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Motivic concentration theorem

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In this short article, given a smooth diagonalizable group scheme G of finite type acting on a smooth quasi-compact separated scheme X, we prove that (after inverting some elements of representation ring of G) all the information concerning the additive invariants of the quotient stack $[X/G]$ is “concentrated” in the subscheme of G-fixed points X^G. Moreover, in the particular case where G is connected and the action has finite stabilizers, we compute the additive invariants of $[X/G]$ using solely the subgroups of roots of unity of G. As an application, we establish a Lefschtez-Riemann-Roch formula for the computation of the additive invariants of proper push-forwards.

1. Introduction and statement of results

A *dg category* $\mathcal{A}$, over a base field k (of characteristic $p \geq 0$), is a category enriched over complexes of k-vector spaces; see §2.1. Every (dg) k-algebra A gives naturally rise to a dg category with a single object. Another source of examples is provided by schemes (or, more generally, by algebraic stacks) since the category of perfect complexes $\mathrm{perf}(X)$ of every quasi-compact quasi-separated k-scheme X (or algebraic stack) admits a canonical dg enhancement $\mathrm{perf}_{\mathrm{dg}}(X)$; see §2.3. Let us denote by $\mathrm{dgcat}(k)$ the category of (essentially small) dg categories.

An *additive invariant* is a functor $E\colon \mathrm{dgcat}(k) \to \mathrm{D}$, with values in an additive category, which inverts Morita equivalences and sends semi-orthogonal decompositions in the sense of Bondal-Orlov [4] to direct sums; see §2.2. Examples of additive invariants include algebraic K-theory and its variants, cyclic homology and its variants, topological Hochschild homology and its

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variants, etc. Given a k-scheme X (or algebraic stack) as above, we will often write $E(X)$ instead of $E(\mathrm{perf}_{\mathrm{dg}}(X))$.

Let G be a smooth diagonalizable group k-scheme of finite type and X a smooth quasi-compact separated k-scheme X equipped with a G-action. In what follows, we will write $[X/G]$ for the associated (global) quotient stack, $G^\vee$ for the group of characters of G, and $R(G) \simeq \mathbb{Z}[G^\vee]$ for the representation ring of G.

As explained below in §2.4, given an additive invariant $E\colon \mathrm{dgcat}(k) \to \mathrm{D}$, the Grothendieck ring $K_0([X/G])$, *i.e.* the G-equivariant Grothendieck ring of X, acts naturally on the object $E([X/G]) \in \mathrm{D}$. By pre-composing this action with the ring homomorphism $R(G) \to K_0([X/G])$ (induced by pull-back along the projection map $X \to \bullet := \mathrm{Spec}(k)$), we hence obtain an action of $R(G)$ on $E([X/G])$. Given a multiplicative set $S \subset R(G)$, consider the following presheaf of abelian groups:

$$S^{-1}E([X/G]) := \mathrm{Hom}_{\mathrm{D}}(-, E([X/G])) \otimes_{R(G)} S^{-1}R(G)\,. \tag{1.1}$$

Note that since $S^{-1}R(G)$ can be written as a filtered colimit of free finite $R(G)$-modules, the presheaf (1.1) belongs to the category $\mathrm{Ind}(\mathrm{D})$ of ind-objects[1] in D.

Let H be a closed diagonalizable subgroup of G. In what follows, we will write X^H for the smooth closed subscheme of H-fixed points (consult [8, Exposé XII §9]) and S_H for the multiplicative set generated by the elements $(1-\chi) \in R(G) \simeq \mathbb{Z}[G^\vee]$, where $\chi \in G^\vee$ is any character of G whose restriction to H is non-trivial.

Under the above notations and assumptions, our first main result is the following:

Theorem 1.2 (Motivic concentration). *We have an isomorphism of ind-objects*

$$E(\iota^*)\colon S_H^{-1}E([X/G]) \xrightarrow{\simeq} S_H^{-1}E([X^H/G])$$

induced by pull-back along the closed immersion $\iota\colon X^H \hookrightarrow X$.

Moreover, its inverse is given by the following composition

$$S_H^{-1}E([X^H/G]) \xrightarrow{(\lambda_{-1}(\mathcal{N})\cdot -)^{-1}} S_H^{-1}E([X^H/G]) \xrightarrow{E(\iota_*)} S_H^{-1}E([X/G])\,,$$

where $\mathcal{N}$ *stands for the conormal bundle of the closed immersion* $\iota\colon X^H \hookrightarrow X$, $\lambda_{-1}(\mathcal{N})$ *for the Grothendieck class* $\sum_j (-1)^j [\bigwedge^j(\mathcal{N})] \in K_0([X^H/G])$, *and*

[1]For the general theory of ind-objects, we invite the reader to consult [2, 3].

$-\cdot-$ for the induced action of the ring $K_0([X^H/G])$ on the ind-object $S_H^{-1}E([X^H/G])$.

Intuitively speaking, Theorem 1.2 shows that (after inverting the multiplicative set S_H) all the information concerning the additive invariants of the quotient stack $[X/G]$ is "concentrated" in the quotient stack $[X^H/G]$. Since Theorem 1.2 holds for *every* additive invariant, we named it the "motivic concentration theorem".

Remark 1.3 (Generalization). Let $\mathcal{H}$ be a flat quasi-coherent sheaf of algebras over $[X/G]$, *i.e.* a G-equivariant flat quasi-coherent sheaf of algebras over X. Given an additive invariant $E\colon \mathrm{dgcat}(k) \to \mathrm{D}$, let us write $E([X/G];\mathcal{H})$ for the object $E(\mathrm{perf}_{\mathrm{dg}}([X/G];\mathcal{H})) \in \mathrm{D}$, where $\mathrm{perf}_{\mathrm{dg}}([X/G];\mathcal{H})$ stands for the canonical dg enhancement of the category of G-equivariant perfect $\mathcal{H}$-modules $\mathrm{perf}([X/G];\mathcal{H})$. As explained in Remark 4.10, Theorem 1.2 holds more generally with $S_H^{-1}E([X/G])$ and $S_H^{-1}E([X^H/G])$ replaced by $S_H^{-1}E([X/G];\mathcal{H})$ and $S_H^{-1}E([X^H/G];\iota^*(\mathcal{H}))$.

Remark 1.4 (Localization at prime ideals). Let ρ be a prime ideal of the representation ring $R(G) \simeq \mathbb{Z}[G^\vee]$. Recall that $G \simeq D(G^\vee)$, where $D(-)$ stands for the classical diagonalizable group scheme construction. On the one hand, similarly to (1.1), we can consider the following presheaf of abelian groups:

$$E([X/G])_{(\rho)} := \mathrm{Hom}_{\mathrm{D}}(-, E([X/G])) \otimes_{R(G)} R(G)_{(\rho)}\,.$$

On the other hand, following Segal [18, Prop. 3.7], we can consider the closed diagonalizable subgroup $H_\rho := D(G^\vee/K_\rho)$ of G (called the *support* of ρ), where $K_\rho := \{\chi \in G^\vee \mid 1-\chi \in \rho \subset \mathbb{Z}[G^\vee]\}$. Note that $S_{H_\rho} \cap \rho = \emptyset$ and that H_ρ is maximal with respect to this property. Therefore, by further inverting the elements $R(G)\backslash(S_{H_\rho} \cup \rho)$, we conclude that Theorem 1.2 holds similarly with $S_H^{-1}E([X/G])$ and $S_H^{-1}E([X^H/G])$ replaced by $E([X/G])_{(\rho)}$ and $E([X^{H_\rho}/G])_{(\rho)}$, respectively.

Given an additive category D, let us write $-\otimes_{\mathbb{Z}}-$ for the canonical action of the category of finite free $\mathbb{Z}$-modules $\mathrm{free}(\mathbb{Z})$ on D. This action extends naturally to an action of $\mathrm{Ind}(\mathrm{free}(\mathbb{Z}))$ on $\mathrm{Ind}(\mathrm{D})$. Our second main result is the following:

Theorem 1.5. *Assume that the base field k (of characteristic $p \geq 0$) contains the l^{th} roots of unity for every prime $l \neq p$ such that $(G^\vee)_{l\text{-torsion}} \neq 0$.*

Under this assumption, we have an isomorphism of ind-objects:

$$S_G^{-1}E([X/G]) \simeq E(X^G) \otimes_{\mathbb{Z}} S_G^{-1}R(G)\,.$$

Note that when G is moreover connected, *i.e.* a k-split torus T, the assumption of Theorem 1.5 is vacuous. In this case, we have an isomorphism of ind-objects

$$S_T^{-1}E([X/T]) \simeq E(X^T) \otimes_{\mathbb{Z}} \mathbb{Z}[t_1^{\pm}, \ldots, t_r^{\pm}][\{(1-t_i^j)^{-1}\}_{i,j}])\,,$$

where r stands for the rank of T, $1 \leq i \leq r$, and $j \neq 0 \in \mathbb{Z}$.

Similarly to Theorem 1.2, Theorem 1.5 shows that (after inverting the multiplicative set S_G) all the information concerning the additive invariants of the quotient stack $[X/G]$ is "concentrated" in the subscheme of G-fixed points X^G.

We now illustrate Theorems 1.2 and 1.5 in several examples:

Example 1.6 (Algebraic K-theory). Algebraic K-theory gives rise to an additive invariant $K\colon \mathrm{dgcat}(k) \to \mathrm{Ho}(\mathrm{Spt})$ with values in the category of spectra; see [19, §2.2.1]. Hence, Theorem 1.2 applied to $E = K$ yields an isomorphism of ind-objects:

$$K(\iota^*)\colon S_H^{-1}K([X/G]) \xrightarrow{\simeq} S_H^{-1}K([X^H/G])\,. \tag{1.7}$$

Consequently, we obtain, in particular, isomorphisms of abelian groups:

$$K_*(\iota^*)\colon S_H^{-1}K_*([X/G]) \xrightarrow{\simeq} S_H^{-1}K_*([X^H/G])\,. \tag{1.8}$$

Several variants of algebraic K-theory such as Karoubi-Villamayor K-theory, homotopy K-theory, and étale K-theory, are also additive invariants; see [19, §2.2.2-§2.2.6]. Hence, isomorphisms similar to (1.7)-(1.8) also hold for all these variants.

The above isomorphisms (1.8) and their explicit inverses, with $S_H^{-1}K_*([X/G])$ and $S_H^{-1}K_*([X^H/G])$ replaced by $K_*([X/G])_{(\rho)}$ and $K_*([X^{H_\rho}/G])_{(\rho)}$ (see Remark 1.4), were originally established by Thomason in [23, Thm. 2.1 and Prop. 3.1] under the weaker assumption that X is a regular algebraic space. Previously, in the particular case of the Grothendieck group, the isomorphism (1.8) and its explicit inverse were established by Nielsen in [16, Thm. 3.2] under the stronger assumptions that X is a smooth projective k-scheme and that k is algebraically closed.

Example 1.9 (Mixed complex). Recall from Kassel [12, §1] that a *mixed complex* is a (right) dg module over the algebra of dual numbers $\Lambda := k[\epsilon]/\epsilon^2$, where $\deg(\epsilon) = -1$ and $d(\epsilon) = 0$. The mixed complex construction gives rise to an additive invariant $\mathrm{C}\colon \mathrm{dgcat}(k) \to \mathcal{D}(\Lambda)$ with values in the derived category of Λ; see [19, §2.2.7]. Hence, Theorem 1.2 applied to $E = \mathrm{C}$ yields an isomorphism of ind-objects:

$$\mathrm{C}(\iota^*)\colon S_H^{-1}\mathrm{C}([X/G]) \xrightarrow{\simeq} S_H^{-1}\mathrm{C}([X^H/G]) \,. \tag{1.10}$$

Cyclic homology and all its variants such as Hochschild homology, negative cyclic homology, and periodic cyclic homology, factor through C. Consequently, an isomorphism similar to (1.10) also holds for all these invariants. To the best of the authors' knowledge, all these isomorphisms are new in the literature.

Example 1.11 (Periodic cyclic homology). Assume that $\mathrm{char}(k) = 0$. Periodic cyclic homology gives rise to an additive invariant $HP_\pm\colon \mathrm{dgcat}(k) \to \mathrm{Vect}_{\mathbb{Z}/2}(k)$ with values in the category of $\mathbb{Z}/2$-graded k-vector spaces; see [19, §2.2.11]. Moreover, thanks to the Hochschild-Kostant-Rosenberg theorem, we have an isomorphism $HP_\pm(Y) \simeq (\bigoplus_{i\,\mathrm{even}} H^i_{\mathrm{dR}}(Y), \bigoplus_{i\,\mathrm{odd}} H^i_{\mathrm{dR}}(Y))$ for every smooth k-scheme Y, where $H^*_{\mathrm{dR}}(-)$ stands for de Rham cohomology. Therefore, Theorem 1.5 applied to $E = HP_\pm$ yields, in particular, an isomorphism of $\mathbb{Z}/2$-graded k-vector spaces:

$$S_G^{-1}HP_\pm([X/G]) \simeq \left(\bigoplus_{i\,\mathrm{even}} H^i_{\mathrm{dR}}(X^G), \bigoplus_{i\,\mathrm{odd}} H^i_{\mathrm{dR}}(X^G)\right) \otimes_{\mathbb{Z}} S_G^{-1}R(G) \,.$$

This description of the periodic cyclic homology of the quotient stack $[X/G]$ in terms of the de Rham cohomology of the subscheme of G-fixed points X^G is, to the best of the authors' knowledge, new in the literature.

Example 1.12 (Topological Hochschild homology). Topological Hochschild homology gives rise to an additive invariant $THH\colon \mathrm{dgcat}(k) \to \mathrm{Ho}(\mathrm{Spt})$; see [19, §2.2.12]. Hence, Theorem 1.2 applied to $E = THH$ yields an isomorphism of ind-objects:

$$THH(\iota^*)\colon S_H^{-1}THH([X/G]) \xrightarrow{\simeq} S_H^{-1}THH([X^H/G]) \,. \tag{1.13}$$

Topological Hochschild homology and all its variants such as topological cyclic homology, topological negative cyclic homology, and topological periodic cyclic homology, are also additive invariants; consult [11, 15][19, §2.2.13].

Consequently, an isomorphism similar to (1.13) also holds for all these variants. To the best of the authors' knowledge, all these isomorphisms are new in the literature.

Example 1.14 (Topological periodic cyclic homology). Assume that k is a perfect field of characteristic $p > 0$. Let $W(k)$ be the ring of p-typical Witt vectors of k and $K := W(k)_{1/p}$ the fraction field of $W(k)$. Periodic cyclic homology gives rise to an additive invariant $TP_{\pm}(-)_{1/p} \colon \mathrm{dgcat}(k) \to \mathrm{Vect}_{\mathbb{Z}/2}(K)$ with values in the category of $\mathbb{Z}/2$-graded K-vector spaces; see [20, Thm. 2.3]. Moreover, following Scholze (see [21, Thm. 5.2]), we have $TP_{\pm}(Y)_{1/p} \simeq (\bigoplus_{i\,\mathrm{even}} H^i_{\mathrm{crys}}(Y), \bigoplus_{i\,\mathrm{odd}} H^i_{\mathrm{crys}}(Y))$ for every smooth proper k-scheme Y, where $H^*_{\mathrm{crys}}(-)$ stands for crystalline cohomology. Therefore, Theorem 1.5 applied to $E = TP_{\pm}(-)_{1/p}$ yields, in particular, an isomorphism of $\mathbb{Z}/2$-graded K-vector spaces:

$$S_G^{-1} TP_{\pm}([X/G])_{1/p} \simeq \left(\bigoplus_{i\,\mathrm{even}} H^i_{\mathrm{crys}}(X^G), \bigoplus_{i\,\mathrm{odd}} H^i_{\mathrm{crys}}(X^G) \right) \otimes_{\mathbb{Z}} S_G^{-1} R(G)\,.$$

Similarly to the above Example 1.11, this description of the topological periodic cyclic homology of the quotient stack $[X/G]$ in terms of the crystalline cohomology of the subscheme of G-fixed points X^G is new in the literature.

Proper push-forwards

The following result is an immediate application of the above Theorems 1.2 and 1.5:

Theorem 1.15 (Motivic Lefschetz-Riemann-Roch formula). *Given a G-equivariant proper map $f \colon X \to Y$, between smooth quasi-compact separated k-schemes, we have the following commutative diagram of ind-objects:*

$$\begin{array}{ccc}
S_H^{-1} E([X/G]) & \xrightarrow{E(\iota^*)} & S_H^{-1} E([X^H/G]) \\
& & \downarrow{\scriptstyle (\lambda_{-1}(\mathcal{N})\cdot -)^{-1}} \\
\downarrow{\scriptstyle E(f_*)} & & S_H^{-1} E([X^H/G]) \\
& & \downarrow{\scriptstyle E((f\circ\iota)_*)} \\
S_H^{-1} E([Y/G]) & = & S_H^{-1} E([Y/G])\,.
\end{array} \tag{1.16}$$

Moreover, in the particular case where X^G consists of a finite set of k-rational points and $Y = \bullet$, the commutative diagram (1.16) *(with $H = G$) reduces to the following commutative diagram of ind-objects*[2]

$$
\begin{array}{ccc}
S_G^{-1}E([X/G]) & \xrightarrow{E(\iota^*)} & \bigoplus_{x \in X^G} S_G^{-1}E([\bullet/G]) \\
 & & \downarrow \bigoplus_{x\in X^G}(\lambda_{-1}(T_x^\vee)\cdot -)^{-1} \\
\downarrow E(f_*) & & \bigoplus_{x \in X^G} S_G^{-1}E([\bullet/G]) \\
 & & \downarrow \nabla \\
S_G^{-1}E([\bullet/G]) & = & S_G^{-1}E([\bullet/G])\,,
\end{array}
\tag{1.17}
$$

where ∇ stands for the co-diagonal map and $T_x^\vee$ for the dual of the tangent bundle of X at the point x. Furthermore, whenever k contains the l^{th} roots of unity for every prime $l \neq p$ such that $(G^\vee)_{l\text{-torsion}} \neq 0$, the ind-object $\bigoplus_{x\in X^G} S_G^{-1}E([\bullet/G])$ in (1.17) *can be replaced by the ind-object $E(k) \otimes_{\mathbb{Z}} \bigoplus_{x\in X^G} S_G^{-1}R(G)$.*

Intuitively speaking, the commutative diagram (1.16), resp. (1.17), shows that after inverting the multiplicative set S_H, resp. S_G, all the information concerning the additive invariants of the push-forward along f, resp. along $X \to \bullet$, is "concentrated" in the quotient stack $[X^H/G]$, resp. in the set of k-rational points X^G.

To the best of the authors' knowledge, Theorem 1.15 is new in the literature. In the particular case where $E = K_0(-)$, the diagram (1.17) was originally established by Nielsen in [16, Prop. 4.5] (under the stronger assumptions that X is a smooth projective k-scheme and that k is algebraically closed) and later by Thomason in [23, Thm. 3.5] (with $S_G^{-1}K_0(-)$ replaced by $K_0(-)_{(\{0\})}$ (see Remark 1.4) under the weaker assumption that X is a regular algebraic space). Note that in this particular case, the diagram (1.17) reduces to the classical Lefschetz-Riemann-Roch formula

$$
\sum_i (-1)^i [H^i(X;\mathcal{F})] = \sum_{x\in X^G} \frac{[\mathcal{F}_x]}{\sum_j (-1)^j [\bigwedge^j (T_x^\vee)]} \quad \text{in } S_G^{-1}R(G)\,, \tag{1.18}
$$

[2]In the particular case where X^H consists of a finite set of k-rational points and G is moreover connected, the G-action on X^H is necessarily trivial. Consequently, in this case, the above diagram (1.17) holds similarly with $S_G^{-1}E(-)$ and X^G replaced by $S_H^{-1}E(-)$ and X^H, respectively.

which computes the G-equivariant Euler characteristic of every G-equivariant perfect complex of $\mathcal{O}_X$-modules $\mathcal{F}$ in terms of the finite set of k-rational points X^G. It is well-known that the formula (1.18) implies many other classical formulas such as the Woods Hole fixed-point formula (see [1]), the Weyl's character formula (see [7, 25]), the Brion's counting formula (see [6]), etc.

Torus actions with finite stabilizers

In this subsection we assume that G is moreover connected, i.e. a k-split torus T, and that the T-action on X has finite (geometric) stabilizers. Let us denote by $\mathcal{C}(T)$ the set of all those subgroups $\mu_n \subset T$ such that $X^{\mu_n} \neq \emptyset$. Note that since the T-action on X has finite stabilizers, the set $\{n \in \mathbb{N} \mid \mu_n \in \mathcal{C}(T)\}$ is finite; in what follows, we will write r for the least common multiple of the elements of this latter set.

Given a subgroup $\mu_n \in \mathcal{C}(T)$, let $\underline{S}_{\mu_n} \subset R(T)_{1/r}$ be the multiplicative set defined as the pre-image of 1 under the following $\mathbb{Z}[1/r]$-algebra homomorphism

$$R(T)_{1/r} \xrightarrow{\text{(a)}} R(\mu_n)_{1/r} \overset{\text{(b)}}{\simeq} \frac{\mathbb{Z}[1/r][t]}{\langle t^n - 1\rangle} \simeq \prod_{d|n} \frac{\mathbb{Z}[1/r][t]}{\langle \Phi_d(t)\rangle} \xrightarrow{\text{(c)}} \frac{\mathbb{Z}[1/r][t]}{\langle \Phi_n(t)\rangle},$$

where (a) is the restriction homomorphism, (b) is induced by the choice of a(ny) generator t of the character group $\mu_n^\vee$, $\Phi_d(t)$ stands for the d^{th} cyclotomic polynomial, and (c) is the projection homomorphism. Under these notations and assumptions, our third main result is the following:

Theorem 1.19. *For every additive invariant $E\colon \mathrm{dgcat}(k) \to \mathrm{D}$, with values in a $\mathbb{Z}[1/r]$-linear category, we have an isomorphism of ind-objects*

$$E([X/T]) \xrightarrow{\simeq} \bigoplus_{\mu_n \in \mathcal{C}(T)} \underline{S}_{\mu_n}^{-1} E([X^{\mu_n}/T]) \tag{1.20}$$

induced by pull-back along the closed immersions $X^{\mu_n} \hookrightarrow X$. Moreover, the direct sum on the right-hand side is finite.

Intuitively speaking, Theorem 1.19 shows that all the information concerning the additive invariants of the quotient stack $[X/T]$ (no invertion is needed!) is "concentrated" in the quotient stacks $[X^{\mu_n}/T]$.

Thanks to Theorem 1.19, the above isomorphism (1.20) holds for algebraic K-theory and all its variants, for cyclic homology and all its variants, for topological Hochschild homology and all its variants, etc. In the particular case of algebraic K-theory such an isomorphism was originally established by Vezzosi-Vistoli in [24, §3] under the weaker assumption that X is a regular algebraic space. The remaining isomorphisms are, to the best of the authors' knowledge, new in the literature.

Proofs

Our proof of Theorem 1.2, resp. Theorem 1.19, is different from the proof of Thomason, resp. of Vezzosi-Vistoli, in algebraic K-theory. Nevertheless, we do borrow some ingredients from their proofs. In fact, using a certain category of subschemes of the quotient stack $[X/G]$ (see §3), we are able to ultimately reduce the proof of Theorem 1.2, resp. Theorem 1.19, to the proof of the K_0-case of Thomason's result, resp. of Vezzosi-Vistoli's result; consult §4 for details. Note, however, that we *cannot* mimic Thomason's arguments, resp. Vezzosi-Vistoli's arguments, because they depend in an essential way on the dévissage property of G-theory (=K-theory for smooth schemes), which does *not* hold for many additive invariants. For example, as explained by Keller in [14, Example 1.11], Hochschild homology, and consequently the mixed complex, do *not* satisfy dévissage.

2. Preliminaries

Throughout the article, k will be a base field of characteristic $p \geq 0$.

2.1. Dg categories

Let $(\mathcal{C}(k), \otimes, k)$ be the category of (cochain) complexes of k-vector spaces. A *dg category* $\mathcal{A}$ is a category enriched over $\mathcal{C}(k)$ and a *dg functor* $F\colon \mathcal{A} \to \mathcal{B}$ is a functor enriched over $\mathcal{C}(k)$; consult Keller's survey [13]. Recall from §1 that $\mathrm{dgcat}(k)$ stands for the category of (essentially small) dg categories.

Let $\mathcal{A}$ be a dg category. The opposite dg category $\mathcal{A}^{\mathrm{op}}$ has the same objects and $\mathcal{A}^{\mathrm{op}}(x,y) := \mathcal{A}(y,x)$. The category $\mathrm{H}^0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and morphisms $\mathrm{H}^0(\mathcal{A})(x,y) := H^0\mathcal{A}(x,y)$, where $H^0(-)$ stands for the 0^{th}-cohomology functor. A *right dg $\mathcal{A}$-module* is a dg functor $M\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k)$ with values in the dg category $\mathcal{C}_{\mathrm{dg}}(k)$ of complexes of k-vector spaces. Let us write $\mathcal{C}(\mathcal{A})$ for the category of right dg $\mathcal{A}$-modules. Following [13, §3.2], the derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ is defined as the localization of $\mathcal{C}(\mathcal{A})$ with respect to the objectwise quasi-isomorphisms.

A dg functor $F\colon \mathcal{A} \to \mathcal{B}$ is called a *Morita equivalence* if the restriction functor $\mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{A})$ is an equivalence of derived categories; see [13, §4.6]. As explained in [19, §1.6], the category $\mathrm{dgcat}(k)$ admits a Quillen model structure whose weak equivalences are the Morita equivalences. Let us denote by $\mathrm{Hmo}(k)$ the associated homotopy category.

The *tensor product* $\mathcal{A} \otimes \mathcal{B}$ of dg categories is defined as follows: the set of objects is the cartesian product of the sets of objects of $\mathcal{A}$ and $\mathcal{B}$ and $(\mathcal{A} \otimes \mathcal{B})((x,w),(y,z)) := \mathcal{A}(x,y) \otimes \mathcal{B}(w,z)$. As explained in [13, §2.3], this construction gives rise to a symmetric monoidal structure on $\mathrm{dgcat}(k)$, which descends to $\mathrm{Hmo}(k)$.

A *dg $\mathcal{A}$-$\mathcal{B}$-bimodule* is a dg functor $\mathrm{B}\colon \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k)$ or, equivalently, a right dg $(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$-module. A standard example is the dg $\mathcal{A}$-$\mathcal{B}$-bimodule

$$ {}_F\mathrm{B} : \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \longrightarrow \mathcal{C}_{\mathrm{dg}}(k) \qquad (x,z) \mapsto \mathcal{B}(z, F(x)) \tag{2.1} $$

associated to a dg functor $F : \mathcal{A} \to \mathcal{B}$.

2.2. Additive invariants

A functor $E\colon \mathrm{dgcat}(k) \to \mathrm{D}$, with values in an additive category, is called an *additive invariant* if it satisfies the following two conditions:

(i) It sends the Morita equivalences (see §2.1) to isomorphisms.

(ii) Let $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{C} \subseteq \mathcal{B}$ be dg categories inducing a semi-orthogonal decomposition $\mathrm{H}^0(\mathcal{B}) = \langle \mathrm{H}^0(\mathcal{A}), \mathrm{H}^0(\mathcal{C}) \rangle$ in the sense of Bondal-Orlov [4]. In this case, the inclusions $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{C} \subseteq \mathcal{B}$ induce an isomorphism $E(\mathcal{A}) \oplus E(\mathcal{C}) \stackrel{\simeq}{\to} E(\mathcal{B})$.

Given small dg categories $\mathcal{A}$ and $\mathcal{B}$, let us write $\mathrm{rep}(\mathcal{A},\mathcal{B})$ for the full triangulated subcategory of $\mathcal{D}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$ consisting of those dg $\mathcal{A}$-$\mathcal{B}$-modules B such that for every object $x \in \mathcal{A}$ the associated right dg $\mathcal{B}$-module $\mathrm{B}(x,-) \in \mathcal{D}(\mathcal{B})$ belongs to the full triangulated subcategory of compact objects $\mathcal{D}_c(\mathcal{B})$. As explained in [19, §1.6.3], there is a natural bijection between $\mathrm{Hom}_{\mathrm{Hmo}(k)}(\mathcal{A},\mathcal{B})$ and the set of isomorphism classes of the category $\mathrm{rep}(\mathcal{A},\mathcal{B})$. Moreover, under this bijection, the composition law of $\mathrm{Hmo}(k)$ corresponds to the (derived) tensor product of bimodules.

The *additivization* $\mathrm{Hmo}_0(k)$ of $\mathrm{Hmo}(k)$ is defined as the category with the same objects as $\mathrm{Hmo}(k)$ and morphisms $\mathrm{Hom}_{\mathrm{Hmo}_0(k)}(\mathcal{A},\mathcal{B}) := K_0\mathrm{rep}(\mathcal{A},\mathcal{B})$.

Since the dg $\mathcal{A}$-$\mathcal{B}$-bimodules (2.1) belong to $\mathrm{rep}(\mathcal{A},\mathcal{B})$, we have the symmetric monoidal functor:

$$\text{(2.2)} \qquad \mathrm{U}\colon \mathrm{dgcat}(k) \longrightarrow \mathrm{Hmo}_0(k) \quad \mathcal{A} \mapsto \mathcal{A} \quad (\mathcal{A} \stackrel{F}{\to} \mathcal{B}) \mapsto [{}_F\mathrm{B}]\,.$$

As explained in [19, §2.3], this functor is the *universal* additive invariant, *i.e.* given any additive category D, pre-composition with U gives rise to an equivalence

$$\text{(2.3)} \qquad \mathrm{Fun}_{\mathrm{additive}}(\mathrm{Hmo}_0(k),\mathrm{D}) \stackrel{\simeq}{\longrightarrow} \mathrm{Fun}_{\mathrm{add}}(\mathrm{dgcat}(k),\mathrm{D})\,,$$

where the left-hand side stands for the category of additive functors and the right-hand side for the category of additive invariants.

2.3. Derived categories of quotient stacks

Let G be an affine group k-scheme of finite type and X a quasi-compact quasi-separated k-scheme equipped with a G-action. Let us denote by $\mathrm{Mod}([X/G])$ the Grothendieck category of G-equivariant $\mathcal{O}_X$-modules and by $\mathrm{Qcoh}([X/G])$, resp. $\mathrm{coh}([X/G])$, the full subcategory of G-equivariant quasi-coherent, resp. coherent, $\mathcal{O}_X$-modules. We will write $\mathcal{D}([X/G]) := \mathcal{D}(\mathrm{Mod}([X/G]))$ for the derived category of the quotient stack $[X/G]$, $\mathcal{D}_{\mathrm{Qcoh}}([X/G]) \subset \mathcal{D}([X/G])$ for the full subcategory of those complexes of G-equivariant $\mathcal{O}_X$-modules whose cohomology belongs to $\mathrm{Qcoh}([X/G])$, $\mathcal{D}^b(\mathrm{coh}([X/G])) \subset \mathcal{D}_{\mathrm{Qcoh}}([X/G])$ for the full subcategory of bounded complexes of G-equivariant coherent $\mathcal{O}_X$-modules, and $\mathrm{perf}([X/G]) \subset \mathcal{D}^b(\mathrm{coh}([X/G]))$ for the full subcategory of perfect complexes of G-equivariant $\mathcal{O}_X$-modules.

Let $\mathcal{E}x$ be an exact category. As explained in [13, §4.4], the *derived dg category* $\mathcal{D}_{\mathrm{dg}}(\mathcal{E}x)$ *of* $\mathcal{E}x$ is defined as the Drinfeld's dg quotient $\mathcal{C}_{\mathrm{dg}}(\mathcal{E}x)/\mathcal{A}c_{\mathrm{dg}}(\mathcal{E}x)$ of the dg category of complexes over $\mathcal{E}x$ by its full dg subcategory of acyclic complexes.

Following the above, we will write $\mathcal{D}_{\mathrm{dg}}([X/G])$ for the dg category $\mathcal{D}_{\mathrm{dg}}(\mathcal{E}x)$, with $\mathcal{E}x := \mathrm{Mod}([X/G])$, and $\mathcal{D}_{\mathrm{Qcoh,dg}}([X/G])$, $\mathcal{D}^b_{\mathrm{dg}}(\mathrm{coh}([X/G]))$, and $\mathrm{perf}_{\mathrm{dg}}([X/G])$, for the corresponding full dg subcategories.

Proposition 2.4 (Trivial action). *Assume that the category* $\mathcal{D}_{\mathrm{Qcoh}}([\bullet/G])$ *is compactly generated. Under this assumption, whenever the* G*-action on* X *is trivial, we have the following Morita equivalence:*

$$\text{(2.5)} \qquad \mathrm{perf}_{\mathrm{dg}}(X) \otimes \mathrm{perf}_{\mathrm{dg}}([\bullet/G]) \longrightarrow \mathrm{perf}_{\mathrm{dg}}([X/G]) (\mathcal{F}, V) \mapsto \mathcal{F} \boxtimes V\,.$$

Remark 2.6. As proved in [10, Thm. A] and [9, Lem. 4.1], the category $\mathcal{D}_{\mathrm{Qcoh}}([\bullet/G])$ is compactly generated if and only if k is of characteristic zero or if k is of positive characteristic and $\overline{G} := G \otimes_k \overline{k}$ does not contains a copy of the additive group $\mathbb{G}_a$.

Proof. Given any $\mathcal{F} \in \mathcal{D}_{\mathrm{Qcoh}}(X)$, any $V \in \mathcal{D}_{\mathrm{Qcoh}}([\bullet/G])$, and any $\mathcal{G} \in \mathcal{D}_{\mathrm{Qcoh}}([X/G])$, we have the following classical tensor-Hom relation:

$$\mathbf{R}\mathrm{Hom}_{[X/G]}(\mathcal{F} \boxtimes V, \mathcal{G}) \simeq \mathbf{R}\mathrm{Hom}_{[\bullet/G]}(V, \mathbf{R}\mathrm{Hom}_X(\mathcal{F}, \mathcal{G}))\,. \tag{2.7}$$

The relation (2.7) implies that if $\mathcal{F}$ and V are compact objects, then $\mathcal{F} \boxtimes V \in \mathcal{D}_{\mathrm{Qcoh}}([X/G])$ is also a compact object. Moreover, if $\mathbf{R}\mathrm{Hom}_{[X/G]}(\mathcal{F} \boxtimes V, \mathcal{G}) = 0$ for every $\mathcal{F}$ and V, then $\mathcal{G}$ is necessarily equal to zero. Furthermore, if $\mathcal{F}$ and V are perfect complexes, then $\mathcal{F} \boxtimes V$ is also a perfect complex. Since the categories $\mathcal{D}_{\mathrm{Qcoh}}(X)$ and $\mathcal{D}_{\mathrm{Qcoh}}([\bullet/G])$ are compactly generated by perfect complexes (consult [5, Thm. 3.1.1] and [10, Thm. A (b)], respectively) the above three facts imply that the category $\mathcal{D}_{\mathrm{Qcoh}}([X/G])$ is also compactly generated by perfect complexes. Finally, given any two perfect complexes $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{perf}(X)$ and any two G-representations $V_1, V_2 \in \mathrm{perf}([\bullet/G])$, note that (2.7) also implies that

$$\mathbf{R}\mathrm{Hom}_{[X/G]}(\mathcal{F}_1 \boxtimes V_1, \mathcal{F}_2 \boxtimes V_2) \simeq \mathbf{R}\mathrm{Hom}_{[\bullet/G]}(V_1, V_2) \otimes \mathbf{R}\mathrm{Hom}_X(\mathcal{F}_1, \mathcal{F}_2)\,.$$

This allows us to conclude that the dg functor (2.5) is a Morita equivalence. □

2.4. Action of the Grothendieck ring

Let G be an affine group k-scheme of finite type and X a quasi-compact quasi-separated k-scheme equipped with a G-action. Since the tensor product $-\otimes_X-$ makes the dg category $\mathrm{perf}_{\mathrm{dg}}([X/G])$ into a commutative monoid and the universal additive invariant (2.2) is symmetric monoidal, we obtain a commutative monoid $\mathrm{U}([X/G])$ in the category $\mathrm{Hmo}_0(k)$. Making use of the following natural ring isomorphism

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Hmo}_0(k)}(\mathrm{U}(k), \mathrm{U}([X/G])) &:= K_0\mathrm{rep}(k, \mathrm{perf}_{\mathrm{dg}}([X/G])) \\ &\simeq K_0(\mathrm{perf}([X/G]))\,,\end{aligned}$$

we hence conclude that the Grothendieck ring $K_0([X/G]) \simeq K_0(\mathrm{perf}([X/G]))$ acts on the object $\mathrm{U}([X/G])$ (and also that the monoid structure of $\mathrm{U}([X/G])$

is $K_0([X/G])$-linear). Concretely, this action can be explicitly described as follows:

$$\begin{aligned} &K_0(\mathrm{perf}([X/G])) \longrightarrow K_0(\mathrm{rep}(\mathrm{perf}_{\mathrm{dg}}([X/G]), \mathrm{perf}_{\mathrm{dg}}([X/G]))) \\ &[\mathcal{F}] \mapsto [{}_{(\mathcal{F}\otimes_X -)}\mathrm{B}]\,. \end{aligned}$$

Given any additive invariant $E\colon \mathrm{dgcat}(k) \to \mathrm{D}$, the equivalence of categories (2.3) implies, by functoriality, that $K_0([X/G])$ acts on the object $E([X/G]) \in \mathrm{D}$.

3. Category of subschemes of a quotient stack

Let G be a smooth affine group k-scheme of finite type and X a smooth quasi-compact separated k-scheme equipped with a G-action. In this section, we construct a certain category[3] $\mathbb{S}\mathrm{ub}_0^G(X)$ of G-stable smooth closed subschemes of X. This category, which is of independent interest, will play a key role in the proof of Theorems 1.2 and 1.19; consult §4 below.

Definition of the category $\mathbb{S}\mathbf{ub}_0^G(X)$

Let $\mathbb{S}\mathrm{ub}^G(X)$ be the category whose objects are the G-stable closed immersions $Y \overset{\tau}{\hookrightarrow} X$, with Y a smooth quasi-compact separated k-scheme. In what follows, in order to simplify the exposition, we will often write Y. Given two objects Y_1 and Y_2, $\mathrm{Hom}_{\mathbb{S}\mathrm{ub}^G(X)}(Y_1, Y_2)$ is defined as the set of isomorphism classes of the full subcategory

$$\begin{aligned} \mathcal{D}^b_{Y_1\times_X Y_2}(\mathrm{coh}([(Y_1\times Y_2)/G])) &\subset \mathcal{D}^b(\mathrm{coh}([(Y_1\times Y_2)/G])) \\ &\simeq \mathrm{perf}([(Y_1\times Y_2)/G]) \end{aligned}$$

of those bounded complexes of G-equivariant coherent $\mathcal{O}_{Y_1\times Y_2}$-modules whose cohomology is (topologically) supported on the closed subscheme $Y_1 \times_X Y_2$; note that since the quotient stack $[(Y_1 \times Y_2)/G]$ is smooth, every bounded complex of G-equivariant coherent $\mathcal{O}_{Y_1\times Y_2}$-modules is perfect.

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[3] In the case of a constant finite group k-scheme G, a related category of G-equivariant smooth "covers" of X was constructed in [22, §5].

Given three objects Y_1, Y_2, and Y_3, the composition law

$$\mathrm{Hom}_{\mathbb{S}\mathrm{ub}^G(X)}(Y_2, Y_3) \times \mathrm{Hom}_{\mathbb{S}\mathrm{ub}^G(X)}(Y_1, Y_2) \longrightarrow \mathrm{Hom}_{\mathbb{S}\mathrm{ub}^G(X)}(Y_1, Y_3)$$

is induced by the classical (derived) "pull-back/push-forward" formula

$$(\mathcal{E}_{23}, \mathcal{E}_{12}) \mapsto (q_{13})_*((q_{23})^*(\mathcal{E}_{23}) \otimes^{\mathbf{L}} (q_{12})^*(\mathcal{E}_{12}))\,, \tag{3.1}$$

where q_{ij} stands for the projection from the triple fiber product onto its ij-factor. Finally, the identity of an object Y is the (isomorphism class of the) G-equivariant structure sheaf $\mathcal{O}_\Delta$ of the diagonal Δ in $Y \times Y$.

The *additivization* $\mathbb{S}\mathrm{ub}_0^G(X)$ of $\mathbb{S}\mathrm{ub}^G(X)$ is defined by formally adding all finite direct sums to the category which has the same objects as $\mathbb{S}\mathrm{ub}^G(X)$ and morphisms

$$\mathrm{Hom}_{\mathbb{S}\mathrm{ub}_0^G(X)}(Y_1, Y_2) := K_0(\mathcal{D}^b_{Y_1 \times_X Y_2}(\mathrm{coh}([(Y_1 \times Y_2)/G])))\,.$$

Note that, since the above formula (3.1) is exact in each one of the variables, the composition law of $\mathbb{S}\mathrm{ub}^G(X)$ extends naturally to $\mathbb{S}\mathrm{ub}_0^G(X)$. Let us denote by

$$\mathbb{U}\colon \mathbb{S}\mathrm{ub}^G(X) \longrightarrow \mathbb{S}\mathrm{ub}_0^G(X)$$

the canonical functor. Note also that thanks to Quillen's dévissage theorem [17, §5] and to the definition of G-theory, we have isomorphisms:

$$\mathrm{Hom}_{\mathbb{S}\mathrm{ub}_0^G(X)}(Y_1, Y_2) \simeq G_0([(Y_1 \times_X Y_2)/G])\,.$$

In particular, we have ring isomorphisms:

$$\mathrm{End}_{\mathbb{S}\mathrm{ub}_0^G(X)}(Y) \simeq G_0([Y/G]) \simeq K_0([Y/G])\,.$$

Relation(s) between the categories $\mathbb{S}\mathrm{ub}_0^G(X)$ and $\mathbf{Hmo}_0(k)$

Given two objects Y_1 and Y_2 of the category $\mathbb{S}\mathrm{ub}^G(X)$, consider the exact functor

$$\mathcal{D}^b_{Y_1 \times_X Y_2}(\mathrm{coh}([(Y_1 \times Y_2)/G])) \longrightarrow \mathrm{rep}(\mathrm{perf}_{\mathrm{dg}}([Y_1/G]), \mathrm{perf}_{\mathrm{dg}}([Y_2/G])$$

that sends a bounded complex of G-equivariant coherent $\mathcal{O}_{Y_1 \times Y_2}$-modules $\mathcal{E}_{12}$ (whose cohomology is (topologically) supported on the closed subscheme

$Y_1 \times_X Y_2$) to the following Fourier-Mukai dg-functor:

$$\Phi_{\mathcal{E}_{12}}\colon \mathrm{perf}_{\mathrm{dg}}([Y_1/G]) \longrightarrow \mathrm{perf}_{\mathrm{dg}}([Y_2/G]) \qquad \mathcal{F} \mapsto (q_2)_*((q_1)^*(\mathcal{F}) \otimes^{\mathbf{L}} \mathcal{E}_{12})\,.$$

By definition of the categories $\mathbb{S}\mathrm{ub}^G(X)$ and $\mathrm{Hmo}(k)$, the above constructions lead to a well-defined functor

$$\mathbb{S}\mathrm{ub}^G(X) \longrightarrow \mathrm{Hmo}(k) \quad Y \mapsto \mathrm{perf}_{\mathrm{dg}}([Y/G]) \quad \mathcal{E}_{12} \mapsto {}_{\Phi_{\mathcal{E}_{12}}}\mathrm{B}\,,$$

which naturally extends to the additive categories:

$$\Psi\colon \mathbb{S}\mathrm{ub}_0^G(X) \longrightarrow \mathrm{Hmo}_0(k) \qquad \mathbb{U}(Y) \mapsto \mathrm{U}([Y/G])\,.$$

Remark 3.2 (Sheaves of algebras). Let $\mathcal{H}$ be a flat quasi-coherent sheaf of algebras over $[X/G]$, *i.e.* a G-equivariant flat quasi-coherent sheaf of algebras over X. Given two objects $Y_1 \overset{\tau_1}{\hookrightarrow} X$ and $Y_2 \overset{\tau_2}{\hookrightarrow} X$ of $\mathbb{S}\mathrm{ub}^G(X)$, consider the exact functor

$$\begin{aligned} &\mathcal{D}^b_{Y_1\times_X Y_2}(\mathrm{coh}([(Y_1 \times Y_2)/G])) \\ &\to \mathrm{rep}(\mathrm{perf}_{\mathrm{dg}}([Y_1/G];\tau_1^*(\mathcal{H})), \mathrm{perf}_{\mathrm{dg}}([Y_2/G];\tau_2^*(\mathcal{H}))) \end{aligned}$$

defined, as above, by the assignment $\mathcal{E}_{12} \mapsto {}_{\Phi_{\mathcal{E}_{12}}}\mathrm{B}$. This leads to a functor

$$\mathbb{S}\mathrm{ub}^G(X) \longrightarrow \mathrm{Hmo}(k) \quad (Y \overset{\tau}{\hookrightarrow} X) \mapsto \mathrm{perf}_{\mathrm{dg}}([Y/G];\tau^*(\mathcal{H})) \quad \mathcal{E}_{12} \mapsto {}_{\Phi_{\mathcal{E}_{12}}}\mathrm{B}\,,$$

which naturally extends to the additive categories:

$$\Psi_{\mathcal{H}}\colon \mathbb{S}\mathrm{ub}_0^G(X) \longrightarrow \mathrm{Hmo}_0(k) \qquad \mathbb{U}(Y \overset{\tau}{\hookrightarrow} X) \mapsto \mathrm{U}([Y/G];\tau^*(\mathcal{H}))\,.$$

Some properties of the category $\mathbb{S}\mathrm{ub}_0^G(X)$ and of the functor Ψ

In what follows, we describe three important properties that will be used in the sequel.

3.0.1. Pull-back and push-forward. Let $Y_1 \overset{\tau_1}{\hookrightarrow} X$ and $Y_2 \overset{\tau_2}{\hookrightarrow} X$ be two objects of the category $\mathbb{S}\mathrm{ub}^G(X)$. Given a G-stable closed immersion $\iota\colon Y_1 \hookrightarrow Y_2$ such that $\tau_2 \circ \iota = \tau_1$, its *pull-back* $\mathbb{U}(\iota^*)\colon \mathbb{U}(Y_2) \to \mathbb{U}(Y_1)$, resp. *push-forward* $\mathbb{U}(\iota_*)\colon \mathbb{U}(Y_1) \to \mathbb{U}(Y_2)$, is defined as the Grothendieck class $[(\iota \times_X \mathrm{id})_*(\mathcal{O}_{Y_1})]$, resp. $[(\mathrm{id} \times_X \iota)_*(\mathcal{O}_{Y_1})]$, of the group $G_0([(Y_2 \times_X Y_1)/G])$, resp. $G_0([(Y_1 \times_X Y_2)/G])$. Note that $\Psi(\mathbb{U}(\iota^*)) = \mathrm{U}(\iota^*)$ and $\Psi(\mathbb{U}(\iota_*)) = \mathrm{U}(\iota_*)$.

3.0.2. K_0-action. Let Y be an object of $\mathbb{S}\mathrm{ub}^G(X)$. The push-forward along the diagonal map $i_\Delta : Y \hookrightarrow Y \times Y$ leads to an exact functor

$$(i_\Delta)_* \colon \mathrm{perf}([Y/G]) \longrightarrow \mathcal{D}^b_\Delta(\mathrm{coh}([(Y \times Y)/G])) \tag{3.3}$$

that sends the tensor product $- \otimes_Y -$ on the left-hand side to the "pull-back/push-forward" formula (3.1) on the right-hand side. Therefore, by applying $K_0(-)$ to (3.3), we obtain an induced ring morphism $K_0([Y/G]) \to \mathrm{End}_{\mathbb{S}\mathrm{ub}_0^G(X)}(\mathbb{U}(Y))$. In other words, we obtain an action of $K_0([Y/G])$ on the object $\mathbb{U}(Y)$.

Lemma 3.4. *The functor Ψ interchanges with the $K_0([Y/G])$-action on $\mathbb{U}(Y)$ (defined above) with the $K_0([Y/G])$-action on $\mathrm{U}([Y/G])$ (defined in §2.4).*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{perf}([Y/G]) & \xrightarrow{(i_\Delta)_*} & \mathcal{D}^b_\Delta(\mathrm{coh}([(Y \times Y)/G])) \\ \| & & \downarrow{\scriptstyle \mathcal{E} \mapsto \Phi_{\mathcal{E}}\mathrm{B}} \\ \mathrm{perf}([Y/G]) & \xrightarrow[\mathcal{F} \mapsto (\mathcal{F} \otimes_Y -)\mathrm{B}]{} & \mathrm{rep}(\mathrm{perf}_{\mathrm{dg}}([Y/G]), \mathrm{perf}_{\mathrm{dg}}([Y/G])) . \end{array} \tag{3.5}$$

By applying $K_0(-)$ to (3.5), we obtain the claimed compatibility. □

3.0.3. K_0-linearity. Let $Y \overset{\tau}{\hookrightarrow} X$ be an object of $\mathbb{S}\mathrm{ub}^G(X)$. By composing the induced ring homomorphism $\tau^* \colon K_0([X/G]) \to K_0([Y/G])$ with the $K_0([Y/G])$-action on $\mathbb{U}(Y)$ described in §3.0.2, we obtain a $K_0([X/G])$-action on $\mathbb{U}(Y)$. A simple verification shows that this $K_0([X/G])$-action is compatible with the morphisms of the category $\mathbb{S}\mathrm{ub}_0^G(X)$. This implies that $\mathbb{S}\mathrm{ub}_0^G(X)$ is a $K_0([X/G])$-linear category. Note that since the projection map $X \to \bullet$ induces a ring homomorphism $R(G) \to K_0([X/G])$, the category $\mathbb{S}\mathrm{ub}_0^G(X)$ is also $R(G)$-linear.

4. Proofs

In this section, making use of the category $\mathbb{S}\mathrm{ub}_0^G(X)$ of G-stable smooth closed subschemes of X (consult §3), we prove Theorems 1.2, 1.5 and 1.19.

Proof of Theorem 1.2

Consider the following morphisms

$$\mathbb{U}(X) \xrightarrow{\mathbb{U}(\iota^*)} \mathbb{U}(X^H) \qquad \mathbb{U}(X^H) \xrightarrow{\mathbb{U}(\iota_*)} \mathbb{U}(X) \tag{4.1}$$

in the category $\mathbb{S}\mathrm{ub}_0^G(X)$. Since both these morphisms are $R(G)$-equivariant (see §3.0.3), they give rise to well-defined morphisms of ind-objects

$$S_H^{-1}\mathbb{U}(X) \xrightarrow{\mathbb{U}(\iota^*)} S_H^{-1}\mathbb{U}(X^H) \qquad S_H^{-1}\mathbb{U}(X^H) \xrightarrow{\mathbb{U}(\iota_*)} S_H^{-1}\mathbb{U}(X) \tag{4.2}$$

in the category $\mathrm{Ind}(\mathbb{S}\mathrm{ub}_0^G(X))$. Under the ring isomorphisms

$$\mathrm{End}_{\mathbb{S}\mathrm{ub}_0^G(X)}(\mathbb{U}(X^H)) \simeq G_0([X^H/G]) \simeq K_0([X^H/G]) \,,$$

the composition $\mathbb{U}(\iota^*) \circ \mathbb{U}(\iota_*)$ of the morphisms (4.1) (which by definition is given by $[\mathcal{O}_{X^H} \otimes^{\mathbf{L}}_X \mathcal{O}_{X^H}]$) corresponds to the Grothendieck class

$$\sum_j (-1)^j \left[\bigwedge^j (I/I^2)\right] \in K_0([X^H/G]),$$

where I stands for the sheaf of ideals associated to the closed immersion $\iota\colon X^H \hookrightarrow X$. Consequently, since $I/I^2 = \mathcal{N}$, the composition $\mathbb{U}(\iota^*) \circ \mathbb{U}(\iota_*)$ of the morphisms (4.2) corresponds to the following morphism of ind-objects

$$S_H^{-1}\mathbb{U}(X^H) \xrightarrow{\lambda_{-1}(\mathcal{N})\cdot -} S_H^{-1}\mathbb{U}(X^H) \,, \tag{4.3}$$

where $- \cdot -$ stands for the induced action of the Grothendieck group $K_0([X^H/G])$ on the ind-object $S_H^{-1}\mathbb{U}(X^H)$ (see §3.0.2).

Lemma 4.4. *The above morphism of ind-objects* (4.3) *is invertible.*

Proof. Thanks to the Yoneda lemma, it is enough to show that (4.3) becomes an isomorphism after application of the functor

$$\mathrm{Hom}_{\mathrm{Ind}(\mathbb{S}\mathrm{ub}_0^G(X))}(S_H^{-1}\mathbb{U}(X^H), -).$$

Recall that $S_H^{-1}R(G)$ can be written as a filtered colimit of free finite $R(G)$-modules. Therefore, it suffices to show that (4.3) becomes an isomorphism after application of the functor $\mathrm{Hom}_{\mathrm{Ind}(\mathbb{S}\mathrm{ub}_0^G(X))}(\mathbb{U}(X^H), -)$. By definition of the category $\mathrm{Ind}(\mathbb{S}\mathrm{ub}_0^G(X))$, this latter claim is equivalent to the invertibility of the following homomorphism of abelian groups:

$$S_H^{-1}K_0([X^H/G]) \longrightarrow S_H^{-1}K_0([X^H/G])\alpha \mapsto \lambda_{-1}(\mathcal{N})\cdot\alpha\,. \tag{4.5}$$

As proved by Thomason in [23, Lem. 3.2], (4.5) is indeed invertible. □

Thanks to Lemma 4.4, we can now consider the following composition:

$$S_H^{-1}\mathbb{U}(X^H) \stackrel{(\lambda_{-1}(\mathcal{N})\cdot-)^{-1}}{\longrightarrow} S_H^{-1}\mathbb{U}(X^H) \stackrel{\mathbb{U}(\iota_*)}{\longrightarrow} S_H^{-1}\mathbb{U}(X)\,. \tag{4.6}$$

Proposition 4.7. *The morphism of ind-objects* $\mathbb{U}(\iota^*)\colon S_H^{-1}\mathbb{U}(X)\to S_H^{-1}\mathbb{U}(X^H)$ *is invertible. Moreover, its inverse is given by the above composition* (4.6)*.*

Proof. Thanks to Lemma 4.4, $S_H^{-1}\mathbb{U}(X^H)$ is a direct summand of $S_H^{-1}\mathbb{U}(X)$. Therefore, using the Yoneda lemma, it is enough to show that $\mathbb{U}(\iota^*)$ becomes an isomorphism after application of the functor

$$\mathrm{Hom}_{\mathrm{Ind}(\mathbb{S}\mathrm{ub}_0^G(X))}(S_H^{-1}\mathbb{U}(X),-).$$

Moreover, similarly to the proof of Lemma 4.4, it suffices to show that $\mathbb{U}(\iota^*)$ becomes an isomorphism after application of the functor

$$\mathrm{Hom}_{\mathrm{Ind}(\mathbb{S}\mathrm{ub}_0^G(X))}(\mathbb{U}(X),-).$$

By definition of the category $\mathrm{Ind}(\mathbb{S}\mathrm{ub}_0^G(X))$, this latter claim is equivalent to the invertibility of the following homomorphism of abelian groups:

$$K_0(\iota^*)\colon S_H^{-1}K_0([X/G]) \longrightarrow S_H^{-1}K_0([X^H/G])\,. \tag{4.8}$$

As proved by Thomason in [23, Thm. 2.1 and Lem. 3.3], (4.8) is indeed invertible.

Finally, note that the composition (4.6) is the right-inverse of $\mathbb{U}(\iota^*)$. Since $\mathbb{U}(\iota^*)$ is invertible, (4.6) is also the left-inverse of $\mathbb{U}(\iota^*)$. □

We now have the ingredients necessary to conclude the proof of Theorem 1.2. As explained in §3.0.1, resp. §3.0.2, resp. §3.0.3, the functor $\Psi\colon \mathbb{S}\mathrm{ub}_0^G(X)\to$

$\mathrm{Hmo}_0(k)$ is compatible with pull-backs and push-forwards, resp. with K_0-actions, resp. with $R(G)$-actions. Moreover, it extends naturally to the categories of ind-objects:

$$\mathrm{Ind}(\Psi)\colon \mathrm{Ind}(\mathbb{S}\mathrm{ub}_0^G(X)) \longrightarrow \mathrm{Ind}(\mathrm{Hmo}_0(k))\,. \tag{4.9}$$

Therefore, by combining the preceding functor (4.9) with Proposition 4.7, we conclude that the morphism of ind-objects

$$\mathrm{U}(\iota^*)\colon S_H^{-1}\mathrm{U}([X/G]) \to S_H^{-1}\mathrm{U}([X^H/G])$$

is invertible and that its inverse is given by the following composition:

$$S_H^{-1}\mathrm{U}([X^H/G]) \overset{(\lambda_{-1}(\mathcal{N})\cdot -)^{-1}}{\longrightarrow} S_H^{-1}\mathrm{U}([X^H/G]) \overset{\mathrm{U}(\iota_*)}{\longrightarrow} S_H^{-1}\mathrm{U}([X/G])\,.$$

This proves Theorem 1.2 in the case of the universal additive invariant. The general case follows now from the equivalence of categories (2.3) and from the fact that every additive functor $\mathrm{Hmo}_0(k) \to \mathrm{D}$ extends naturally to the categories of ind-objects.

Remark 4.10 (Generalization). Let $\mathcal{H}$ be a flat quasi-coherent sheaf of algebras over $[X/G]$, *i.e.* a G-equivariant flat quasi-coherent sheaf of algebras over X. A proof similar to the above one, with Ψ replaced by the functor $\Psi_{\mathcal{H}}$ (see Remark 3.2), allows us to conclude that Theorem 1.2 holds more generally with $S_H^{-1}E([X/G])$ and $S_H^{-1}E([X^H/G])$ replaced by $S_H^{-1}E([X/G];\mathcal{H})$ and $S_H^{-1}E([X^H/G];\iota^*(\mathcal{H}))$.

Proof of Theorem 1.5

Note first that since $R(G) \simeq \mathbb{Z}[G^\vee]$ and $G^\vee$ is a finitely generated abelian group, the abelian group $R(G)$ belongs to $\mathrm{Ind}(\mathrm{free}(\mathbb{Z}))$.

Since $\overline{G} := G \otimes_k \overline{k}$ does not contains a copy of the additive group $\mathbb{G}_a$ (in any caracteristic) and the G-action on X^G is trivial, Proposition 2.4 and Remark 2.6 yield a Morita equivalence $\mathrm{perf}_{\mathrm{dg}}(X^G) \otimes \mathrm{perf}_{\mathrm{dg}}([\bullet/G]) \to \mathrm{perf}_{\mathrm{dg}}([X^G/G])$. Therefore, using the fact that the universal additive invariant (2.2) is symmetric monoidal, we obtain an induced isomorphism

$$\mathrm{U}(X^G) \otimes \mathrm{U}([\bullet/G]) \overset{\simeq}{\longrightarrow} \mathrm{U}([X^G/G])\,. \tag{4.11}$$

Recall from §2.2-§2.4 that the object $\mathrm{U}([\bullet/G]) \in \mathrm{Hmo}_0(k)$ carries a canonical commutative monoid structure and that we have natural ring isomorphisms:

$$(4.12) \qquad \mathrm{Hom}_{\mathrm{Hmo}_0(k)}(\mathrm{U}(k), \mathrm{U}([\bullet/G])) \simeq K_0([\bullet/G]) \simeq R(G) \simeq \mathbb{Z}[G^\vee]\,.$$

Using the characters of G, we hence obtain an induced morphism of ind-objects:

$$(4.13) \qquad \mathrm{U}(k) \otimes_{\mathbb{Z}} R(G) \longrightarrow \mathrm{U}([\bullet/G])\,.$$

Proposition 4.14. *The above morphism of ind-objects* (4.13) *is invertible.*

Proof. Note that, thanks to the ring isomorphisms (4.12), by applying the functor $\mathrm{Hom}_{\mathrm{Ind}(\mathrm{Hmo}_0(k))}(\mathrm{U}(k), -)$ to (4.13) we obtain an isomorphism. Hence, in order to prove that (4.13) is invertible, it is enough to show that $\mathrm{U}([\bullet/G])$ is isomorphic to a (possibly infinite) direct sum of copies of $\mathrm{U}(k)$.

Recall that we have an isomorphism $G \simeq \mathbb{G}_m^{\times r} \times \mu_{l_1^{\nu_1}} \times \cdots \times \mu_{l_s^{\nu_s}}$ for some prime numbers $l_1, \ldots, l_s$ and non-integers $r, \nu_1, \ldots, \nu_s$. The multiplicative group k-scheme $\mathbb{G}_m$ is semi-simple. Moreover, the simple $\mathbb{G}_m$-representations V are parametrized by the group of characters $\mathbb{G}_m^\vee$ and we have $\mathrm{End}_{\mathbb{G}_m}(V) \simeq k$. Since, by assumption, k contains the l^{th} roots of unity, with $l = l_1, \ldots, l_s$, the group k-schemes $\mu_{l_1^{\nu_1}}, \ldots, \mu_{l_v^{\nu_s}}$ are isomorphic to the constant finite group k-schemes $C_{l_1^{\nu_1}}, \ldots, C_{l_s^{\nu_s}}$, respectively. In particular, they are semi-simple. Moreover, the simple μ_{l^ν}-representations V are parametrized by the group of characters $\mu_{l^\nu}^\vee$ and we have $\mathrm{End}_{\mu_{l^\nu}}(V) \simeq k$. These considerations imply that the group k-scheme G is also semi-simple and that the dg category $\mathrm{perf}_{\mathrm{dg}}([\bullet/G])$ is Morita equivalent to the disjoint union $\coprod_{\chi \in G^\vee} k$. Consequently, since $\mathrm{rep}(\coprod_{\chi\in G^\vee} k, \mathcal{B}) \simeq \prod_{\chi \in G^\vee} \mathrm{rep}(k, \mathcal{B})$ for every dg category $\mathcal{B}$ and the functor $K_0(-)$ preserves arbitrary products, we obtain canonical isomorphisms:

$$\mathrm{Hom}_{\mathrm{Hmo}_0(k)}(\mathrm{U}([\bullet/G]), \mathcal{B}) \simeq \prod_{\chi \in G^\vee} \mathrm{Hom}_{\mathrm{Hmo}_0(k)}(\mathrm{U}(k), \mathcal{B})\,.$$

This shows not only that the (possibly infinite) direct sum $\bigoplus_{\chi\in G^\vee} \mathrm{U}(k)$ exists in the category $\mathrm{Hmo}_0(k)$, but moreover that $\mathrm{U}([\bullet/G]) \simeq \bigoplus_{\chi \in G^\vee} \mathrm{U}(k)$. □

The above isomorphisms (4.11) with (4.13) yield an isomorphism of ind-objects $\mathrm{U}(X^G) \otimes_{\mathbb{Z}} R(G) \xrightarrow{\simeq} \mathrm{U}([X^G/G])$. Under this latter isomorphism, the natural action of $R(G)$ on the right-hand side corresponds to the canonical

$R(G)$-action on $R(G)$. Consequently, we obtain an induced isomorphism of ind-objects:

$$\mathrm{U}(X^G) \otimes_{\mathbb{Z}} S_G^{-1}R(G) \xrightarrow{\simeq} S_G^{-1}\mathrm{U}([X^G/G]) \,. \tag{4.15}$$

Finally, by combining (4.15) with the (inverse of the) isomorphism of ind-objects $S_G^{-1}\mathrm{U}([X/G]) \xrightarrow{\simeq} S_G^{-1}\mathrm{U}([X^G/G])$ provided by Theorem 1.2, we obtain an isomorphism of ind-objects $S_G^{-1}\mathrm{U}([X/G]) \simeq \mathrm{U}(X^G) \otimes_{\mathbb{Z}} S_G^{-1}R(G)$. This proves Theorem 1.5 in the case of the universal additive invariant. The general case follows now from the equivalence of categories (2.3) and from the fact that the natural extension of every additive functor $\mathrm{Hmo}_0(k) \to \mathrm{D}$ to the categories of ind-objects is compatible with the induced action $- \otimes_{\mathbb{Z}} -$ of the category $\mathrm{Ind}(\mathrm{free}(\mathbb{Z}))$.

Proof of Theorem 1.19

Let us denote by $\mathrm{Hmo}_0(k)_{1/r}$, resp. by $\mathbb{S}\mathrm{ub}_0^G(X)_{1/r}$, the $\mathbb{Z}[1/r]$-linear category obtained by tensoring the abelian groups of morphisms of $\mathrm{Hmo}_0(k)$, resp. of $\mathbb{S}\mathrm{ub}_0^G(X)$, with $\mathbb{Z}[1/r]$. In the same vein, let us denote by $\Psi_{1/r}$: $\mathbb{S}\mathrm{ub}_0^G(X)_{1/r} \to \mathrm{Hmo}_0(k)_{1/r}$ the induced $\mathbb{Z}[1/r]$-linear functor.

Lemma 4.16. *The set of ind-objects $\{\underline{S}_{\mu_n}^{-1}\mathbb{U}(X)_{1/r} \,|\, \mu_n \in \mathcal{C}(T)\}$ is finite.*

Proof. Note first that the ind-object $\underline{S}_{\mu_n}^{-1}\mathbb{U}(X)_{1/r}$ is trivial if and only if we have $\mathrm{End}_{\mathrm{Ind}(\mathbb{S}\mathrm{ub}_0^G(X)_{1/r})}(\underline{S}_{\mu_n}^{-1}\mathbb{U}(X)_{1/r}) = 0$. Recall that $\underline{S}_{\mu_n}^{-1}R(T)_{1/r}$ can be written as a filtered colimit of free finite $R(T)_{1/r}$-modules. Hence, by definition of the category $\mathrm{Ind}(\mathbb{S}\mathrm{ub}_0^G(X)_{1/r})$, the ind-object $\underline{S}_{\mu_n}^{-1}\mathbb{U}(X)_{1/r}$ is trivial if and only if we have:

$$\mathrm{Hom}_{\mathrm{Ind}(\mathbb{S}\mathrm{ub}_0^G(X)_{1/r})}(\mathbb{U}(X)_{1/r}, \underline{S}_{\mu_n}^{-1}\mathbb{U}(X)_{1/r}) \simeq \underline{S}_{\mu_n}^{-1}K_0([X/T])_{1/r} = 0 \,.$$

The proof follows now from the fact that the following set of $\mathbb{Z}[1/r]$-modules

$$\{\underline{S}_{\mu_n}^{-1}K_0([X/T])_{1/r} \,|\, \mu_n \in \mathcal{C}(T)\}$$

is finite; consult Vezzosi-Vistoli [24, Prop. 3.4(ii)]. □

Consider the following canonical morphism of ind-objects:

$$\mathbb{U}(X)_{1/r} \longrightarrow \bigoplus_{\mu_n \in \mathcal{C}(T)} \underline{S}_{\mu_n}^{-1}\mathbb{U}(X)_{1/r} \,. \tag{4.17}$$

Note that, thanks to Lemma 4.16, the direct sum on the right-hand side is finite.

Proposition 4.18. *The above morphism of ind-objects* (4.17) *is invertible.*

Proof. Thanks to the Yoneda lemma, since $\underline{S}_{\mu_n}^{-1}R(T)_{1/r}$ can be written as a filtered colimit of free finite $R(T)_{1/r}$-modules, it suffices to show that (4.17) becomes an isomorphism after application of the functor

$$\mathrm{Hom}_{\mathrm{Ind}(\mathbb{S}\mathrm{ub}_0^G(X))}(\mathbb{U}(X)_{1/r}, -).$$

By definition of the category $\mathrm{Ind}(\mathbb{S}\mathrm{ub}_0^G(X)_{1/r})$, this latter claim is equivalent to the invertibility of the following homomorphism of $\mathbb{Z}[1/r]$-modules:

$$K_0([X/T])_{1/r} \longrightarrow \bigoplus_{\mu_n \in \mathcal{C}(T)} \underline{S}_{\mu_n}^{-1} K_0([X/T])_{1/r}\,. \tag{4.19}$$

As proved by Vezzosi-Vistoli in [24, Prop. 3.4(ii)], (4.19) is indeed invertible. □

Recall from §3.0.2 and §3.0.3 that the functor Ψ is compatible with K_0-actions and $R(T)$-actions, respectively. The same holds for its $\mathbb{Z}[1/r]$-linearization $\Psi_{1/r}$ and for the induced functor $\mathrm{Ind}(\Psi_{1/r})$ between the categories of ind-objects. Therefore, by applying this latter functor to (4.17), we obtain an isomorphism of ind-objects:

$$\mathrm{U}([X/T])_{1/r} \xrightarrow{\simeq} \bigoplus_{\mu_n \in \mathcal{C}(T)} \underline{S}_{\mu_n}^{-1} \mathrm{U}([X/T])_{1/r}\,. \tag{4.20}$$

Lemma 4.21. *For every $\mu_n \in \mathcal{C}(T)$, we have an isomorphism of ind-objects*

$$\underline{S}_{\mu_n}^{-1}\mathrm{U}([X/T])_{1/r} \xrightarrow{\simeq} \underline{S}_{\mu_n}^{-1}\mathrm{U}([X^{\mu_n}/T])_{1/r} \tag{4.22}$$

induced by pull-back along the closed immersion $X^{\mu_n} \hookrightarrow X$.

Proof. In order to simplify the exposition, let us still denote by S_{μ_n} the image of the multiplicative set $S_{\mu_n} \subset R(T)$ (see §1) in the $\mathbb{Z}[1/r]$-linearized representation ring $R(T)_{1/r}$. Thanks to Theorem 1.2, we have an isomorphism of ind-objects

$$S_{\mu_n}^{-1}\mathrm{U}([X/T])_{1/r} \xrightarrow{\simeq} S_{\mu_n}^{-1}\mathrm{U}([X^{\mu_n}/T])_{1/r} \tag{4.23}$$

induced by pull-back along the closed immersion $X^{\mu_n} \hookrightarrow X$. Let $\chi \in T^\vee$ be a character of T whose restriction to μ_n is non-trivial. As explained by Thomason in the proof of [24, Lem. 3.6], the image of $1-\chi$ under the $\mathbb{Z}[1/r]$-algebra homomorphism $R(T)_{1/r} \to \underline{S}_{\mu_n}^{-1}R(T)_{1/r}$ is invertible. Consequently, we obtain an induced $R(T)_{1/r}$-linear homomorphism $S_{\mu_n}^{-1}R(T)_{1/r} \to \underline{S}_{\mu_n}^{-1}R(T)_{1/r}$. Therefore, by applying the functor $-\otimes_{S_{\mu_n}^{-1}R(T)_{1/r}} \underline{S}_{\mu_n}^{-1}R(T)_{1/r}$ to (4.23), we obtain the searched isomorphism of ind-objects (4.22). □

Finally, by combining (4.20) with (4.22), we obtain an isomorphism of ind-objects

$$\mathrm{U}([X/T])_{1/r} \xrightarrow{\simeq} \bigoplus_{\mu_n \in \mathcal{C}(T)} \underline{S}_{\mu_n}^{-1}\mathrm{U}([X^{\mu_n}/T])_{1/r}$$

induced by pull-back along the closed immersions $X^{\mu_n} \hookrightarrow X$. This proves Theorem 1.19 in the case of the universal additive invariant. The general case follows now from the equivalence of categories (2.3) and from the fact that every additive functor $\mathrm{Hmo}_0(k) \to \mathrm{D}$, with values in a $\mathbb{Z}[1/r]$-linear category, extends naturally to a $\mathbb{Z}[1/r]$-linear functor $\mathrm{Hmo}_0(k)_{1/r} \to \mathrm{D}$ and, consequently, to a $\mathbb{Z}[1/r]$-linear functor $\mathrm{Ind}(\mathrm{Hmo}_0(k)_{1/r}) \to \mathrm{Ind}(\mathrm{D})$.

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