Math. Res. Lett.
Volume 27, Number 2, 551–563, 2020

# Separating invariants for Hopf algebras of small dimensions

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In this paper we obtain a finite set $S$ of separating invariants for the variety of Hopf algebras of a fixed dimension. In dimension $p^2$ where $p$ is a prime or when dimension is $< 18$, except $8, 12, 16$, these invariants determine isomorphism classes of Hopf algebras, *i.e.,* two Hopf algebras of a given dimension are isomorphic if and only if each of the invariants in $S$ take the same values on both the Hopf algebras.

## 1. Introduction

The aim of this article is to study the general isomorphism problem for finite dimensional Hopf algebras. The isomorphism problem for Hopf algebras is to be able to obtain an explicit decision procedure for determining if two given Hopf algebras are isomorphic or not. This problem has been settled for finite dimensional semisimple Hopf algebra in [3]. The idea there was to consider the different possible semi-simple Hopf algebra structures that can be given to an $n$-dimensional vector space over $\mathbb{C}$. By making a choice of basis for the underlying vector space, these Hopf algebra structures can be represented by their associated structure constants. The set of points so obtained form a closed subvariety of an affine space. The action of the general linear group $\mathrm{GL}_n(\mathbb{C})$ on the $n$-dimensional vector space by change of basis then induces an action also on the associated closed subvariety. The orbits for this action correspond to isomorphism classes of Hopf algebras. They show in [3] that the generators of the ring of invariants corresponding to this action on the closed subvariety give a collection of polynomials which separate isomorphism classes of semi-simple Hopf algebras of dimension $n$ (See [3, Theorem 11]). Here, we extend this result to finite dimensional Hopf algebras of certain dimensions.

We first obtain a set of separating invariants for the variety of Hopf algebras of a fixed dimension (See Theorem 4.1). This is done uniformly for all dimensions. In the case of semi-simple Hopf algebras, this can further be simplified to a subcollection of complete invariants, listed in [3, Theorem 4.1]. This is possible because the separating invariants separate closed orbits. This is ensured in the semi-simple case by a result of Ştefan [1]. This is not true uniformly for arbitrary dimensions, when working with all Hopf algebras of a given dimension. So in the general case, we need the condition of rigidity, instead. It is known by [7] that the orbit of a Hopf algebra is open if and only if it is rigid, in the sense of Gerstenharber. Makhlouf proved in [9] that in dimension $p^2$ where $p$ is a prime and in dimensions $< 14$, the Hopf algebras are rigid except in dimensions $8, 12$. This result can be easily extended to dimension $n < 18$, except $n = 16$. This along with the finiteness of the number of orbits implies that the orbits in these dimensions are closed. This sets the stage for the methods of [3] to be applied to get a set of complete invariants in these cases. In the non-semisimple case, the antipode satisfies a weaker set of relations so the invariants in this case differ from that of [3].

In Section 2, we recall various definitions and background on Hopf algebras. We then give a description in Section 3 of picture invariants for the ring of invariants for mixed tensors. With the aid of this, we list a separating set of invariants for the variety of Hopf algebras in Section 4. As a corollary, we also obtain a complete set of invariants for the Hopf algebras of certain small dimensions.

## 2. Preliminaries

### 2.1. Hopf algebra

A *bialgebra* over a field $K$ is an algebra with unit $(H, \mu, \eta)$ which is also a co-algebra with co-unit $(H, \Delta, \epsilon)$ such that the co-multiplication $\Delta$ and co-unit $\epsilon$ are algebra maps. This means that the following diagrams commute: (associativity and co-associativity)

$$\begin{array}{ccc} H\otimes H\otimes H & \xrightarrow{\mu\otimes id} & H\otimes H \\ \downarrow{\scriptstyle id\otimes\mu} & & \downarrow{\scriptstyle \mu} \\ H\otimes H & \xrightarrow{\mu} & H \end{array} \qquad \begin{array}{ccc} H & \xrightarrow{\Delta} & H\otimes H \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle id\otimes\Delta} \\ H\otimes H & \xrightarrow{\Delta\otimes id} & H\otimes H\otimes H \end{array}$$

(unit and co-unit)

$$
\begin{array}{ccc}
K \otimes H & \xrightarrow{\eta \otimes id} & H \otimes H \\
\simeq \uparrow & & \downarrow \mu \\
H & \xrightarrow{\quad id \quad} & H \\
\downarrow \simeq & & \mu \uparrow \\
H \otimes K & \xrightarrow{id \otimes \eta} & H \otimes H
\end{array}
\qquad
\begin{array}{ccc}
H \otimes H & \xrightarrow{\epsilon \otimes id} & K \otimes H \\
\Delta \uparrow & & \downarrow \simeq \\
H & \xrightarrow{\quad id \quad} & H \\
\downarrow \Delta & & \mu \uparrow \\
H \otimes H & \xrightarrow{id \otimes \epsilon} & H \otimes K
\end{array}
$$

(co-multiplication is an algebra homomorphism)

$$
\begin{array}{ccccc}
H \otimes H & \xrightarrow{\qquad \mu \qquad} & H & \xrightarrow{\qquad \Delta \qquad} & H \otimes H \\
\downarrow \Delta \otimes \Delta & & & & \downarrow id \\
H \otimes H \otimes H \otimes H & \xrightarrow{\sigma} & H \otimes H \otimes H \otimes H & \xrightarrow{\mu \otimes \mu} & H \otimes H
\end{array}
$$

where $\sigma : H \otimes H \otimes H \otimes H \to H \otimes H \otimes H \otimes H$ is given by $h_1 \otimes h_2 \otimes h_3 \otimes h_4 \mapsto h_1 \otimes h_3 \otimes h_2 \otimes h_4$. (co-unit is an algebra homomorphism; equivalently, unit is a co-algebra homomorphism)

$$
\begin{array}{ccc}
K & \xrightarrow{\simeq} & K \otimes K \\
\downarrow \eta & & \downarrow \eta \otimes \eta \\
H & \xrightarrow{\Delta} & H \otimes H
\end{array}
\qquad
\begin{array}{ccc}
H \otimes H & \xrightarrow{\mu} & H \\
\downarrow \epsilon \otimes \epsilon & & \downarrow \epsilon \\
K \otimes K & \xrightarrow{\simeq} & K
\end{array}
$$

$$
\begin{array}{ccc}
K & \xrightarrow{\eta} & H \\
& \searrow^{id} & \downarrow \epsilon \\
& & K
\end{array}
$$

The vector space $\mathrm{Hom}_K(H, H)$ becomes an algebra with respect to the convolution product given by

$$(f \star g)(x) := \mu \circ (f \otimes g) \circ \Delta(x).$$

The map $\eta\epsilon$ is the identity with respect to $\star$. A bialgebra is called a *Hopf algebra* if the idenity map has a two-sided inverse in $\mathrm{Hom}_K(H, H)$ with respect to the convolution product. This inverse is called the *antipode* of $H$.

This corresponds to the commutativity of the following diagram:

$$\begin{array}{ccc} H\otimes H & \xrightarrow{\iota\otimes id} & H\otimes H \\ \Delta\uparrow & & \downarrow\mu \\ H & \xrightarrow{\quad\eta\epsilon\quad} & H \\ \downarrow\Delta & & \mu\uparrow \\ H\otimes H & \xrightarrow{id\otimes\iota} & H\otimes H \end{array}$$

The classification of Hopf algebras over an algebraically closed field $K$ of characteristic 0 in low dimensions and various isolated cases is known. The following is a summary of classification data in small dimensions:

| Dimension | Classification |
|---|---|
| $p$, prime | $K[\mathbb{Z}_p]$ [16] |
| $p^2$, $p$ prime | $K[\mathbb{Z}_{p^2}]$; $K[\mathbb{Z}_p \times \mathbb{Z}_p]$; Taft algebras $T_{p^2}(q)$ for $q$ a primitive $p$-the root of unity. [12] |
| $pq$, $p$, $q$ distinct primes such that either<br>a.) $p < q < 4p + 11$, or<br>b.) $p$, $q$ are twin primes, or<br>c.) $p = 2$ | $H$ is semisimple, hence isomorphic to $K[\mathbb{Z}_{pq}]$ or its dual. [13] |
| 8 | $K[G]$, $\lvert G\rvert = 8$; $K[G]^*$, $G$ is non-abelian of order 8, $A_8$, $A_{C_2}$, $A'_{C_4}$, $A''_{C_4}$, $A'''_{C_{4,q}}$, $(A''_{C_4})^*$, $A_{C_2\times C_2}$ [2] |
| 12 | $K[G]$, $\lvert G\rvert = 12$; $K[D_6]^*$, $Al_4$, $Al_4^*$, $A_+$, $A_-$; $A_0$, $A_1$, $B_0$, $B_1$, $A_1^*$ [11] |
| 16 | [5, Theorem 1.3] |

From the above classification data it is clear that the number of isomorphism classes of Hopf algebras of dimension $n$, when $n$ is one of the numbers listed above, is finite. From now on we assume that the underlying field $K$ is **algebraically closed of characteristic** 0.

### 2.2. Variety of $n$-dimensional Hopf algebras

Let $V$ be a finite dimensional vector space over $K$ of dimension $n$. Let $T := \mathrm{Hom}(V \otimes V, V) \times \mathrm{Hom}(V, V \otimes V) \times V \times V^* \times \mathrm{Hom}(V, V)$. With respect to a choice of a basis of $V$, $T$ can be identified with the affine space of dimension $d = 2n^3 + n^2 + 2n$. Under this identification an element $(\mu, \Delta, \eta, \epsilon, \iota) \in T$ corresponds to the structure constants

$$(\mu_{11}^1, \mu_{12}^1, \ldots, \mu_{nn}^n, \Delta_1^{11}, \Delta_1^{12}, \ldots, \Delta_n^{nn}, \eta^1, \ldots, \eta_n, \epsilon_1, \ldots, \epsilon_n, \iota_1^1, \iota_2^1, \ldots, \iota_n^n) \in \mathbb{A}^d.$$

Such a $d$-tuple corresponds to an $n$-dimensional Hopf algebra if and only if the multiplication, co-multiplication, unit, counit, antipode maps, $\mu, \Delta, \eta, \epsilon, \iota$ (respectively) given by these constants satisfy the commutative diagrams described above. In other words, a $d$-tuple gives a Hopf algebra structure on $V$, if the structure constants satisfy the following equations:

$$\begin{aligned}
\mu_{jk}^t \mu_{tl}^i &= \mu_{kl}^t \mu_{jt}^i \\
\Delta_i^{jt} \Delta_t^{kl} &= \Delta_i^{tl} \Delta_t^{jk} \\
\eta^t \mu_{it}^j &= \delta_i^j = \eta^t \mu_{ti}^j \\
\Delta_i^{tj} \epsilon_t &= \delta_i^j = \Delta_i^{jt} \epsilon_t \\
\mu_{ij}^t \Delta_t^{kl} &= \Delta_i^{pq} \Delta_j^{rs} \mu_{pr}^k \mu_{qs}^l \\
\eta^t \Delta_t^{ij} &= \eta^i \eta^j \\
\mu_{ij}^t \epsilon_t &= \epsilon_i \epsilon_j \\
\eta^t \epsilon_t &= 1 \\
\Delta_i^{jk} \iota_j^t \mu_{tk}^r &= \epsilon^r \eta_i = \Delta_i^{jk} \iota_k^t \mu_{jt}^r
\end{aligned}$$

Here we have used the notation that a repeated upper and lower index indicates a sum over 1 to $n$ of that index. Let $H(V)$ denote the set of all elements in $\mathbb{A}^d$ satisfying the above equations. Then $H(V)$ is an affine subvariety of $T \simeq \mathbb{A}^d$ where $d = 2n^3 + 2n + n^2$. We follow the convention that the $d$-tuple associated to a Hopf algebra structure $H$ on $V$ is also denoted by $H$.

The group $G := GL(V)$ acts naturally on $T$ via change of basis. This induces an action on $\mathbb{A}^d$ described as follows:

$$\begin{aligned}
g.\mu_{jk}^t &= g_p^t (g^{-1})_j^q (g^{-1})_k^r \mu_{qr}^p \\
g.\Delta_i^{jk} &= (g^{-1})_i^p g_q^j g_r^k \Delta_p^{qr} \\
g.\eta^i &= g_p^i \eta^p \\
g.\epsilon_i &= (g^{-1})_i^p \epsilon_p \\
g.\iota_i^j &= (g^{-1})_i^p g_q^j \iota_p^q
\end{aligned}$$

This action preserves the subvariety $H(V)$; further, two points of $H(V)$ lie in the same $G$-orbit if and only if they correspond to isomorphic Hopf algebra structures on $V$. Thus the orbits for this action are in one-one correspondence with the isomorphism classes of Hopf algebras of dimension $n$. By the $G$-orbit of a Hopf algebra $H$ we shall interchangeably mean the $G$-orbit of $H$ in $H(V)$ or the isomorphism class of $H$.

### 2.3. Deformations, rigidity of Hopf algebras and open orbits

Orbit of a Hopf algebra under the $GL(V)$ action described above, need not be closed. In order to separate orbits by regular maps we need to ensure that the orbits are closed. Since orbits are disjoint and there are only finitely many in the above mentioned dimensions, if we can show that the orbits are open then it will imply that they are also closed. This is not true for all the dimensions mentioned above. Whether an orbit is open or not is captured by a certain rigidity condition which we recall shortly.

A *deformation* of a Hopf algebra $(H, \mu, \Delta, \eta, \epsilon, \iota)$ over $K$ is a Hopf algebra structure on $H \otimes_K K[[t]]$ given by the data $H_t := (H[[t]], \mu_t, \Delta_t, \eta_t, \epsilon_t, \iota_t)$ such that $\mu_t$ is expressible in the form $\mu_0 + \mu_1 t + \mu_2 t^2 + \cdots$ where $\mu_0 = \mu$ and each $\mu_i, i > 0$ are extended $K[[t]]$-bilinearly from $K$-bilinear maps $\mu_i : H \otimes H \to H$; similarly, $\Delta_t = \Delta_0 + \Delta_1 t + \Delta_2 t^2 + \cdots$ where $\Delta_0 = \Delta$ and $\Delta_i$, $i > 0$ are extended $K[[t]]$-linearly from $K$-bilinear maps $\Delta_i : H \to H \otimes H$. The maps $\eta_t$ and $\epsilon_t$ are extended $K[[t]]$-linearly from $\eta$ and $\epsilon$ respectively. A *null deformation* of $H$ is one in which the $\mu_i$ and $\Delta_i$ are 0 for all $i > 0$.

**Remark 2.1.** *The deformation of a Hopf algebra is determined completely by the deformation of its underlying bialgebra ([8]).*

Two deformations $H_t$ and $H'_t$ are equivalent if there exists a $K[[t]]$-linear map $\phi_t : H_t \to H'_t$ of the form $\phi_t = id + \phi_1 t + \phi_2 t^2 + \cdots$, which is a Hopf algebra isomorphism and $\phi_i : H \to H$ are $k$-linear maps extended $K[[t]]$-linearly. A Hopf algebra $H$ is said to be *rigid* over $K$ if it has no non-trivial deformations. By the Remark 2.1, it follows that the Hopf algebra $H$ is rigid if and only if the underlying bialgebra is so. The notion of rigidity as introduced above, is stronger than the notion of geometric rigidity. A Hopf algebra $H$ is said to be *geometrically rigid* if the orbit of $H$ is open. In characteristic 0, these two notions coincide. ([7, §9]; See also [6]).

By [9, Theorem 5.1, Theorem 5.3,] the Hopf algebras listed in the above table except $A_{C'_4}$ in dimension 8 and $A_0$ and $B_1$ in dimension 12 are rigid; in dimension 14 and 15, all the Hopf algebras are semisimple ([13]), hence rigid by [1, Corollary 1.5]; in dimension 16 the rigidity is not known yet. All Hopf algebras of dimension $p^2$, where $p$ is a prime, are rigid. Hence, in dimension $p^2$, where $p$ is a prime or in dimensions $\leq 17$, excluding $8, 12, 16$, the orbits are open. Being finitely many and disjoint, in all these cases, they are also closed. Hence in the above mentioned dimensions the separating invariants separate the orbits.

## 3. Invariants of mixed tensors and pictures

For non-negative integers $t$ and $b$, set $V_b^t := V^{*\otimes b} \otimes V^{\otimes t}$. For $s \in \mathbb{N}$ and given non-negative integers $t_i, b_i$, for $i = 1, \ldots, s$, we wish to describe a generating set for the ring of polynomial $\mathrm{GL}(V)$-invariant functions on the space $W = V_{b_1}^{t_1} \times \cdots \times V_{b_s}^{t_s}$ of several tensors.

Note that the coordinate ring of $W$, $K[W]$, can be identified with the symmetric algebra of the dual $W^*$

$$\begin{aligned} Sym_K(W^*) &= \oplus_{j\geq 0} Sym_K^j(W^*) \\ &= \oplus_{j\geq 0} \oplus_{\{(m_1,\cdots,m_s):\sum m_i = j\}} \otimes_{i=1}^s Sym_K^{m_i}((V_{b_i}^{t_i})^*). \end{aligned}$$

The natural surjection from $((V_{b_i}^{t_i})^*)^{\otimes m_i} \to Sym_K^{m_i}((V_{b_i}^{t_i})^*)$ leads to a $GL(V)$ - equivariant map $\otimes_{i=1}^s((V_{b_i}^{t_i})^*)^{\otimes m_i} \to \otimes_{i=1}^s Sym_K^{m_i}((V_{b_i}^{t_i})^*)$. This induces a surjection from the space of GL(V)-invariants of $(V_M^N)^*$ to

$$(\otimes_{i=1}^s Sym_K^{m_i}((V_{b_i}^{t_i})^*))^{\mathrm{GL(V)}},$$

where $N = \sum m_i t_i,\ \ M = \sum m_i b_i$. This followed by the observation that non-zero GL(V)-invariants for $(V_M^N)^*$ exist only if $N = M$, and in that case, $(V_N^N)^* \simeq \mathrm{End}(V^{\otimes N})$, enables us to use [14, Theorem 1]. The space of $GL(V)$-invariants of $(V_N^N)^*$ is spanned by the elements $T_\sigma : v_1 \otimes \cdots \otimes v_N \otimes \alpha^1 \otimes \cdots \otimes \alpha^N \mapsto \langle v_1, \alpha^{\sigma(1)}\rangle.\ldots.\langle v_N, \alpha^{\sigma(N)}\rangle$ as $\sigma$ varies over $\mathcal{S}_N$ such that $\sigma$ has no decreasing sub-sequence of length exceeding $n$ and hence, we also obtain a spanning set for $(\otimes_{i=1}^s Sym_K^{m_i}((V_{b_i}^{t_i})^*))^{\mathrm{GL(V)}}$ (by the surjection above).

In [3, §3], the notion of a 'picture invariant' is introduced, to describe the invariants obtained above. Picture invariants span the space of invariant polynomial functions ([3, Proposition 7]) on $W$.

We recall from [3] the definition of picture invariants. Choose a basis $v_1, \ldots, v_n$ for $V$ and let $v^1, \ldots, v^n$ be the dual basis of $V^*$. Let $T_{l_1\cdots l_t}^{u_1\cdots u_b}$ be the co-ordinate function on $V_b^t$ that is 1 on the basis element $v_{u_1} \otimes \cdots \otimes v_{u_t} \otimes v^{l_1} \otimes \cdots \otimes v^{l_b} \in V_b^t$ and 0 on the other basis elements. The ring of polynomial functions on the space $V_b^t$ can be identified with the polynomial ring

$K[T_{l_1\cdots l_b}^{u_1\cdots u_t} \mid l_l,\ldots,l_b,u_1,\ldots,u_t \in \{1,\ldots,n\}]$ in $n^{t+b}$ variables. More generally, for $W$ (as defined above) the co-ordinate ring can be identified with the polynomial ring $K[T(i)_{l_1\cdots l_{b_i}}^{u_1\cdots u_{t_i}}]$ where the indices $l_1,\ldots,l_{b_i},u_1,\ldots,u_{t_i}$, varies over $\{1,\ldots,n\}$, for each $i=1,\ldots,s$. This is a polynomial ring in $\sum_{i=1}^s n^{t_i+b_i}$ variables. For non-negative integers $m_1,\ldots,m_s$ such that

$$\sum_{i=1}^{s} m_i t_i = \sum_{i=1}^{s} m_i b_i = N$$

and given a $\sigma \in \mathcal{S}_N$, by the *picture invariant* $P(\sigma)$ on $W$ associated to the element $T_\sigma$, we mean the following element of $\otimes_{i=1}^s Sym^{m_i}(V_{b_i}^{t_i})^*$:

$$\prod_{i=1}^{s}\left(\prod_{j=1}^{m_i} T(i)_{r_{(\sum_{p<i} m_p b_p+(j-1)b_i+1)},\ldots,r_{(\sum_{p<i} m_p b_p+jb_i)}}^{r_{\sigma(\sum_{p<i} m_p b_p+(j-1)b_i+1)},\ldots,r_{\sigma(\sum_{p<i} m_p b_p+jb_i)}}\right)$$

where $r_1,\ldots,r_N$ are just place-holder indices.

**Example 3.1.** Let $W = V_2^1 \oplus V_1^2 \oplus V_0^1$ and $m_1 = 2,\ m_2 = 1,\ m_3 = 1$. Let $\sigma = (12)(45)$. The invariant $T_\sigma$ takes value 1 on the basis elements $v_{r_1} \otimes v_{r_2} \otimes v^{r_2} \otimes v_{r_3} \otimes v_{r_4} \otimes v^{r_1} \otimes v_{r_5} \otimes v^{r_3} \otimes v^{r_5} \otimes v^{r_4}$ as $r_1,\ldots,r_5$ vary over indices $\{1,\ldots,n\}$ and 0 on all the other basis elements. The picture invariant corresponding to this permutation is $T(1)_{r_1r_2}^{r_2}T(1)_{r_3r_4}^{r_1}T(2)_{r_5}^{r_3r_5}T(3)^{r_4}$.

By passing through the various identifications described above, we observe that

**Proposition 3.2.** ([15, Theorem 4.4.1]; Compare [3, Proposition 7]) *The picture invariants $P(\sigma)$ with underlying permutation $\sigma \in S_N$ having no decreasing subsequences of length exceeding $n$, as $N$ varies, span the ring of $GL_K(V)$-invariant polynomial functions on the space $V_{b_1}^{t_1} \times \cdots \times V_{b_s}^{t_s}$ of several tensors.* □

## 4. Separating invariants for Hopf algebras

Let $X$ be a $G$-variety. Two elements $w, w' \in X$ is said be *separated by polynomial invariants* if there exists $f \in K[X]^G$ such that $f(w) \neq f(w')$. A subset $S$ of $K[X]^G$ is said to be a *separating set of invariants* if whenever $w, w' \in X$ can be separated by invariants there is an element $f \in S$ such that $f(w) \neq f(w')$.

Given a $N \in \mathbb{N}$, let $\lambda \vdash N$. Denote the set of standard tableaux of shape $\lambda$ by $\mathrm{Std}(\lambda)$. Then we associate a picture invariant $\mathcal{P}(a,b,c,p,q)$ to a pair $p,\ q \in \mathrm{Std}(\lambda)$ and every triple of non-negative integers $(a,b,c)$ such that $a+b+c \leq N$. This is the picture invariant $P(RSK(p,q)) \in Sym^a(V_2^1)^* \otimes Sym^b(V_1^2)^* \otimes Sym^{a'}(V_0^1)^* \otimes Sym^{b'}(V_0^1)^* \otimes Sym^c(V_1^1)^*$ where $a' = N-(a+b+c)-a,\ b' = N-(a+b+c)-b$, as described in Proposition 3.2. Here $RSK(p,q)$ is the permutation obtained by applying the RSK-algorithm ([14, §2.4]) to the pair of standard tableaux $p,q$.

**Theorem 4.1.** *With notations as in § 2.2, the set* $S = \{\mathcal{P}(a,b,c,p,q) \mid a,b, c \geq 0, a+b+c \leq N, p,q \in \mathrm{Std}(\lambda), \lambda \vdash N, N < C(n)\}$ *forms a separating set of invariants for* $H(V)$*. Here* $C(n) = 9n^5(2n+1)^{2n^2}$.

*Proof.* First of all, the surjection from $K[T]^G \to K[H(V)]^G$ maps separating invariants to separating invariants. Hence, it suffices to show that $S$ is a separating set for $K[T]$. By Proposition 3.2, we know that the set of picture invariants $P(\sigma)$ as $\sigma$ varies over permutations with no decreasing subsequence of length bigger than $n$ generate the ring of invariants $K[T]$. Such permutations $\sigma$ are given under the RSK-algorithm by pairs of standard tableaux of shape $\lambda \vdash N$, where $\lambda$ has at most $n$ parts. Now by [4, Theorem 4.7.4] we observe that the invariants of degree $< C(n)$ generate the ring of invariants. This, one observes by noting that the dimension of $K[T]^G$ is bounded by $2n^3+2n+n^2$ and an upper bound for $\gamma(K[T]^G)$ may be calculated to be $(n+1)(2n+1)^{n^2}$. Now, for each $N < C(n)$, and for each triple of non-negative integers $a,b,c$ such that $a+b+c \leq N$ we obtain invariants lying in the multi-homogeneous component $Sym^a(V_2^1)^* \otimes Sym^b(V_1^2)^* \otimes Sym^{a'}(V_0^1)^* \otimes Sym^{b'}(V_0^1)^* \otimes Sym^c(V_1^1)^*$ of $K[T]$ where $a' = N-(a+b+c)-a$ and $b' = N-(a+b+c)-b$. As the triple $(a,b,c)$ varies over all possible permissible choices, we get all the invariants in $K[T]^G$. Thus, for each $N < C(n)$ and each triple $(a,b,c)$ as above, associated to each pair of standard tableaux $p,q \in \mathrm{Std}(\lambda)$ we get picture invariants $\mathcal{P}(a,b,c,p,q)$ which generate the ring of invariants $K[T]^G$. Hence, $S$ is a separating set of invariants. □

### 4.1. Complete invariants of Hopf algebras in small dimensions

As seen in Section 2, every Hopf algebra $H$ of dimension $n$ is associated with a $d$-tuple in $\mathbb{A}^d$ which, by abuse of notation, we also denote by $H$. With this convention, it makes sense to "evaluate" an invariant from $K[H(V)]^{GL(V)}$ at a Hopf algebra $H$.

**Corollary 1.** *Let* $n < 18$ *or* $n \in \{p^2, p \text{ is a prime}\}$ *and* $n \neq 8, 12, 16$*. Then two Hopf algebras* $H_1$ *and* $H_2$ *of dimension* $n$ *are isomorphic if and only if* $\mathcal{P}(a,b,c,p,q)(H_1) = \mathcal{P}(a,b,c,p,q)(H_2)$ *for all* $a,b,c \geq 0, a+b+c \leq N, p,q \in \mathrm{Std}(\lambda), \lambda \vdash N, N < C(n)\}$*, where* $C(n) = 9n^5(2n+1)^{2n^2}$*.*

*Proof.* This follows from Theorem 4.1, by noting that in the dimensions mentioned above the orbits of the Hopf algebras are closed, as indicated in Section 2.3 □

### 4.2. Additional remarks

The above discussion of invariants aims at obtaining a separating set of invariants for general finite dimensional Hopf algebras, not necessarily semi-simple. These invariants give a complete set of invariants in small dimensions, in particular in dimension $p^2$ ($p$ is a prime), where not all the Hopf algebras are semi-simple. However, in dimensions where all the Hopf algebras are semi-simple, for example when dimension is $pq$ where $p$, $q$ are twin primes, these invariants can be reduced to the invariants $\mathcal{I}(\mathbf{r},\mathbf{s},\sigma)$, defined in [3, §4]. We briefly explain this now.

When $H$ is a semi-simple Hopf algebra of dimension $n$ ($H^*$ denote its dual), the following pictorial relations (in the notation of Kuperberg, refer [3, §2]) additionally hold:

$$\longrightarrow nS \longrightarrow \;=\; \begin{array}{ccc} \longrightarrow \mu \longleftarrow & \Delta & \longrightarrow \\ \downarrow & \uparrow & \\ \phi & h & \end{array} \qquad\qquad h \rightarrow \phi = n \neq 0.$$

Here we follow the notations of §2.1. Further, $h \in H$ (resp. $\phi \in H^*$) denotes the trace of the left regular representation in $H^*$ (resp. $H$). In terms of the structure constants of $\mu$ and $\Delta$, the elements $h$, $\phi$ can be expressed as $h_i = \Delta_j^{ij}$ and $\phi_i = \mu_{ij}^j$. These elements are left integrals in $H$ and $H^*$ respectively. In view of the above relations, the antipode map $S$ can be expressed in terms of $\mu$, $\Delta$; further all the additional relations given in [3] hold. We are therefore in a position to apply [3, Proposition 10] to reduce the picture invariants $\mathcal{P}(a,b,c,p,q)$ to an invariant of the form $\mathcal{I}(\mathbf{r},\mathbf{s},\sigma)$ for a suitable $\mathbf{r},\mathbf{s} \vdash N$, $\sigma \in S_N$.

The invariants $\mathcal{I}(\mathbf{r},\mathbf{s},\sigma)$ coincide with the $(0,0)$-*basic invariants* of [10] defined as the scalars obtained by taking elements of the form $h^{\otimes a} \otimes \phi^{\otimes b}$ for some $a$ and $b$ and then applying comultiplication, multiplication repeatedly to the tensor factors to obtain an element of $H^{\otimes m} \otimes (H^*)^{\otimes m}$ for some $m$

and then permuting the tensor factors and finally pairing them. As discussed in [10, §10], some of these $(0,0)$-basic invariants have other representation theoretic interpretations. For example, the Frobenius-Schur indicator $\nu_n$ of an irreducible character $\psi_i$ of $H$ can be expressed as $\psi_i(\mu_n\Delta_n(h))/n$, giving rise to more complicated relations between some of the $(0,0)$-basic invariants and the higher Frobenius-Schur indicators. In particular, the basic invariant $\phi(h_1h_{m+1}\dots h_{(n-1)m+1}h_2h_{m+2}\dots h_{(n-1)m+2}\dots h_mh_{2m}\dots h_{nm})$, which is the same as $\mathcal{I}(nm,nm,\sigma)$ for $\sigma = \left(\begin{smallmatrix} 1 & 2 & \dots & n & n+1 & n+2 & \dots & nm \\ 1 & m+1 & \dots & (n-1)m+1 & 2 & m+2 & \dots & nm \end{smallmatrix}\right)$, can be shown to be $\dim(H)\sum_{i,j}\nu_n(\psi_i)\nu_m(\tau_j)\psi_i(S(\tau_i))$ where $\psi_i$ and $\tau_j$ are the irreducible characters of $H$ and $H^*$ respectively and $\nu_n$, $\nu_m$ are the Frobenius-Schur indicators. Similarly, in [10] Meir also exhibits that the Reshetikhin-Turaev invariants, which are given by the traces of the elements of the braid group on $n$-strings on the representation $D(H)^{\otimes n}$ where $D(H)$ is the Drinfeld double of $H$, are also basic invariants. So by the above discussion and the fact that the picture invariants $\mathcal{P}(a,b,c,p,q)$ can be reduced to the form $\mathcal{I}(\mathbf{r},\mathbf{s},\sigma)$ in the case when $H$ is semisimple, it follows that the above representation theoretic invariants of $H$ are related to the picture invariants $P(a,b,c,p,q)$, following the same line of argument as in [10].

When $H$ is non-semisimple, the analogues of the higher Frobenius-Schur indicators and Reshetikhin-Turaev invariants are not yet well-understood. It would, therefore, be interesting to directly relate the picture invariants $\mathcal{P}(a,b,c,p,q)$ to these invariants without the reduction to the form $\mathcal{I}(\mathbf{r},\mathbf{s},\sigma)$, but this seems to be difficult because of the lingering antipode map appearing in $\mathcal{P}(a,b,c,p,q)$. Unless the antipode can be replaced by the above pictorial equation in terms of $\mu$ and $\Delta$, a systematic description of $\mathcal{P}(a,b,c,p,q)$ to obtain such a relation seems hard to achieve.

## Acknowledgements

The author would like to thank Prof. Abdenacer Makhlouf for the expository talks on deformation theory given at IIT-K and for his invigorating discussions on the topic. Also, the author acknowledges the referee for bringing to notice the pertinent paper of Meir [10] and for indicating possible relations of the picture invariants with already known invariants.

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Received November 11, 2018
Accepted October 26, 2019