1/p_2$, $1 < p_1, p_2 \le \infty$.

For the moments we assume Lemmas 3.2 and 3.3. Then by interpolating (3.1), (3.2), and (3.3) we have

$$\sum_{i,j\ge 1} \|\mathcal{M}^0_{i,j}(f,g)\|_{L^p} \le \|f\|_{L^{p_1}}\|g\|_{L^{p_2}}$$

if $1/p = 1/p_1 + 1/p_2$ and $(1/p_1, 1/p_2)$ lies in the open region stated in Theorem 1. The point $\overrightarrow{P_4} = \left(\frac{2d-2}{2d-1}, \frac{2d-2}{2d-1}\right)$ is obtained by interpolating (3.2) and (3.3). The points $\overrightarrow{P_5} = \left(\frac{3d-2}{2d+2}, \frac{1}{2}\right)$ and $\overrightarrow{P_6} = \left(\frac{1}{2}, \frac{3d-2}{2d+2}\right)$ are obtained by interpolating (3.1) and (3.2). The points $\overrightarrow{Q_5} = \left(1, \frac{d-3}{d-2}\right)$ and $\overrightarrow{Q_6} = \left(\frac{d-3}{d-2}, 1\right)$ are obtained by interpolating (3.1) and (3.3). We refer to [10] for the interpolation arguments.

Proof of Lemma 3.2. We set

$$\begin{aligned}
&\mathcal{M}^0_{ij}(f,g)(x)\\
&= \sup_{1\le t\le 2}\left|\int_{\mathbb{R}^{2d}} \widehat{f}(\xi)\,\widehat{g}(\eta)\,\widehat{\sigma_{2d}}(t\xi,t\eta)\widehat{\psi}(2^{-i}\xi)\widehat{\psi}(2^{-j}\eta)\,\mathrm{e}^{2\pi\mathrm{i}x\cdot(\xi+\eta)}\,d\xi\,d\eta\right|\\
&= \sup_{1\le t\le 2}\left|T^t_{ij}(f,g)(x)\right|.
\end{aligned}$$

Let $\phi \in \mathcal{S}(\mathbb{R})$ be a smooth function supported in $\{1/2 < t < 3\}$ and $\phi(t) = 1$ on $1 \le t \le 2$. For $f, g \in \mathcal{S}(\mathbb{R}^d)$ and $1 \le t \le 2$, we have

$$\left(T^t_{ij}(f,g)(x)\right)^2 = \int_{\frac{1}{2}}^{t} \frac{d}{ds}\left(\phi(s)T^s_{ij}(f,g)(x)\right)^2 ds.$$

Hence for $1 \le t \le 2$

$$\left|T_{ij}^t(f,g)(x)\right|^2 \le 2\left(\int_{\frac{1}{2}}^2 \left|\phi(s)T_{ij}^s(f,g)(x)\right|^2\, ds\right)^{\frac{1}{2}} \times \left(\int_{\frac{1}{2}}^2 \left|\frac{d}{ds}\left(\phi(s)T_{ij}^s(f,g)(x)\right)\right|^2\, ds\right)^{\frac{1}{2}},$$

and

$$\mathcal{M}_{ij}^0(f,g)(x) \le \sqrt{2}\left(\int_{\frac{1}{2}}^2 \left|\phi(s)T_{ij}^s(f,g)(x)\right|^2\, ds\right)^{\frac{1}{4}} \times \left(\int_{\frac{1}{2}}^2 \left|\frac{d}{ds}\left(\phi(s)T_{ij}^s(f,g)(x)\right)\right|^2\, ds\right)^{\frac{1}{4}}.$$

By Hölder's inequality for any $p \ge 1/2$ we have

$$\|\mathcal{M}_{ij}^0(f,g)\|_{L^p} \le \sqrt{2}\left[\int_{\mathbb{R}^d}\left(\int_{\frac{1}{2}}^2 \left|\phi(s)T_{ij}^s(f,g)(x)\right|^2\, ds\right)^{\frac{p}{2}} dx\right]^{\frac{1}{2p}} \times \left[\int_{\mathbb{R}^d}\left(\int_{\frac{1}{2}}^2 \left|\frac{d}{ds}\left(\phi(s)T_{ij}^s(f,g)(x)\right)\right|^2\, ds\right)^{\frac{p}{2}} dx\right]^{\frac{1}{2p}}.$$

For (3.1), we apply Lemma 2.1 and (2.24) of Lemma 2.3 together with (2.1) and the asymptotic expansion (2.2) of the Bessel function for $N > 2d+4$ in (2.2), and we obtain that

$$\left\|\left(\int_{\frac{1}{2}}^2 \left|\phi(s)T_{ij}^s(f,g)(x)\right|^2\, ds\right)^{\frac{1}{2}}\right\|_{L_x^{\frac{2}{3}}} \le C\big(\max(2^i,2^j)\big)^{\frac{-d+3}{2}}\,\|f*\psi_i\|_{L^{p_1}}\|g*\psi_j\|_{L^{p_2}},$$

and

$$\left\|\left(\int_{\frac{1}{2}}^2 \left|\frac{d}{ds}\left(\phi(s)T_{ij}^s(f,g)(x)\right)\right|^2\, ds\right)^{\frac{1}{2}}\right\|_{L_x^{\frac{2}{3}}} \le C\big(\max(2^i,2^j)\big)^{\frac{-d+5}{2}}\,\|f*\psi_i\|_{L^{p_1}}\|g*\psi_j\|_{L^{p_2}},$$

for $1 \le p_1, p_2 \le 2$ with $3/2 = 1/p_1 + 1/p_2$. Similarly for (3.2), we apply Lemma 2.1 and (2.25) of Lemma 2.3 together with (2.1) and the asymptotic expansion (2.2) of the Bessel function for $N > 2d + 4$ in (2.2), and we obtain that

$$\begin{aligned}
&\left\| \left(\int_{\frac{1}{2}}^{2} \left| \phi(s) T_{ij}^{s}(f,g)(x) \right|^2 \, ds \right)^{\frac{1}{2}} \right\|_{L^1_x} \\
&\le C \big(\max(2^i, 2^j) \big)^{\frac{-2d+2}{2}} \|f * \psi_i\|_{L^2} \|g * \psi_j\|_{L^2}, \\
&\left\| \left(\int_{\frac{1}{2}}^{2} \left| \frac{d}{ds} \left(\phi(s) T_{ij}^{s}(f,g)(x) \right) \right|^2 \, ds \right)^{\frac{1}{2}} \right\|_{L^1_x} \\
&\le C \big(\max(2^i, 2^j) \big)^{\frac{-2d+4}{2}} \|f * \psi_i\|_{L^2} \|g * \psi_j\|_{L^2}.
\end{aligned}$$

□

For the proof of Lemma 3.3 we begin with the following lemma.

Lemma 3.4. *Let $y', z' \in \mathbb{R}^d$. For each $\epsilon > 0, \delta > 0$, define*

$$E_{\epsilon,\delta}(y', z') := \left\{ (y, z) \in \mathbb{R}^d \times \mathbb{R}^d : |y|^2 + |z|^2 = 1, \, |y - y'| < \epsilon, \, |z - z'| < \delta \right\} \subset \mathbb{S}^{2d-1}.$$

Then

$$\left| E_{\epsilon,\delta}(y', z') \right| \le C \max \left(\epsilon^{d-1} \delta^d, \, \epsilon^d \delta^{d-1} \right)$$

where $|E_{\epsilon,\delta}(y', z')|$ denotes the surface measure of the set $E_{\epsilon,\delta}(y', z')$, and the constant C does not depend on y', z', ϵ, and δ.

Proof of Lemma 3.4. For each $1 \le i \le 2d$, let

$$\begin{aligned}
\Sigma_+^i &:= \left\{ x = (x_1, \cdots, x_{2d}) \in \mathbb{S}^{2d-1} : x_i > 0 \right\}, \\
\Sigma_-^i &:= \left\{ x = (x_1, \cdots, x_{2d}) \in \mathbb{S}^{2d-1} : x_i < 0 \right\}.
\end{aligned}$$

Then by using a partition of unity we have

$$d\sigma_{2d} = \sum_{i=1}^{2d} d\mu_\pm^i$$

where $d\mu_\pm^i$ are compactly supported surface measures on $\Sigma_\pm^i$. Note that the surfaces $\Sigma_\pm^i$ can be given as graphs $x_i = \gamma(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{2d}) := \pm\sqrt{1-(|x|^2 - x_i^2)}$. For each $1 \le i \le 2d$, let $X_i = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{2d})$, then $\mu_\pm^i$ is the measure given by

$$\langle f, \mu_\pm^i \rangle = \int_{\mathbb{R}^{2d-1}} f\big(x_1, \cdots, x_{i-1}, \gamma(X_i), x_{i+1}, \cdots, x_{2d}\big)\, \omega(X_i)\, dX_i$$

for some smooth function ω. Then we have

$$\mu_\pm^i\big(E_{\epsilon,\delta}(y', z')\big) \lesssim \begin{cases} \epsilon^{d-1}\delta^d, & \text{if } 1 \le i \le d; \\ \epsilon^d \delta^{d-1}, & \text{if } d+1 \le i \le 2d. \end{cases}$$

Thus

$$\sigma_{2d}\big(E_{\epsilon,\delta}(y', z')\big) \lesssim \sum_{i=1}^{2d} \mu_\pm^i\big(E_{\epsilon,\delta}(y', z')\big) \lesssim \max\big(\epsilon^{d-1}\delta^d,\, \epsilon^d\delta^{d-1}\big).$$

□

Proof of Lemma 3.3. We write

(3.4)
$$\begin{aligned} T_{ij}^t(f, g)(x) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \widehat{f}(\xi)\, \widehat{g}(\eta)\, \widehat{\sigma_{2d}}(t\xi, t\eta)\widehat{\psi}(2^{-i}\xi)\widehat{\psi}(2^{-j}\eta)\ \mathrm{e}^{2\pi \mathrm{i} x\cdot(\xi+\eta)}\, d\xi\, d\eta \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\, g(x-z) K_{i,j}^t(y, z)\, dy\, dz, \end{aligned}$$

where

$$K_{ij}^t(y, z) := \int_{\mathbb{S}^{2d-1}} \Big[2^{id}\psi\big(2^i(y - ty')\big)\Big]\ \Big[2^{jd}\psi\big(2^j(z - tz')\big)\Big]\ d\sigma(y', z').$$

For each positive integers i', j' and $1 \le t \le 2$ we define

$$\begin{aligned} E_{0,0}^t(y, z) &:= \Big\{(y', z') \in \mathbb{S}^{2d-1} : |y - ty'| < 2^{-i},\, |z - tz'| < 2^{-j}\Big\}, \\ E_{i',0}^t(y, z) &:= \Big\{(y', z') \in \mathbb{S}^{2d-1} : 2^{-i+i'-1} \le |y - ty'| < 2^{-i+i'},\, |z - tz'| < 2^{-j}\Big\}, \\ E_{0,j'}^t(y, z) &:= \Big\{(y', z') \in \mathbb{S}^{2d-1} : |y - ty'| < 2^{-i},\, 2^{-j+j'-1} \le |z - tz'| < 2^{-j+j'}\Big\}, \\ E_{i',j'}^t(y, z) &:= \Big\{(y', z') \in \mathbb{S}^{2d-1} : 2^{-i+i'-1} \le |y - ty'| < 2^{-i+i'}, \\ &\qquad\qquad 2^{-j+j'-1} \le |z - tz'| < 2^{-j+j'}\Big\}. \end{aligned}$$

Then

$$\mathbb{S}^{2d-1} = \bigcup_{i'=0}^{\infty} \bigcup_{j'=0}^{\infty} E^t_{i',j'}(y,z).$$

By Lemma 3.4 for each $i', j' \geq 0$ we have

$$\sigma_{2d}(E^t_{i',j'}(y,z)) \lesssim (2^{-i+i'})^d (2^{-j+j'})^{d-1} + (2^{-i+i'})^{d-1}(2^{-j+j'})^d \tag{3.5}$$

uniformly in $1 \leq t \leq 2$. If $|y| \lesssim 1$ and $|z| \lesssim 1$, then by (3.5) for any $N > 0$

(3.6)
$$\begin{aligned}
|K^t_{ij}(y,z)| &\leq C_N \int_{\mathbb{S}^{2d-1}} \frac{2^{id}}{(1+2^i|y-ty'|)^N} \frac{2^{jd}}{(1+2^j|z-tz'|)^N} \, d\sigma_{2d}(y',z') \\
&\leq C_N \sum_{i',j'\geq 0} \int_{E^t_{i',j'}(y,z)} \frac{2^{id}}{(1+2^i|y-ty'|)^N} \frac{2^{jd}}{(1+2^j|z-tz'|)^N} \, d\sigma_{2d}(y',z') \\
&\leq C_N \sum_{i',j'\geq 0} 2^{id+jd-i'N-j'N} \, \sigma_{2d}(E^t_{i',j'}(y,z)) \\
&\leq C_N \max(2^i, 2^j).
\end{aligned}$$

If $|y| \gtrsim 1$ and $|z| \lesssim 1$, then by Lemma 3.4

$$\begin{aligned}
|K^t_{ij}(y,z)| &\leq C_N \int_{\mathbb{S}^{2d-1}} \frac{2^{id}}{(1+2^i|y-ty'|)^N} \frac{2^{jd}}{(1+2^j|z-tz'|)^N} \, d\sigma_{2d}(y',z') \\
&\leq C_N \int_{\mathbb{S}^{2d-1}} \frac{2^{id}}{(1+2^i|y|)^N} \frac{2^{jd}}{(1+2^j|z-tz'|)^N} \, d\sigma_{2d}(y',z') \\
&\leq C_N \frac{2^{id}}{(1+2^i|y|)^N} \sum_{j'\geq 0} \int_{|z-tz'|\lesssim 2^{-j+j'}} 2^{jd} \, 2^{-j'N} \, d\sigma_{2d}(y',z') \\
&\leq C_N \frac{2^{id}}{(1+2^i|y|)^N} \sum_{j'\geq 0} 2^{jd} \, 2^{-j'N} \left[(2^{-j+j'})^{d-1} + (2^{-j+j'})^d \right] \\
&\leq C_N \, 2^j \, \frac{2^{id}}{(1+2^i|y|)^N}.
\end{aligned}$$

Similarly if $|y| \lesssim 1$ and $|z| \gtrsim 1$, then we have

$$|K^t_{ij}(y,z)| \leq C_N 2^i \frac{2^{jd}}{(1+2^j|z|)^N}.$$

If $|y| \gtrsim 1$, $|z| \gtrsim 1$, then

$$
\begin{aligned}
|K_{ij}^t(y,z)| &\le C_N \int_{\mathbb{S}^{2d-1}} \frac{2^{id}}{(1+2^i|y-ty'|)^N} \frac{2^{jd}}{(1+2^j|z-tz'|)^N}\, d\sigma_{2d}(y',z') \\
&\le C_N \int_{\mathbb{S}^{2d-1}} \frac{2^{id}}{(1+2^i|y|)^N} \frac{2^{jd}}{(1+2^j|z|)^N}\, d\sigma_{2d}(y',z') \\
&\le C_N \frac{2^{id}}{(1+2^i|y|)^N} \frac{2^{jd}}{(1+2^j|z|)^N}. \qquad (3.7)
\end{aligned}
$$

By (3.4) and (3.6) through (3.7), if $|t| \lesssim 1$, then we obtain

$$
|K_{ij}^t(y,z)| \le \max(2^i,2^j) \left(\chi_{|y|\lesssim 1} + \frac{2^{id}}{(1+2^i|y|)^N}\right)\left(\chi_{|z|\lesssim 1} + \frac{2^{jd}}{(1+2^j|z|)^N}\right),
$$

and so

$$
|T_{ij}^t(f,g)(x)| \le C \max(2^i,2^j)\, \mathrm{M}(f)(x)\, \mathrm{M}(g)(x). \qquad \square
$$

4. The general case $t > 0$

In this section by using the results for $\mathcal{M}^0$ in Section 3, we prove that $\mathcal{M}$ is bounded from $L^{p_1}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ with $1/p = 1/p_1 + 1/p_2$ when $(1/p_1, 1/p_2)$ lies in the open region stated in Theorem 1. The proof for the general case is similar to that in [7]. For each integer l, note that

$$
\widehat{\varphi}(2^{-l}\xi) + \widehat{\varphi}(2^{-l}\eta) - \widehat{\varphi}(2^{-l}\xi)\widehat{\varphi}(2^{-l}\eta) + \sum_{i,j\ge 1} \widehat{\psi}(2^{-i-l}\xi)\widehat{\psi}(2^{-j-l}\eta) \equiv 1.
$$

Thus for each $l \in \mathbb{Z}$, we define $E_l = [2^{-l}, 2^{-l+1}]$ and if we let

$$
\mathcal{M}^l(f,g)(x) = \sup_{t\in E_l} \left| \int_{\mathbb{R}^{2d}} \widehat{f}(\xi)\, \widehat{g}(\eta)\, \widehat{\sigma_{2d}}(t\xi, t\eta)\, \mathrm{e}^{2\pi \mathrm{i} x\cdot(\xi+\eta)}\, d\xi\, d\eta \right|,
$$

then

$$
\begin{aligned}
\mathcal{M}^l(f,g)(x) \le{}& \mathcal{M}^l_{00}(f,g)(x) + \mathcal{M}^l_{0\infty}(f,g)(x) + \mathcal{M}^l_{\infty 0}(f,g)(x) \\
&+ \sum_{i,j\ge 1} \mathcal{M}^l_{ij}(f,g)(x)
\end{aligned}
$$

where

$$\mathcal{M}^l_{00}(f,g)(x) = \sup_{t\in E_l} \left| \int_{\mathbb{R}^{2d}} \widehat{f}(\xi)\,\widehat{g}(\eta)\,\widehat{\sigma_{2d}}(t\xi,t\eta)\widehat{\varphi}(2^{-l}\xi)\widehat{\varphi}(2^{-l}\eta)\,\mathrm{e}^{2\pi \mathrm{i}x\cdot(\xi+\eta)}\,d\xi\,d\eta \right|,$$

$$\mathcal{M}^l_{0\infty}(f,g)(x) = \sup_{t\in E_l} \left| \int_{\mathbb{R}^{2d}} \widehat{f}(\xi)\,\widehat{g}(\eta)\,\widehat{\sigma_{2d}}(t\xi,t\eta)\widehat{\varphi}(2^{-l}\xi)\,\mathrm{e}^{2\pi \mathrm{i}x\cdot(\xi+\eta)}\,d\xi\,d\eta \right|,$$

$$\mathcal{M}^l_{\infty 0}(f,g)(x) = \sup_{t\in E_l} \left| \int_{\mathbb{R}^{2d}} \widehat{f}(\xi)\,\widehat{g}(\eta)\,\widehat{\sigma_{2d}}(t\xi,t\eta)\widehat{\varphi}(2^{-l}\eta)\,\mathrm{e}^{2\pi \mathrm{i}x\cdot(\xi+\eta)}\,d\xi\,d\eta \right|,$$

$$\mathcal{M}^l_{i,j}(f,g)(x) = \sup_{t\in E_l} \left| \int_{\mathbb{R}^{2d}} \widehat{f}(\xi)\,\widehat{g}(\eta)\,\widehat{\sigma_{2d}}(t\xi,t\eta)\widehat{\psi}(2^{-i-l}\xi)\widehat{\psi}(2^{-j-l}\eta)\,\mathrm{e}^{2\pi \mathrm{i}x\cdot(\xi+\eta)}\,d\xi\,d\eta \right|.$$

Now we have

$$\begin{aligned}\mathcal{M}(f,g)(x) \le \sup_{l\in\mathbb{Z}} \mathcal{M}^l_{00}(f,g)(x) + \sup_{l\in\mathbb{Z}} \mathcal{M}^l_{0\infty}(f,g)(x) \\ + \sup_{l\in\mathbb{Z}} \mathcal{M}^l_{\infty 0}(f,g)(x) + \sum_{i,j\ge 1} \sup_{l\in\mathbb{Z}} \mathcal{M}^l_{ij}(f,g)(x).\end{aligned}$$

By scaling $\xi \to 2^l\xi$, $\eta \to 2^l\eta$, we have

$$\mathcal{M}^l_{ij}(f,g)(x) = \mathcal{M}^0_{ij}\left(f_{\mathrm{dil}(2^{-l})},\, g_{\mathrm{dil}(2^{-l})}\right)(2^l x) \quad \text{for all } 0 \le i,j \le \infty, \tag{4.1}$$

where $f_{\mathrm{dil}(2^{-l})}(x) := f(2^{-l}x)$ and $g_{\mathrm{dil}(2^{-l})}(x) := g(2^{-l}x)$.

Lemma 4.1. *We have*

1) $\sup_{l\in\mathbb{Z}} \mathcal{M}^l_{00}(f,g)(x) \lesssim \mathrm{M}(f)(x)\mathrm{M}(g)(x)$,
2) $\sup_{l\in\mathbb{Z}} \mathcal{M}^l_{0\infty}(f,g)(x) \lesssim \mathrm{M}(f)(x)\, \big(\sup_{t>0} \int_{\mathbb{S}^{2d-1}} |g|(x-tz)\, d\sigma_{2d}(y,z)\big)$,
3) $\sup_{l\in\mathbb{Z}} \mathcal{M}^l_{\infty 0}(f,g)(x) \lesssim \big(\sup_{t>0} \int_{\mathbb{S}^{2d-1}} |f|(x-ty)\, d\sigma_{2d}(y,z)\big)\, \mathrm{M}(g)(x)$,
4) $\sup_{l\in\mathbb{Z}} \mathcal{M}^l_{ij}(f,g)(x) \lesssim \max(2^i,2^j)\,\mathrm{M}(f)(x)\mathrm{M}(g)(x)$.

Proof of Lemma 4.1. First by using $f_{\mathrm{dil}(2^{-l})}(x) = f(2^{-l}x)$ we note that

$$\begin{aligned}\mathrm{M}\big(f_{\mathrm{dil}(2^{-l})}\big)(2^l x) &= \sup_{r>0} \frac{1}{|B(0,r)|} \int_{|y|<r} \left|f_{\mathrm{dil}(2^{-l})}(2^l x - y)\right| dy \\ &= \sup_{r>0} \frac{1}{|B(0,r)|} \int_{|y|<r} |f(x-2^{-l}y)|\, dy \\ &= \sup_{r>0} \frac{2^{ld}}{|B(0,r)|} \int_{|y|<2^{-l}r} |f(x-y)|\, dy = \mathrm{M}(f)(x).\end{aligned}$$

Thus by (3.3), Lemma 3.1, and (4.1) we have

$$\begin{aligned}\mathcal{M}^l_{00}(f,g)(x) &= \mathcal{M}^0_{00}\left(f_{\mathrm{dil}(2^{-l})}, g_{\mathrm{dil}(2^{-l})}\right)(2^l x)\\ &\lesssim \mathrm{M}\big(f_{\mathrm{dil}(2^{-l})}\big)(2^l x)\,\mathrm{M}\big(g_{\mathrm{dil}(2^{-l})}\big)(2^l x)\\ &= \mathrm{M}(f)(x)\,\mathrm{M}(g)(x),\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}^l_{ij}(f,g)(x) &= \mathcal{M}^0_{ij}\left(f_{\mathrm{dil}(2^{-l})}, g_{\mathrm{dil}(2^{-l})}\right)(2^l x)\\ &\lesssim \max(2^i,2^j)\,\mathrm{M}\big(f_{\mathrm{dil}(2^{-l})}\big)(2^l x)\,\mathrm{M}\big(g_{\mathrm{dil}(2^{-l})}\big)(2^l x)\\ &= \max(2^i,2^j)\,\mathrm{M}(f)(x)\,\mathrm{M}(g)(x).\end{aligned}$$

Therefore we have (1) and (4). For (2), by Lemma 3.1 and (4.1)

$$\begin{aligned}&\mathcal{M}^l_{0\infty}(f,g)(x) = \mathcal{M}^0_{0\infty}\left(f_{\mathrm{dil}(2^{-l})}, g_{\mathrm{dil}(2^{-l})}\right)(2^l x)\\ &\lesssim \mathrm{M}\big(f_{\mathrm{dil}(2^{-l})}\big)(2^l x)\sup_{1\le t\le 2}\left(\int_{\mathbb{S}^{2d-1}}\left|g_{\mathrm{dil}(2^{-l})}(2^l x-tz)\right|d\sigma_{2d}(y,z)\right)\\ &= \mathrm{M}f(x)\sup_{2^{-l}\le t\le 2^{-l+1}}\left(\int_{\mathbb{S}^{2d-1}}|g(x-tz)|\,d\sigma_{2d}(y,z)\right).\end{aligned}$$

The proof of (3) is the same as that of (2). □

Corollary 4.1. *Let $d\ge 2$, then*

1) $\|\sup_{l\in\mathbb{Z}}\mathcal{M}^l_{00}(f,g)\|_{L^p}\le C\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}$,
2) $\|\sup_{l\in\mathbb{Z}}\mathcal{M}^l_{0\infty}(f,g)\|_{L^p}\le C\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}$,
3) $\|\sup_{l\in\mathbb{Z}}\mathcal{M}^l_{\infty 0}(f,g)\|_{L^p}\le C\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}$,
4) $\|\sup_{l\in\mathbb{Z}}\mathcal{M}^l_{ij}(f,g)\|_{L^p}\le C\max(2^i,2^j)\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}$,

when $1/p=1/p_1+1/p_2$, $1<p_1,p_2\le\infty$.

Proof of Corollary 4.1. If $d\ge 2$, then the maximal operators

$$\sup_{t>0}\left(\int_{\mathbb{S}^{2d-1}}|f(x-ty)|\,d\sigma_{2d}(y,z)\right),$$
$$\text{and}\quad \sup_{t>0}\left(\int_{\mathbb{S}^{2d-1}}|g(x-tz)|\,d\sigma_{2d}(y,z)\right)$$

are bounded on $L^p(\mathbb{R}^d)$ when $1<p\le\infty$ (see [1, 15]). And so the results follow immediately from Lemma 4.1. □

By (4.1) if $1/p = 1/p_1 + 1/p_2$, then for all $l \in \mathbb{Z}$, we have

$$\|\mathcal{M}_{ij}^l\|_{L^{p_1}\times L^{p_2}\to L^p} = \|\mathcal{M}_{ij}^0\|_{L^{p_1}\times L^{p_2}\to L^p} := C(i,j;p,p_1,p_2). \tag{4.2}$$

For a fixed positive integer $\mathcal{N}$, define the operator

$$\mathbf{M}_{\mathcal{N}}(f,g)(x) := \sup_{|l|\le\mathcal{N}} \mathcal{M}^l(f,g)(x),$$

and let $A_{\mathcal{N}}(p,p_1,p_2)$ be such that

$$\|\mathbf{M}_{\mathcal{N}}(f,g)\|_{L^p} \le A_{\mathcal{N}}(p,p_1,p_2)\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}. \tag{4.3}$$

Since $\mathrm{M}_{\mathcal{N}}(f,g)(x) \uparrow \mathcal{M}(f,g)(x)$, by the monotone convergence theorem, we need to prove that $A_{\mathcal{N}}(p,p_1,p_2)$ is actually bounded by a constant independent of $\mathcal{N}$. Define the vector-valued operator

$$\mathbb{M}_{ij}^{\mathcal{N}} : \{f_l : |l|\le\mathcal{N}\} \times \{g_l : |l|\le\mathcal{N}\} \to \left\{\mathcal{M}^l\big(f_l*\psi_{i+l},\, g_l*\psi_{j+l}\big) : |l|\le\mathcal{N}\right\}.$$

Then by (4.3) we have

$$\begin{aligned}
&\left\|\mathbb{M}_{ij}^{\mathcal{N}}\left(\{f_l\}_{|l|\le\mathcal{N}} \times \{g_l\}_{|l|\le\mathcal{N}}\right)\right\|_{L^p(\ell^\infty)} \\
&= \left\|\sup_{|l|\le\mathcal{N}} \mathcal{M}^l\big(f_l*\psi_{i+l},\, g_l*\psi_{j+l}\big)\right\|_{L^p} \\
&\lesssim \left\|\sup_{|l|\le\mathcal{N}} \mathcal{M}^l\left(\mathrm{M}(f_l),\mathrm{M}(g_l)\right)\right\|_{L^p} \\
&\le \left\|\sup_{|l|\le\mathcal{N}} \mathcal{M}^l\left(\mathrm{M}\big(\sup_{|l|\le\mathcal{N}}|f_l|\big),\mathrm{M}\big(\sup_{|l|\le\mathcal{N}}|g_l|\big)\right)\right\|_{L^p} \\
&\le A_{\mathcal{N}}(p,p_1,p_2)\left\|\mathrm{M}\left(\sup_{|l|\le\mathcal{N}}|f_l|\right)\right\|_{L^{p_1}}\left\|\mathrm{M}\left(\sup_{|l|\le\mathcal{N}}|g_l|\right)\right\|_{L^{p_2}} \\
&\lesssim A_{\mathcal{N}}(p,p_1,p_2)\left\|\sup_{|l|\le\mathcal{N}}|f_l|\right\|_{L^{p_1}}\left\|\sup_{|l|\le\mathcal{N}}|g_l|\right\|_{L^{p_2}} \\
&= A_{\mathcal{N}}(p,p_1,p_2)\left\|\{f_l\}_{|l|\le\mathcal{N}}\right\|_{L^{p_1}(\ell^\infty)}\left\|\{g_l\}_{|l|\le\mathcal{N}}\right\|_{L^{p_2}(\ell^\infty)}.
\end{aligned} \tag{4.4}$$

Also by (4.2) we have

$$
\begin{aligned}
&\left\|\mathbb{M}_{ij}^{\mathcal{N}}\left(\{f_l\}_{|l|\le\mathcal{N}}\times\{g_l\}_{|l|\le\mathcal{N}}\right)\right\|_{L^p(\ell^p)} \\
&= \left\|\left\{\mathcal{M}^l(f_l*\psi_{i+l},g_l*\psi_{j+l}) : |l|\le N\right\}\right\|_{L^p(\ell^p)} \\
&\le C(i,j;p,p_1,p_2)\left(\sum_{|l|\le\mathcal{N}}\|f_l\|_{L^{p_1}}^p\|g_l\|_{L^{p_2}}^p\right)^{\frac{1}{p}} \\
&\le C(i,j;p,p_1,p_2)\left\|\{f_l\}_{|l|\le\mathcal{N}}\right\|_{L^{p_1}(\ell^{p_1})}\left\|\{g_l\}_{|l|\le\mathcal{N}}\right\|_{L^{p_2}(\ell^{p_2})} \\
&\le C(i,j;p,p_1,p_2)\left\|\{f_l\}_{|l|\le\mathcal{N}}\right\|_{L^{p_1}(\ell^1)}\left\|\{g_l\}_{|l|\le\mathcal{N}}\right\|_{L^{p_2}(\ell^1)}.
\end{aligned}
\tag{4.5}
$$

By interpolating (4.4) and (4.5) we have

$$
\begin{aligned}
&\left\|\mathbb{M}_{ij}^{\mathcal{N}}\left(\{f_l\}_{|l|\le\mathcal{N}}\times\{g_l\}_{|l|\le\mathcal{N}}\right)\right\|_{L^p(\ell^{2p})} \\
&\le\left[A_{\mathcal{N}}(p,p_1,p_2)\right]^{\frac12}\left[C(i,j;p,p_1,p_2)\right]^{\frac12}\left\|\{f_l\}_{|l|\le\mathcal{N}}\right\|_{L^{p_1}(\ell^2)}\left\|\{g_l\}_{|l|\le\mathcal{N}}\right\|_{L^{p_2}(\ell^2)}.
\end{aligned}
$$

We finally apply Littlewood-Paley Theory to obtain

$$
\begin{aligned}
\left\|\sup_{|l|\le\mathcal{N}}\mathcal{M}_{ij}^l(f,g)\right\|_{L^p} &= \left\|\sup_{|l|\le\mathcal{N}}\mathcal{M}^l(f*\psi_{i+l},g*\psi_{j+l})\right\|_{L^p} \\
&= \left\|\mathbb{M}_{ij}^{\mathcal{N}}\left(\{f*\psi_{i+l}\}_{|l|\le\mathcal{N}}\times\{g*\psi_{j+l}\}_{|l|\le\mathcal{N}}\right)\right\|_{L^p(\ell^\infty)} \\
&\le \left\|\mathbb{M}_{ij}^{\mathcal{N}}\left(\{f*\psi_{i+l}\}_{|l|\le\mathcal{N}}\times\{g*\psi_{j+l}\}_{|l|\le\mathcal{N}}\right)\right\|_{L^p(\ell^{2p})} \\
&\le \left[A_{\mathcal{N}}(p,p_1,p_2)\right]^{\frac12}\left[C(i,j;p,p_1,p_2)\right]^{\frac12} \\
&\quad\times\left\|\{f*\psi_{i+l}\}_{|l|\le\mathcal{N}}\right\|_{L^{p_1}(\ell^2)}\left\|\{g*\psi_{j+l}\}_{|l|\le\mathcal{N}}\right\|_{L^{p_2}(\ell^2)} \\
&\le \left[A_{\mathcal{N}}(p,p_1,p_2)\right]^{\frac12}\left[C(i,j;p,p_1,p_2)\right]^{\frac12}\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}.
\end{aligned}
$$

Thus if $p\ge1$, then we obtain

$$
A_{\mathcal{N}}(p,p_1,p_2)\lesssim\left[A_{\mathcal{N}}(p,p_1,p_2)\right]^{\frac12}\sum_{i,j\ge1}\left[C(i,j;p,p_1,p_2)\right]^{\frac12}+1,
$$

which implies that

$$A_{\mathcal{N}}(p,p_1,p_2) \lesssim \left(\sum_{i,j\geq 1} \left[C(i,j;p,p_1,p_2)\right]^{\frac{1}{2}} \right)^2 + 1.$$

Otherwise if $0 < p < 1$, then we obtain

$$[A_{\mathcal{N}}(p,p_1,p_2)]^p \lesssim [A_{\mathcal{N}}(p,p_1,p_2)]^{\frac{p}{2}} \sum_{i,j\geq 1} \left[C(i,j;p,p_1,p_2)\right]^{\frac{p}{2}} + 1,$$

which implies that

$$A_{\mathcal{N}}(p,p_1,p_2) \lesssim \left(\sum_{i,j\geq 1} \left[C(i,j;p,p_1,p_2)\right]^{\frac{p}{2}} \right)^{\frac{2}{p}} + 1.$$

Therefore $\mathcal{M}$ is bounded from $L^{p_1}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ if $1/p = 1/p_1 + 1/p_2$ and $(1/p_1, 1/p_2)$ lies in the open region stated in Theorem 1.

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