Semi-group structure of all endomorphisms of a projective variety admitting a polarized endomorphism

Sheng Meng and De-Qi Zhang

Let X be a projective variety admitting a polarized (or more generally, int-amplified) endomorphism. We show: there are only finitely many contractible extremal rays; and when X is $\mathbb{Q}$-factorial normal, every minimal model program is equivariant relative to the monoid $\mathrm{SEnd}(X)$ of all surjective endomorphisms, up to finite index.

Further, when X is rationally connected and smooth, we show: there is a finite-index submonoid G of $\mathrm{SEnd}(X)$ such that G acts via pullback as diagonal (and hence commutative) matrices on the Neron-Severi group; the full automorphisms group $\mathrm{Aut}(X)$ has finitely many connected components; and every amplified endomorphism is int-amplified.

1. Introduction

We work over an algebraically closed field k which has characteristic zero (unless otherwise indicated), and is uncountable (only used to guarantee the birational invariance of the rational connectedness property). Let f be a surjective endomorphism of a projective variety X. We say that f is q-*polarized* if $f^*L \sim qL$ (linear equivalence) for some ample Cartier divisor L and integer $q > 1$. We say that f is *amplified* (resp. *int-amplified*), if $f^*L - L$ is ample for some Cartier (resp. ample Cartier) divisor L. The notion of amplified endomorphisms f was first defined by Krieger and Reschke (cf. [15]). Fakhruddin showed that such f has a countable Zariski-dense subset of periodic points (cf. [10, Theorem 5.1]).

We refer to [14] for the definitions of log canonical (lc), klt or terminal singularities. A sequence $X = X_1 \dashrightarrow X_2 \dashrightarrow \cdots$ of log MMP (= *log minimal model program*) consists of the contraction $X_i \dashrightarrow X_{i+1}$ of a $(K_{X_i} + \Delta_i)$-negative extremal ray for some log canonical pair (X_i, Δ_i) with Δ_i being an $\mathbb{R}$-Cartier effective divisor, which is of divisorial, flip or Fano type. An

MMP is a log MMP with $\Delta = 0$. We refer to [12, Theorem 1.1] for the cone theorem of lc pairs and [1, Corollary 1.2] for the existence of lc flips.

A submonoid G of a monoid Γ is said to be of *finite-index* in Γ if there is a chain $G = G_0 \leq G_1 \leq \cdots \leq G_r = \Gamma$ of submonoids and homomorphisms $\rho_i : G_i \to F_i$ such that $\mathrm{Ker}(\rho_i) = G_{i-1}$ and all F_i are finite *groups*.

Theorem 1.1 below is the most crucial result of the paper. First, an lc pair (X, Δ) may have infinitely many $(K_X + \Delta)$-negative extremal rays. Theorem 1.1 below implies that this case will never happen if we assume X admits a polarized (or int-amplified) endomorphism (see also Theorem 4.5 for a more general result).

Let $\mathrm{SEnd}(X)$ be the set of *all surjective endomorphisms* on X.

Theorem 1.1 below says that every finite sequence of MMP starting from a $\mathbb{Q}$-factorial normal X is equivariant (cf. Definition 2.1) relative to $\mathrm{SEnd}(X)$, up to finite-index. Note that $\mathrm{SEnd}(X)$ is usually a huge infinite set; and also the image $h_*(R)$ of a K_X-negative extremal ray R in the closed effective 1-cycle cone $\overline{\mathrm{NE}}(X)$ under a map $h \in \mathrm{SEnd}(X)$, may not be K_X-negative anymore. So we have to deal with general (not necessarily K_X-negative) contractible extremal rays R of $\overline{\mathrm{NE}}(X)$ in the sense of Definition 4.1. Our Theorem 1.1 below asserts the finiteness of these rays R, without even assuming K_X being $\mathbb{Q}$-Cartier.

Theorem 1.1. *(cf. Theorems 4.5 and 4.7) Let X be a (not necessarily normal or $\mathbb{Q}$-Gorenstein) projective variety with a polarized (or int-amplified) endomorphism. Then:*

(1) *X has only finitely many (not necessarily K_X-negative) contractible extremal rays in the sense of Definition 4.1.*

(2) *Suppose X is $\mathbb{Q}$-factorial normal. Then any finite sequence of MMP starting from X is G-equivariant for some finite-index submonoid G of $\mathrm{SEnd}(X)$.*

We extend the results in [19] and [18] about equivariant MMP from being relative to a single polarized or int-amplified endomorphism to the whole $\mathrm{SEnd}(X)$ up to finite-index. When X is a point, every endomorphism of X is regarded as being polarized. A normal projective variety X is said to be *Q-abelian* if there is a finite surjective morphism $\pi : A \to X$ étale in codimension 1 with A being an abelian variety.

Theorem 1.2. *Let $f : X \to X$ be an int-amplified endomorphism of a $\mathbb{Q}$-factorial klt projective variety X. Then there exist a finite-index submonoid*

G of $\mathrm{SEnd}(X)$, a Q-abelian variety Y, and a G-equivariant (cf. Definition 2.1) relative MMP over Y

$$X = X_0 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow \cdots \dashrightarrow X_r = Y$$

(i.e. $g \in G = G_0$ descends to $g_i \in G_i$ on each X_i), with every $X_i \dashrightarrow X_{i+1}$ a divisorial contraction, a flip or a Fano contraction, of a K_{X_i}-negative extremal ray, such that:

(1) There is a finite quasi-étale Galois cover $A \to Y$ from an abelian variety A such that $G_Y := G_r$ lifts to a submonoid G_A of $\mathrm{SEnd}(A) \leq \mathrm{End}_{\mathrm{variety}}(A)$.

(2) If g in G is polarized (resp. int-amplified), then so are its descending g_i on X_i and the lifting to A of g_r on $X_r = Y$.

(3) If g in G is amplified and its descending g_i on X_i is int-amplified for some i, then g is int-amplified.

(4) If K_X is pseudo-effective, then $X = Y$ and it is Q-abelian.

(5) If K_X is not pseudo-effective, then for each i, $X_i \to Y$ is equi-dimensional holomorphic with every fibre (irreducible) rationally connected. The $X_{r-1} \to X_r = Y$ is a Fano contraction.

(6) For any subset $H \subseteq G$ and its descending $H_Y \subseteq \mathrm{SEnd}(Y)$, H acts via pullback on $\mathrm{NS}_{\mathbb{Q}}(X)$ or $\mathrm{NS}_{\mathbb{C}}(X)$ as commutative diagonal matrices with respect to a suitable basis if and only if so does H_Y.

Let $\mathrm{Pol}(X)$ be the set of *all polarized endomorphisms* on X, and let $\mathrm{IAmp}(X)$ be the set of *all int-amplified endomorphisms* on X. In general, they are not semigroups, i.e., they may not be closed under composition; see Example 1.7. When X is rationally connected and smooth, Theorem 1.4 below gives the assertion that if g and h are in $\mathrm{Pol}(X)$ (resp. $\mathrm{IAmp}(X)$) then $g^M \circ h^M$ remains in $\mathrm{Pol}(X)$ (resp. $\mathrm{IAmp}(X)$) for some $M > 0$ depending only on X. For general X, Corollary 1.3 says that the same assertion on X is reduced to that on the base of the end product Y of the MMP starting from X, or the quasi-étale abelian variety cover A of Y.

Corollary 1.3. *We use the notation and assumption in Theorem 1.2. For g, h in $G \subseteq \mathrm{SEnd}(X)$, let $\tau = g \circ h$, $\tau_Y = g_Y \circ h_Y$ the descending to Y and $\tau_A = g_A \circ h_A$ the lifting to A. Then we have:*

(I) *Suppose both g^*, h^* are diagonalizable on $\mathrm{NS}_{\mathbb{C}}(X)$ (resp. both g^*, h^* are diagonalizable on $\mathrm{NS}_{\mathbb{Q}}(X)$; both g, h are in $\mathrm{Pol}(X)$; one of g, h is in $\mathrm{IAmp}(X)$). Then* (Ia) *and* (Ib) *below are equivalent.*

(Ia) τ^* *is diagonalizable on $\mathrm{NS}_{\mathbb{C}}(X)$ (resp. τ^* is diagonalizable on $\mathrm{NS}_{\mathbb{Q}}(X)$; $\tau \in \mathrm{Pol}(X)$; $\tau \in \mathrm{IAmp}(X)$).*

(Ib) τ_Y^* *is diagonalizable on $\mathrm{NS}_{\mathbb{C}}(Y)$ (resp. τ_Y^* is diagonalizable on $\mathrm{NS}_{\mathbb{Q}}(Y)$; $\tau_Y \in \mathrm{Pol}(Y)$; $\tau_Y \in \mathrm{IAmp}(Y)$).*

(II) τ_Y *is in $\mathrm{Pol}(Y)$ (resp. $\mathrm{IAmp}(Y)$) if and only if τ_A is in $\mathrm{Pol}(A)$ (resp. $\mathrm{IAmp}(A)$).*

(III) *Suppose that both g^* and h^* are diagonalizable on $\mathrm{NS}_{\mathbb{C}}(X)$. Then $g^*h^* = h^*g^*$ on $\mathrm{NS}_{\mathbb{C}}(X)$ if and only if $g_Y^*h_Y^* = h_Y^*g_Y^*$ on $\mathrm{NS}_{\mathbb{C}}(Y)$.*

By the results in [19] and [18], we know that the building blocks of polarized (or more generally int-amplified) endomorphisms are those on Abelian varieties and rationally connected varieties. Indeed, if X has mild singularities, is non-uniruled and admits a polarized (resp. int-amplified) endomorphism, then X is a Q-abelian variety: there is a finite Galois cover $A \to X$ étale in codimension one such that f lifts to a polarized (resp. int-amplified) endomorphism on the abelian variety A; if X is uniruled, then a polarized (resp. int-amplified) f descends to a polarized (resp. int-amplified) endomorphism on the base Y of a special maximal rationally connected fibration $X \dashrightarrow Y$, and Y is non-uniruled, hence it is a Q-abelian variety; see [19, Proposition 1.6], [20, Corollary 4.20]. Therefore, the essential building blocks we have to study are those polarized (resp. int-amplifed) endomorphisms on rationally connected varieties.

Our next Theorem 1.4 gives the structure of the monoid $\mathrm{SEnd}(X)$ for a rationally connected X. The second assertion below says that the surjective endomorphisms on a rationally connected variety admitting a polarized (or int-amplifed) endomorphism, act as diagonal (and hence commutative) matrices on the Neron-Severi group, up to finite-index.

Though $\mathrm{Pol}(X)$ and $\mathrm{IAmp}(X)$ may not be subsemigroups of $\mathrm{SEnd}(X)$, the third and fourth assertions below say that they are semigroups "up to finite-index"; it also answers affirmatively [23, Question 4.15], "up to finite-index", when X is rationally connected and smooth. By Example 1.7, this extra "up to finite-index" assumption is necessary.

The fourth assertion below also says that the pullback action of $\mathrm{SEnd}(X)$ on $\mathrm{NS}_{\mathbb{Q}}(X)$ is determined by that of $\mathrm{IAmp}(X)$, up to finite-index (*hence the importance of studying int-amplified endomorphisms*). For a subset S of a

semigroup H and an integer $M \geq 1$, denote by $\langle S^{[M]} \rangle := \{ s_1^M \cdots s_r^M \mid r \geq 1, \, s_i \in S \}$.

Theorem 1.4. *(cf. Theorem 6.2) Let X be a rationally connected smooth projective variety admitting a polarized (or int-amplified) endomorphism f. We use the notation $X = X_0 \dashrightarrow \cdots \dashrightarrow X_r = Y$ and the finite-index submonoid $G \leq \mathrm{SEnd}(X)$ as in Theorem 1.2. Then there is an integer $M \geq 1$ depending only on X such that:*

(1) *The Y in Theorem 1.2 is a point.*

(2) *$G^*|_{\mathrm{NS}_{\mathbb{Q}}(X)}$ is a commutative diagonal monoid with respect to a suitable $\mathbb{Q}$-basis B of $\mathrm{NS}_{\mathbb{Q}}(X)$. Further, for every g in G, the representation matrix $[g^*|_{\mathrm{NS}_{\mathbb{Q}}(X)}]_B$ relative to B, is equal to $\mathrm{diag}[q_1, q_2, \ldots]$ with integers $q_i \geq 1$.*

(3) *$G \cap \mathrm{Pol}(X)$ is a subsemigroup of G, and consists exactly of those g in G such that $[g^*|_{\mathrm{NS}_{\mathbb{Q}}(X)}]_B = \mathrm{diag}[q, \ldots, q]$ for some integer $q \geq 2$. Further,*
$$G \cap \mathrm{Pol}(X) \supseteq \langle \mathrm{Pol}(X)^{[M]} \rangle.$$

(4) *$G \cap \mathrm{IAmp}(X)$ is a subsemigroup of G, and consists exactly of those g in G such that $[g^*|_{\mathrm{NS}_{\mathbb{Q}}(X)}]_B = \mathrm{diag}[q_1, q_2, \ldots]$ with integers $q_i \geq 2$. Further,*
$$G\,(G \cap \mathrm{IAmp}(X)) = G \cap \mathrm{IAmp}(X) \supseteq \langle \mathrm{IAmp}(X)^{[M]} \rangle;$$
any h in $\mathrm{SEnd}(X)$ has $(h^M)^* = (g_1^*)^{-1} g_2^*$ on $\mathrm{NS}_{\mathbb{Q}}(X)$ for some g_i in $G \cap \mathrm{IAmp}(X)$.

(5) *We have $h^M \in G$ and that $h^*|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is diagonalizable for every $h \in \mathrm{SEnd}(X)$.*

Let $\mathrm{Aut}(X)$ be the group of all automorphisms of X, and $\mathrm{Aut}_0(X)$ its neutral connected component. By applying Theorem 1.4, we have the following result.

Theorem 1.5. *(cf. Theorem 6.3) Let X be a rationally connected smooth projective variety admitting a polarized (or int-amplified) endomorphism. Then we have:*

(1) *$\mathrm{Aut}(X)/\mathrm{Aut}_0(X)$ is a finite group. More precisely, $\mathrm{Aut}(X)$ is a linear algebraic group (with only finitely many connected components).*

(2) *Every amplified endomorphism of X is int-amplified.*

(3) *X has no automorphism of positive entropy (nor amplified automorphism).*

Remark 1.6.

(1) The assumption of X being rationally connected smooth in Theorems 1.4 and 1.5 can be weakened as in Theorems 6.2 and 6.3.

(2) Let X be a projective variety with $f \in \mathrm{IAmp}(X)$ and $g \in \mathrm{SEnd}(X)$. Then both $f^i \circ g$ and $g \circ f^i$ are in $\mathrm{IAmp}(X)$ when $i \geq N$ for some $N > 0$; see [18, Proposition 1.4]. However, this N may depend on f and g.

Example 1.7. Let $X := \mathbb{P}^1 \times \mathbb{P}^1$. We define endomorphisms f, g on X as:

$$f([a_1 : b_1], [a_2 : b_2]) = ([a_2 : b_2], [a_1^4 : b_1^4]),$$
$$g([a_1 : b_1], [a_2 : b_2]) = ([a_2^4 : b_2^4], [a_1 : b_1]).$$

Denote by $h = g \circ f$. Then

$$h([a_1 : b_1], [a_2 : b_2]) = ([a_1^{16} : b_1^{16}], [a_2 : b_2]).$$

Note that $f^2([a_1 : b_1], [a_2 : b_2]) = g^2([a_1 : b_1], [a_2 : b_2]) = ([a_1^4 : b_1^4], [a_2^4 : b_2^4])$. Clearly, f and g are then 2-polarized, but h is not int-amplified. Note also that the set of preperiodic points of f and g are the same.

The difference with early papers. In [19] for polarized $f \in \mathrm{SEnd}(X)$ and [18] for int-amplified f, it was proved that the single f, replaced by a power, fixes a K_X-negative extremal ray. In this paper, we prove that there are only finitely many (not necessarily K_X-negative) contractible extremal rays. This guarantees the MMP is $\mathrm{SEnd}(X)$-equivariant; and even the whole monoid $\mathrm{SEnd}(X)$ (all up to finite-index) is diagonalizable (and hence commutative) over $\mathrm{NS}_{\mathbb{Q}}(X)$ when X is smooth rationally connected.

Even when X has Picard number one, the following question is still open when $n \geq 4$.

Question 1.8. *Let X be a rationally connected smooth projective variety of dimension $n \geq 1$ which admits a polarized endomorphism. Is X (close to be) a toric variety?*

2. Preliminaries

Throughout this section, we work over an arbitrary algebraically closed field k.

Terminology and notation. Let X be a projective variety. A Cartier divisor is always integral, unless otherwise indicated.

Let $n := \dim(X)$. We can regard $\mathrm{N}^1(X) := \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ as the space of *numerically equivalent classes* of $\mathbb{R}$-Cartier divisors. Denote by $\mathrm{N}_r(X)$ the space of *weakly numerically equivalent classes* of r-cycles with $\mathbb{R}$-coefficients (cf. [19, Definition 2.2]). Denote by $\overline{\mathrm{NE}}(X)$ the cone of the closure of effective real 1-cycles in $\mathrm{N}_1(X)$. When X is normal, we also call $\mathrm{N}_{n-1}(X)$ the space of *weakly numerically equivalent classes* of Weil $\mathbb{R}$-divisors. In this case, $\mathrm{N}^1(X)$ can be regarded as a subspace of $\mathrm{N}_{n-1}(X)$ (cf. [25, Lemma 3.2]). For $\mathbb{K} := \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$, denote by $\mathrm{NS}_{\mathbb{K}}(X) := \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{K}$.

Definition 2.1. Let

$$(*) : X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_r$$

be a finite sequence of dominant rational maps of projective varieties. Let $f : X_1 \to X_1$ be a surjective endomorphism. We say the sequence $(*)$ is f-*equivariant* if the following diagram is commutative

$$
\begin{array}{ccccc}
X_1 & \dashrightarrow & X_2 & \dashrightarrow \cdots \dashrightarrow & X_r \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_r} \\
X_1 & \dashrightarrow & X_2 & \dashrightarrow \cdots \dashrightarrow & X_r
\end{array}
$$

where $f_1 = f$ and all f_i are surjective endomorphisms. Let G be a subset of $\mathrm{SEnd}(X_1)$. We say the sequence $(*)$ is G-*equivariant* if $(*)$ is g-equivariant for any $g \in G$.

Definition 2.2. Let $f : X \to X$ be a surjective endomorphism of a projective variety X.

(1) f is *q-polarized* if $f^*L \sim qL$ for some ample Cartier divisor L and integer $q > 1$.

(2) f is *amplified* if $f^*L - L = H$ for some Cartier divisor L and ample divisor H.

(3) f is *int-amplified* if $f^*L - L = H$ for some ample Cartier divisors L and H.

(4) f is *separable* if the induced field extension $f^* : k(X) \to k(X)$ is separable where $k(X)$ is the function field of X.

Let $f : X \to X$ be a surjective endomorphism of a projective variety X of dimension $n \geq 1$. Let L be a Cartier divisor of X. Then

$$(f^s)^*L - L = f^*L' - L' = \sum_{i=0}^{s-1}(f^i)^*(f^*L - L)$$

where $L' = \sum_{i=0}^{s-1}(f^i)^*L$. Therefore, f is amplified (resp. int-amplified) if and only if so is f^s for some (or all) $s > 0$. Suppose further $q := (\deg f)^{\frac{1}{n}}$ is rational (and hence an integer). If $(f^s)^*L \sim q'L$ for some ample Cartier divisor L and $q' > 0$, then $q' = (\deg f^s)^{\frac{1}{n}} = q^s$ and $f^*L'' \sim qL''$ where $L'' = \sum_{i=0}^{s-1} q^{s-i}(f^i)^*L$. Therefore, f is polarized if and only if so is f^s for some (or all) $s > 0$.

Definition 2.3. Let X be a projective variety.

(1) $\mathrm{SEnd}(X)$ is the monoid of surjective endomorphisms of X.

(2) $\mathrm{Pol}(X)$ is the set of polarized endomorphisms of X.

(3) $\mathrm{IAmp}(X)$ is the set of int-amplified endomorphisms of X.

We thank the referee to point out that the assumption of normality in the below two lemmas is necesssary and give the reference.

Lemma 2.4. *(cf. [5, Theorem, Page 220]) Let $f : X \to Y$ be a finite surjective morphism of two varieties with Y being normal. Then f is an open map.*

By the above lemma, one easily gets the following result.

Lemma 2.5. *(cf. [4, Lemma 7.2]) Let $f : X \to Y$ be a finite surjective morphism of two varieties with Y being normal. Let S be a subset of Y. Then $f^{-1}(\overline{S}) = \overline{f^{-1}(S)}$.*

Next we prepare some useful lemmas about (int-)amplified endomorphisms.

Lemma 2.6. *Let $f : X \to X$ be an int-amplified endomorphism of a normal projective variety X of dimension n. Suppose $f^*Z \equiv_w aZ$ (weakly numerical equivalence) for some real number a and effective r-cycle $Z \in \mathrm{N}_r(X)$ with $r < n$. Then either $Z = 0$ or $a > 1$.*

Proof. Let H be any ample Cartier divisor. By [18, Lemma 3.11],

$$0 = \lim_{i \to +\infty} Z \cdot \frac{(f^i)^*(H^r)}{(\deg f)^i} = \lim_{i \to +\infty} \frac{1}{a^i}(f^i)^*Z \cdot \frac{(f^i)^*(H^r)}{(\deg f)^i} = \lim_{i \to +\infty} \frac{1}{a^i} Z \cdot H^r.$$

Suppose $Z \neq 0$. Since Z is effective, $Z \cdot H^r > 0$ and $a > 0$. Therefore, $a > 1$. □

Lemma 2.7. *Let $\pi : X \dashrightarrow Y$ be a dominant map of projective varieties. Let $f : X \to X$ and $g : Y \to Y$ be two surjective endomorphisms such that $g \circ \pi = \pi \circ f$. Suppose f is amplified. Then $\mathrm{Per}(g)$ is Zariski dense in Y.*

Proof. Let U be an open dense subset of X such that $\pi|_U$ is well defined. By [10, Theorem 5.1], $\mathrm{Per}(f) \cap U$ is Zariski dense in X and hence $\pi(\mathrm{Per}(f) \cap U)$ is Zariski dense in Y. Note that $\pi(\mathrm{Per}(f) \cap U) \subseteq \mathrm{Per}(g)$. So the lemma is proved. □

Lemma 2.8. *Let $\pi : X \dashrightarrow Y$ be a dominant map of projective varieties. Let $f : X \to X$ be an amplified endomorphisms such that $\pi = \pi \circ f$. Then $\dim(Y) = 0$.*

Proof. We may assume X is over the field k which is uncountable by taking the base change. Let U be an open dense subset of X such that $\pi|_U$ is well-defined. Let W be the graph of π and $p_1 : W \to X$ and $p_2 : W \to Y$ the two projections. For any closed point $y \in Y$, denote by $X_y := p_1(p_2^{-1}(y))$ and $U_y := U \cap X_y$. Note that $U_{y_1} \cap U_{y_2} = \emptyset$ if $y_1 \neq y_2$. By assumption, $f^{-1}(X_y) = X_y$. Then for some $s_y > 0$, $f^{-s_y}(X_y^i) = X_y^i$ for every irreducible component X_y^i of X_y, and $f^{s_y}|_{X_y^i}$ is amplified (cf. [18, Lemma 2.3]). If $U_y \neq \emptyset$, then $\mathrm{Per}(f) \cap U_y = \mathrm{Per}(f|_{X_y}) \cap U_y = \bigcup_i \mathrm{Per}(f^{s_y}|_{X_y^i}) \cap U_y \neq \emptyset$ by [10, Theorem 5.1]. Suppose $\dim(Y) > 0$. There are uncountably many $y \in Y$ such that $U_y \neq \emptyset$ and $\mathrm{Per}(f) \supseteq \bigcup_{y \in Y}(\mathrm{Per}(f) \cap U_y)$. In particular, $\mathrm{Per}(f)$ is uncountable, a contradiction to [18, Lemma 2.4]. □

We don't know whether the "amplified" property is preserved via an equivariant descending. Nevertheless, the following result is enough during the proof of Theorem 1.2.

Lemma 2.9. *Let $\pi : X \dashrightarrow Y$ be a dominant map of projective varieties, $f : X \to X$ and $g : Y \to Y$ two surjective endomorphisms such that $g \circ \pi = \pi \circ f$, and Z a closed subvariety of Y such that $g(Z) = Z$. Suppose f is amplified, $\dim(Z) > 0$ and π is well defined over an open dense subset $U \subseteq X$ such that $\pi|_U^{-1}(Z) \neq \emptyset$. Then $g|_Z \notin \mathrm{Aut}_0(Z)$.*

Proof. Let W be the graph of π and $p_1 : W \to X$ and $p_2 : W \to Y$ the two projections. Denote by $X' := p_1(p_2^{-1}(Z))$. Then $f(X') \subseteq X'$. Since $\pi|_U^{-1}(Z) \neq \emptyset$, there exists at least one irreducible component X'_i of X' dominating Z via π. If X'_i dominates Z, then $f(X'_i)$ dominates Z. Replacing f by some positive power, we may assume $f(X'_i) = X'_i$ for some X'_i dominating Z. Note that $f|_{X'_i}$ is still amplified (cf. [18, Lemma 2.3]). Therefore, it suffices for us to consider the case when $Z = Y$.

Suppose the contrary that $g \in \mathrm{Aut}_0(Y)$. Let G be the closure of the group generated by g in $\mathrm{Aut}_0(Y)$. Let $\tau : Y \dashrightarrow Y' := Y/G$. Then $\tau = \tau \circ g$. By Lemma 2.8, Y' is a point. Then the orbit Gy is open dense in Y for some $y \in Y$. By Lemma 2.7, we may assume $y \in \mathrm{Per}(g)$. Then Gy is a finite set and hence $\dim(Y) = 0$, a contradiction. $\square$

3. Totally periodic subvarieties

Throughout this section, we work over an arbitrary algebraically closed field k.

Let $f : X \to X$ be a surjective endomorphism of a normal projective variety X and S a subset of X. Here, a subset S of X is always a set of closed points. We say S is *f-invariant* (resp. *f-periodic*) if $f(S) = S$ (resp. $f^r(S) = S$ for some $r \geq 1$). We say S is *f^{-1}-invariant* (resp. *f^{-1}-periodic*) if $f^{-1}(S) = S$ (resp. $f^{-r}(S) = S$ for some $r \geq 1$).

Lemma 3.1. *Let $f : X \to X$ be a surjective endomorphism of a projective variety X and Z a Zariski closed subset of X. Then Z is f^{-1}-periodic if and only if so is any irreducible component of Z.*

Proof. Let $Z = \bigcup_{1 \leq i \leq n} Z_i$ be the irreducible decomposition of Z. If $f^{-s_i}(Z_i) = Z_i$ for some $s_i > 0$, then $f^{-s}(Z) = Z$ with $s = \prod_{i=1}^n s_i$.

Suppose $f^{-s}(Z) = Z$ for some $s > 0$. Then f^{-s} induces a permutation on the finite set $\{Z_i\}_{i=1}^n$. Therefore, $f^{-sn!}(Z_i) = Z_i$ for each i. $\square$

Definition 3.2. Let $f : X \to Y$ be a separable finite surjective morphism of two normal projective varieties. Denote by R_f the *ramification divisor* of f (cf. [22, Lemma 4.4]), and Σ_f the union of the prime divisors in R_f.

Lemma 3.3. *Let $f : X \to X$ be an int-amplified separable endomorphism of a normal projective variety X. Let Z be an f^{-1}-periodic irreducible closed subvariety such that $Z \subsetneq X$. Then $f^{-i}(Z) \subseteq \mathrm{Sing}(X) \cup \Sigma_f$ for some $i \geq 0$.*

Proof. We may assume $\dim(X) > 0$. Suppose $f^{-m}(Z) = Z$ for some $m > 0$. Let $Z_i = f^{-i}(Z)$, which is irreducible. If $Z_i \not\subset \mathrm{Sing}(X) \cup \Sigma_f$ for each i, then $Z_i = f^* Z_{i-1}$ by the purity of branch loci and hence $(f^m)^* Z = Z$. By Lemma 2.6, $Z = 0$, a contradiction. □

Following the proof of [19, Lemma 6.1], [4, Lemma 6.2] and [18, Lemma 8.1], we have the key lemma below. As shown in [4, Remark 6.3], the following condition (2) is necessary.

Lemma 3.4. *Let $f : X \to X$ be an int-amplified separable endomorphism of a projective variety X over the field k of characteristic $p \geq 0$. Assume $A \subseteq X$ is an irreducible closed subvariety with $f^{-i} f^i(A) = A$ for all $i \geq 0$. Assume further either one of the following conditions.*

(1) *A is a prime divisor of X.*

(2) *$p > 0$ and co-prime with $\deg f$.*

(3) *$p = 0$.*

Then $M(A) := \{ f^i(A) \,|\, i \in \mathbb{Z} \}$ is a finite set.

Proof. The proof follows from the proof of [19, Lemma 6.1], [4, Lemma 6.2] and [18, Lemma 8.1]. The only thing we need to check is that if condition (2) holds and Z is an f^{-1}-invariant closed subvariety of X, then p and $\deg f|_Z$ are co-prime. Let $d_1 = \deg f$ and $d_2 = \deg f|_Z$. Then $f_* Z = d_2 Z$. Suppose $f^* Z = aZ$ for some integer $a > 0$. By the projection formula, $ad_2 = d_1$. Then p and d_2 are co-prime. □

Lemma 3.5. *Let $f : X \to X$ be an int-amplified separable endomorphism of a projective variety X over the field k of characteristic $p \geq 0$. Assume $A \subseteq X$ is a Zariski closed subset with $f^{-i} f^i(A) = A$ for all $i \geq 0$. Assume further either one of the following conditions.*

(1) *A is a reduced divisor of X.*

(2) *$p > 0$ and co-prime with $\deg f$.*

(3) *$p = 0$.*

Then each irreducible component A_k of A is f^{-1}-periodic. In particular, A is f^{-1}-periodic.

Proof. Choose $i_0 \geq 0$ such that $A' := f^{i_0}(A), f(A'), f^2(A'), \ldots$ all have the same number of irreducible components. Then $f^{-i}f^i(A'_k) = A'_k$ for every irreducible component A'_k of A'. Now the lemma follows from Lemmas 3.4 and 3.1. □

We use Proposition 3.6 below in proving the results in the introduction. As kindly informed by Professors Dinh and Sibony, when $k = \mathbb{C}$, this kind of result (with a complete proof) first appeared in [8, Section 3.4]; [7, Theorem 3.2] is a more general form including Proposition 3.6 below, requiring a weaker condition and dealing with also dominant meromorphic self-maps of Kähler manifolds; see comments in [9, page 615] for the history of these results.

Here we offer a slightly more algebraic proof and it works also over any algebraically closed field k with $p = \text{char } k$ co-prime to $\deg f$ (so that the usual ramification divisor formula is applicable to f and its restrictions to subvarieties stable under the action of the powers of f). The assumption that $p = \text{char } k$ and $\deg f$ are co-prime is necessary; see Example 3.7.

Proposition 3.6. *(see [8, Section 3.4], [7, Theorem 3.2] and comments in [9, page 15]; see also [2]) Let $f : X \to X$ be an int-amplified endomorphism of a projective variety X over the field k of characteristic $p \geq 0$. Suppose either $p = 0$, or p and $\deg f$ are co-prime. Then there are only finitely many f^{-1}-periodic Zariski closed subsets.*

Proof. By taking normalization, we may assume X is normal. If S is an f^{-1}-periodic Zariski closed subsets, then each irreducible component of S is f^{-1}-periodic by Lemma 3.1. So it suffices to show that X has only finitely many f^{-1}-periodic irreducible closed subvarieties.

We prove by induction on $\dim(X)$. It is trivial if $\dim(X) = 0$. Suppose the contrary that there are infinitely many f^{-1}-periodic proper closed subvarieties of the same dimension d. Then we may find an infinite sequence of f^{-1}-periodic proper closed subvarieties S_i of the same dimension d with $S_i \subseteq \text{Sing}(X) \cup \Sigma_f$ by Lemma 3.3. Let Y be the closure of the union of S_i. Then $Y \subseteq \text{Sing}(X) \cup \Sigma_f$. By Lemma 2.5, for any $j \geq 0$,

$$f^{-j}f^j(Y) = f^{-j}f^j(\overline{\bigcup S_i}) = f^{-j}(\overline{f^j(\bigcup S_i)})$$
$$= f^{-j}(\overline{\bigcup f^j(S_i)}) = \overline{f^{-j}(\bigcup f^j(S_i))} = \overline{\bigcup f^{-j}f^j(S_i)} = \overline{\bigcup S_i} = Y.$$

Let Y_k be the irreducible component of Y. By Lemma 3.5, we may assume $f^{-1}(Y_k) = Y_k$ after replacing f by some positive power. Note that $f|_{Y_k}$ is int-amplified and $\dim(Y_k) < \dim(X)$. Then for each k, Y_k contains finitely many S_i by induction. This is a contradiction.   □

Example 3.7. Let $X := \mathbb{P}^3_k$ with $p = \text{char } k = 3$. Let $f : X \to X$ via

$$f([a : b : c : d]) = [a^3 + acd : b^3 + bcd : c^3 + c^2 d : d^3 - cd^2].$$

Then f is 3-polarized and separable. Let $X_1 := \{c = 0, d = 0\} \cong \mathbb{P}^1$. Then $f^{-1}(X_1) = X_1$ and $f|_{X_1}([a : b]) = [a^3 : b^3]$ which is a geometric Frobenius of $\mathbb{P}^1$. Note that $f|_{X_1}$ is polarized and bijective. When a is a $(3m - 1)$-th root of unity for some $m > 0$ and b is a $(3n - 1)$-th root of unity for some $n > 0$, the point $[a : b : 0 : 0]$ is f-periodic and hence f^{-1}-periodic. In particular, there are infinitely many f^{-1}-periodic closed points in X.

A Zariski-open subset of Zariski-closed subvariety of X is called a *subvariety* of X.

Corollary 3.8. *Let $f : X \to X$ be an int-amplified separable endomorphism of a projective variety X over the field k of characteristic $p \geq 0$. Suppose either $p = 0$, or p and $\deg f$ are co-prime. Then X has only finitely many (not necessarily closed) f^{-1}-periodic subvarieties.*

Proof. By taking normalization, we may assume X is normal. If A is f^{-1}-periodic, then so are $\overline{A}$ and $\overline{A} - A$ by Lemma 2.5. Note that $\overline{A} - A$ is a Zariski closed subset of X. If X has infinitely many f^{-1}-periodic subvarieties S_i, then we may assume $\overline{S_i} - S_i \neq \emptyset$ with $\overline{S_i} = \overline{S_j}$ for any i, j by Proposition 3.6. If $\overline{S_i} - S_i = \overline{S_j} - S_j$, then $S_i = S_j$. Hence, X has infinitely many f^{-1}-periodic Zariski closed subsets $\overline{S_i} - S_i$, a contradiction to Proposition 3.6.   □

4. Equivariant MMP and proof of Theorem 1.1

In this section, we work over an algebraically closed field k of characteristic 0. We prove Theorems 4.5 and 4.7 which include Theorem 1.1.

Let X be a projective variety and let C be a curve. Denote by $R_C := \mathbb{R}_{\geq 0}[C]$ the ray generated by $[C]$ in $\overline{\text{NE}}(X)$. Denote by Σ_C the union of curves whose classes are in R_C.

Definition 4.1. Let X be a projective variety. Let C be a curve such that R_C is an extremal ray in $\overline{\mathrm{NE}}(X)$. We say C or R_C is *contractible* if there is a surjective morphism $\pi : X \to Y$ to a projective variety Y such that the following hold.

(1) $\pi_* \mathcal{O}_X = \mathcal{O}_Y$.

(2) Let C' be a curve in X. Then $\pi(C')$ is a point if and only if $[C'] \in R_C$.

(3) Let D be a $\mathbb{Q}$-Cartier divisor of X. Then $D \cdot C = 0$ if and only if $D \equiv \pi^* D_Y$ (numerical equivalence) for some $\mathbb{Q}$-Cartier divisor D_Y of Y.

If R_C is an extremal ray contracted by π, then Σ_C equals $\mathrm{Exc}(\pi)$ which is Zariski closed in X; here $\mathrm{Exc}(\pi)$ is the exceptional locus of π (i.e. the subset of X along which π is not an isomorphism).

When (X, Δ) is lc, every $(K_X + \Delta)$-negative extremal ray R_C is contractible.

Lemma 4.2. *(cf. [24, Lemma 2.11]) Let X be a projective variety and let R_C be a ray of $\overline{\mathrm{NE}}(X)$ generated by some curve C. Let $h \in \mathrm{SEnd}(X)$. Then we have:*

(1) *$h_*(R_C) = R_{h(C)}$ and $h^*(R_C) = R_{C'}$ for any curve C' with $h(C') = C$.*

(2) *$h(\Sigma_C) = \Sigma_{h(C)}$ and $h^{-1}(\Sigma_C) = \Sigma_{C'}$ for any curve C' with $h(C') = C$.*

(3) *R_C is extremal if and only if so is $R_{h(C)}$ for some $h \in \mathrm{SEnd}(X)$, if and only if so is $R_{h(C)}$ for any $h \in \mathrm{SEnd}(X)$.*

Suppose R_C is extremal.

(4) *If $R_{h(C)}$ is contractible, then so is R_C.*

Proof. Let $h \in \mathrm{SEnd}(X)$. Note that h_* and h^* are invertible linear self-maps of $\mathrm{N}_1(X)$ and $h_*^{\pm}(\overline{\mathrm{NE}}(X)) = h^{*\pm}(\overline{\mathrm{NE}}(X)) = \overline{\mathrm{NE}}(X)$. Note that $h_* C = (\deg h|_C) h(C)$. So $h_*(R_C) = R_{h(C)}$. Since

$$(h_* \circ h^*)|_{\mathrm{N}_1(X)} = (\deg h)\,\mathrm{id}, \quad h_*(R_{C'}) = R_{h(C')} = R_{h(C)}$$

implies $h^*(R_C) = R_{C'}$ for any curve C' with $h(C') = C$. So (1) is proved.

For any curve E with $[E] \in R_C$, $[h(E)] \in R_{h(C)}$. Then

$$h(\Sigma_C) = h \left(\bigcup_{[E] \in R_C} E \right) = \bigcup_{[E] \in R_C} h(E) \subseteq \Sigma_{h(C)}.$$

For any curve F with $[F] \in R_{h(C)}$, there is some curve F_1 such that $h(F_1) = F$. Note that $R_{F_1} = h^*(R_F) = h^*(R_{h(C)}) = R_C$ by (1). So $[F_1] \in R_C$ and hence $h(\Sigma_C) = \Sigma_{h(C)}$. Similarly, $h^{-1}(\Sigma_C) = \Sigma_{C'}$. So (2) is proved.

By (1), $R_{h(C)} = h_*(R_C)$. Note that the set of extremal rays are stable under the actions h_* and h^*. So (3) is straightforward.

For (4), suppose $R_{h(C)}$ is extremal and contractible by $\pi : X \to Y$. Taking the Stein factorization of $\pi \circ h$, we have $\pi' : X \to Y'$ and $\tau : Y' \to Y$ such that $\pi'_*\mathcal{O}_X = \mathcal{O}_Y$ and τ is a finite surjective morphism. We claim that π' is the contraction of R_C. For any curve C' on X, $\pi'(C')$ is a point if and only if $\pi(h(C'))$ is a point; if and only if $[h(C')] \in R_{h(C)}$; if and only if $[C'] \in R_C$ by (1). Let D' be a $\mathbb{Q}$-Cartier divisor of X such that $D' \cdot C = 0$. Since $h^*|_{\mathrm{NS}_{\mathbb{Q}}(X)}$ is invertible, $D' \equiv h^*D$ for some $\mathbb{Q}$-Cartier divisor D. By the projection formula, $D \cdot h(C) = 0$. Since π is the contraction of $h(C)$, $D \equiv \pi^*D_Y$ for some $\mathbb{Q}$-Cartier divisor D_Y of Y. Then $D' \equiv h^*(\pi^*D_Y) = \pi'^*(\tau^*D_Y)$. So the claim is proved. $\square$

Lemma 4.3. *Let $f : X \to X$ be an int-amplified endomorphism of a projective variety. Let $h \in \mathrm{SEnd}(X)$. Let R_C be a contractible extremal ray of $\overline{\mathrm{NE}}(X)$ and F an irreduicble component of Σ_C. Then we have:*

(1) *$h^i(\Sigma_C)$ and $h^i(F)$ are f^{-1}-periodic for any $i \in \mathbb{Z}$.*

(2) *Σ_C and F are h^{-1}-periodic.*

Proof. Let $h \in \mathrm{SEnd}(X)$, $C' = h(C)$ and $C = h(\widetilde{C})$ for some curve $\widetilde{C}$. Since R_C is contractible, Σ_C is Zariski closed in X. By Lemma 4.2, $h(\Sigma_C) = \Sigma_{C'}$ and $h^{-1}(\Sigma_C) = \Sigma_{\widetilde{C}}$ are Zariski closed in X; and for any $j \geq 0$, $f^{-j}f^j(\Sigma_{C'}) = \Sigma_{C'}$ and $f^{-j}f^j(\Sigma_{\widetilde{C}}) = \Sigma_{\widetilde{C}}$. By Lemma 3.5, $h(\Sigma_C)$ and $h^{-1}(\Sigma_C)$ are both f^{-1}-periodic. By Lemma 3.1, $h(F)$ and $h^{-1}(F)$ are then f^{-1}-periodic. So (1) is proved.

Note that there are only finitely many f^{-1}-periodic Zariski closed subsets in X by Proposition 3.6. We have $h^m(F) = h^n(F)$ for some $m < n < 0$. So $h^{m-n}(F) = F$ and Σ_C is h^{-1}-periodic by Lemma 3.1. So (2) is proved. $\square$

Following [19, Lemma 6.2], we may further have the following stronger result.

Lemma 4.4. *Let $f : X \to X$ be an int-amplified endomorphism of a projective variety X. Let $E \subseteq X$ be a Zariski closed subset and let $\mathcal{R}_E$ be the set of all contractible extremal rays R_C with $\Sigma_C = E$. Then we have*

(1) *$\mathcal{R}_E$ is a finite set with $\sharp\mathcal{R}_E \leq \dim(E)$.*

(2) *Let F be an irreducible component of E. Then*

$$\mathcal{R}_E^F := \{R_{h(C)} \mid R_C \in \mathcal{R}_E, h \in \operatorname{SEnd}(X), h^{-1}(F) = F\}$$

is a finite set with $\sharp \mathcal{R}_E^F \leq \dim(F)$.

Proof. We assume that $\mathcal{R}_E$ is non-empty. Let $R_C \in \mathcal{R}_E$ (we may assume $C \subseteq F$). We have a contraction $\pi_C : X \to Y_C$ and a linear exact sequence

$$0 \to \operatorname{NS}_{\mathbb{C}}(Y_C) \xrightarrow{\pi_C^*} \operatorname{NS}_{\mathbb{C}}(X) \xrightarrow{\cdot C} \mathbb{C} \to 0.$$

So $\pi_C^* \operatorname{NS}_{\mathbb{C}}(Y_C)$ is a subspace in $\operatorname{NS}_{\mathbb{C}}(X)$ of codimension 1. Let F be an irreducible component of E. Let $j : F \hookrightarrow X$ be the inclusion map. For any $\mathbb{C}$-Cartier divisor D of X, denote by $D|_F := j^*D \in \operatorname{NS}_{\mathbb{C}}(F)$ the pullback. Let

$$\operatorname{NS}_{\mathbb{C}}(X)|_F := j^*(\operatorname{NS}_{\mathbb{C}}(X))$$

which is a subspace of $\operatorname{NS}_{\mathbb{C}}(F)$. Denote by

$$L_C := \{D|_F \ : \ D \in \operatorname{NS}_{\mathbb{C}}(X), D \cdot C = 0\}.$$

Then $L_C = j^*\pi_C^*(\operatorname{NS}_{\mathbb{C}}(Y_C))$ is a subspace in $\operatorname{NS}_{\mathbb{C}}(X)|_F$ of codimension at most 1. Note that for an ample divisor H in X, $H|_F \cdot C = H \cdot C \neq 0$. Therefore, $H|_F \notin L_C$ and hence L_C has codimension 1 in $\operatorname{NS}_{\mathbb{C}}(X)|_F$. Denote by

$$S := \{D|_F \in \operatorname{NS}_{\mathbb{C}}(X)|_F \ : \ (D|_F)^{\dim(F)} = 0\}.$$

We claim that S is a hypersurface (an algebraic set defined by a non-zero polynomial) in the complex affine space $\operatorname{NS}_{\mathbb{C}}(X)|_F$ and each L_C is an irreducible component of S in the sense of Zariski topology. Indeed, let $\{e_1, \ldots, e_k\}$ be a fixed basis of $\operatorname{NS}_{\mathbb{C}}(X)|_F$. Then

$$S = \left\{(x_1, \ldots, x_k) \ : \ \left(\sum_{i=1}^k x_i e_i\right)^{\dim(F)} = 0\right\}$$

is determined by a homogeneous polynomial of degree $\dim(F)$ and the coefficient of the term $\prod_i x_i^{\ell_i}$ is the intersection number $e_1^{\ell_1} \cdots e_k^{\ell_k}$. Note that for an ample divisor H in X, $H|_F \in \operatorname{NS}_{\mathbb{C}}(X)|_F$ and $(H|_F)^{\dim(F)} = H^{\dim(F)} \cdot F > 0$. So $e_1^{\ell_1} \cdots e_k^{\ell_k} \neq 0$ for some ℓ_i. In particular, S is determined by a non-zero

polynomial. Since $\dim(\pi_C(F)) < \dim(F)$, $\pi_{C*}F = 0$. For any $P \in \mathrm{NS}_{\mathbb{C}}(Y)$, we have

$$(\pi_C^* P|_F)^{\dim(F)} = (\pi_C^* P)^{\dim(F)} \cdot F = P^{\dim(F)} \cdot \pi_{C*}F = 0$$

by the projection formula. So $\pi_C^* P|_F \in S$. Hence $L_C \subseteq S$. Since L_C and S have the same dimension, each L_C is an irreducible component of S. The claim is proved.

Let $h \in \mathrm{SEnd}(X)$ such that $h^{-1}(F) = F$. The pullback h^* induces an automorphism of $\mathrm{NS}_{\mathbb{C}}(X)|_F$. Note that $h^* F = aF$ (as cycles) for some $a > 0$, and $(h^* D)^{\dim(F)} \cdot F = \frac{\deg h}{a} D^{\dim(F)} \cdot F$. Hence, $D \in S$ if and only if $h^* D \in S$. This implies that S is h^*-invariant. By the projection formula, $L_{h(C)} = (h^*)^{-1}(L_C)$ is also an irreducible component of S. Note that S has at most $\dim(F)$ irreducible components. So (2) follows from the claim below. Clearly, (1) follows from (2) directly.

Let $g, g' \in \mathrm{SEnd}(X)$ such that $g^{-1}(F) = g'^{-1}(F) = F$. Let $R_C \in \mathcal{R}_E$ and let $C' \subseteq F$ be another (not necessarily contractible or extremal) curve. We claim that $R_{g(C)} = R_{g'(C')}$ if and only if $L_{g(C)} = L_{g'(C')}$. Suppose $L_{g(C)} = L_{g'(C')}$. Let C_1 be some curve such that $g(C_1) = g'(C')$. By the projection formula, $L_{g(C)} = (g^*)^{-1}(L_C)$ and $L_{g'(C')} = (g^*)^{-1}(L_{C_1})$. Then $L_C = L_{C_1}$. Let H be an ample Cartier divisor of Y. Then $\pi_C^* H \cdot C_1 = 0$ implies that $\pi_C(C_1)$ is a point and hence $R_C = R_{C_1}$. Therefore, $R_{g(C)} = R_{g(C_1)} = R_{g'(C')}$ by Lemma 4.2. Another direction is trivial. So the claim is proved. $\square$

Theorem 4.5. *Let $f : X \to X$ be an int-amplified endomorphism of a projective variety X. Let $\mathcal{R}_{contr}$ be the set of all contractible extremal rays R_C. Then we have:*

(1) *$\mathcal{R}_{contr}$ is a finite set.*

(2) *The set*

$$\widetilde{\mathcal{R}}_{contr} := \{(h_*)^i(R_C) \,|\, R_C \in \mathcal{R}_{contr}, h \in \mathrm{SEnd}(X), i \in \mathbb{Z}\}$$

is finite.

(3) *There is a finite-index submonoid H of $\mathrm{SEnd}(X)$ such that $h_*(R) = h^*(R) = R$ for any $R \in \widetilde{\mathcal{R}}_{contr}$ and $h \in H$.*

Proof. We use the notation in Lemma 4.4. Let P_f be the set of f^{-1}-periodic Zariski closed subsets, which is finite by Proposition 3.6. For any $R_C \in \mathcal{R}_{contr}$, $\Sigma_C \in P_f$ by Lemma 4.3. Then $\mathcal{R}_{contr} = \bigcup_{E \in P_f} \mathcal{R}_E$ is finite by Lemma 4.4. So (1) is proved.

Let $\widetilde{\mathcal{R}}^0_{contr} := \{h_*(R_C) \,|\, R_C \in \mathcal{R}_{contr}, h \in \mathrm{SEnd}(X)\}$. We first claim that $\widetilde{\mathcal{R}}^0_{contr}$ is finite. Suppose the contrary that $\widetilde{\mathcal{R}}^0_{contr}$ is infinite. Since $\mathcal{R}_{contr}$ is finite by (1), there exist some $R_C \in \mathcal{R}_{contr}$ and infinitely many $h_j \in \mathrm{SEnd}(X)$ with $j > 0$ such that the set $\{h_{j*}(R_C)\}_{j=1}^\infty$ is infinite. Let F be an irreducible component of Σ_C. By Lemma 4.3, $h_j^{-1}(F) \in P_f$ and $h_1^{-s}(F) = F$ for some $s > 0$. Note that P_f is finite. So we may assume $h_j^{-1}(F) = h_1^{-1}(F)$ for any $j > 0$. Let $\widetilde{h}_j := h_j \circ h_1^{s-1}$. Then $\widetilde{h}_j^{-1}(F) = h_1^{-s}(F) = F$. For any $j_1, j_2 > 0$, $(\widetilde{h}_{j_1})_*(R_C) = (\widetilde{h}_{j_2})_*(R_C)$ implies $(h_{j_1})_*(R_C) = (h_{j_2})_*(R_C)$ by Lemma 4.2. In particular, the set $\{(\widetilde{h}_j)_*(R_C)\}_{j=1}^\infty$ is infinite. However, this contradicts Lemma 4.4.

Since $\widetilde{\mathcal{R}}^0_{contr}$ is finite, for any $h \in \mathrm{SEnd}(X)$ and $R_C \in \mathcal{R}$, $(h^m)_*(R_C) = (h^n)_*(R_C)$ for some $0 < m < n$. By Lemma 4.2, for any $i > 0$, $(h_*)^{-i}(R_C) = (h_*)^{k(n-m)-i}(R_C) = (h^{k(n-m)-i})_*(R_C) \in \widetilde{\mathcal{R}}^0_{contr}$ for $k \gg 1$. Then $\widetilde{\mathcal{R}} = \widetilde{\mathcal{R}}^0_{contr}$ is finite. So (2) is proved.

Note that the monoid action of $\mathrm{SEnd}(X)$ on $\widetilde{\mathcal{R}}_{contr}$ via $(h', h_*(R_C)) \rightarrow (h' \circ h)_*(R_C)$ is well defined. So (3) is proved. $\qquad\square$

Theorem 4.6. *Let (X, Δ) be an lc pair. Let $f : X \rightarrow X$ be an int-amplified endomorphism. Then we have:*

(1) *The set $\mathcal{R}_{neg}$ of $(K_X + \Delta)$-negative extremal rays in $\overline{\mathrm{NE}}(X)$ is finite.*

(2) *The set*

$$\widetilde{\mathcal{R}}_{neg} := \{(h_*)^i(R) \,|\, R \in \mathcal{R}_{neg}, h \in \mathrm{SEnd}(X), i \in \mathbb{Z}\}$$

   is finite.

(3) *There is a finite-index submonoid H of $\mathrm{SEnd}(X)$ such that $h_*(R) = h^*(R) = R$ for any $R \in \widetilde{\mathcal{R}}_{neg}$ and $h \in H$.*

Proof. Let $R \in \mathcal{R}_{neg}$. Since (X, Δ) is lc, $R = R_C$ for some curve C and R is contractible by the Cone theorem in [12, Theorem 1.1]. Then we are done by Theorem 4.5. $\qquad\square$

Theorem 4.7. *Let $f : X \rightarrow X$ be an int-amplified endomorphism of a $\mathbb{Q}$-factorial normal projective variety X. Then any finite sequence of MMP starting from X is G-equivariant for some finite-index submonoid G of $\mathrm{SEnd}(X)$.*

Proof. By [18, Theorem 1.6] (see also [3, Corollary 1.3]), X is lc. Let $X := X_1 \dashrightarrow \cdots \dashrightarrow X_s$ be a sequence of MMP. By [18, Theorem 8.2], replacing

f by a positive power, we may assume the above sequence is f-equivariant and $f_i := f|_{X_i}$ is int-amplified.

We show the theorem by induction on s. Suppose $X := X_1 \dashrightarrow \cdots \dashrightarrow X_{s-1}$ is G-equivariant. By Theorem 4.6, replacing G by its finite-index submonoid, we may assume $h^*(R) = R$ for any $h \in G|_{X_{s-1}}$ and any $K_{X_{s-1}}$-negative extremal ray R. If $\pi_{s-1} : X_{s-1} \dashrightarrow X_s$ is a divisorial contraction or a Fano contraction, then π_{s-1} is $G|_{X_{s-1}}$-equivariant. If π_{s-1} is a flip, then π_{s-1} is $G|_{X_{s-1}}$-equivariant by further applying [24, Lemma 3.6] (cf. [19, Lemma 6.6]). □

5. Proof of Theorem 1.2 and Corollary 1.3

Throughout this section, we work over characteristic 0. First, we prepare the following lemmas which are frequently used in the proof of our main theorems.

Lemma 5.1. *Let $f : X \to X$ be a surjective endomorphism of a projective variety. Then all the eigenvalues of $f^*|_{\mathrm{NS}_{\mathbb{C}}(X)}$ are algebraic integers.*

Proof. The action $f^*|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is induced by $f^*|_{\mathrm{NS}(X)}$. Note that $\mathrm{NS}(X)$ is a $\mathbb{Z}$-module of finite rank. The lemma follows. □

Lemma 5.2. *Let $\pi : X \to Y$ be a surjective morphism of two projective varieties such that π is not a finite morphism and $\pi^*\mathrm{NS}_{\mathbb{Q}}(Y)$ is a codimension-1 subspace of $\mathrm{NS}_{\mathbb{Q}}(X)$. Let $f : X \to X$ and $g : Y \to Y$ be surjective endomorphisms such that $\pi \circ f = g \circ \pi$. Then $f^*|_{\mathrm{NS}_{\mathbb{Q}}(X)/\mathrm{NS}_{\mathbb{Q}}(Y)} = q\,\mathrm{id}$ for some integer $q > 0$.*

Proof. Note that $\mathrm{NS}_{\mathbb{Q}}(X)/\mathrm{NS}_{\mathbb{Q}}(Y)$ is 1-dimensional. Then $f^*|_{\mathrm{NS}_{\mathbb{Q}}(X)/\mathrm{NS}_{\mathbb{Q}}(Y)} = q\,\mathrm{id}$ for some $q \in \mathbb{Q}$. By Lemma 5.1, q is then an integer. Let H be an ample Cartier divisor on X. Then $f^*H - qH \in \pi^*\mathrm{NS}_{\mathbb{Q}}(Y)$. Suppose $q \le 0$. Then $f^*H - qH$ is ample on X. Since π is not finite, there is no ample class in $\pi^*\mathrm{NS}_{\mathbb{Q}}(Y)$. So we get a contradiction. □

Lemma 5.3. *Let (X, Δ) be a $\mathbb{Q}$-factorial lc pair. Let $\pi : X \dashrightarrow Y$ be either a divisorial contraction, a flip, or a Fano contraction of a $K_X + \Delta$-negative extremal ray. Let $f : X \to X$ and $g : Y \to Y$ be surjective endomorphisms such that $g \circ \pi = \pi \circ f$. Suppose there are a dominant map $\tau : W \dashrightarrow X$ and an amplified endomorphism $h : W \to W$ such that $f \circ \tau = \tau \circ h$. Suppose further g is int-amplified. Then f is int-amplified.*

Proof. During the proof, we may always replacing f, g and h by suitable positive powers. If π is birational, then f is int-amplified by [18, Lemma 3.6]. Suppose π is a Fano contraction and f is not int-amplified. Then $f^*D \equiv D$ for some $D \in \mathrm{NS}_{\mathbb{Q}}(X) \backslash \pi^* \mathrm{NS}_{\mathbb{Q}}(Y)$ by [18, Proposition 3.3] and Lemma 5.2. We may assume D is π-ample. Since g is int-amplified, $\mathrm{Per}(g)$ is Zariski dense in Y by [10, Theorem 5.1]. Let $y \in \mathrm{Per}(g)$ be general and we may assume $g(y) = y$. Then $F := \pi^{-1}(y)$ is irreducible. Also $f(F) = F$. Suppose τ is well defined over an open dense subset $U \subseteq W$. Since F is over general point, $\tau|_U^{-1}(F) \neq \emptyset$. Note that $\dim(F) > 0$, $D|_F$ is ample and $(f|_F)^*(D|_F) \equiv D|_F$. Then we may assume $f|_F \in \mathrm{Aut}_0(F)$ (cf. [19, Theorem 1.2], [16, Proposition 2.2], [11, Theorem 4.8]). However, this contradicts Lemma 2.9. $\square$

We recall [18, Lemma 9.2] about the diagonalizable criterion for the pullback action.

Lemma 5.4. *Let (X, Δ) be a $\mathbb{Q}$-factorial lc pair. Let $\pi : X \dashrightarrow Y$ be either a divisorial contraction, a flip, or a Fano contraction of a $K_X + \Delta$-negative extremal ray. Let $f : X \to X$ and $g : Y \to Y$ be surjective endomorphisms such that $g \circ \pi = \pi \circ f$. Suppose $g^*|_{\mathrm{NS}_{\mathbb{C}}(Y)}$ is diagonalizable. Then so is $f^*|_{\mathrm{NS}_{\mathbb{C}}(X)}$.*

Proof. If π is a flip, then $\mathrm{NS}_{\mathbb{C}}(X) = \pi^* \mathrm{NS}_{\mathbb{C}}(Y)$ and hence $f^*|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is diagonalizable. If π is a divisorial contraction with E being the π-exceptional prime divisor, then $f^*E = \lambda E$ for some integer $\lambda \geq 1$. Note that $-E$ is π-ample by [14, Lemma 2.62]. Its class $[E] \in \mathrm{NS}_{\mathbb{C}}(X) \backslash \pi^* \mathrm{NS}_{\mathbb{C}}(Y)$. Note that $\pi^* \mathrm{NS}_{\mathbb{C}}(Y)$ is a codimension-1 subspace of $\mathrm{NS}_{\mathbb{C}}(X)$. Then $f^*|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is diagonalizable. If π is a Fano contraction, then $f^*|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is diagonalizable by [18, Lemma 9.2]. $\square$

Lemma 5.5. *Let (X, Δ) be an lc pair and let $\pi : X \to Y$ be a Fano contraction of a $(K_X + \Delta)$-negative extremal ray. Let $m := \dim(X)$, $n := \dim(Y)$ and $d := \dim(X) - \dim(Y)$. Let $D \in \mathrm{NS}_{\mathbb{C}}(X)$ and $H_1, \ldots, H_n \in \mathrm{NS}_{\mathbb{C}}(Y)$ such that $D^d \cdot \pi^*H_1 \cdots \pi^*H_n = 0$. Then either $D \in \pi^* \mathrm{NS}_{\mathbb{C}}(Y)$ or $H_1 \cdots H_n = 0$.*

Proof. Let C be some curve contracted by π. Suppose $D \notin \pi^* \mathrm{NS}_{\mathbb{C}}(Y)$. This is equivalent to saying $D \cdot C \neq 0$. Let A be a very ample Cartier divisor of X. Then $(D - aA) \cdot C = 0$ for some $a \neq 0$ and $E := D - aA \in \pi^* \mathrm{NS}_{\mathbb{C}}(Y)$. Let $Z := A_1 \cap \cdots \cap A_d$ where $A_1, \ldots, A_d$ are general members in the linear system $|A|$. Note that Z is pure n-dimensional and every irreducible component of Z dominates Y. In particular, $\pi_*(A^d) = bY$ for some $b > 0$. By the projection formula, we have $0 = D^d \cdot \pi^*H_1 \cdots \pi^*H_n = (E +$

$aA)^d \cdot \pi^* H_1 \cdots \pi^* H_n = (a^d) A^d \cdot \pi^* H_1 \cdots \pi^* H_n = (a^d b) H_1 \cdots H_n$. Therefore, $H_1 \cdots H_n = 0$. $\qquad\square$

Next, we provide a submonoid version of Lemma 5.4.

Lemma 5.6. *Let X be a $\mathbb{Q}$-factorial lc projective variety and let (X, Δ) be lc. Let $\pi : X \dashrightarrow Y$ be either a divisorial contraction, a flip, or a Fano contraction of a $K_X + \Delta$-negative extremal ray. Let G be a subset of $\mathrm{SEnd}(X)$ such that π is G-equivariant. Suppose $(G|_Y)^*|_{\mathrm{NS}_{\mathbb{C}}(Y)}$ is diagonalizable. Then so is $G^*|_{\mathrm{NS}_{\mathbb{C}}(X)}$.*

Proof. If π is a flip, the lemma is trivial. If π is a divisorial contraction with E being the π-exceptional prime divisor, then $[E] \in \mathrm{NS}_{\mathbb{C}}(X) \backslash \pi^* \mathrm{NS}_{\mathbb{C}}(Y)$ is a common eigenvector of $h^*|_{\mathrm{NS}_{\mathbb{C}}(X)}$ for any $h \in G$, and the lemma also holds. Next we assume π is a Fano contraction and regard $\mathrm{NS}_{\mathbb{C}}(Y)$ as a 1-codimensional subspace of $\mathrm{NS}_{\mathbb{C}}(X)$. Note that for any $h \in G$, $h^*|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is diagonalizable by Lemma 5.4.

Let $f, g \in G$. Suppose $f^* x_1 = a x_1$ for some $x_1 \in \mathrm{NS}_{\mathbb{C}}(X) \backslash \mathrm{NS}_{\mathbb{C}}(Y)$ and $a \neq 0$. Let $x_2, \ldots, x_k$ be a basis of $\mathrm{NS}_{\mathbb{C}}(Y)$ such that $x_2, \ldots, x_k$ are eigenvectors of $h^*|_{\mathrm{NS}_{\mathbb{C}}(Y)}$ for any $h \in G$. Suppose $f^* x_i = a_i x_i$ with $a_i \neq 0$. We may assume that $a_i = a$ if and only if $i \leq r$ for some $r \geq 1$. Let $g^* x_1 = b x_1 + y$ for some $b \neq 0$ and $y \in \mathrm{NS}_{\mathbb{C}}(Y)$. Write $y = \sum_{i=2}^{k} s_i x_i$ where $s_i \in \mathbb{C}$. Since $g^*|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is diagonalizable, $s_i \neq 0$ implies $g^* x_i \neq b x_i$. Then for each $i \leq r$ such that $s_i \neq 0$, we may replace x_1 by $x_1 + t_i x_i$ for some suitable t_i, such that, finally $f^* x_1 = a x_1$ and $g^* x_1 = b x_1 + \sum_{i=r+1}^{k} s_i x_i$.

Next we claim $y = 0$. Set $m := \dim(X)$, $n := \dim(Y)$ and $d := m - n$. Suppose $y \neq 0$. Then $y \cdot C \neq 0$ on Y for some $C = x_2^{\ell_2} \cdots x_k^{\ell_k}$ with $\sum_{i=2}^{k} \ell_i = n - 1$. So $x_j \cdot C \neq 0$ on Y for some $j > r$ and hence $x_1^d \cdot y \cdot C \neq 0$ and $x_1^d \cdot x_j \cdot C \neq 0$ on X by Lemma 5.5. Let $f^* C = e C$ and $g^* C = e' C$ for some non-zero e and e'. By the projection formula,

$$
\begin{aligned}
(\deg g) x_1^{d+1} \cdot C &= (g^* x_1)^{d+1} \cdot g^* C \\
&= (b^{d+1} e') x_1^{d+1} \cdot C + ((d+1) b^d e') x_1^d \cdot y \cdot C.
\end{aligned}
$$

Since $x_1^d \cdot y \cdot C \neq 0$, we have $x_1^{d+1} \cdot C \neq 0$. On the other hand, by the projection formula,

$$
(\deg f) x_1^d \cdot x_j \cdot C = (f^* x_1)^d \cdot f^* x_j \cdot f^* C = (a^d a_j e) x_1^d \cdot x_j \cdot C.
$$

Since $x_1^d \cdot x_j \cdot C \neq 0$, we have $\deg f = a^d a_j e$. By the projection formula again,

$$
(\deg f) x_1^{d+1} \cdot C = (f^* x_1)^{d+1} \cdot f^* C = (a^{d+1} e) x_1^{d+1} \cdot C.
$$

Since $a_j \neq a$, $\deg f = a^d a_j e \neq a^{d+1} e$. Hence $x_1^{d+1} \cdot C = 0$, a contradiction. So $y = 0$ as claimed.

Now $y = 0$ implies $f^*|_{\mathrm{NS}_{\mathbb{C}}(X)} \circ g^*|_{\mathrm{NS}_{\mathbb{C}}(X)} = g^*|_{\mathrm{NS}_{\mathbb{C}}(X)} \circ f^*|_{\mathrm{NS}_{\mathbb{C}}(X)}$. So $G^*|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is a commutative set. Since $G^*|_{\mathrm{NS}_{\mathbb{C}}(X)}$ consists of diagonalizable elements, $G^*|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is diagonalizable by [13, Section 15.4]. □

Proof of Theorem 1.2. By [18, Theorem 1.10], we have an f-equivariant relative MMP

$$X = X_0 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow \cdots \dashrightarrow X_r = Y$$

over Y, with Y being Q-abelian.

By Theorem 4.7, this MMP is also G-equivaraint for some finite-index submonoid G of $\mathrm{SEnd}(X)$. Since Y is $\mathbb{Q}$-abelian, any surjective endomorphism $g_r \in \mathrm{SEnd}(Y)$ is quasi-étale. By [21, Lemma 2.12] or [4, Lemma 8.1 and Corollary 8.2], G_r lifts to a subsemigroup G_A of $\mathrm{SEnd}(A) \leq \mathrm{End}_{\mathrm{variety}}(A)$. So (1) is proved.

(2) follows from [19, Theorem 3.11 and Corollary 3.12] and [18, Lemmas 3.5 and 3.6]. (3) follows from Lemma 5.3. (4) and (5) follow directly from [18, Theorem 1.10].

For (6), one direction is trivial and the case over $\mathbb{C}$ has been shown by Lemma 5.6. Suppose $H_Y^*|_{\mathrm{NS}_{\mathbb{Q}}(Y)}$ is diagonalizable. Then $H^*|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is diagonalizable by Lemma 5.6 and hence $H^*|_{\mathrm{NS}_{\mathbb{Q}}(X)}$ is commutative. Let $h \in H$ and λ be an eigenvalue of $h^*|_{\mathrm{NS}_{\mathbb{C}}(X)}$. Then λ is either an eigenvalue of $h_i^*|_{\mathrm{NS}_{\mathbb{Q}}(X_i)/\mathrm{NS}_{\mathbb{Q}}(X_{i+1})}$ or an eigenvalue of $h_r^*|_{\mathrm{NS}_{\mathbb{Q}}(Y)}$. In particular, $\lambda \in \mathbb{Q}$. So $h^*|_{\mathrm{NS}_{\mathbb{Q}}(X)}$ is diagonalizable for any $h \in G$. By [13, Section 15.4], $H^*|_{\mathrm{NS}_{\mathbb{Q}}(X)}$ is diagonalizable. □

Proof of Corollary 1.3. For (I), we see that (Ia) implies (Ib) by [19, Theorem 3.11] and [18, Lemma 3.5]. Conversely, the diagonalizable case has been shown by Lemma 5.4. Suppose g is q_g-polarized, h is q_h-polarized and τ_Y is q-polarized for some integers $q_g \geq 2, q_h \geq 2, q \geq 2$. For each i, $g_i := g|_{X_i}$ is q_g-polarized and $h_i := h|_{X_i}$ is q_h-polarized by [19, Lemma 3.10 and Theorem 3.11]. Since $\tau_Y^*|_{\mathrm{NS}_{\mathbb{C}}(Y)}$ is diagonalizable by [19, Proposition 2.9], $\tau^*|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is diagonalizable by Lemma 5.4. Let λ be an eigenvalue of $\tau^*|_{\mathrm{NS}_{\mathbb{C}}(X)}$. Then λ is either an eigenvalue of $\tau_i^*|_{\mathrm{NS}_{\mathbb{C}}(X_i)/\mathrm{NS}_{\mathbb{C}}(X_{i+1})}$ for some i or an eigenvalue of $\tau_Y^*|_{\mathrm{NS}_{\mathbb{C}}(Y)}$. Suppose that λ is an eigenvalue of $\tau_Y^*|_{\mathrm{NS}_{\mathbb{C}}(Y)}$ with $\dim(Y) > 0$. Note that

$$\deg \tau_Y = q^{\dim(Y)} = (\deg h_Y) \cdot (\deg g_Y) = q_h^{\dim(Y)} \cdot q_g^{\dim(Y)}.$$

So $|\lambda| = q = q_h \cdot q_g$ by [21, Lemma 2.1]. Suppose that λ is an eigenvalue of $\tau_i^*|_{\mathrm{NS}_{\mathbb{C}}(X_i)/\mathrm{NS}_{\mathbb{C}}(X_{i+1})}$. By Lemma 5.2,

$$\tau_i^*|_{\mathrm{NS}_{\mathbb{C}}(X_i)/\mathrm{NS}_{\mathbb{C}}(X_{i+1})} = (h_i^* \circ g_i^*)|_{\mathrm{NS}_{\mathbb{C}}(X_i)/\mathrm{NS}_{\mathbb{C}}(X_{i+1})} = (q_h \text{ id}) \circ (q_g \text{ id}) = q \text{ id}.$$

Then $\lambda = q$. Therefore, $\tau^*|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is diagonalizable with all the eigenvalues being of the same modulus. Applying [19, Proposition 2.9] and [21, Lemma 2.3], τ is q-polarized.

Suppose either g or h is int-amplified. Suppose τ_Y is int-amplified. Let λ be an eigenvalue of $\tau^*|_{\mathrm{NS}_{\mathbb{C}}(X)}$. If λ is an eigenvalue of $\tau_Y^*|_{\mathrm{NS}_{\mathbb{C}}(Y)}$, then $|\lambda| > 1$ by [18, Proposition 3.3]. Suppose λ is an eigenvalue of $\tau_i^*|_{\mathrm{NS}_{\mathbb{C}}(X_i)/\mathrm{NS}_{\mathbb{C}}(X_{i+1})}$ for some i. By Lemma 5.2, $g_i^*|_{\mathrm{NS}_{\mathbb{C}}(X_i)/\mathrm{NS}_{\mathbb{C}}(X_{i+1})} = a$ id for some integer $a \geq 1$ and $h_i^*|_{\mathrm{NS}_{\mathbb{C}}(X_i)/\mathrm{NS}_{\mathbb{C}}(X_{i+1})} = b$ id for some integer $b \geq 1$. Since either g or h is int-amplified, either $a > 1$ or $b > 1$ by [18, Proposition 3.3]. In particular, $\lambda = ab > 1$. By [18, Proposition 3.3] again, τ is int-amplified.

(II) follows from [19, Corollary 3.12] and [18, Lemma 3.6].

(III) follows from Theorem 1.2 by applying $H := \{f, g\}$. $\qquad\square$

6. Proof of Theorems 1.4 and 1.5

In this section, we work over characteristic 0. We prove Theorems 6.2 and 6.3 which include Theorems 1.4 and 1.5 as special cases.

Definition 6.1. Let X be a normal projective variety.

(1) $q(X) := h^1(X, \mathcal{O}_X) = \dim H^1(X, \mathcal{O}_X)$ (the irregularity).

(2) $\tilde{q}(X) := q(\tilde{X})$ with $\tilde{X}$ a smooth projective model of X.

(3) $q^\natural(X) := \sup\{\tilde{q}(Y) \,|\, Y \to X \text{ is finite surjective and étale in codimension } 1\}$.

(4) $\pi_1^{\mathrm{alg}}(X_{\mathrm{reg}})$ is the algebraic fundamental group of the smooth locus X_{reg} of X.

Theorem 6.2. *Let X be a $\mathbb{Q}$-factorial klt projective variety admitting an int-amplified endomorphism f. We use the notation $X = X_0 \dashrightarrow \cdots \dashrightarrow X_r = Y$ and the finite-index submonoid $G \leq \mathrm{SEnd}(X)$ as in Theorem 1.2. Suppose futher either $q^\natural(X) = 0$ or $\pi_1^{\mathrm{alg}}(X_{\mathrm{reg}})$ is finite. Then there is an integer $M \geq 1$ depending only on X such that:*

(1) *The Y in Theorem 1.2 is a point.*

(2) $G^*|_{\mathrm{NS}_{\mathbb{Q}}(X)}$ is a commutative diagonal monoid with respect to a suitable $\mathbb{Q}$-basis B of $\mathrm{NS}_{\mathbb{Q}}(X)$. Further, for every g in G, the representation matrix $[g^*|_{\mathrm{NS}_{\mathbb{Q}}(X)}]_B$ relative to B, is equal to $\mathrm{diag}[q_1, q_2, \ldots]$ with integers $q_i \geq 1$.

(3) $G \cap \mathrm{Pol}(X)$ is a subsemigroup of G, and consists exactly of those g in G such that $[g^*|_{\mathrm{NS}_{\mathbb{Q}}(X)}]_B = \mathrm{diag}[q, \ldots, q]$ for some integer $q \geq 2$. Further,

$$G \cap \mathrm{Pol}(X) \supseteq \langle \mathrm{Pol}(X)^{[M]} \rangle.$$

(4) $G \cap \mathrm{IAmp}(X)$ is a subsemigroup of G, and consists exactly of those g in G such that $[g^*|_{\mathrm{NS}_{\mathbb{Q}}(X)}]_B = \mathrm{diag}[q_1, q_2, \ldots]$ with integers $q_i \geq 2$. Further,

$$G\,(G \cap \mathrm{IAmp}(X)) = G \cap \mathrm{IAmp}(X) \supseteq \langle \mathrm{IAmp}(X)^{[M]} \rangle;$$

any h in $\mathrm{SEnd}(X)$ has $(h^M)^* = (g_1^*)^{-1} g_2^*$ on $\mathrm{NS}_{\mathbb{Q}}(X)$ for some g_i in $G \cap \mathrm{IAmp}(X)$.

(5) We have $h^M \in G$ and that $h^*|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is diagonalizable for every $h \in \mathrm{SEnd}(X)$.

Proof. We apply Theorem 1.2 and use the notation there. Note that $\pi : X \to Y$ is equi-dimensional and π has irreducible fibres. So (1) follows from [4, Lemma 11.1] and the proof of [19, Lemma 9.1].

The first half of (2) has been shown in Theorem 1.2. For any $g \in G$, let λ be an eigenvalue of $g^*|_{\mathrm{NS}_{\mathbb{Q}}(X)}$. Then λ is an eigenvalue of $g_j^*|_{\mathrm{NS}_{\mathbb{Q}}(X_j)/\mathrm{NS}_{\mathbb{Q}}(X_{j+1})}$ for some j. By Lemma 5.2, λ is a positive integer. So (2) is proved.

By Corollary 1.3, $G \cap \mathrm{Pol}(X)$ and $G \cap \mathrm{IAmp}(X)$ are both semigroups. For any $g \in G \cap \mathrm{Pol}(X)$, $[g^*|_{\mathrm{NS}_{\mathbb{Q}}(X)}]_B = \mathrm{diag}[q, \ldots, q]$ for some integer $q \geq 2$ by (2) and [21, Lemma 2.1]. For any $g \in G \cap \mathrm{IAmp}(X)$, $[g^*|_{\mathrm{NS}_{\mathbb{Q}}(X)}]_B = \mathrm{diag}[q_1, q_2, \ldots]$ with integers $q_i \geq 2$ by (2) and [18, Proposition 3.3]. Note that $\langle \mathrm{SEnd}(X)^{[M]} \rangle \subseteq G$ for some $M > 0$. So $G \cap \mathrm{Pol}(X) \supseteq \langle \mathrm{Pol}(X)^{[M]} \rangle$ and $G \cap \mathrm{IAmp}(X) \supseteq \langle \mathrm{IAmp}(X)^{[M]} \rangle$. Since G is a monoid, $G\,(G \cap \mathrm{IAmp}(X)) = G \cap \mathrm{IAmp}(X)$ by Corollary 1.3. For any $h \in \mathrm{SEnd}(X)$, $g_2 := h^M \circ f^M \in G \cap \mathrm{IAmp}(X)$. Let $g_1 := f^M$, which is in $G \cap \mathrm{IAmp}(X)$. Then $(h^M)^* = (g_1^*)^{-1} \circ g_2^*$ on $\mathrm{NS}_{\mathbb{Q}}(X)$. So (3) and (4) are proved. (5) is clear. $\square$

Theorem 6.3. *Let X be a $\mathbb{Q}$-factorial klt projective variety admitting an int-amplified endomorphism f. Suppose futher either $q^\natural(X) = 0$ or $\pi_1^{\mathrm{alg}}(X_{\mathrm{reg}})$ is finite. Then we have:*

(1) $\mathrm{Aut}(X)/\mathrm{Aut}_0(X)$ *is a finite group. Further,* $\mathrm{Aut}_0(X)$ *is a linear algebraic group.*

(2) *Every amplified endomorphism of* X *is int-amplified.*

(3) X *has no automorphism of positive entropy (nor amplified automorphism).*

Proof. By Theorem 6.2, we may run MMP $X = X_0 \dashrightarrow \cdots \dashrightarrow X_r = Y$ as in Theorem 1.2, with Y being a point. Moreover, for some $M > 0$, $(g^M)^*|_{\mathrm{NS}_{\mathbb{Q}}(X)}$ = id for any $g \in \mathrm{Aut}(X)$ since g has inverse. Let H be an ample Cartier divisor of X and let $H_g := \sum_{i=0}^{M-1}(g^i)^*H$. Then H_g is ample and $g^*H_g \equiv H_g$. Thus $[\mathrm{Aut}(X) : \mathrm{Aut}_0(X)] < \infty$ (cf. [19, Theorem 1.2], [16, Proposition 2.2], [11, Theorem 4.8]).

Let $X' \to X$ be an $\mathrm{Aut}(X)$-equivariant resolution of X. By Theorems 1.2 and 6.2, X and hence X' are rationally connected. So X' has trivial $\mathrm{Alb}(X')$. In particular, $\mathrm{Aut}_0(X')$ and hence $\mathrm{Aut}_0(X)$ are linear (cf. [17]). Therefore, (1) is proved.

(2) follows from Lemma 5.3; see also Theorem 1.2. (3) follows from (1) and (2); see also [18, Lemma 3.10]. $\qquad\square$

Proof of Theorems 1.4 and 1.5. By [6, Corollary 4.18], $\pi_1^{\mathrm{alg}}(X_{\mathrm{reg}})$ is trivial when X is a rationally connected smooth projective variety. Then Theorem 1.4 follows from Theorem 6.2 and Theorem 1.5 follows from Theorem 6.3. $\qquad\square$

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References

[1] C. Birkar, *Existence of log canonical flips and a special LMMP*, Publ. Math. Inst. Hautes Études Sci. **115** (2012), 325–368.

[2] J. -Y. Briend and J. Duval, *Erratum: Deux caractérisations de la mesure d'équilibre d'un endomorphisme de $P^k(\mathbb{C})$*, Publ. Math. Inst. Hautes Études Sci. No. 109 (2009), 295–296.

[3] A. Broustet and A. Höring, *Singularities of varieties admitting an endomorphism*, Math. Ann. **360** (2014), no. 1-2, 439–456.

[4] P. Cascini, S. Meng and D.-Q. Zhang, *Polarized endomorphisms of normal projective threefolds in arbitrary characteristic*, Math. Ann., `https://doi.org/10.1007/s00208-019-01877-6`.

[5] V. I. Danilov and V. V. Shokurov, Algebraic Geometry. I: Algebraic Varieties and Schemes, pp. 167–297, Encyclopaedia Math. Sci. **23**, Springer, Berlin, (1994).

[6] O. Debarre, Higher-Dimensional Algebraic Geometry, Universitext. Springer-Verlag, New York, (2001).

[7] T.-C. Dinh, *Analytic multiplicative cocycles over holomorphic dynamical systems*, Complex Var. Elliptic Equ. **54** (2009), no. 3-4, 243–251.

[8] T.-C. Dinh and N. Sibony, *Dynamique des applications d'allure polynomiale*, J. Math. Pures Appl. **82** (2003), 367–423.

[9] T. -C. Dinh and N. Sibony, *Equidistribution speed for endomorphisms of projective spaces*, Math. Ann. **347** (2010), no. 3, 613–626.

[10] N. Fakhruddin, *Questions on self-maps of algebraic varieties*, J. Ramanujan Math. Soc. **18** (2003), no. 2, 109–122.

[11] A. Fujiki, *On automorphism groups of compact Kähler manifolds*, Invent. Math. **44** (1978), 225–258.

[12] O. Fujino, *Fundamental theorems for the log minimal model program*, Publ. Res. Inst. Math. Sci. **47** (2011), no. 3, 727–789.

[13] J. E. Humphreys, Linear Algebraic Groups, Graduate Texts in Mathematics, Volume **21**, Springer, (1975).

[14] J. Kollár and S. Mori, Birational Geometry of Algebraic Varieties, Cambridge Tracts in Math. **134**, Cambridge Univ. Press, (1998).

[15] H. Krieger and P. Reschke, *Cohomological conditions on endomorphisms of projective varieties*, Bull. Soc. Math. France **145** (2017), no. 3, 449–468.

[16] D. I. Lieberman, *Compactness of the Chow scheme: applications to automorphisms and deformations of Kähler manifolds*, *Fonctions de plusieurs variables complexes, III* (Sém. François Norguet, 1975–1977), pp. 140–186, Lecture Notes in Math. **670**, Springer, Berlin, (1978).

[17] H. Matsumura, *On algebraic groups of birational transformations*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **34** (1963), 151–155.

[18] S. Meng, *Building blocks of amplified endomorphisms of normal projective varieties*, Math. Z. **294** (2020), no. 3, 1727–1747.

[19] S. Meng and D. -Q. Zhang, *Building blocks of polarized endomorphisms of normal projective varieties*, Adv. Math. **325** (2018), 243–273.

[20] N. Nakayama, *Intersection sheaves over normal schemes*, J. Math. Soc. Japan **62** (2010), no. 2, 487–595.

[21] N. Nakayama and D.-Q. Zhang, *Polarized endomorphisms of complex normal varieties*, Math. Ann. **346** (2010), no. 4, 991–1018.

[22] S. Okawa, *Extensions of two Chow stability criteria to positive characteristics*, Michigan Math. J. **60** (2011), no. 3, 687–703.

[23] X. Yuan and S. Zhang, *The arithmetic Hodge index theorem for adelic line bundles*, Math. Ann. **367** (2017), no. 3-4, 1123–1171.

[24] D.-Q. Zhang, *Polarized endomorphisms of uniruled varieties*, Compos. Math. **146** (2010), no. 1, 145–168.

[25] D.-Q. Zhang, *n-dimensional projective varieties with the action of an abelian group of rank $n-1$*, Trans. Amer. Math. Soc. **368** (2016), no. 12, 8849–8872.

Department of Mathematics, National University of Singapore
Singapore 119076, Republic of Singapore
and Korea Institute For Advanced Study, 85 Hoegiro, Dongdaemungu
Seoul 02455, Republic of Korea
E-mail address: math1103@outlook.com, ms@u.nus.edu

Department of Mathematics, National University of Singapore
Singapore 119076, Republic of Singapore
E-mail address: matzdq@nus.edu.sg