Variational estimates for operators over some thin subsets of primes

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We establish $\ell^p(\mathbb{Z})$ boundedness of r-variational seminorm for operators of Radon type along subsets of prime numbers of the form $\{p \in \mathbb{P} : \{\varphi_1(p)\} < \psi(p)\}$. As an application we obtain the corresponding pointwise ergodic theorems.

1. Introduction

Given a dynamical system $(X, \mathcal{B}, \mu, T)$, that is a σ-finite measure space $(X, \mathcal{B}, \mu)$ with an invertible measure preserving transformation $T : X \to X$, and any polynomial $P : \mathbb{Z} \to \mathbb{Z}$ of degree $d \geq 1$ having integer coefficients and without a constant term, we are interested in the pointwise convergence for $f \in L^s(X, \mu)$, $s > 1$, of the averages

$$\mathscr{A}_N f(x) = \frac{1}{|\mathbf{P} \cap [1, N]|} \sum_{p \in \mathbf{P} \cap [1, N]} f\left(T^{P(p)} x\right)$$

where $\mathbf{P}$ is a thin subset of prime numbers $\mathbb{P}$, i.e. a subset of $\mathbb{P}$ such that

$$\lim_{N \to \infty} \frac{|\mathbf{P} \cap [1, N]|}{|\mathbb{P} \cap [1, N]|} = 0.$$

Our principle example is the set

$$\mathbf{P} = \big\{p \in \mathbb{P} : p = \lfloor h(n) \rfloor \text{ for some } n \in \mathbb{N}\big\}$$

where h is a regularly-varying function of index $c \in [1, 2)$, for instance $x^c \log^A(x)$ for some $A > 0$. In this context, we also study pointwise convergence of the truncated discrete Hilbert transform with an appropriate weight function ω,

$$\mathscr{H}_N f(x) = \sum_{p \in \pm \mathbf{P} \cap [1,N]} f\left(T^{P(p)} x\right) \frac{\omega(|p|)}{p}.$$

The problem we are interested in may be stated as follows: for a subset $\mathbf{A} \subseteq \mathbb{N}$, $s \geq 1$ and any polynomial P having integer coefficients and without a constant term, determine whether for any function $f \in L^s(X, \mu)$, the limit

$$(1) \qquad \lim_{N \to \infty} \frac{1}{|\mathbf{A} \cap [1, N]|} \sum_{n \in \mathbf{A} \cap [1,N]} f\left(T^{P(n)} x\right)$$

exists for μ-almost all x.

Pointwise convergence of ergodic averages was initially observed by Birkhoff in [1] where the author considered $\mathbf{A} = \mathbb{N}$, $P(n) = n$ and $s \geq 1$. The higher degree polynomials required a new approach discovered by Bourgain in 80's. In the series of papers, [3–5], Bourgain proved the pointwise convergence for $\mathbf{A} = \mathbb{N}$, any polynomial P having integer coefficients, and $s > 1$. The restriction to the range $s > 1$, in Bourgain's theorem is essential. In fact, Buczolich and Mauldin [6], and LaVictoire [16] showed that in the case of $P(n) = n^k$, $k \geq 2$, the pointwise convergence (1) for a function in $L^1(X, \mu)$ may fail on a large set.

Considering averages over prime numbers, in [2] Bourgain proved their pointwise convergence for $P(n) = n$ and functions in $L^2(X, \mu)$. Later, in [31], Wierdl extended this result to all $s > 1$, (see also [5, Section 9]). Again the restriction $s > 1$, is essential as LaVictoire showed in [16]. The case of higher degree polynomials, at least for functions in $L^2(X, \mu)$, was investigated by Nair in [24]. In [25], Nair also studied $s > 1$ but his proof of Lemma 14 contains an error. The general case $s > 1$, I have covered in the recent paper [28]. Finally, a subclass of thin subsets of primes discussed in this article were previously studied by Mirek in [18].

The initial study of pointwise convergence for the truncated discrete Hilbert transform goes back to Cotlar [8] where $\mathbf{A} = \mathbb{N}$ and $P(n) = n$ was considered. The case with a general polynomial P was a more delicate issue

recently resolved by Mirek, Stein and myself in [20]. On the other hand, the truncated discrete Hilbert transform along prime numbers was the subject of the article written by Mirek and myself [21], see also [23]. Ultimately, the general polynomials I have considered in the recent paper [28].

Classical proofs of pointwise convergence proceeds in two steps: The first, is to establish the convergence for a class of functions dense in $L^p(X, \mu)$. To extend the result to all functions, one needs L^p-boundedness of the corresponding maximal function. Nevertheless, finding the dense class may be a difficult task. To overcome this, one can show the r-variational estimates, see Theorem A and Theorem B for details. This approach to study discrete operators has already been used in many papers, see [7, 12, 13, 20, 22, 23, 28, 32].

Before stating the results, let us define thin subsets of $\mathbb{P}$ we are interested in.

Definition 1. Let $\mathcal{L}$ be a family of slowly varying functions $L : [1, \infty) \to (0, \infty)$ such that

$$L(x) = \exp\left(\int_1^x \frac{\vartheta(t)}{t}\, \mathrm{d}t\right)$$

where $\vartheta \in \mathcal{C}^\infty([1, \infty))$ is a real function satisfying

$$\lim_{x \to \infty} \vartheta(x) = 0, \qquad \text{and} \qquad \lim_{x \to \infty} x^n \vartheta^{(n)}(x) = 0, \qquad \text{for every } n \in \mathbb{N}.$$

Let us distinguish a subfamily $\mathcal{L}_0$ of $\mathcal{L}$.

Definition 2. Let $\mathcal{L}_0$ be a family of slowly varying functions $L : [1, \infty) \to (0, \infty)$ such that $\lim_{x \to \infty} L(x) = \infty$ and

$$L(x) = \exp\left(\int_1^x \frac{\vartheta(t)}{t}\, \mathrm{d}t\right)$$

where $\vartheta \in \mathcal{C}^\infty([1, \infty))$ is positive decreasing real function satisfying

$$\lim_{x \to \infty} \vartheta(x) = 0, \qquad \text{and} \qquad \lim_{x \to \infty} \frac{x^n \vartheta^{(n)}(x)}{\vartheta(x)} = 0, \qquad \text{for every } n \in \mathbb{N},$$

and for every $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that $1 \leq C_\epsilon \vartheta(x) x^\epsilon$.

Lastly, we define the subfamily $\mathcal{R}_c$ of regularly varying functions.

Definition 3. For every $c \in (0,2)$, let $\mathcal{R}_c$ be a family of increasing convex regularly-varying functions $h : [1,\infty) \to [1,\infty)$ of the form

$$h(x) = x^c L(x)$$

where $L \in \mathcal{L}_0$, if $c = 1$, and $L \in \mathcal{L}$ otherwise.

Let us fix two functions $h_1 \in \mathcal{R}_{c_1}$ and $h_2 \in \mathcal{R}_{c_2}$ for $c_1, c_2 \in [1,2)$. In the whole article it is assumed that $\gamma_1 = 1/c_1$ and $\gamma_2 = 1/c_2$ satisfy

(i) if $d = 1$,
$$\begin{cases} (1 - \gamma_1) + 15(1 - \gamma_2) < 1, \\ 3(1 - \gamma_1) + 12(1 - \gamma_2) < 2, \end{cases}$$

(ii) if $d = 2$,
$$\begin{cases} 3(1 - \gamma_1) + 62(1 - \gamma_2) < 3, \\ 4(1 - \gamma_1) + 32(1 - \gamma_2) < 3, \end{cases}$$

(iii) if $d \in \{3, \ldots, 9\}$,
$$\begin{cases} \frac{1}{3\cdot 2^d}(1 - \gamma_1) + \left(1 + \frac{1}{6(2^d-1)}\right)(1 - \gamma_2) < \frac{1}{3\cdot 2^d}, \\ (1 - \gamma_2) < \frac{1}{4\cdot 2^d}, \end{cases}$$

(iv) if $d \geq 10$,
$$\frac{2}{3d(d+1)^2}(1 - \gamma_1) + \left(1 + \frac{1}{3d(d+1)}\right)(1 - \gamma_2) < \frac{2}{3d(d+1)^2}.$$

Let φ_1 and φ_2 be the inverse of h_1 and h_2, respectively. By [18, Lemma 2.20], if $c_j = 1$ then there is a positive real decreasing function σ_j satisfying $\sigma_j(2x) \simeq \sigma_j(x)$ and $\sigma_j(x) \gtrsim x^{-\epsilon}$ for any $\epsilon > 0$, such that for each $k \in \mathbb{N}$, [1]

$$(2) \qquad \varphi_j^{(k)}(x) \simeq \frac{\varphi_j(x)\sigma_j(x)}{x^k}.$$

We set $\sigma_j \equiv 1$ whenever $c_j > 1$. In this article, we are interested in sets of the form

$$\mathbf{P}_+ = \{p \in \mathbb{P} : \{\varphi_1(p)\} < \psi(p)\},$$

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[1] We write $A \lesssim B$ if there is an absolute constant $C > 0$ such that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$ hold simultaneously then we write $A \simeq B$.

and

$$\mathbf{P}_- = \left\{ p \in \mathbb{P} : \{-\varphi_1(p)\} < \psi(p) \right\}$$

where $\psi : [1, \infty) \to (0, \infty)$ is a positive function such that $\psi(x) \leq \frac{1}{2}$ for all $x \geq 1$, and

(3)
$$\lim_{x \to +\infty} \frac{\psi^{(k)}(x)}{\varphi_2^{(k+1)}(x)} = 1,$$

for $k = 0, \ldots, d + 2$ where $d \geq 1$ is the degree of the polynomial P. The sets $\mathbf{P}_-$ and $\mathbf{P}_+$ are intersections with primes numbers of sets studied in [14].

Let us observe that, if $h_1 = h_2 = h$ is the inverse function to φ and $\psi(x) = \varphi(x+1) - \varphi(x)$ then

$$\mathbf{P}_- = \left\{ p \in \mathbb{P} : p = \lfloor h(n) \rfloor \text{ for some } n \in \mathbb{N} \right\}.$$

Indeed, we have the following chain of equivalences

$$
\begin{aligned}
& \mathbb{P} \ni p = \lfloor h(n) \rfloor \quad \text{for some } n \in \mathbb{N} \\
\Longleftrightarrow\ & h(n) - 1 < p \leq h(n) < p + 1 \\
\Longleftrightarrow\ & \varphi(p) \leq n < \varphi(p+1), \quad \text{since } \varphi \text{ is increasing} \\
\Longleftrightarrow\ & 0 \leq n - \varphi(p) < \varphi(p+1) - \varphi(p) = \psi(p) \leq \tfrac{1}{2} \\
\Longleftrightarrow\ & 0 \leq \{-\varphi(p)\} < \psi(p) \\
\Longleftrightarrow\ & p \in \mathbf{P}_-.
\end{aligned}
$$

In particular, the sets $\mathbf{P}_-$ are a generalization of those considered by Leitmann [17] and Mirek [18].

For any $r \geq 1$, the r-variational seminorm V_r of a sequence $(a_n : n \in \mathbb{N})$ of complex numbers is defined by

$$V_r(a_n : n \in \mathbb{N}) = \sup_{k_0 < \cdots < k_J} \left(\sum_{j=1}^{J} |a_{k_j} - a_{k_{j-1}}|^r \right)^{1/r}.$$

Observe that, if $V_r(a_n : n \in \mathbb{N}) < \infty$ for any $r \geq 1$, then the sequence $(a_n : n \in \mathbb{N})$ convergences. Therefore, we can deduce the pointwise ergodic theorems from the following two statements.

Theorem A. *Let* $\mathbf{P} \in \{\mathbf{P}_-, \mathbf{P}_+\}$. *For every* $s > 1$ *there is* $C_s > 0$ *such that for all* $r > 2$ *and any* $f \in L^s(X, \mu)$,

$$\left\| V_r\big(\mathscr{A}_N f : N \in \mathbb{N}\big)\right\|_{L^s} \le C_s \frac{r}{r-2} \|f\|_{L^s}.$$

Moreover, the constant C_s *is independent of coefficients of the polynomial* P.

Theorem B. *Let* $\mathbf{P} \in \{\mathbf{P}_-, \mathbf{P}_+\}$. *For every* $s > 1$ *there is* $C_s > 0$ *such that for all* $r > 2$ *and any* $f \in L^s(X, \mu)$,

$$\left\| V_r\big(\mathscr{H}_N f : N \in \mathbb{N}\big)\right\|_{L^s} \le C_s \frac{r}{r-2} \|f\|_{L^s}.$$

Moreover, the constant C_s *is independent of coefficients of the polynomial* P.

We point out that Theorem B allows us to define ergodic counterpart of the singular integral operator. Namely, for $f \in L^s(X, \mu)$, $s > 1$, we set

$$\mathscr{H} f(x) = \lim_{N \to \infty} \mathscr{H}_N f(x),$$

for μ-almost all $x \in X$.

In view of the Calderón transference principle while proving Theorem A and Theorem B we may assume that we deal with the model dynamical system, namely, the integers $\mathbb{Z}$ with the counting measure and the shirt operator. As usual, r-variations are divided into to two parts: short and long variations. By choosing long variations to be over the set $Z_\rho = \{\lfloor 2^{k^\rho} \rfloor : k \in \mathbb{N}\}$ for some $\rho \in (0, 1)$, we make short variations easier to handle. Indeed, bounding short variations is reduced to estimating $\ell^1(\mathbb{Z})$-norm of convolution kernels, which is a consequence of the asymptotic of some exponential sums over $\mathbf{P}$ combined with the prime number theorem or the Mertens theorem. For long variations, we replace the operators modeled on $\mathbf{P}$ by operators modeled on $\mathbb{P}$. For this step, we need to establish a decay of ℓ^2-norm of the corresponding difference. In view of the Plancherel's theorem, it is a consequence of estimates for some exponential sums over $\mathbf{P}$, see Section 2. Lastly, variational estimates for the operators modeled on $\mathbb{P}$ are proved in [28, Theorem C].

2. Exponential sums

In this section we develop estimates on exponential sums that are essential to our argument. The main tools is van der Corput's lemma in the classical form as well as the one recently obtained by Heath-Brown (see [11, Theorem 1]).

Lemma 1 ([29], [27, Theorem 5.11, Theorem 5.13]). *Suppose that $N \geq 1$ and $k \geq 2$ are two integers and $a \leq b \leq a + N$. Let $F \in \mathcal{C}^k(a, b)$ be a real-valued function such that*

$$\eta \lesssim |F^{(k)}(x)| \lesssim r\eta, \quad \text{for all } x \in (a, b),$$

for some $\eta > 0$ and $r \geq 1$. Then

$$\left| \sum_{a \leq n \leq b} e^{2\pi i F(n)} \right| \lesssim N \left(\eta^{\frac{1}{2^k - 2}} + N^{-\frac{2}{2^k}} + (N^k \eta)^{-\frac{2}{2^k}} \right).$$

The implied constant depends only on r.

Lemma 2 ([11]). *Suppose that $N \geq 1$ and $k \geq 3$ are two integers and $a \leq b \leq a + N$. Let $F \in \mathcal{C}^k(a, b)$ be a real-valued function such that*

$$\eta \lesssim |F^{(k)}(x)| \lesssim r\eta, \quad \text{for all } x \in (a, b),$$

for some $\eta > 0$ and $r \geq 1$. Then for every $\epsilon > 0$,

$$\left| \sum_{a \leq n \leq b} e^{2\pi i F(n)} \right| \lesssim N^{1+\epsilon} \left(\eta^{\frac{1}{k(k-1)}} + N^{-\frac{1}{k(k-1)}} + (N^k \eta)^{-\frac{2}{k^2(k-1)}} \right)$$

where the implied constant depends only on r, k and ϵ.

Notice that the exponents in Lemma 2 are improved for $n \geq 10$. In fact, the second term in the bracket has smaller exponent in Lemma 1 for $2 \leq n \leq 5$, while the third term for $2 \leq n \leq 9$. To benefit from this observation, we take the minimum of both estimates.

We start by investigating some exponential sums over integers in arithmetic progression.

Proposition 2.1. *For $m \in \mathbb{Z} \setminus \{0\}$, $\tau \in \{0, 1\}$ and $j \geq 1$, we set*

$$T(K) = \sum_{1 \leq k \leq K} \exp \left(2\pi i \big(\xi P(jk) + m(\varphi_1(jk) - \tau \psi(jk)) \big) \right).$$

Then [2]

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[2] We write $A \lesssim_\delta B$ to indicate that the implied constant depends on δ.

(i) *if $d \geq 1$ then for each $\epsilon > 0$,*

$$\left|T(K)\right| \lesssim_\epsilon |m|^{\frac{1}{2(2^d-1)}}(jK)^{1+\epsilon}\big(\varphi_1(jK)\sigma_1(jK)\big)^{-\frac{1}{2^d}},$$

(ii) *if $d \geq 2$ then for each $\epsilon > 0$,*

$$\left|T(K)\right| \lesssim_\epsilon |m|^{\frac{1}{d(d+1)}}(jK)^{1+\epsilon}\big(\varphi_1(jK)\sigma_1(jK)\big)^{-\frac{2}{d(d+1)^2}}.$$

The implied constants are independent of j, m, τ, K and ξ.

Proof. For the proof, let us define $F : [1, \infty) \to \mathbb{R}$ by

$$F(t) = \xi P(jt) + m\big(\varphi_1(jt) - \tau\psi(jt)\big).$$

By (2) and (3),

$$\psi^{(d+1)}(x) \simeq \varphi_2^{(d+2)}(x) \simeq \frac{\varphi_2(x)\sigma_2(x)}{x^{d+2}},$$

and since $\gamma_2 \leq 1 \leq 1 + \gamma_1$, we have

$$\frac{\varphi_2(x)\sigma_2(x)}{x\varphi_1(x)\sigma_1(x)} = o(1),$$

thus

$$\psi^{(d+1)}(x) = o\left(\frac{\varphi_1(x)\sigma_1(x)}{x^{d+1}}\right).$$

Hence, by (2), for $t \in [X, 2X]$, we obtain

$$\left|F^{(d+1)}(t)\right| = j^{d+1}|m| \cdot \left|\varphi_1^{(d+1)}(jt) - \tau\psi^{(d+1)}(jt)\right|$$
$$\simeq \frac{j^{d+1}|m|\varphi_1(jX)\sigma_1(jX)}{(jX)^{d+1}}.$$

For $X < X' \leq 2X$, we set

$$T(X, X') = \sum_{X < k \leq X'} e^{2\pi i F(k)}.$$

Then

(4) $$\left|T(K)\right| \lesssim (\log K) \max_{\substack{X < X' \leq K \\ X' \leq 2X}} \left|T(X, X')\right|.$$

Since for each $\delta > 0$ satisfying $\delta < \gamma_1^{-1}$, and $\delta \leq 1$ if $\gamma_1 = 1$, a function $x \mapsto x(\varphi_1(x)\sigma_1(x))^{-\delta}$ is increasing, see [19, Lemma 2.6], by Lemmas 1 and 2, we

obtain respectively

$$
\begin{aligned}
\left|T(X, X')\right| &\lesssim X\left(\frac{j^{d+1}|m|\varphi_1(jX)\sigma_1(jX)}{(jX)^{d+1}}\right)^{\frac{1}{2(2^d-1)}} + X^{1-\frac{1}{2^d}} \\
&\quad + X\left(X^{d+1}\frac{j^{d+1}|m|\varphi_1(jX)\sigma_1(jX)}{(jX)^{d+1}}\right)^{-\frac{1}{2^d}} \\
&\lesssim (|m|j)^{\frac{1}{2(2^d-1)}}X^{1-\frac{d}{2(2^d-1)}} + X^{1-\frac{1}{2^d}} + X\left(|m|\varphi_1(jX)\sigma_1(jX)\right)^{-\frac{1}{2^d}} \\
&\lesssim |m|^{\frac{1}{2(2^d-1)}}jX\left(\varphi_1(jX)\sigma_1(jX)\right)^{-\frac{1}{2^d}},
\end{aligned}
$$

and

$$
\begin{aligned}
\left|T(X, X')\right| &\lesssim X^{1+\epsilon}\left(\frac{j^{d+1}|m|\varphi_1(jX)\sigma_1(jX)}{(jX)^{d+1}}\right)^{\frac{1}{d(d+1)}} + X^{1+\epsilon-\frac{1}{d(d+1)}} \\
&\quad + X^{1+\epsilon}\left(X^{d+1}\frac{j^{d+1}|m|\varphi_1(jX)\sigma(jX)}{(jX)^{d+1}}\right)^{-\frac{2}{d(d+1)^2}} \\
&\lesssim (|m|j)^{\frac{1}{d(d+1)}}X^{1+\epsilon-\frac{1}{d+1}} + X^{1+\epsilon-\frac{1}{d(d+1)}} \\
&\quad + X^{1+\epsilon}\left(|m|\varphi_1(jX)\sigma_1(jX)\right)^{-\frac{2}{d(d+1)^2}} \\
&\lesssim |m|^{\frac{1}{d(d+1)}}(jX)^{1+\epsilon}\left(\varphi_1(jX)\sigma_1(jX)\right)^{-\frac{2}{d(d+1)^2}}.
\end{aligned}
$$

Now, using (4) we easily finish the proof. $\qquad\square$

Let us turn to estimating the exponential sums over prime numbers. To regularize them we use von Mangoldt's function defined as

$$
\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m, \text{ for some } p \in \mathbb{P} \text{ and } m \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}
$$

The classical way to handle von Mangoldt's function is to use Vaughan's identity (see [30], see also [9, Lemma 4.12]), which states that for any $n > u \geq 1$,

$$
(5) \qquad \Lambda(n) = \sum_{\substack{j,k>u \\ jk=n}} \Lambda(k)a_j + \sum_{\substack{j\leq u \\ jk=n}} \mu(j)\log(k) - \sum_{\substack{j\leq u^2 \\ jk=n}} b_j
$$

where

$$
a_j = \sum_{\substack{d>u \\ d\ell=j}} \mu(d), \qquad b_j = \sum_{\substack{d,\ell\leq u \\ d\ell=j}} \mu(d)\Lambda(\ell),
$$

and $\mu(n)$ is the Möbius function defined for $n = p_1^{m_1} \cdots p_k^{m_k}$ where p_j are distinct prime numbers, as

$$\mu(n) = \begin{cases} (-1)^k & \text{if } m_1 = \cdots = m_k, \\ 0 & \text{otherwise.} \end{cases}$$

Let us observe that for any $\epsilon > 0$,

$$\sum_{J \leq j \leq 2J} |a_j|^2 \lesssim_\epsilon J^{1+\epsilon}, \qquad \text{and} \qquad \sum_{J \leq j \leq 2J} |b_j|^2 \lesssim_\epsilon J^{1+\epsilon}.$$

Theorem 1. *For $m \in \mathbb{Z} \setminus \{0\}$, $\tau \in \{0,1\}$ and $1 \leq X \leq X' \leq 2X$, we set*

$$S(X, X') = \sum_{X < n \leq X'} \exp\Big(2\pi i \big(\xi P(n) + m(\varphi_1(n) - \tau\psi(n)) \big) \Big) \Lambda(n).$$

Then for each $\epsilon > 0$,

(i) if $d = 1$,

$$\big| S(X, X') \big| \lesssim_\epsilon X^{1+\epsilon} \Big(|m|^{\frac{1}{4}} X^{-\frac{1}{12}} + |m|^{\frac{1}{14}} \big(\varphi_1(X)\sigma_1(X) \big)^{-\frac{1}{14}}$$
$$+ X^{\frac{1}{12}} \big(\varphi_1(X)\sigma_1(X) \big)^{-\frac{1}{4}} \Big),$$

(ii) if $d = 2$,

$$\big| S(X, X') \big| \lesssim_\epsilon X^{1+\epsilon} \Big(|m|^{\frac{1}{12}} X^{-\frac{1}{16}} + |m|^{\frac{1}{30}} \big(\varphi_1(X)\sigma_1(X) \big)^{-\frac{1}{20}}$$
$$+ X^{\frac{1}{32}} \big(\varphi_1(X)\sigma_1(X) \big)^{-\frac{1}{8}} \Big),$$

(iii) if $d \in \{3, \ldots, 9\}$,

$$\big| S(X, X') \big| \lesssim_\epsilon X^{1+\epsilon} \Big(X^{-\frac{1}{4 \cdot 2^d}} + |m|^{\frac{1}{4(2^d - 1)}} X^{-\frac{d-1}{8(2^d - 1)}}$$
$$+ |m|^{\frac{1}{6(2^d - 1)}} \big(\varphi_1(X)\sigma_1(X) \big)^{-\frac{1}{3 \cdot 2^d}} \Big),$$

(iv) if $d \geq 10$,

$$\big| S(X, X') \big| \lesssim_\epsilon X^{1+\epsilon} \Big(X^{-\frac{1}{4d(d+1)}} + |m|^{\frac{1}{2d(d+1)}} X^{-\frac{d-1}{4d(d+1)}}$$
$$+ |m|^{\frac{1}{3d(d+1)}} \big(\varphi_1(X)\sigma_1(X) \big)^{-\frac{2}{3d(d+1)^2}} \Big).$$

The implied constants are independent of m, τ, X, X' and ξ.

Proof. To simplify notation, let $F : [1, \infty) \to \mathbb{R}$ stand for

$$F(t) = \xi P(t) + m(\varphi_1(t) - \tau \psi(t)).$$

Fix $1 \leq u \leq X^{\frac{1}{3}}$ whose value will be determined later. By Vaughan's identity (5), we can write

$$S(X, X') = \Sigma_1 - \Sigma_{21} - \Sigma_{22} + \Sigma_3$$

where

$$\Sigma_1 = \sum_{j \leq u} \mu(j) \sum_{X/j < k \leq X'/j} e^{2\pi i F(jk)} \log(k),$$

$$\Sigma_{21} = \sum_{j \leq u} b_j \sum_{X/j < k \leq X'/j} e^{2\pi i F(jk)},$$

$$\Sigma_{22} = \sum_{u < j \leq u^2} b_j \sum_{X/j < k \leq X'/j} e^{2\pi i F(jk)},$$

$$\Sigma_3 = \sum_{u < j \leq X'/u} a_j \sum_{\substack{X/j < k \leq X'/j \\ k > u}} e^{2\pi i F(jk)} \Lambda(k).$$

Therefore, our aim is reduced to bounding each term separately.
The estimate for Σ_1 and Σ_{21}. For $1 \leq j \leq u$ we set

$$T_j(K) = \sum_{X/j < k \leq K} e^{2\pi i F(jk)}.$$

By the partial summation, we can write

$$\sum_{X/j < k \leq X'/j} e^{2\pi i F(jk)} \log(k) = T_j(X'/j) \log(X'/j) - \int_X^{X'} T_j(t/j) \frac{\mathrm{d}t}{t},$$

thus

$$\left| \Sigma_1 \right| \lesssim (\log X) \sum_{j \leq u} \max_{X/j \leq K \leq X'/j} \left| T_j(K) \right|.$$

Moreover, since

$$|b_j| \leq \sum_{\ell | j} \Lambda(\ell) = \log(j),$$

we have

$$\big|\Sigma_{21}\big| \lesssim (\log X) \sum_{j \leq u} \max_{X/j \leq K \leq X'/j} \big|T_j(K)\big|.$$

Therefore, by Proposition 2.1(i), we obtain

$$|\Sigma_1| + |\Sigma_{21}| \lesssim (\log X) \sum_{j \leq u} \max_{X/j \leq K \leq X'/j} |m|^{\frac{1}{2(2^d-1)}}(jK)^{1+\epsilon}\big(\varphi_1(jK)\sigma_1(jK)\big)^{-\frac{1}{2^d}}$$

$$(6) \qquad \lesssim u|m|^{\frac{1}{2(2^d-1)}} X^{1+2\epsilon}\big(\varphi_1(X)\sigma_1(X)\big)^{-\frac{1}{2^d}}.$$

Similarly, Proposition 2.1(ii) gives

$$(7) \qquad |\Sigma_1| + |\Sigma_{21}| \lesssim u|m|^{\frac{1}{d(d+1)}} X^{1+2\epsilon}\big(\varphi_1(X)\sigma_1(X)\big)^{-\frac{2}{d(d+1)^2}}.$$

The estimate for Σ_{22} and Σ_3. Controlling Σ_{22} and Σ_3 requires more work. First, let us dyadically split the defining sums to get

$$(8) \quad |\Sigma_{22}| \lesssim (\log X)^2 \max_{\substack{u \leq J < J' \leq 2J \\ J' \leq u^2}} \max_{\substack{X/u^2 \leq K < K' \leq 2K \\ K' \leq X'/u}} \bigg| \sum_{\substack{J < j \leq J' \\ X < jk \leq X'}} \sum_{K < k \leq K'} e^{2\pi i F(jk)} b_j \bigg|,$$

and
(9)

$$|\Sigma_3| \lesssim (\log X)^2 \max_{\substack{u \leq J < J' \leq 2J \\ J' \leq u^2}} \max_{\substack{X/u^2 \leq K < K' \leq 2K \\ K' \leq X'/u}} \bigg| \sum_{\substack{J < j \leq J' \\ X < jk \leq X'}} \sum_{K < k \leq K'} e^{2\pi i F(jk)} \Lambda(k) a_j \bigg|.$$

To be able to deal with both cases simultaneously, let us consider two sequences of complex numbers $(A_j : j \in \mathbb{N})$ and $(B_k : k \in \mathbb{N})$, such that for each $\epsilon > 0$,

$$(10) \qquad \sum_{J \leq j \leq 2J} |A_j|^2 \lesssim_\epsilon J^{1+\epsilon}, \qquad \text{and} \qquad \sum_{K \leq k \leq 2K} |B_k|^2 \lesssim_\epsilon K^{1+\epsilon},$$

and study exponential sums of a form

$$\sum_{\substack{J < j \leq J' \\ X < jk \leq X'}} \sum_{K < k \leq K'} e^{2\pi i F(jk)} A_j B_k$$

where $J < J' \leq 2J$ and $K < K' \leq 2K$. Without loss of generality we may assume that $K \leq J$. By Cauchy–Schwarz inequality and (10), we have

$$\left| \sum_{\substack{J<j\leq J' \\ X<jk\leq X'}} \sum_{K<k\leq K'} e^{2\pi i F(jk)} A_j B_k \right|^2 \lesssim J^{1+\epsilon} \sum_{J<j\leq J'} \left| \sum_{\substack{K<k\leq K' \\ X<jk\leq X'}} e^{2\pi i F(jk)} B_k \right|^2.$$

To estimate the right-hand side, we expand the square and rearrange terms to get

$$(11) \qquad \left| \sum_{\substack{K<k\leq K' \\ X<jk\leq X'}} e^{2\pi i F(jk)} B_k \right|^2$$

$$= \sum_{|r|\leq K} \sum_{\substack{K<k,k+r\leq K' \\ X<jk,j(k+r)\leq X'}} \exp\left(2\pi i\big(F(jk) - F(j(k+r))\big)\right) B_k \overline{B_{k+r}}.$$

Therefore,

$$(12) \qquad \left| \sum_{\substack{J<j\leq J' \\ X<jk\leq X'}} \sum_{K<k\leq K'} e^{2\pi i F(jk)} A_j B_k \right|^2$$

$$\lesssim J^{1+\epsilon} \sum_{|r|\leq K} \sum_{K<k,k+r\leq K'} |B_k||B_{k+r}||U_{k,k+r}|$$

where for $K < k, k' < K'$, we have set

$$U_{k,k'} = \sum_{j\in\mathcal{J}_{k,k'}} \exp\left(2\pi i\big(F(jk) - F(jk')\big)\right),$$

and $\mathcal{J}_{k,k'} = \big(\max\{X/k, X/k', J\}, \min\{X'/k, X'/k', J'\}\big] \cap \mathbb{Z}$. To estimate $U_{k,k'}$, we are going to apply van der Corput's lemma. Let us fix $k \neq k'$. Setting $G(t) = F(tk) - F(tk')$ for $t \in \mathcal{J}_{k,k'}$, we can write

$$\left| G^{(d+1)}(t) \right| \simeq |m| \left| \big(\varphi_1^{(d+1)}(tk)k^{d+1} - \varphi_1^{(d+1)}(tk')(k')^{d+1}\big) \right.$$
$$\left. - \tau\big(\psi^{(d+1)}(tk)k^{d+1} - \psi^{(d+1)}(tk')(k')^{d+1}\big) \right|.$$

By the mean value theorem, for some x between tk and tk' we have

$$
\varphi_1^{(d+1)}(tk)(tk)^{d+1} - \varphi_1^{(d+1)}(tk')(tk')^{d+1}
$$
$$
= \left(\varphi_1^{(d+2)}(x)x^{d+1} + (d+1)\varphi_1^{(d+1)}(x)x^d\right)(k-k')t,
$$

thus, by (2), we obtain

$$
\left|\varphi_1^{(d+1)}(tk)(tk)^{d+1} - \varphi_1^{(d+1)}(tk')(tk')^{d+1}\right| \simeq \frac{\varphi_1(JK)\sigma_1(JK)}{JK}|k-k'|J.
$$

Similarly, we get

$$
\left|\psi^{(d+1)}(tk)(tk)^{d+1} - \psi^{(d+1)}(tk')(tk')^{d+1}\right| \simeq \frac{\varphi_2(JK)\sigma_2(JK)}{(JK)^2}|k-k'|J.
$$

Since

$$
\frac{\varphi_2(JK)\sigma_2(JK)}{JK\varphi_1(JK)\sigma_1(JK)} = o(1),
$$

we conclude that for $t \in \mathcal{J}_{k,k'}$,

$$
\left|G^{(d+1)}(t)\right| \simeq |m| \cdot |k-k'|\frac{\varphi_1(JK)\sigma_1(JK)}{JK}J^{-d}.
$$

Now, by Lemma 1, we get

$$
\begin{aligned}
\left|U_{k,k'}\right| &\lesssim J\left(|m| \cdot |k-k'|\frac{\varphi_1(JK)\sigma_1(JK)}{JK}J^{-d}\right)^{\frac{1}{2(2^d-1)}} + J^{1-\frac{1}{2^d}} \\
&\quad + J\left(J^{d+1}|m| \cdot |k-k'|\frac{\varphi_1(JK)\sigma_1(JK)}{JK}J^{-d}\right)^{-\frac{1}{2^d}} \\
&\lesssim J^{1-\frac{d}{2(2^d-1)}}\left(|m| \cdot |k-k'|\right)^{\frac{1}{2(2^d-1)}} + J^{1-\frac{1}{2^d}} \\
&\quad + JK^{\frac{1}{2^d}}\left(|m| \cdot |k-k'|\right)^{-\frac{1}{2^d}}\left(\varphi_1(JK)\sigma_1(JK)\right)^{-\frac{1}{2^d}}.
\end{aligned}
$$

(13)

By Cauchy–Schwarz inequality and (10), we obtain

$$
\begin{aligned}
&\sum_{1\leq|r|\leq K}\sum_{K<k,k+r\leq K'}|B_k||B_{k+r}|J^{1-\frac{d}{2(2^d-1)}}|mr|^{\frac{1}{2(2^d-1)}} \\
&\lesssim J^{1-\frac{d}{2(2^d-1)}}|m|^{\frac{1}{2(2^d-1)}}\sum_{1\leq|r|\leq K}|r|^{\frac{1}{2(2^d-1)}}\sum_{K<k\leq K'}|B_k|^2 \\
&\lesssim J^{1-\frac{d}{2(2^d-1)}}|m|^{\frac{1}{2(2^d-1)}}K^{1+\frac{1}{2(2^d-1)}}K^{1+\epsilon}.
\end{aligned}
$$

Analogously, we show that

$$\sum_{1 \le |r| \le K} \sum_{K < k, k+r \le K'} |B_k||B_{k+r}|J^{1-\frac{1}{2d}} \lesssim J^{1-\frac{1}{2d}}K^{2+\epsilon},$$

and

$$\sum_{1 \le |r| \le K} \sum_{K < k, k+r \le K'} |B_k||B_{k+r}|JK^{\frac{1}{2d}}|mr|^{-\frac{1}{2d}}\big(\varphi_1(JK)\sigma_1(JK)\big)^{-\frac{1}{2d}}$$

$$\lesssim J|m|^{-\frac{1}{2d}}K^{2+\epsilon}\big(\varphi_1(JK)\sigma_1(JK)\big)^{-\frac{1}{2d}}.$$

Therefore,

$$\sum_{1 \le |r| \le K} \sum_{K < k, k+r \le K'} |B_k||B_{k+r}||U_{k,k'}|$$

$$\lesssim JK^{2+\epsilon}\Big(|m|^{\frac{1}{2(2^d-1)}}J^{-\frac{d}{2(2^d-1)}}K^{\frac{1}{2(2^d-1)}}$$

$$+ J^{-\frac{1}{2d}} + |m|^{-\frac{1}{2d}}\big(\varphi_1(JK)\sigma_1(JK)\big)^{-\frac{1}{2d}}\Big).$$

Since for $r = 0$, we have

$$\sum_{K < k \le K'} |B_k|^2|U_{k,k}| \lesssim JK^{1+\epsilon},$$

by (12), we can estimate

$$(14) \qquad \left|\sum_{\substack{J < j \le J' \\ X < jk \le X'}} \sum_{K < k \le K'} e^{2\pi i F(jk)}A_j B_k\right|^2$$

$$\lesssim J^{2+\epsilon}K^{2+\epsilon}\Big(K^{-1} + |m|^{\frac{1}{2(2^d-1)}}J^{-\frac{d}{2(2^d-1)}}K^{\frac{1}{2(2^d-1)}}$$

$$+ J^{-\frac{1}{2d}} + \big(\varphi_1(JK)\sigma_1(JK)\big)^{-\frac{1}{2d}}\Big),$$

provided that $K \le J$. We are now going to apply (14) to derive the estimates for Σ_{22} and Σ_3. Let us recall that $u \le J \le u^2$, $X/u^2 \le K \le X'/u$, $u^3 < X$ and $X < JK \le 2X$, thus

$$u \le \min\{J, K\} \le \sqrt{3X} \le \sqrt{3}\max\{J, K\}.$$

Hence, (14) applied to (8) and (9) results in

$$
(15) \qquad |\Sigma_{22}| + |\Sigma_3| \lesssim X^{1+\epsilon}\Big(u^{-\frac{1}{2}} + |m|^{\frac{1}{4(2^d-1)}} X^{-\frac{d-1}{8(2^d-1)}}
$$
$$
+ X^{-\frac{1}{2^{d+2}}} + \big(\varphi_1(X)\sigma_1(X)\big)^{-\frac{1}{2^{d+1}}}\Big).
$$

For $d \in \{1,2\}$, we improve the estimate (15), by applying to (11) the Weyl–van der Corput's inequality, see [9, Lemma 2.5]. For each $1 \le R \le K$, we have

$$
\sum_{\substack{J<j\le J'}} \Bigg| \sum_{\substack{K<k\le K'\\ X<jk\le X'}} e^{2\pi i F(jk)} B_k \Bigg|^2
$$
$$
\le \left(1 + \frac{K}{R}\right) \sum_{|r|\le R} \left(1 - \frac{|r|}{R}\right) \sum_{K\le k,k+r\le K'} |B_k||B_{k+r}||U_{k,k'}|.
$$

For $d = 1$, we take $R = K^{\frac{1}{3}}$. Then, by (13), we get

$$
\sum_{\substack{J<j\le J'}} \Bigg| \sum_{\substack{K<k\le K'\\ X<jk\le X'}} e^{2\pi i F(jk)} B_k \Bigg|^2
$$
$$
\lesssim JK^{1+\epsilon}(K+R)\Big(R^{-1} + |m|^{\frac{1}{2}} J^{-\frac{1}{2}} R^{\frac{1}{2}} + K^{\frac{1}{2}} R^{-\frac{1}{2}} \big(\varphi_1(JK)\sigma_1(JK)\big)^{-\frac{1}{2}}\Big)
$$
$$
\lesssim JK^{2+\epsilon}\Big(K^{-\frac{1}{3}} + |m|^{\frac{1}{2}} J^{-\frac{1}{2}} K^{\frac{1}{6}} + K^{\frac{1}{3}} \big(\varphi_1(JK)\sigma_1(JK)\big)^{-\frac{1}{2}}\Big).
$$

Therefore,

$$
(16) \qquad |\Sigma_{22}| + |\Sigma_3| \lesssim X^{1+\epsilon}\Big(u^{-\frac{1}{6}} + |m|^{\frac{1}{4}} X^{-\frac{1}{12}} + X^{\frac{1}{12}}\big(\varphi_1(X)\sigma_1(X)\big)^{-\frac{1}{4}}\Big).
$$

Similarly, for $d = 2$; we set $R = K^{\frac{1}{2}}$ which entails that

$$
\sum_{\substack{J<j\le J'}} \Bigg| \sum_{\substack{K<k\le K'\\ X<jk\le X'}} e^{2\pi i F(jk)} B_k \Bigg|^2
$$
$$
\lesssim JK^{1+\epsilon}(K+R)\Big(R^{-1} + |m|^{\frac{1}{6}} J^{-\frac{1}{3}} R^{\frac{1}{6}} + J^{-\frac{1}{4}}
$$
$$
+ |m|^{-\frac{1}{4}} K^{\frac{1}{4}} R^{-\frac{1}{4}} \big(\varphi_1(JK)\sigma_1(JK)\big)^{-\frac{1}{4}}\Big)
$$
$$
\lesssim JK^{2+\epsilon}\Big(K^{-\frac{1}{2}} + |m|^{\frac{1}{6}} J^{-\frac{1}{3}} K^{\frac{1}{12}} + |m|^{-\frac{1}{4}} K^{\frac{1}{8}} \big(\varphi_1(JK)\sigma_1(JK)\big)^{-\frac{1}{4}}\Big),
$$

and hence

$$(17) \quad |\Sigma_{22}| + |\Sigma_3| \lesssim X^{1+\epsilon} \left(u^{-\frac{1}{4}} + |m|^{\frac{1}{12}} X^{-\frac{1}{16}} + X^{\frac{1}{32}} \left(\varphi_1(X) \sigma_1(X) \right)^{-\frac{1}{8}} \right).$$

Next, let us observe that for $d \geq 2$, while estimating $U_{k,k'}$, instead of Lemma 1 we can use Lemma 2. This leads to

$$\left| U_{k,k'} \right| \lesssim J^{1+\epsilon-\frac{1}{d+1}} \left(|m| \cdot |k - k'| \right)^{\frac{1}{d(d+1)}} + J^{1+\epsilon-\frac{1}{d(d+1)}}$$
$$+ J^{1+\epsilon} K^{\frac{2}{d(d+1)^2}} \left(|m| \cdot |k - k'| \right)^{-\frac{2}{d(d+1)^2}} \left(\varphi_1(JK) \sigma_1(JK) \right)^{-\frac{2}{d(d+1)^2}},$$

and

$$\sum_{J < j \leq J'} \left| \sum_{\substack{K < k \leq K' \\ X < jk \leq X'}} e^{2\pi i F(jk)} B_k \right|^2$$
$$\lesssim J^{1+\epsilon} K^{2+\epsilon} \left(K^{-1} + |m|^{\frac{1}{d(d+1)}} J^{-\frac{1}{d+1}} K^{\frac{1}{d(d+1)}} \right.$$
$$\left. + J^{-\frac{1}{d(d+1)}} + \left(\varphi_1(JK) \sigma_1(JK) \right)^{-\frac{2}{d(d+1)^2}} \right),$$

which entails that

$$(18) \quad |\Sigma_{22}| + |\Sigma_3| \lesssim X^{1+\epsilon} \left(u^{-\frac{1}{2}} + |m|^{\frac{1}{2d(d+1)}} X^{-\frac{d-1}{4d(d+1)}} \right.$$
$$\left. + X^{-\frac{1}{4d(d+1)}} + \left(\varphi_1(X) \sigma_1(X) \right)^{-\frac{1}{d(d+1)^2}} \right).$$

Conclusion. In view of the estimates (6) and (15), by selecting

$$u = |m|^{-\frac{2}{6(2^d-1)}} \left(\varphi_1(X) \sigma_1(X) \right)^{\frac{2}{3} \cdot \frac{1}{2^d}},$$

we obtain

$$\left| S(X, X') \right| \lesssim X^{1+\epsilon} \left(u|m|^{\frac{1}{2(2^d-1)}} \left(\varphi_1(X) \sigma_1(X) \right)^{-\frac{1}{2^d}} \right.$$
$$+ u^{-\frac{1}{2}} + |m|^{\frac{1}{4(2^d-1)}} X^{-\frac{d-1}{8(2^d-1)}}$$
$$\left. + X^{-\frac{1}{2^{d+2}}} + \left(\varphi_1(X) \sigma_1(X) \right)^{-\frac{1}{2^{d+1}}} \right)$$
$$\lesssim X^{1+\epsilon} \left(|m|^{\frac{1}{6(2^d-1)}} \left(\varphi_1(X) \sigma_1(X) \right)^{-\frac{1}{3 \cdot 2^d}} \right.$$
$$\left. + |m|^{\frac{1}{4(2^d-1)}} X^{-\frac{d-1}{8(2^d-1)}} + X^{-\frac{1}{2^{d+2}}} \right).$$

Analogously, setting

$$u = |m|^{-\frac{2}{3d(d+1)}} \big(\varphi_1(X)\sigma_1(X)\big)^{\frac{4}{3d(d+1)^2}},$$

from (7) and (18), we get

$$\big|S(X, X')\big| \lesssim X^{1+\epsilon}\Big(|m|^{\frac{1}{3d(d+1)}}\big(\varphi_1(X)\sigma_1(X)\big)^{-\frac{2}{3d(d+1)^2}}$$
$$+ |m|^{\frac{1}{2d(d+1)}} X^{-\frac{d-1}{4d(d+1)}} + X^{-\frac{1}{4d(d+1)}}\Big).$$

For $d = 1$, we take

$$u = |m|^{-\frac{3}{7}} \big(\varphi_1(X)\sigma_1(X)\big)^{\frac{3}{7}},$$

and use (6) together with (16), to get

$$\big|S(X, X')\big| \lesssim X^{1+\epsilon}\Big(|m|^{\frac{1}{14}}\big(\varphi_1(X)\sigma_1(X)\big)^{-\frac{1}{14}}$$
$$+ |m|^{\frac{1}{4}} X^{-\frac{1}{12}} + X^{\frac{1}{12}}\big(\varphi_1(X)\sigma_1(X)\big)^{-\frac{1}{4}}\Big).$$

Lastly, for $d = 2$ and

$$u = |m|^{-\frac{2}{15}} \big(\varphi_1(X)\sigma_1(X)\big)^{\frac{1}{5}},$$

by (6) and (17), we obtain

$$\big|S(X, X')\big| \lesssim X^{1+\epsilon}\Big(|m|^{\frac{1}{30}}\big(\varphi_1(X)\sigma_1(X)\big)^{-\frac{1}{20}}$$
$$+ |m|^{\frac{1}{12}} X^{-\frac{1}{16}} + X^{\frac{1}{32}}\big(\varphi_1(X)\sigma_1(X)\big)^{-\frac{1}{8}}\Big),$$

which concludes the proof of theorem.   □

The reasoning for $\mathbf{P}_+$ and $\mathbf{P}_-$ are similar, therefore to simplify the notation we are going to write

$$\mathbf{P} = \mathbf{P}_+ = \big\{p \in \mathbb{P} : \{\varphi_1(p)\} < \psi(p)\big\}.$$

For $N \in \mathbb{N}$ we set $\mathbb{P}_N = \mathbb{P} \cap [1, N]$ and $\mathbf{P}_N = \mathbf{P} \cap [1, N]$. In what follows, we need a characterization of the sets $\mathbf{P}$. The proof follows a line parallel to [14, Lemma 2.2].

Lemma 3. $p \in \mathbf{P}$ *if and only if* $p \in \mathbb{P}$ *and* $\lfloor \varphi_1(p) \rfloor - \lfloor \varphi_1(p) - \psi(p) \rfloor = 1$.

Proof. We begin with the forward implication; it suffices to show that if $p \in \mathbf{P}$, then the integer

$$\lfloor \varphi_1(p) \rfloor - \lfloor \varphi_1(p) - \psi(p) \rfloor,$$

belongs to $\left(0, \frac{3}{2}\right)$. By definition, if $p \in \mathbf{P}$ then $0 \le \varphi_1(p) - \lfloor \varphi_1(p) \rfloor < \psi(p)$, thus

$$-\varphi_1(p) \le -\lfloor \varphi_1(p) \rfloor < \psi(p) - \varphi_1(p),$$

if and only if

$$\varphi_1(p) \ge \lfloor \varphi_1(p) \rfloor > \varphi_1(p) - \psi(p),$$

from where it follows that

$$\lfloor \varphi_1(p) \rfloor - \lfloor \varphi_1(p) - \psi(p) \rfloor > \{\varphi_1(p) - \psi(p)\} \ge 0.$$

In view of $\lfloor \varphi_1(p) - \psi(p) \rfloor \ge \varphi_1(p) - \psi(p) - 1$, we obtain

$$\lfloor \varphi_1(p) \rfloor - \lfloor \varphi_1(p) - \psi(p) \rfloor \le \lfloor \varphi_1(p) \rfloor - \varphi_1(p) + \psi(p) + 1$$
$$\le \psi(p) + 1 < \tfrac{3}{2}.$$

We now turn to the reverse implication; if $p \in \mathbb{P}$ and $\lfloor \varphi_1(p) \rfloor = 1 + \lfloor \varphi_1(p) - \psi(p) \rfloor$, then we have

$$0 \le \varphi_1(p) - \lfloor \varphi_1(p) \rfloor = \varphi_1(p) - 1 - \lfloor \varphi_1(p) - \psi(p) \rfloor$$
$$< \varphi_1(p) - 1 + 1 + \psi(p) - \varphi_1(p) = \psi(p).$$

Consequently, we get $\{\varphi_1(p)\} < \psi(p)$, as desired. $\qquad\square$

We are now ready to prove the main theorem of this section.

Theorem 2. *For each $\epsilon > 0$, satisfying*

    (i) if $d = 1$,

$$\begin{cases} (1 - \gamma_1) + 15(1 - \gamma_2) + 84\epsilon < 1, \\ 3(1 - \gamma_1) + 12(1 - \gamma_2) + 60\epsilon < 2 \end{cases}$$

    (ii) if $d = 2$,

$$\begin{cases} 3(1 - \gamma_1) + 62(1 - \gamma_2) + 360\epsilon < 3, \\ 4(1 - \gamma_1) + 32(1 - \gamma_2) + 160\epsilon < 3 \end{cases}$$

(iii) if $d \in \{3, \ldots, 9\}$,

$$\begin{cases} \frac{1}{3 \cdot 2^d}(1 - \gamma_1) + \left(1 + \frac{1}{6(2^d - 1)}\right)(1 - \gamma_2) + 6\epsilon < \frac{1}{3 \cdot 2^d}, \\ (1 - \gamma_2) + 4\epsilon < \frac{1}{4 \cdot 2^d}, \end{cases}$$

(iv) if $d \geq 10$,

$$(19) \quad \frac{2}{3d(d+1)^2}(1 - \gamma_1) + \left(1 + \frac{1}{3d(d+1)}\right)(1 - \gamma_2) + 6\epsilon < \frac{2}{3d(d+1)^2},$$

we have

$$\sum_{p \in \mathbf{P}_N} e^{2\pi i \xi P(p)} \log(p) = \sum_{p \in \mathbb{P}_N} e^{2\pi i \xi P(p)} \log(p)\psi(p) + \mathcal{O}\left(\varphi_2(N)N^{-\epsilon}\right).$$

Proof. We treat $d \geq 10$ only since similar arguments apply to the other cases. Let us introduce the "sawtooth" function $\Phi(x) = \{x\} - 1/2$. Notice that, in view of Lemma 3 we have

$$\lfloor \varphi_1(n) \rfloor - \lfloor \varphi_1(n) - \psi(n) \rfloor = \psi(n) + \Phi\big(\varphi_1(n) - \psi(n)\big) - \Phi\big(\varphi_1(n)\big).$$

Hence, we may write

$$\sum_{p \in \mathbf{P}_N} e^{2\pi i \xi P(p)} \log(p) = \sum_{p \in \mathbb{P}_N} e^{2\pi i \xi P(p)} \log(p)\psi(p)$$
$$+ \sum_{p \in \mathbb{P}_N} e^{2\pi i \xi P(p)} \log(p)\big(\Phi\big(\varphi_1(p) - \psi(p)\big) - \Phi\big(\varphi_1(p)\big)\big).$$

Since

$$\frac{1}{2} - \gamma_2 + 2\epsilon = (1 - \gamma_2) - \frac{1}{2} + 2\epsilon < 0,$$

by the prime number theorem we get

$$\sum_{p \in \mathbb{P}_N} e^{2\pi i \xi P(p)} \log(p)\big(\Phi\big(\varphi_1(p) - \psi(p)\big) - \Phi\big(\varphi_1(p)\big)\big)$$
$$= \sum_{n=1}^{N} e^{2\pi i \xi P(n)}\Lambda(n)\big(\Phi\big(\varphi_1(n) - \psi(n)\big) - \Phi\big(\varphi_1(n)\big)\big) + \mathcal{O}\left(\varphi_2(N)N^{-\epsilon}\right).$$

Next, we claim that

$$\sum_{n=1}^{N} e^{2\pi i \xi P(n)}\Lambda(n)\big(\Phi\big(\varphi_1(n) - \psi(n)\big) - \Phi\big(\varphi_1(n)\big)\big) = \mathcal{O}\left(\varphi_2(N)N^{-\epsilon}\right).$$

To see this, let us expand Φ into its Fourier series, i.e.,

$$\Phi(x) = \sum_{0<|m|\leq M} \frac{1}{2\pi i m} e^{-2\pi i m x} + \mathcal{O}\left(\min\left\{1, \frac{1}{M\|x\|}\right\}\right),$$

for some $M > 0$ where $\|x\| = \min\{|x-n| : n \in \mathbb{Z}\}$ is the distance of $x \in \mathbb{R}$ to the nearest integer. Next, we split the resulting sum into three parts,

$$I_1 = \sum_{0<|m|\leq M} \frac{1}{2\pi i m} \sum_{n=1}^{N} e^{2\pi i(\xi P(n) - m\varphi_1(n))} \left(e^{2\pi i m\psi(n)} - 1\right) \Lambda(n),$$

and

$$I_2 = \mathcal{O}\left(\sum_{n=1}^{N} \min\left\{1, \frac{1}{M\|\varphi_1(n) - \psi(n)\|}\right\} \Lambda(n)\right),$$

$$I_3 = \mathcal{O}\left(\sum_{n=1}^{N} \min\left\{1, \frac{1}{M\|\varphi_1(n)\|}\right\} \Lambda(n)\right).$$

In this way, our aim is reduced to showing that each term I_1, I_2 and I_3 belongs to $\mathcal{O}\left(\varphi_2(N)N^{-\epsilon}\right)$.

The estimate for I_1. Let $\phi_m(x) = e^{2\pi i m\psi(x)} - 1$. Using (3), we easily see that

$$(20) \qquad |\phi_m(x)| \lesssim \frac{|m|\varphi_2(x)}{x}, \quad \text{and} \quad |\phi'_m(x)| \lesssim \frac{|m|\varphi_2(x)}{x^2}.$$

Let us first estimate the inner sum in I_1. By dyadic splitting we get

$$(21) \qquad \left|\sum_{n=1}^{N} \exp\left(2\pi i\left(\xi P(n) - m\varphi_1(n)\right)\right) \phi_m(n)\Lambda(n)\right|$$

$$\lesssim (\log N) \max_{\substack{X<X'\leq 2X \\ X'\leq N}} \left|\sum_{X<n\leq X'} \exp\left(2\pi i\left(\xi P(n) - m\varphi_1(n)\right)\right) \phi_m(n)\Lambda(n)\right|.$$

Now, by the partial summation, we have

$$\left|\sum_{X<n\leq X'} \exp\left(2\pi i\left(\xi P(n) - m\varphi_1(n)\right)\right) \phi_m(n)\Lambda(n)\right|$$

$$\leq |S(X,X')| \cdot |\phi_m(X')| + \int_X^{X'} |S(X,x)| \cdot |\phi'_m(x)| \, \mathrm{d}x$$

where

$$S(X, x) = \sum_{X < n \leq x} \exp\left(2\pi i\left(\xi P(n) - m\varphi_1(n)\right)\right)\Lambda(n).$$

It follows from Theorem 1(iv) and estimates (20) that

$$\left|\sum_{X < n \leq X'} \exp\left(2\pi i\left(\xi P(n) - m\varphi_1(n)\right)\right)\phi_m(n)\Lambda(n)\right|$$

$$\lesssim |m| \max_{X \in [1,N]} X^{\epsilon - \frac{1}{4d(d+1)}} \varphi_2(X) + |m|^{1 + \frac{1}{2d(d+1)}} \max_{X \in [1,N]} X^{\epsilon - \frac{d-1}{4d(d+1)}} \varphi_2(X)$$

$$+ |m|^{1 + \frac{1}{3d(d+1)}} \max_{X \in [1,N]} X^{\epsilon} \varphi_2(X) \left(\varphi_1(X)\sigma_1(X)\right)^{-\frac{2}{3d(d+1)^2}}$$

$$\lesssim |m| N^{\epsilon} \varphi_2(N) \left(N^{-\frac{1}{4d(d+1)}} + |m|^{\frac{1}{2d(d+1)}} N^{-\frac{d-1}{4d(d+1)}}\right.$$

$$\left. + |m|^{\frac{1}{3d(d+1)}} \left(\varphi_1(N)\sigma_1(N)\right)^{-\frac{2}{3d(d+1)^2}}\right),$$

and hence by (21), for each $\epsilon > 0$,

$$\frac{1}{|m|}\left|\sum_{n=1}^{N} \exp\left(2\pi i\left(\xi P(n) - m\varphi_1(n)\right)\right)\phi_m(n)\Lambda(n)\right|$$

$$\lesssim N^{\epsilon} \varphi_2(N) \left(N^{-\frac{1}{4d(d+1)}} + |m|^{\frac{1}{2d(d+1)}} N^{-\frac{d-1}{4d(d+1)}}\right.$$

$$\left. + |m|^{\frac{1}{3d(d+1)}} \left(\varphi_1(N)\sigma_1(N)\right)^{-\frac{2}{3d(d+1)^2}}\right).$$

Now, by summing up over $m \in \{1, \ldots, M\}$ we arrive at the conclusion that

$$(22) \qquad |I_1| \lesssim M N^{\epsilon} \varphi_2(N) \left(N^{-\frac{1}{4d(d+1)}} + M^{\frac{1}{2d(d+1)}} N^{-\frac{d-1}{4d(d+1)}}\right.$$

$$\left. + M^{\frac{1}{3d(d+1)}} \left(\varphi_1(N)\sigma_1(N)\right)^{-\frac{2}{3d(d+1)^2}}\right).$$

The estimates for I_2 and I_3. Let us consider I_2. Since (see [10, Section 2])

$$(23) \qquad\qquad \min\left\{1, \frac{1}{M\|x\|}\right\} = \sum_{m \in \mathbb{Z}} c_m e^{2\pi i m x}$$

where

$$(24) \qquad\qquad |c_m| \lesssim \min\left\{\frac{\log M}{M}, \frac{1}{|m|}, \frac{M}{|m|^2}\right\},$$

we have

$$\sum_{n=1}^{N} \min\left\{1, \frac{1}{M\|\varphi_1(n) - \psi(n)\|}\right\} \Lambda(n)$$

$$\leq (\log N) \sum_{m\in\mathbb{Z}} |c_m| \left|\sum_{n=1}^{N} e^{2\pi im(\varphi_1(n) - \psi(n))}\right|$$

$$\lesssim \frac{\log M}{M} N(\log N)$$

$$+ (\log N)\left(\sum_{0<|m|<M} \frac{1}{|m|} + \sum_{|m|>M} \frac{M}{|m|^2}\right)\left|\sum_{n=1}^{N} e^{2\pi im(\varphi_1(n) - \psi(n))}\right|.$$

By Proposition 2.1(i), we get

$$\left|\sum_{n=1}^{N} e^{2\pi im(\varphi_1(n) - \psi(n))}\right| \lesssim |m|^{\frac{1}{2}} \sup_{X\in[1,N]} X^{1+\frac{1}{2}\epsilon} \big(\varphi_1(X)\sigma_1(X)\big)^{-\frac{1}{2}}$$

$$\lesssim |m|^{\frac{1}{2}} N^{1+\frac{1}{2}\epsilon} \big(\varphi_1(N)\sigma_1(N)\big)^{-\frac{1}{2}},$$

thus

(25) $$|I_2| \lesssim M^{-1}(\log M)N^{1+\frac{1}{2}\epsilon} + M^{\frac{1}{2}} N^{1+\epsilon}\big(\varphi_1(N)\sigma_1(N)\big)^{-\frac{1}{2}}.$$

Arguments similar to the above leads to the same bounds for I_3.

Conclusion. From estimates (22) and (25), we conclude that

$$|I_1| + |I_2| + |I_3| \lesssim MN^{\epsilon}\varphi_2(N)\Big(N^{-\frac{1}{4d(d+1)}} + M^{\frac{1}{2d(d+1)}} N^{-\frac{d-1}{4d(d+1)}}$$

$$+ M^{\frac{1}{3d(d+1)}} \big(\varphi_1(N)\sigma_1(N)\big)^{-\frac{2}{3d(d+1)^2}}\Big)$$

$$+ M^{-1}(\log M)N^{1+\frac{1}{2}\epsilon} + M^{\frac{1}{2}} N^{1+\epsilon}\big(\varphi_1(N)\sigma_1(N)\big)^{-\frac{1}{2}}.$$

Take $M = N^{1+2\epsilon}\varphi_2(N)^{-1}$. As it may be easily verified, if ϵ satisfies (19) then

$$\frac{1}{2}(1 + 2\epsilon - \gamma_2) + 1 + \epsilon - \frac{1}{2}\gamma_1 + \frac{1}{2}\epsilon - \gamma_2 + \epsilon$$

$$\leq \frac{3}{2}(1 - \gamma_2) + \frac{1}{2}(1 - \gamma_1) - \frac{1}{2} + 4\epsilon \leq 0,$$

thus

$$M^{\frac{1}{2}} N^{1+\epsilon}\big(\varphi_1(N)\sigma_1(N)\big)^{-\frac{1}{2}} = \mathcal{O}\big(\varphi_2(N)N^{-\epsilon}\big).$$

Since

$$\left(\frac{2}{3d(d+1)^2} - 6\epsilon\right)\frac{3d(d+1)}{3d(d+1)+1}$$
$$< \frac{1}{4d(d+1)} - 5\epsilon < \left(\frac{d-1}{4d(d+1)} - 5\epsilon\right)\frac{2d(d+1)}{2d(d+1)+1},$$

for the other terms, we obtain

$$(1+2\epsilon-\gamma_2)+\epsilon - \frac{1}{4d(d+1)} + \epsilon \leq (1-\gamma_2) - \frac{1}{4d(d+1)} + 5\epsilon \leq 0,$$

and

$$\left(1+\frac{1}{2d(d+1)}\right)(1+2\epsilon-\gamma_2)+\epsilon - \frac{d-1}{4d(d+1)} + \epsilon$$
$$\leq \left(1+\frac{1}{2d(d+1)}\right)(1-\gamma_2) - \frac{d-1}{4d(d+1)} + 5\epsilon \leq 0.$$

Finally,

$$\left(1+\frac{1}{3d(d+1)}\right)(1+2\epsilon-\gamma_2)+\epsilon - \frac{2}{3d(d+1)^2}\gamma_1 + \frac{2}{3d(d+1)^2}\epsilon + \epsilon$$
$$\leq \left(1+\frac{1}{3d(d+1)}\right)(1-\gamma_2) + \frac{2}{3d(d+1)^2}(1-\gamma_1) - \frac{2}{3d(d+1)^2} + 6\epsilon \leq 0.$$

Consequently,

$$|I_1| + |I_2| + |I_3| \lesssim \varphi_2(N)N^{-\epsilon},$$

which completes the proof. □

Theorem 3. *For each $\epsilon > 0$, satisfying*

(i) *if $d = 1$,*

$$\begin{cases}(1-\gamma_1)+15(1-\gamma_2)+164\epsilon < 1, \\ 3(1-\gamma_1)+12(1-\gamma_2)+120\epsilon < 2\end{cases}$$

(ii) *if $d = 2$,*

$$\begin{cases}3(1-\gamma_1)+52(1-\gamma_2)+720\epsilon < 3, \\ 4(1-\gamma_1)+32(1-\gamma_2)+320\epsilon < 3\end{cases}$$

(iii) if $d \in \{3, \ldots, 9\}$,

$$\begin{cases} \frac{1}{3 \cdot 2^d}(1 - \gamma_1) + \left(1 + \frac{1}{6(2^d - 1)}\right)(1 - \gamma_2) + 12\epsilon < \frac{1}{3 \cdot 2^d}, \\ (1 - \gamma_2) + 8\epsilon < \frac{1}{4 \cdot 2^d}, \end{cases}$$

(iv) if $d \geq 10$,

$$\frac{2}{3d(d+1)^2}(1 - \gamma_1) + \left(1 + \frac{1}{3d(d+1)}\right)(1 - \gamma_2) + 12\epsilon < \frac{2}{3d(d+1)^2},$$

we have

$$(26) \qquad \sum_{p \in \mathbf{P}_N} e^{2\pi i \xi P(p)} \frac{\log(p)}{p\psi(p)} = \sum_{p \in \mathbb{P}_N} e^{2\pi i \xi P(p)} \frac{\log(p)}{p} + \mathcal{O}(N^{-\epsilon}),$$

and

$$(27) \qquad \sum_{p \in \mathbf{P}_N} e^{2\pi i \xi P(p)} \frac{\log(p)}{\psi(p)} = \sum_{p \in \mathbb{P}_N} e^{2\pi i \xi P(p)} \log(p) + \mathcal{O}(N^{1-\epsilon}).$$

Proof. Set

$$S(N) = \sum_{p \in \mathbf{P}_N} e^{2\pi i \xi P(p)} \log(p), \quad \text{and} \quad U(N) = \sum_{p \in \mathbb{P}_N} e^{2\pi i \xi P(p)} \log(p)\psi(p).$$

In view of Theorem 2,

$$(28) \qquad \qquad \left|S(N) - U(N)\right| \lesssim C\varphi_2(N)N^{-2\epsilon}.$$

Notice that by the partial summation we have

$$\sum_{p \in \mathbf{P}_N} e^{2\pi i \xi P(p)} \frac{\log(p)}{p\psi(p)} = \sum_{n=2}^{N} \frac{1}{n\psi(n)}(S(n) - S(n-1))$$

$$(29) \qquad \qquad = \frac{S(N)}{N\psi(N)} + \sum_{n=2}^{N-1} \left(\frac{1}{n\psi(n)} - \frac{1}{(n+1)\psi(n+1)}\right) S(n).$$

Similarly, we get

$$\sum_{p \in \mathbb{P}_N} e^{2\pi i \xi P(p)} \frac{\log(p)}{p} = \sum_{n=2}^{N} \frac{1}{n\psi(n)} \big(U(n) - U(n-1)\big)$$

(30)
$$= \frac{U(N)}{N\psi(N)} + \sum_{n=2}^{N-1} \left(\frac{1}{n\psi(n)} - \frac{1}{(n+1)\psi(n+1)} \right) U(n).$$

Therefore, by subtracting (30) from (29), we arrive at the conclusion that

$$\left| \sum_{p \in \mathbf{P}_N} e^{2\pi i \xi P(p)} \frac{\log(p)}{p\psi(p)} - \sum_{p \in \mathbb{P}_N} e^{2\pi i \xi P(p)} \frac{\log(p)}{p} \right|$$

$$\lesssim \frac{|S(N) - U(N)|}{N\psi(N)} + \sum_{n=2}^{N-1} \left| \frac{1}{n\psi(n)} - \frac{1}{(n+1)\psi(n+1)} \right| \cdot |S(n) - U(n)|.$$

Since, by (3) and (2),

$$\frac{1}{N\psi(N)} \lesssim \frac{1}{\varphi_2(N)\sigma_2(N)},$$

and

$$\left| \frac{1}{(n+1)\psi(n+1)} - \frac{1}{n\psi(n)} \right| \leq \sup_{x \in [n,n+1]} \left| \frac{1}{x^2\psi(x)} + \frac{\psi'(x)}{x\psi(x)^2} \right|$$

$$\lesssim \frac{1}{n\varphi_2(n)\sigma_2(n)},$$

the estimate (28) gives

$$\frac{|S(N) - U(N)|}{N\psi(N)} \lesssim N^{-\epsilon},$$

and

$$\left| \frac{1}{n\psi(n)} - \frac{1}{(n+1)\psi(n+1)} \right| \cdot |S(n) - U(n)| \lesssim n^{-1-\epsilon}.$$

Hence,

$$\left| \sum_{p \in \mathbf{P}_N} e^{2\pi i \xi P(p)} \frac{\log(p)}{p\psi(p)} - \sum_{p \in \mathbb{P}_N} e^{2\pi i \xi P(p)} \frac{\log(p)}{p} \right| \lesssim N^{-\epsilon} + \sum_{n=2}^{N-1} n^{-1-\epsilon} \lesssim N^{-\epsilon},$$

which concludes the proof of (26). Similar considerations apply to (27). $\square$

The following theorem generalizes the results obtained in [17, 26] and [19].

Theorem 4.
$$|\mathbf{P}_N| = \left(\int_2^N \frac{\psi(x)}{\log(x)} \, dx \right) (1 + o(1)).$$

Proof. Set
$$W_N = \sum_{p \in \mathbf{P}_N} \log(p), \qquad \text{and} \qquad V_N = \sum_{p \in \mathbb{P}_N} \log(p)\psi(p).$$

Let ϵ satisfy hypotheses of Theorem 2, then

(31)
$$W_N = V_N + \mathcal{O}\big(\varphi_2(N)N^{-\epsilon}\big).$$

By the partial summation we have

$$|\mathbf{P}_N| = \sum_{n=2}^N \big(W_n - W_{n-1}\big)\frac{1}{\log(n)}$$
$$= W_N \frac{1}{\log N} + \sum_{n=2}^{N-1} W_n \left(\frac{1}{\log(n)} - \frac{1}{\log(n+1)} \right),$$

and

$$\sum_{p \in \mathbb{P}_N} \psi(p) = V_N \frac{1}{\log N} + \sum_{n=2}^{N-1} V_n \left(\frac{1}{\log(n)} - \frac{1}{\log(n+1)} \right).$$

Therefore, by (31), we obtain

$$\left| |\mathbf{P}_N| - \sum_{p \in \mathbb{P}_N} \psi(p) \right| \leq |W_N - V_N| \frac{1}{\log N}$$
$$+ \sum_{n=2}^{N-1} |W_n - V_n| \left(\frac{1}{\log(n)} - \frac{1}{\log(n+1)} \right)$$
$$\lesssim \varphi_2(N)N^{-\epsilon} + \sum_{n=2}^{N-1} \varphi_2(n)n^{-1-\epsilon},$$

and thus

$$|\mathbf{P}_N| = \sum_{p \in \mathbb{P}_N} \psi(p) + \mathcal{O}\big(\varphi_2(N)N^{-\epsilon}\big).$$

Setting

$$\vartheta(N) = \sum_{p \in \mathbb{P}_N} \log(p),$$

by the summation by parts, we obtain

$$\sum_{p \in \mathbb{P}_N} \psi(p) = \sum_{n=2}^{N} \big(\vartheta(n) - \vartheta(n-1)\big) \frac{\psi(n)}{\log(n)}$$

$$= \vartheta(N) \frac{\psi(N)}{\log N} - \vartheta(2) \frac{2}{\log 2} + \sum_{n=2}^{N-1} \vartheta(n) \left(\frac{\psi(n)}{\log n} - \frac{\psi(n+1)}{\log(n+1)} \right),$$

and

$$\sum_{n=2}^{N} \frac{\psi(n)}{\log n} = N \frac{\psi(N)}{\log N} - 2 \frac{\psi(2)}{\log 2} + \sum_{n=2}^{N-1} n \left(\frac{\psi(n)}{\log n} - \frac{\psi(n+1)}{\log(n+1)} \right).$$

The prime number theorem implies that

(32) $$\vartheta(N) = N \Big(1 + \mathcal{O}\big(N^{-2\epsilon}\big) \Big).$$

Moreover, by (3) and (2),

$$\left| \frac{\psi(n)}{\log(n)} - \frac{\psi(n+1)}{\log(n+1)} \right| \leq \sup_{x \in [n,n+1]} \left| \frac{\psi'(x) \log(x) - \psi(x) x^{-1}}{(\log(n))^2} \right|$$

$$\lesssim \varphi_2(n) n^{-2+\epsilon}.$$

Hence,

$$\left| \sum_{p \in \mathbb{P}_N} \psi(p) - \sum_{n=2}^{N} \frac{\psi(n)}{\log n} \right| \leq \big| \vartheta(N) - N \big| \frac{\psi(N)}{\log N}$$

$$+ \sum_{n=2}^{N-1} \big| \vartheta(n) - n \big| \left| \frac{\psi(n)}{\log(n)} - \frac{\psi(n+1)}{\log(n+1)} \right|$$

$$\lesssim \varphi_2(N) N^{-\epsilon}.$$

Finally,

$$\left| \sum_{n=2}^{N-1} \frac{\psi(n)}{\log(n)} - \int_2^N \frac{\psi(x)}{\log(x)} \, \mathrm{d}x \right| \lesssim \sum_{n=2}^{N-1} \int_0^1 \left| \frac{\psi(n)}{\log n} - \frac{\psi(n+t)}{\log(n+t)} \right| \mathrm{d}t$$

$$\lesssim \sum_{n=2}^{N-1} \varphi_2(n) n^{-2+\epsilon}$$

Thus

$$\sum_{n=2}^{N} \frac{\psi(n)}{\log(n)} = \int_2^N \frac{\psi(x)}{\log(x)} \, \mathrm{d}x + \mathcal{O}\big(\varphi_2(N) N^{-1+\epsilon}\big)$$

Now, using (3), we get

$$\int_2^N \frac{\psi(x)}{\log(x)} \, \mathrm{d}x \geq \frac{1}{\log N} \int_2^N \psi(x) \, \mathrm{d}x$$

$$\gtrsim \frac{1}{\log N} \int_2^N \varphi_2'(x) \, \mathrm{d}x$$

$$\gtrsim \frac{\varphi_2(N)}{\log(N)},$$

which completes the proof.   □

3. Variational estimates

To deal with r-variational estimates for averaging operators and truncated discrete Hilbert transform, we apply the method used in [32] and [23, Section 4]. For $\rho \in (0,1)$ we set $Z_\rho = \left\{ \lfloor 2^{k^\rho} \rfloor : k \in \mathbb{N} \right\}$ and define long r-variations by

$$V_r^L(a_n : n \in \mathbb{N}) = V_r(a_n : n \in Z_\rho).$$

Then the corresponding short variations are given by

$$V_r^S(a_n : n \in \mathbb{N}) = \left(\sum_{k \geq 1} V_r\big(a_n : n \in [N_{k-1}, N_k)\big)^r \right)^{\frac{1}{r}}$$

where $N_k = \lfloor 2^{k^\rho} \rfloor$. Observe that

$$V_r(a_n : n \in \mathbb{N}) \lesssim V_r^L(a_n : n \in \mathbb{N}) + V_r^S(a_n : n \in \mathbb{N}).$$

3.1. Averaging operators

In this section we prove Theorem A for the model dynamical system.

Given a function f on $\mathbb{Z}$ we set

$$\mathcal{A}_N f(x) = \frac{1}{|\mathbf{P}_N|} \sum_{p \in \mathbf{P}_N} f(x - P(p)).$$

While studying r-variations we may replace the operators $\mathcal{A}_N$ by the weighted averages $\mathcal{M}_N$,

$$\mathcal{M}_N f(x) = \frac{1}{\Psi_N} \sum_{p \in \mathbf{P}_N} f(x - P(p)) \frac{\log p}{\psi(p)}$$

where

$$\Psi_N = \sum_{p \in \mathbf{P}_N} \frac{\log(p)}{\psi(p)}.$$

Indeed, since ψ is decreasing the ratio of weights in $\mathcal{A}_N$ and $\mathcal{M}_N$ is monotonically decreasing, thus by [23, Proposition 5.2], there is $C > 0$ such that for all $r > 2$,

$$V_r(\mathcal{A}_N f(x) : N \in \mathbb{N}) \leq C \cdot V_r(\mathcal{M}_N f(x) : N \in \mathbb{N})$$

where the constant C is independent of f, x and r. Therefore, it is enough to show the following theorem.

Theorem 5. *For each $s > 1$ there is $C_s > 0$ such that for all $r > 2$ and $f \in \ell^s(\mathbb{Z})$,*

$$\left\| V_r(\mathcal{M}_N f : N \in \mathbb{N}) \right\|_{\ell^s} \leq C_s \frac{r}{r-2} \|f\|_{\ell^s}.$$

Proof. We start with short variations. Let us denote by m_n the convolution kernel corresponding to $\mathcal{M}_n$. Then for each $x \in \mathbf{P}_{N_{k-1}}$,

$$\sum_{n=N_{k-1}}^{N_k - 1} |m_{n+1}(x) - m_n(x)| = \left(\Psi_{N_{k-1}}^{-1} - \Psi_{N_k}^{-1} \right) \frac{\log x}{\psi(x)}.$$

On the other hand, for $x \in \mathbf{P}_{N_k} \setminus \mathbf{P}_{N_{k-1}}$,

$$\sum_{n=N_{k-1}}^{N_k - 1} |m_{n+1}(x) - m_n(x)| \leq 2 \Psi_{N_{k-1}}^{-1} \frac{\log x}{\psi(x)}.$$

Therefore,

$$\left\| \sum_{n=N_{k-1}}^{N_k-1} |m_{n+1}-m_n| \right\|_{\ell^1} \leq \left(\Psi_{N_{k-1}}^{-1} - \Psi_{N_k}^{-1}\right)\Psi_{N_{k-1}} + 2\Psi_{N_{k-1}}^{-1}\left(\Psi_{N_{k-1}} - \Psi_{N_k}\right).$$

Let $\epsilon > 0$ satisfy the hypotheses of Theorem 3. By (32) and (27), we get

$$\Psi_N = \vartheta(N) + \mathcal{O}\left(N^{1-\epsilon}\right) = N + \mathcal{O}\left(N^{1-\epsilon}\right),$$

and thus

$$\Psi_{N_k} - \Psi_{N_{k-1}} \lesssim N_k - N_{k-1} + N_{k-1}^{1-\epsilon} \lesssim k^{\rho-1}N_{k-1} \lesssim k^{\rho-1}\Psi_{N_{k-1}}.$$

Therefore, by Young's inequality,

$$\left\| \sum_{n=N_{k-1}}^{N_k-1} |\mathcal{M}_{n+1}f - \mathcal{M}_n f| \right\|_{\ell^s} \leq \left\| \sum_{n=N_{k-1}}^{N_k-1} |m_{n+1}-m_n| \right\|_{\ell^1} \|f\|_{\ell^s}$$

$$\tag{33} \lesssim k^{\rho-1}\|f\|_{\ell^s}.$$

Let $u = \min\{2, s\}$. By monotonicity and Minkowski's inequality, we get

$$\left\| \left(\sum_{k\geq 1} V_r\left(\mathcal{M}_n f : n \in [N_{k-1}, N_k)\right)^r\right)^{\frac{1}{r}} \right\|_{\ell^s}$$

$$\leq \left\| \left(\sum_{k\geq 1} \left(\sum_{n=N_{k-1}}^{N_k-1} |\mathcal{M}_{n+1}f - \mathcal{M}_n f|\right)^u\right)^{\frac{1}{u}} \right\|_{\ell^s}$$

$$\leq \left(\sum_{k\geq 1} \left\| \sum_{n=N_{k-1}}^{N_k-1} |\mathcal{M}_{n+1}f - \mathcal{M}_n f| \right\|_{\ell^s}^u\right)^{\frac{1}{u}},$$

which together with (33) gives

$$\left\| \left(\sum_{k\geq 1} V_r\left(\mathcal{M}_n f : n \in [N_{k-1}, N_k)\right)^r\right)^{\frac{1}{r}} \right\|_{\ell^s} \lesssim \left(\sum_{k\geq 1} k^{-u(1-\rho)}\right)^{\frac{1}{u}}\|f\|_{\ell^s}.$$

We notice that the last sum is finite whenever $0 < \rho < \frac{u-1}{u}$.

To control long r-variations over the set Z_ρ, for any $\rho \in (0,1)$, we replace $\mathcal{M}_N$ by a weighted average over prime numbers

$$M_N f(x) = \frac{1}{\vartheta(N)} \sum_{p \in \mathbb{P}_N} f\big(x - P(p)\big) \log(p).$$

Since both $\mathcal{M}_N$ and M_N are averaging operators, we have

(34) $$\big\|\mathcal{M}_N f - M_N f\big\|_{\ell^s} \leq \big\|\mathcal{M}_N f\big\|_{\ell^s} + \big\|M_N f\big\|_{\ell^s} \leq 2\|f\|_{\ell^s}.$$

On the other hand, by Plancherel's Theorem

$$
\begin{aligned}
&\big\|\mathcal{M}_N f - M_N f\big\|_{\ell^2} \\
&\leq \sup_{\xi \in [0,1]} \left| \frac{1}{\Psi_N} \sum_{p \in \mathbf{P}_N} e^{2\pi i \xi P(p)} \frac{\log(p)}{\psi(p)} - \frac{1}{\vartheta(N)} \sum_{p \in \mathbb{P}_N} e^{2\pi i \xi P(p)} \log(p) \right| \cdot \|f\|_{\ell^2},
\end{aligned}
$$

which together with (27) and (32), implies that there is $\delta > 0$, such that

(35) $$\big\|\mathcal{M}_N f - M_N f\big\|_{\ell^2} \lesssim N^{-\delta} \|f\|_{\ell^2}.$$

Now, interpolating between (34) and (35), one can find $\delta_s > 0$ such that

$$\big\|\mathcal{M}_N f - M_N f\big\|_{\ell^s} \lesssim N^{-\delta_s} \|f\|_{\ell^s}.$$

Hence,

$$
\begin{aligned}
\big\|V_r\big(\mathcal{M}_N f - M_N f : N \in Z_\rho\big)\big\|_{\ell^s} &\lesssim \sum_{N \in Z_\rho} \big\|\mathcal{M}_N f - M_N f\big\|_{\ell^s} \\
&\lesssim \left(\sum_{N \in Z_\rho} N^{-\delta_s} \right) \|f\|_{\ell^s},
\end{aligned}
$$

which is bounded. Finally, by [28, Theorem C],

$$\big\|V_r\big(M_N f : N \in \mathbb{N}\big)\big\|_{\ell^s} \lesssim \frac{r}{r-2} \|f\|_{\ell^s},$$

and the theorem follows. $\qquad\square$

3.2. Variational Hilbert transform

In this section we show Theorem B for the model dynamical system and the truncated discrete Hilbert transform defined as

$$\mathcal{H}_N f(x) = \sum_{p \in \pm \mathbf{P}_N} f\big(x - P(p)\big) \frac{\log(|p|)}{p\psi(|p|)}.$$

Theorem 6. *For each $s > 1$ there is $C_s > 0$ such that for all $r > 2$, and $f \in \ell^s(\mathbb{Z})$,*

$$\big\|V_r(\mathcal{H}_n f : n \in \mathbb{N})\big\|_{\ell^s} \leq C_s \frac{r}{r - 2} \|f\|_{\ell^s}.$$

Proof. Let h_n denote the convolution kernel corresponding to $\mathcal{H}_n$. Then for each $x \in \mathbf{P}_{N_k} \setminus \mathbf{P}_{N_{k-1}}$,

$$\sum_{n=N_{k-1}}^{N_k - 1} |h_{n+1}(x) - h_n(x)| \leq \frac{\log(x)}{x\psi(x)},$$

otherwise the sum equals zero. Let us recall that the Mertens theorem says (see [15, §55])

$$(36) \qquad \sum_{p \in \mathbb{P}_N} \frac{\log(p)}{p} = \log(N) - B_3 + \mathcal{O}\Big(\exp\big(- \sqrt[14]{\log(N)}\big)\Big)$$

where B_3 is the Mertens constant. Hence, by taking in (26), $\xi = 0$, we get

$$\sum_{p \in \mathbf{P}_{N_k} \setminus \mathbf{P}_{N_{k-1}}} \frac{\log(p)}{p\psi(p)} = \sum_{p \in \mathbb{P}_{N_k} \setminus \mathbb{P}_{N_{k-1}}} \frac{\log(p)}{p} + \mathcal{O}\big(N_{k-1}^{-\delta}\big)$$

$$= \log N_k - \log N_{k-1} + \mathcal{O}\big(N_{k-1}^{-\delta}\big).$$

Therefore, by the mean value theorem,

$$\sum_{p \in \mathbf{P}_{N_k} \setminus \mathbf{P}_{N_{k-1}}} \frac{\log(p)}{p\psi(p)} \lesssim k^{-1+\rho},$$

and hence, we can estimate

$$\bigg\| \sum_{n=N_{k-1}}^{N_k - 1} |h_n - h_{n-1}| \bigg\|_{\ell^1} \lesssim \sum_{p \in \mathbf{P}_{N_k} \setminus \mathbf{P}_{N_{k-1}}} \frac{\log(p)}{p\psi(p)} \lesssim k^{\rho-1}.$$

Now, by Young's inequality, we conclude that

$$\left\| \sum_{n=N_{k-1}}^{N_k-1} |\mathcal{H}_{n+1}f - \mathcal{H}_n f| \right\|_{\ell^s} \leq \left\| \sum_{n=N_{k-1}}^{N_k-1} |h_{n+1} - h_n| \right\|_{\ell^1} \cdot \|f\|_{\ell^s}$$

(37)
$$\lesssim k^{-1+\rho}\|f\|_{\ell^s}.$$

Taking $u = \min\{2, s\}$, by monotonicity and Minkowski's inequality, we get

$$\left\| \left(\sum_{k\geq 1} V_r\big(\mathcal{H}_n f : n \in [N_{k-1}, N_k)\big)^r \right)^{\frac{1}{r}} \right\|_{\ell^s}$$

$$\leq \left\| \left(\sum_{k\geq 1} \left(\sum_{n=N_{k-1}}^{N_k-1} |\mathcal{H}_{n+1}f - \mathcal{H}_n f| \right)^u \right)^{\frac{1}{u}} \right\|_{\ell^s}$$

$$\leq \left(\sum_{k\geq 1} \left\| \sum_{n=N_{k-1}}^{N_k-1} |\mathcal{H}_{n+1}f - \mathcal{H}_n f| \right\|_{\ell^s}^u \right)^{\frac{1}{u}},$$

which together with (37), for $0 < \rho < \frac{u-1}{u}$, entails that

$$\left\| \left(\sum_{k\geq 1} V_r\big(\mathcal{H}_n f : n \in [N_{k-1}, N_k)\big)^r \right)^{\frac{1}{r}} \right\|_{\ell^s} \lesssim \left(\sum_{k\geq 1} k^{-(1-\rho)u} \right)^{\frac{1}{u}} \|f\|_{\ell^s} \lesssim \|f\|_{\ell^s}.$$

Let us now turn to estimating the long r-variations. Let $\epsilon > 0$ satisfy the hypotheses of Theorem 3. We are going to replace the operators $\mathcal{H}_N$, by

$$H_N f(x) = \sum_{p\in\pm\mathbb{P}_N} f\big(x - P(p)\big) \frac{\log(|p|)}{p}.$$

To do so, let us observe that Theorem 3 implies that

$$\sum_{p\in\mathbf{P}_N} \frac{\log(p)}{p\psi(p)} \lesssim \sum_{p\in\mathbb{P}_N} \frac{\log(p)}{p} + N^{-\epsilon} \lesssim \log N$$

where the last estimate follows from (36). Hence, by Young's inequality we obtain
(38)

$$\big\| \mathcal{H}_N f - H_N f \big\|_{\ell^s} \lesssim \left(\sum_{p\in\mathbf{P}_N} \frac{\log(p)}{p\psi(p)} + \sum_{p\in\mathbb{P}_N} \frac{\log(p)}{p} \right) \cdot \|f\|_{\ell^s} \lesssim (\log N)\|f\|_{\ell^s}.$$

For $s = 2$, by the Plancherel's theorem and Theorem 3,

$$(39) \qquad \left\| \mathcal{H}_N f - H_N f \right\|_{\ell^2} \leq \sup_{\xi \in [0,1]} \left| \sum_{p \in \pm \mathbf{P}_N} e^{2\pi i \xi P(p)} \frac{\log(|p|)}{p \psi(|p|)} \right.$$

$$\left. - \sum_{p \in \pm \mathbb{P}_N} e^{2\pi i \xi P(p)} \frac{\log(|p|)}{p} \right| \cdot \|f\|_{\ell^2}$$

$$\lesssim N^{-\epsilon} \|f\|_{\ell^2}.$$

Hence, by interpolation between (38) and (39), we obtain

$$\left\| \mathcal{H}_N f - H_N f \right\|_{\ell^s} \lesssim N^{-\delta_s} \|f\|_{\ell^s},$$

for some $\delta_s > 0$. Therefore,

$$\left\| V_r \big(\mathcal{H}_N f - H_N f : N \in Z_\rho \big) \right\|_{\ell^s} \leq \sum_{N \in Z_\rho} \left\| \mathcal{H}_N f - H_N f \right\|_{\ell^s}$$

$$\lesssim \left(\sum_{N \in Z_\rho} N^{-\delta_s} \right) \|f\|_{\ell^s},$$

which is bounded. Finally, the estimate

$$\left\| V_r (H_N f(x) : N \in \mathbb{N}) \right\|_{\ell^s} \leq C_s \frac{r}{r-2} \|f\|_{\ell^s}$$

follows by [28, Theorem C]. □

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