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Saddle hyperbolicity implies hyperbolicity for polynomial automorphisms of $\mathbb{C}^2$

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We prove that for a polynomial diffeomorphism of $\mathbb{C}^2$, uniform hyperbolicity on the set of saddle periodic points implies that saddle points are dense in the Julia set. In particular f satisfies Smale's Axiom A on $\mathbb{C}^2$.

1. Introduction

Let f be a polynomial automorphism of $\mathbb{C}^2$ with non-trivial dynamics. For such a dynamical system there are two natural definitions for the Julia set. The first one is in terms of normal families: $J = J^+ \cap J^-$ is the set of points at which which neither $(f^n)_{n\geq 0}$ nor $(f^{-n})_{n\geq 0}$ is locally equicontinuous. The second one is the closure J^* of the set of saddle periodic orbits. The inclusion $J^* \subset J$ is obvious, and whether the reverse inclusion holds is one of the major open questions in higher dimensional holomorphic dynamics.

Following Bedford and Smillie [BS1], we say that f is *hyperbolic* if J is a hyperbolic set for f. Under this assumption we have a rather satisfactory understanding of the global dynamics of f. Indeed it was shown in [BS1] that under this assumption the forward and backward Julia sets J^+ and J^- (see §2.1 below for precise definitions) are laminated by stable and unstable manifolds, that the Fatou set is the union of finitely many cycles of attracting basins, that f satisfies Smale's Axiom A on $\mathbb{C}^2$ and finally that $J = J^*$. It was shown by Buzzard and Jenkins [BJ] that f is structurally stable on $\mathbb{C}^2$. There are also tentative models for a description of the topological dynamics on J (see Ishii [I] for a survey).

On the other hand it is sometimes more natural to postulate that f is uniformly hyperbolic on J^*. One reason is that this information can be read off from the periodic points of f. This happens for instance in the study of the stability/bifurcation dichotomy for families of polynomial automorphisms [DL, BD]. The global consequences of hyperbolicity on J^* are then less easy to analyze, in particular it does not *a priori* imply a uniform laminar structure on $J^{\pm}$.

The main result of this paper is that these two notions actually coincide.

Main Theorem. *Let f be a polynomial automorphism of $\mathbb{C}^2$ with non-trivial dynamics. If f is hyperbolic on J^*, then $J = J^*$.*

In particular, if f is hyperbolic on J^*, then it is hyperbolic in the sense of [BS1].

Recall that the Jacobian $\mathrm{Jac}(f)$ of a polynomial automorphism is a non-zero constant: thus f is dissipative when $|\mathrm{Jac}(f)| < 1$ and conservative when $|\mathrm{Jac}(f)| = 1$.

This result was first announced in the dissipative case in [F], but the published proof is not correct[1], and it has remained an intriguing open problem since then. Recently, Guerini and Peters [GP] managed to establish the result under the more stringent assumption that f is *substantially dissipative*, that is $|\mathrm{Jac}(f)| < d^{-2}$, where d is the dynamical degree (see §2.1 for this notion). Observe that only *quasi-hyperbolicity* on J^* is assumed in [GP] while our approach seems to require the full strength of hyperbolicity.

The proof of the main theorem starts with the dissipative case (Section 3). We assume by contradiction that f is dissipative, hyperbolic on J^* and that $J \neq J^*$. In a first stage we show that for some $p \in J^*$, J^- intersects $W^s(p)$ along a non-trivial relatively open subset, which is an unexpected property in the dissipative setting (for instance in the substantially dissipative case, the main point of [GP] is to show that $J^- \cap W^s(p)$ is totally disconnected). The main input here is the ergodic closing lemma that we obtained in a previous work [Du2]. In a second stage we use the results of [BS6] on the properties of stable slices of J^- together with some potential-theoretic ideas to actually derive a contradiction.

The conservative case is treated in Section 4 by a perturbative argument. If f is conservative and hyperbolic on J^*, we can find a holomorphic family (f_λ) with $f_0 = f$ containing dissipative parameters, on which J^* moves under a holomorphic motion. Again we assume that $J^*(f) \neq J(f)$, and use the extension properties of the holomorphic motion of J^* obtained in [DL] to derive a contradiction from the previously established dissipative case.

[1] A first problem happens in the proof of [F, Thm. 2], which corresponds to Step 1 in our proof. Indeed in the construction of the "queer" disk V, the sequence (y_n) is contained in $W^s(J^*)$ but not a priori in $W^s_{\mathrm{loc}}(J^*)$, hence one cannot directly deduce that $G^+(y_n) \geq c$. Also, Lemma 6 is not correct: local product structure does not allow to transport whole components of $W^s(x) \cap J$ to components of $W^s(y) \cap J$ when x and y belong to the same global unstable manifold; in particular the boundedness of such a component is *not* an invariant property.

2. Preliminaries

In this section we recall some basic facts on the dynamics of polynomial automorphisms of $\mathbb{C}^2$ and hyperbolic dynamics, and establish a few preliminary results.

2.1. Vocabulary and basic facts

Let f be a polynomial diffeomorphism of $\mathbb{C}^2$ with non-trivial dynamics. This is the case exactly when the *dynamical degree*

$$d = \lim_{n\to\infty} (\deg(f^n))^{1/n}$$

is larger than 1. By [FM] there exists a polynomial change of coordinates in which f is expressed as a composition of Hénon mappings $(z, w) \mapsto (p_i(z) + a_i w, a_i z)$. We fix such coordinates from now on. The degree of f is $d = \prod \deg(p_i) \geq 2$ and the relation $\deg(f^n) = d^n$ holds so that d coincides with the dynamical degree of the original map.

In these adapted coordinates, let

$$V_R^- = \left\{(z, w) \in \mathbb{C}^2,\ |w| \geq R,\ |z| < |w|\right\}$$
$$\text{and} \quad V_R^+ = \left\{(z, w) \in \mathbb{C}^2,\ |z| \geq R,\ |w| < |z|\right\}$$

and fix $R_0 > 0$ so large that for $R \geq R_0$ $f(V_R^+) \subset V_{2R}^+$ and $f^{-1}(V_R^-) \subset V_{2R}^-$. Hence the points of V_R^+ (resp. V_R^-) escape under forward (resp. backward) iteration. We denote by $\mathbb{B}$ the bidisk $D(0, R_0)^2$. The non-wandering set of f is contained in $\mathbb{B}$.

An object (subset, current, or subvariety) in $\mathbb{B}$ is said to be *vertical* (resp. *horizontal*) if its closure in $\overline{\mathbb{B}}$ is disjoint from $\{|z| = R_0\}$ (resp. $\{|w| = R_0\}$). A vertical subvariety has a *degree*, which is the number of intersection points with a generic horizontal line.

Here are some standard facts and notation (see e.g. [BS1, BS2, BLS]):

- $K^\pm$ is the set of points with bounded forward orbits under $f^{\pm 1}$ and $K = K^+ \cap K^-$. Note that K^+ is vertical in $\mathbb{B}$ and $f(\mathbb{B} \cap K^+) \subset K^+$. Similarly, K^- is horizontal and $f^{-1}(\mathbb{B} \cap K^-) \subset K^-$.
- The complement of K^+ is denoted by U^+ and the complement of K^- is U^-.
- $J^\pm = \partial K^\pm$ are the forward and backward Julia sets. If f is dissipative then $K^- = J^-$.

- $J = J^+ \cap J^-$ is the Julia set.
- $J^* \subset J$ is the closure of the set of saddle periodic points. It is also the support of the unique measure of maximal entropy $\log d$.

The *dynamical Green functions* $G^\pm$ are defined by

$$G^\pm(z,w) = \lim_{n\to+\infty} d^{-n} \log^+ \left\| f^{\pm n}(z,w) \right\|$$

(where $\log^+(x) = \max(\log(x), 0)$). These are non-negative continuous plurisubharmonic functions on $\mathbb{C}^2$, such that $K^\pm = \{G^\pm = 0\}$ and $G^\pm$ is pluriharmonic on $U^{+/-} := \left\{G^{+/-} > 0\right\}$. We let $T^\pm = dd^c G^\pm$. The maximum principle implies that $\mathrm{Supp}(T^\pm) = J^\pm$.

The *restriction* $T^+|_D$ of T^+ to a complex submanifold D is a positive measure on D locally defined by $\Delta(G^+|_D)$ and since G^+ is continuous this measure coincides[2] with the wedge product $T^+ \wedge [D]$. A useful remark is that if x belongs to K^+ and $D \subset \mathbb{C}^2$ is a holomorphic disk through x along which G^+ is harmonic, then $D \subset K^+$ and $(f^n|_\Delta)_{n\geq 1}$ is a normal family.

If p is a saddle periodic point or more generally if it belongs to a hyperbolic saddle set, it admits stable and unstable manifolds $W^{s/u}(p)$. Each of them is an immersed Riemann surface biholomorphic to $\mathbb{C}$ and by [BS2, FS] $W^s(p)$ (resp. $W^u(p)$) is dense in J^+ (resp. J^-). A key point in the present paper is to analyse the topological properties of sets of the form $K^- \cap W^s(p)$ or $K^+ \cap W^u(p)$. Following [DL], we define the *intrinsic topology* to be the topology induced on a stable (resp. unstable) manifold by the biholomorphism $W^s \simeq \mathbb{C}$, and the corresponding concepts of boundary, interior, etc. will be labelled with the subscript i: ∂_{i}, $\mathrm{Int}_{\mathrm{i}}$, etc.

The following basic lemma will be used several times.

Lemma 2.1 ([DL, Lemma 5.1]). *Let p be a saddle periodic point. Then the boundary of $W^s(p) \cap J^-$ relative to the intrinsic topology in $W^s(p)$ is contained in J^*.*

We denote by $W^s_{\mathbb{B}}(p)$ the connected component of $W^s(p) \cap \mathbb{B}$ containing p (and accordingly for W^u). Likewise, $W^s_\delta(p)$ is the connected component of $W^s(p) \cap B(p,\delta)$ containing p, and $W^s_{\mathrm{loc}}(p)$ denotes an unspecified open neighborhood of p in $W^s(p)$.

By [BLS], the currents $T^\pm$ have geometric structure, related to the decomposition of $J^\pm$ into stable and unstable manifolds. By *lamination* by

[2]It is standard to define $d^c = \frac{i}{2\pi}(\overline{\partial} - \partial)$. Accordingly, Δ here is $1/2\pi$ times the ordinary Laplacian.

Riemann surfaces we mean a closed subset $\mathcal{L}$ of some open set $\Omega \subset \mathbb{C}^2$ such that every $p \in \mathcal{L}$ admits a neighborhood B biholomorphic to a bidisk, such that in the corresponding coordinates, a neighborhood of p in $\mathcal{L}$ is a union of disjoint graphs (that is, a holomorphic motion) over the first coordinate in B. A positive current S is *uniformly laminar* if there is a lamination of $\mathrm{Supp}(S)$ by Riemann surfaces and in the corresponding local coordinates S is locally expressed as $\int [\Delta_a]\, d\nu(a)$. These disks will be said *subordinate* to S.

A holomorphic disk D is *subordinate* to T^+ if there exists a non-zero uniformly laminar current $S \leq T^+$ such that D is subordinate to S. By [Du1, Prop. 2.3], if p is any saddle point, then any relatively compact disk $D \subset W^s(p)$ is subordinate to T^+.

2.2. Stable (dis)connectivity

It was shown in [BS6] that the connectivity properties of sets of the form $K^+ \cap W^u(p)$ (resp. $K^- \cap W^s(p)$) carry deep information on the geometry of the Julia set. We say that f is *stably connected* if $U^+ \cap W^s(p)$ is simply connected for some (and then any) saddle point p, and *stably disconnected* otherwise. Equivalently, f is stably disconnected if for some saddle point p, $W^s(p) \cap K^-$ admits a compact component relative to the intrinsic topology. This actually implies the stronger property that most components of $W^s(p) \cap K^-$ are points (see the proof of Lemma 3.2 below for more details).

By [BS6, Cor. 7.4], a dissipative polynomial automorphism is always stably disconnected. It was observed in [Du1] that this implies a strong non-extremality property for the current $T^+|_{\mathbb{B}}$: there exists a decomposition $T^+|_{\mathbb{B}} = \sum_{k=1}^{\infty} T_k^+$ where T_k^+ is an average of integration currents over a family of disjoint vertical disks of degree k (see [Du1, Thm. 2.4]).

Lemma 2.2. *Let f be dissipative and hyperbolic on J^* and let $q \in J^*$. Then q belongs to the support of $T^+|_{W^u(q)}$ and for $(T^+|_{W^u(q)})$-a.e. q' near q, $W^s_{\mathbb{B}}(q')$ is a vertical manifold of finite degree in $\mathbb{B}$.*

Proof. The first assertion easily follows from the fact that $(f^n)_{n\geq 0}$ cannot be a normal family on $W^u_{\mathrm{loc}}(q)$ (see [BLS, Lemma 2.8]). The second one is a consequence of [Du1, Thm. 2.4]. Indeed as observed above $T^+|_{\mathbb{B}}$ admits a decomposition $T^+|_{\mathbb{B}} = \sum_{k=1}^{\infty} T_k^+$ where T_k^+ is made of vertical disks of degree k. Thus $T^+|_{W^u_{\mathrm{loc}}(q)} = T^+ \wedge [W^u_{\mathrm{loc}}(q)] = \sum_k T_k^+ \wedge [W^u_{\mathrm{loc}}(q)]$. Now if Γ is a leaf of some T_k^+ intersecting $W^u_{\mathrm{loc}}(q)$ at q', then since $W^u_{\mathrm{loc}}(q)$ is subordinate to T^-, q' belongs to J^* and Γ is a manifold through q' along which forward iterates are bounded, hence $\Gamma = W^s_{\mathbb{B}}(q')$. $\square$

2.3. Hyperbolicity and local product structure

Let us recall some generalities from hyperbolic dynamics, specialized to our situation. A *(saddle) hyperbolic set* for f is a compact invariant set $\Lambda \subset \mathbb{C}^2$ such that $T\mathbb{C}^2|_\Lambda$ admits a hyperbolic splitting, i.e. $T\mathbb{C}^2|_\Lambda = E^s \oplus E^u$, where E^s and E^u are continuous line bundles such that E^s (resp. E^u) is uniformly contracted (resp. expanded) by df. Then there exists $\delta_1 > 0$ such that $W^s_{\delta_1}(\Lambda) := \bigcup_{p\in\Lambda} W^s_{\delta_1}(x)$ and $W^u_{\delta_1}(\Lambda) := \bigcup_{p\in\Lambda} W^u_{\delta_1}(x)$ form laminations in the δ_1-neighborhood of Λ.

A hyperbolic set is *locally maximal* if there exists an open neighborhood $\mathcal{N}$ of Λ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^{-n}(\mathcal{N})$. It has *local product structure* if there exists $0 < \delta_2 \leq \delta_1$ such that if $p, q \in \Lambda$ are such that $d(p,q) < \delta_2$ then $W^s_{\delta_1}(p) \cap W^u_{\delta_1}(q)$ consists of exactly one point belonging to Λ. It turns out that these two properties are equivalent (see [Y, §4.1]).

We will use the following consequence of the shadowing lemma.

Proposition 2.3. *Let Λ be a compact locally maximal hyperbolic set for a polynomial diffeomorphism f of $\mathbb{C}^2$. Then there exist positive constants η, α and A such that for every $n \geq 0$; if x is such that $\{x, \ldots, f^n(x)\} \subset \Lambda_\eta$ then there exists $y \in \Lambda$ such that x is $Ae^{-\alpha n}$-close to the local stable manifold of y.*

A similar result holds for negative iterates: if $\{f^{-n}(x), \ldots, x\} \subset \Lambda_\eta$ then there exists $z \in \Lambda$ such that x is $Ae^{-\alpha n}$-close to the local unstable manifold of z.

The following corollary is well-known.

Corollary 2.4. *If Λ is a compact locally maximal hyperbolic set, then*

$$W^s(\Lambda) := \left\{ x \in \mathbb{C}^2, f^n(x) \underset{n\to\infty}{\longrightarrow} \Lambda \right\} = \bigcup_{p\in\Lambda} W^s(p)$$

and similarly

$$W^u(\Lambda) = \bigcup_{p\in\Lambda} W^u(p).$$

Note however that $W^s(\Lambda)$, being an increasing union of laminations, doesn't need to have a lamination structure (this is already false when Λ is a hyperbolic fixed point).

Proof of Proposition 2.3 (sketch). This is very classical. Given an orbit segment $\{x, \ldots, f^n(x)\}$ as in the statement of the proposition, let $y^{(0)}$ (resp.

$y^{(n)}$) be a point in Λ such that $d(x, y^{(0)}) < \eta$ (resp. $d(x, y^{(n)}) < \eta$). Then define a η-pseudo-orbit $(y^{(k)})_{k\in\mathbb{Z}}$ as follows

$$y^{(k)} = \begin{cases} f^k\big(y^{(0)}\big) & \text{for } k < 0; \\ f^k(x) & \text{for } 0 \leq k \leq n; \\ f^{k-n}\big(y^{(n)}\big) & \text{for } k > n. \end{cases}$$

Then if η is small enough by local maximality and the shadowing lemma there exists a unique $y \in \Lambda$ such that for every $k \in \mathbb{Z}$, $d\big(f^k(y), y^{(k)}\big) < C\eta$ (where C is some constant depending on (f, Λ), see [Y, §4.1]). In particular for $0 \leq k \leq n$ we have $d\big(f^k(x), f^k(y)\big) < C\eta$ and it follows from standard graph transform estimates that $d(x, W^s_\delta(y)) \leq Ae^{-\alpha n}$. □

The next result is a simple application of the techniques of [BLS].

Proposition 2.5. *If J^* is hyperbolic then it has local product structure. Furthermore global stable and unstable manifolds intersect only in J^*:*

$$W^s(J^*) \cap W^u(J^*) = J^*.$$

Proof. Hyperbolicity implies that for some $\delta > 0$, if p and q are close enough, $W^s_\delta(p) \cap W^u_\delta(q)$ consists of a single point r. We have to show that $r \in J^*$. Indeed, $W^s_\delta(p)$ (resp. $W^u_\delta(q)$) is a disk subordinate to T^+ (resp. T^-) so there exists a non-trivial uniformly laminar current $S^+ \leq T^+$ (resp. $S^- \leq T^-$) with $W^s_\delta(p)$ (resp $W^u_\delta(q)$) as a leaf. By [BLS, Lem. 8.2], S^+ and S^- have continuous potentials, so the wedge product $S^+ \wedge S^-$ is well defined, and geometric intersection theory of uniformly laminar currents [BLS, Lem. 8.3] implies that $r \in \mathrm{Supp}(S^+ \wedge S^-)$. Since $S^+ \wedge S^- \leq T^+ \wedge T^-$ we conclude that $r \in J^*$.

The proof of the second assertion is similar. By local product structure,

$$W^s(J^*) = \bigcup_{n\geq 0} f^{-n}(W^s_\delta(J^*)),$$

hence if $r \in W^s(J^*)$, there exists $p \in J^*$ such that $r \in W^s(p)$ so r belongs to a disk subordinate to T^+, and likewise $r \in W^u(q)$ so it belongs to a disk subordinate to T^+. Observe that these two disks are distinct: indeed otherwise we would have $W^s(p) = W^u(q)$ which is impossible because $W^s(p) \cap W^u(q)$ is contained in K which is bounded in $\mathbb{C}^2$. So r is an isolated intersection between $W^s(p)$ and $W^u(q)$ for the leafwise topology. If this intersection is transverse, we argue as above to conclude that r belongs to J^*. If it is a tangency, then by [BLS, Lem. 6.4] for $q' \in J^*$ close to q, we get transverse intersections between $W^s(p)$ and $W^u(q')$ close to r and conclude as in the transverse case. □

2.4. Stability

A theory of stability and bifurcations for polynomial automorphisms of $\mathbb{C}^2$ was developed in [DL], centered on the notion of *weak J^*-stability.*

A *branched holomorphic motion* over a complex manifold Λ in $\mathbb{C}^2$ is a family of holomorphic graphs over Λ in $\Lambda \times \mathbb{C}^2$. It is a holomorphic motion (i.e. an unbranched branched holomorphic motion!) when these graphs are disjoint. A holomorphic family $(f_\lambda)_{\lambda\in\Lambda}$ of polynomial automorphisms of dynamical degree d is *weakly J^*-stable* if the sets $J^*(f_\lambda)$ move under a branched holomorphic motion, and *J^*-stable* if this motion is unbranched. Note that if f_{λ_0} is uniformly hyperbolic on $J^*(f_{\lambda_0})$, then it is J^*-stable near λ_0 in any holomorphic family containing f_{λ_0}.

A number of properties of weakly J^*-stable families are established in [DL], including extension properties of the branched holomorphic motion of J^* to K (and more generally to $J^+ \cup J^-$), that will be used in Section 4. These properties hold under the standing assumption that the family (f_λ) is *substantial* [3]: this means that either all members of the family are dissipative, or that no relation of a certain form between multipliers of periodic points persistently holds in the parameter space Λ. Without entering into the details, let us just note that by [BHI, Thm. 1.4] any open subset of the family of all polynomial automorphisms of dynamical degree d is substantial.

3. Proof of the main theorem: the dissipative case

The proof is by contradiction so assume that f is a dissipative polynomial diffeomorphism of $\mathbb{C}^2$, that is uniformly hyperbolic on J^*, and that $J^* \subsetneq J$.

Step 1. *There exists $p \in J^*$ and a holomorphic disk $\Delta \subset W^s(p)$ such that $G^-|_\Delta \equiv 0$.*

The purpose of the remaining steps 2 and 3 will be to show that such a "queer" component of $W^s(p) \cap J^-$ actually does not exist.

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[3] There is an unfortunate terminological conflict here: this should not be confused with the notion of substantial dissipativity mentioned in the introduction.

Proof. We first claim that it is enough to show that there exists $p \in J^*$ and $q \in W^s(p)$ such that $q \in J \setminus J^*$. Indeed observe that $W^s(p) \cap J = W^s(p) \cap J^- = W^s(p) \cap K^-$. By Lemma 2.1 we have that $\partial_{\mathrm{i}}(W^s(p) \cap J^-) \subset J^*$. Hence if q belongs to $W^s(p) \cap (J \setminus J^*)$, it belongs to the intrinsic interior $\mathrm{Int}_{\mathrm{i}}(W^s(p) \cap J^-)$, hence $G^- \equiv 0$ in a neighborhood of q in $W^s(p)$.

Let now $x \in J \setminus J^*$. By Corollary 2.4, if $\omega(x) \subset J^*$ then $x \in W^s(J^*)$ and if $\alpha(x) \subset J^*$ then $x \in W^u(J^*)$. Since $W^u(J^*) \cap W^s(J^*) = J^*$, we infer that either $\omega(x) \not\subset J^*$ or $\alpha(x) \not\subset J^*$. In either case we will show that both $W^s(J^*) \cap (J \setminus J^*)$ and $W^u(J^*) \cap (J \setminus J^*)$ are non-empty. Thus by symmetry it is enough to deal with the case where $\omega(x) \not\subset J^*$.

Choose η so small that Proposition 2.3 holds for J^* and $\omega(x)$ is not contained in $\overline{\mathcal{N}}$, where $\mathcal{N} := (J^*)_\eta$ is the η-neighborhood of J^*.

Consider the sequence of Cesarò averages $\nu_n = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)}$. By the ergodic closing lemma of [Du2], every cluster value of the sequence (ν_n) is supported on J^*. It follows that the asymptotic proportion of iterates of x belonging to $\mathcal{N}$ tends to 1, i.e.

$$\frac{1}{n} \# \left\{ 0 \leq k \leq n-1,\ f^k(x) \in \mathcal{N} \right\} \underset{n\to\infty}{\longrightarrow} 1.$$

Indeed if a positive proportion of iterates stayed outside $\mathcal{N}$, any cluster limit of ν_n would have to give positive mass to $\mathcal{N}^{\complement}$.

We thus infer that there are arbitrary long strings $\{x_i, \ldots, x_{i+n}\}$ in the orbit of x that are entirely contained in $\mathcal{N}$. Indeed, if on the contrary the length of such a string were uniformly bounded by some n_0, then the density of iterates outside $\mathcal{N}$ would be bounded below by $1/(n_0+1)$. Therefore for every n there exists i_n such that $\{x_{i_n}, \ldots, x_{i_n+n}\} \subset \mathcal{N}$. Choose i_n to be minimal with this property. Since $\omega(x) \not\subset \mathcal{N}$, there exists $j > i$ such that $x_j \notin \mathcal{N}$. So finally for every n we can find $i_n < j_n$ such that $j_n - i_n \geq n$, $\{x_{i_n}, \ldots, x_{j_n}\} \subset \mathcal{N}$, $x_{i_n-1} \notin \mathcal{N}$ and $x_{j_n+1} \notin \mathcal{N}$.

Let p (resp. p') be a cluster value of (x_{i_n-1}) (resp. (x_{j_n+1})). The points p and p' belong to J because x does, but not to J^* because they lie outside $\mathcal{N}$. It follows from Proposition 2.3 that $p \in W^s(q)$ for some $q \in J^*$ and $p' \in W^u(q')$ for some $q' \in J^*$. The proof is complete. □

Remark 3.1. If f is substantially dissipative i.e. $|\mathrm{Jac}(f)| < \deg(f)^{-2}$, then the contradiction readily follows from this first step. Indeed Wiman's theorem together with uniform hyperbolicity imply that the vertical degree of components of stable manifolds in some large bidisk $\mathbb{B}$ is uniformly bounded (see [GP, Prop. 4.2] or [LP, Lem. 5.1]), and it follows that $J^- \cap W^s(x)$ is totally disconnected for every x (see [Du1, Thm. 2.10] or [GP, Thm. 4.3]), which contradicts the conclusion of Step 1.

In the second and third steps we do not use the assumption that $J \setminus J^* \neq \emptyset$.

Step 2. *For every $p \in J^*$, if Ω is a component of* $\mathrm{Int}_{\mathrm{i}}(W^s(p) \cap J^-)$*, then Ω is unbounded for the leafwise topology.*

Proof. Note first that by the maximum principle, any component of $\mathrm{Int}_{\mathrm{i}}(W^s(p) \cap J^-) = \mathrm{Int}_{\mathrm{i}}(W^s(p) \cap K^-)$ is simply connected, so Ω is a topological disk. Assume by contradiction that Ω is bounded for the leafwise topology. Then iterating forward a few times if needed, we can suppose that Ω is entirely contained in a local product structure box.

More precisely for small $\delta > 0$, we can fix holomorphic local coordinates (z, w) near p in which $p = (0, 0)$, $W^s_{2\delta}(p) = \{z = 0\}$ and $W^u_{2\delta}(p) = \{w = 0\}$, and assume $\overline{\Omega}$ is contained in $W^s_\delta(p)$. Note that by Lemma 2.1, $\partial_{\mathrm{i}}\Omega \subset J^*$. We can assume that for every $q \in W^s_\delta(p) \cap J^*$, $W^u_{2\delta}(q)$ contains a graph over the disk $D(0, \delta)$ in the first coordinate, with slope bounded by 1/2. Then if $|z_0| \leq \delta$, the holonomy h^u_{0,z_0} along local unstable leaves is well defined on $W^s_\delta(p) \cap J^*$ and maps $W^s_\delta(p) \cap J^*$ into $\{z = z_0\} \cap J^-$. This holonomy is a holomorphic motion so by Slodkowski's theorem [S] it extends to a holomorphic motion of $W^s_\delta(p)$. In particular the motion of $\partial_{\mathrm{i}}\Omega$ extends to a motion of Ω and it makes sense to speak about $h^u_{0,z_0}(\Omega)$. This is an open subset of $\{z = z_0\}$, which is topologically a disk and whose boundary is contained in J^-. Thus for every $n \geq 0$, $f^{-n}(\partial(h^u_{0,z_0}(\Omega)))$ in contained in $\mathbb{B}$ and by the maximum principle, the same holds for $f^{-n}(h^u_{0,z_0}(\Omega))$.

Finally, $\mathcal{O} := \bigcup_{|z_0|<\delta} h^u_{0,z_0}(\Omega)$ is an open set whose negative iterates remain in $\mathbb{B}$, hence it is contained in the Fatou set of f^{-1}. But since f is dissipative, this Fatou set is empty, which is the desired contradiction. □

Step 3. *The unstable holonomy preserves the decomposition*

$$W^s(p) = (W^s(p) \cap J^-) \sqcup (W^s(p) \cap U^-).$$

To make this statement precise, observe that for every $p \in J^*$, the components of the complement of $\partial_{\mathrm{i}}(W^s(p) \cap J^-)$ in $W^s(p)$ can be divided into two types: components of $\mathrm{Int}_{\mathrm{i}}(W^s(p) \cap J^-)$ and components of $W^s(p) \cap U^-$ (note that since U^- is open in $\mathbb{C}^2$, $W^s(p) \cap U^-$ is open for the intrinsic topology as well). Consider as above local coordinates (z, w) near p in which

$p = (0,0)$, $W^s_{2\delta}(p) = \{z = 0\}$ and $W^u_{2\delta}(p) = \{w = 0\}$. The unstable holonomy h^u_{0,z_0} is initially only defined for points of $W^s_\delta(p) \cap J^* = \partial_{\mathrm{i}}(W^s_\delta(p) \cap J^-)$, however by Słodkowski's theorem it can be extended to $W^s_\delta(p)$. By Step 2, components of $\mathrm{Int}_{\mathrm{i}}(W^s(p) \cap J^-)$ are leafwise unbounded so they cannot be contained in $W^s_{2\delta}(p)$. Obviously, the same holds for components of $W^s(p) \cap U^-$.

If q belongs to $J^* \cap W^u_\delta(p)$, the extended holonomy $h^u_{p,q}$ defines a homeomorphism $W^s_\delta(p) \to h^u_{p,q}(W^s_\delta(p))$. By local product structure this homeomorphism preserves J^* so any component of $W^s_\delta(p) \setminus J^*$ is mapped onto a component of $h^u_{p,q}(W^s_\delta(p)) \setminus J^*)$, which is itself contained in a component of $W^s(q) \setminus J^*$. The claim of Step 3 is that the extended holonomy $h^u_{p,q}$ preserves the type of components.

Since it doesn't make sense to transport a whole leafwise unbounded component by unstable holonomy, to prove this assertion we need to find a criterion that recognizes the type of a component just from local topological properties near a point of its boundary. As already said the maximum principle implies that any component of $\mathrm{Int}_{\mathrm{i}}(W^s(p) \cap J^-)$ is simply connected. Thus Step 3 follows from:

Lemma 3.2. *If Ω is a component of $W^s(p) \cap U^-$, then Ω is not simply connected near any point of $\partial\Omega$, more precisely: if $q \in \partial\Omega$ and N is any neighborhood of q, there is a loop in $N \cap \Omega$, homotopic to a point in N, and enclosing a component of $W^s(p) \cap J^-$*

Proof. Since f is dissipative by [BS6, Cor. 7.4] it is stably disconnected. It follows that almost every unstable component of K^+ is a point (see [BS6, Thm. 7.1] and also [Du1, Thm. 2.10]). More specifically, if μ is the unique measure of maximal entropy, then for μ-a.e. x, the measure $T^-|_{W^s(x)}$ (which is locally given by the wedge product $T^- \wedge [W^s(x)]$) gives full mass to the point components of $J^- \cap W^s(x)$. Obviously by Lemma 2.1 every such point component belongs to J^* so we can transport it to nearby stable manifolds by unstable holonomy. In addition, the measure $T^-|_{W^s(x)}$ is holonomy invariant (see [BS1, Thm. 6.5] or [BLS, Thm. 4.5]) so if x is such that $T^- \wedge [W^s_\delta(x)]$ gives full mass to point components, then the same holds for nearby x'. Thus we conclude that this property holds for *every* $p \in J^*$: $T^-|_{W^s(p)}$ gives full mass to the point components of $J^- \cap W^s(p)$.

Lemma 3.3. *Let $p \in J^*$ and Ω be a component of $W^s(p) \cap U^-$ such that Ω is locally simply connected near some $q \in \partial\Omega$. Then $\partial\Omega$ has positive $(T^-|_{W^s(p)})$-measure.*

This proves Lemma 3.2. Indeed, assuming that Ω is locally simply connected near some $q \in \partial\Omega$, Lemma 3.3 asserts that $T^-|_{W^s(p)}$ carries positive mass on a non-trivial continuum so f cannot be stably disconnected. On the other hand f must be stably disconnected because it is dissipative, and we reach a contradiction. □

The idea of Lemma 3.3 is as follows: every neighborhood of q in $\partial\Omega$ has positive harmonic measure when viewed from Ω. But the harmonic measure viewed from Ω is absolutely continuous with respect to $T^- \wedge [W^s(p)]$, hence the result. The formalization of this argument requires some elementary potential theory, for which we refer the reader to Doob's classical monograph[4] [Do].

In particular we shall use the formalism of *sweeping* (or *balayage*). Let D be a smoothly bounded domain in $\mathbb{C}$, A a non-polar compact subset of D and ν a positive measure on D. The swept measure $\rho_{\nu,D,A}$ of ν on A is the distribution on A of the exit point of the Brownian motion in $D \setminus A$ whose starting point is distributed according to ν. In particular its mass is lower than that of ν since a positive proportion of Brownian paths escape from ∂D. If $G_{\nu,D}$ is the Green potential of ν in D, that is the unique negative subharmonic function on D such that $G_{\nu,D}|_{\partial D} = 0$ and $\Delta G_{\nu,D} = \nu$, then the swept measure of ν on A is $\Delta R_{\nu,D,A}$, where

$$(1)\quad R_{\nu,D,A}(z) = \sup\{u(z),\ u \leq 0 \text{ subharmonic on } D \text{ and } u \leq G_{\nu,D} \text{ on } A\}$$

(see Sections 1.III.4, 1.X and 2.IX.14 in [Do]). If ν and ν' have their supports disjoint from A, then the corresponding swept measures are mutually absolutely continuous (as follows from instance from Theorem 1.X.2 in [Do]).

Proof. We first claim that we can shift q slightly so that the assumptions of the lemma hold and in addition $W^s_{\mathbb{B}}(q')$ is of bounded vertical degree. Indeed q belongs to J^* and for $q' \in W^u_\delta(q) \cap J^*$, there is a component of $W^s_\delta(q') \setminus J^*$ corresponding to Ω under unstable holonomy, which is locally simply connected near q'. Since G^- is continuous, if q' is close enough to q, it takes positive values on that component, so we infer that the property that Ω is a component of U^- is open. Now by Lemma 2.2, for $(T^+|_{W^u_\delta})$-a.e. q', $W^s_{\mathbb{B}}(q')$ is a vertical manifold in $\mathbb{B}$ of finite degree which establishes our claim. Without loss of generality rename q' into q. For every $g_0 < \min_{\partial\mathbb{B}} G^-$, the component of $\{G^- < g_0\}$ containing q in $W^s(q)$ is relatively compact for the intrinsic topology. We fix such a g_0 which is not a critical value of G^-

[4] Note that Doob works with superharmonic functions so all inequalities have to be reversed.

and let D be the corresponding component, which is a smoothly bounded topological disk. From now on we work exclusively in D.

By assumption there is a neighborhood N of q in D and a component U of $\{G^- > 0\} \cap N$ that is simply connected. We have to show that $\partial U \cap N$ has positive mass relative to ΔG^-. We choose N to be closed so that $\partial U \cap N$ is compact. First, observe that for every $z_0 \in U \cap N$, the probability that the Brownian motion issued from z_0 hits $\partial U \cap N$ before leaving U is positive. Therefore the swept measure $\rho_{\delta_{z_0}, D, \partial U \cap N}$ has positive mass and to prove the lemma it is enough to show that is absolutely continuous with respect to ΔG^-. Recall that the measure class of the swept measure does not depend on the starting point so we can replace δ_{z_0} by an arbitrary positive measure on $D \setminus (D \cap K^-)$. Let $0 < g_1 < g_0$ and $\mu_{g_1} := \Delta(\max(G^-, g_1))$ be the natural measure induced by G^- on the level set $\{G^- = g_1\}$. We choose μ_{g_1} for the initial distribution of Brownian motion. Since $G^- \equiv g_0$ on ∂D, the Green function $G_{\Delta G^-, D}$ of the restriction of ΔG^- to D is equal to $G^- - g_0$, and likewise

$$G_{\mu_{g_1}, D} = \max(G^-, g_1) - g_0.$$

Thus from (1) we get that

$$\begin{aligned} R_{\mu_{g_1}, D, K^- \cap D} &= \sup\left\{u(z),\ u \text{ s.h.} \leq 0 \text{ on } D \text{ and } u \leq G_{\mu_{g_1}, D} \text{ on } K^- \cap D\right\} \\ &= \sup\left\{u(z),\ u \text{ s.h.} \leq 0 \text{ on } D \text{ and } u \leq g_1 - g_0 \text{ on } K^- \cap D\right\} \\ &= |g_1 - g_0| \, \frac{G^- - g_0}{g_0} \end{aligned}$$

and finally

$$\rho_{\mu_{g_1}, D, K^- \cap D} = \frac{g_0 - g_1}{g_0} \Delta G^-.$$

The proof is complete. □

Step 4. *Conclusion.*

We just have to assemble the three previous steps. Assume as before by contradiction that f is dissipative, uniformly hyperbolic on J^* and $J^* \subsetneq J$. Then by Step 1 there exists $p \in J^*$ and a "queer" component Ω of $W^s(p) \setminus J^*$ along which $G^- \equiv 0$. Pick $q \in \partial\Omega$. By Lemma 2.1, $q \in J^*$ so we can follow $\Omega \cap W^s_\delta(q)$ using the holonomy along local unstable manifolds. Then for $q' \in W^u_{\rm loc}(q)$ near q, the holonomy image $h_{q,q'}(\Omega \cap W^s_\delta(q))$ is contained in a queer component of $W^s(q') \setminus J^*$, which must be leafwise unbounded by Step 2. On the other hand by Lemma 2.2, for generic q' in $W^u_{\rm loc}(q)$ (relative to the transverse measure $T^+|_{W^u_{\mathrm{loc}}(q)}$) $W^s_{\mathbb{B}}(q')$ is of bounded degree, in particular any component of $K^- \cap W^s_{\mathbb{B}}(q')$ is leafwise bounded. This contradiction finishes the proof. □

4. Proof of the main theorem: the conservative case

Again the proof is by contradiction, so assume that f is a conservative polynomial automorphism of $\mathbb{C}^2$ such that J^* is a hyperbolic set and $J^* \subsetneq J$. We will use a perturbative argument and the dissipative case of the theorem to reach a contradiction.

Assume that f is written as a product of Hénon mappings $f = h_1 \circ \cdots \circ h_k$ and let $(f_\lambda)_{\lambda \in B}$ be a parameterization of a neighborhood of f in the space of such products, that is, the space of coefficients of the h_i, and such that $f_0 = f$. We can assume that B is a ball in $\mathbb{C}^N$ for some N. Since $\lambda \mapsto \mathrm{Jac}(f_\lambda)$ is an open map, there exist parameters arbitrary close to 0 for which f_λ is dissipative. As already said, by [BHI, Thm. 1.4] there is no persistent relation between multipliers of periodic orbits so the family is substantial in the sense of [DL].

Since f_0 is hyperbolic on $J^*(f_0)$ the family (f_λ) is J^*-stable in a neighborhood of the origin, that is, $J^*(f_\lambda)$ moves under a holomorphic motion. Reducing the parameter space we can assume that J^* is hyperbolic throughout B. Pick a point $p = p(0) \in J(f_0) \setminus J^*(f_0)$. It was shown in [DL, Thm. 5.12] that in a (weakly) J^*-stable family, the motion of J^* extends to a branched holomorphic motion of K. Thus there exists a holomorphic continuation $p(\lambda)$ of $p(0)$ such that for every $\lambda \in B$, $p(\lambda)$ belongs to $K(f_\lambda)$. Furthermore for every $\lambda \in B$, $p(\lambda)$ is disjoint from $J^*(f_\lambda)$. Indeed if for some $\lambda_0 \in B$ we had $p(\lambda_0) \in J^*(f_{\lambda_0})$, then by [DL, Lem 4.10] $p(\lambda)$ would have to coincide throughout the family (f_λ) with the natural continuation of $p(\lambda_0)$ as a point of the hyperbolic set J^*, which is not the case since $p(0) \notin J^*(f_0)$.

Let now $\lambda_1 \in B$ be such that f_{λ_1} is dissipative. Then by the first part of the proof $J(f_{\lambda_1}) = J^*(f_{\lambda_1})$, and $K(f_{\lambda_1}) \setminus J(f_{\lambda_1})$ is non-empty since it contains $p(\lambda_1)$. For a dissipative hyperbolic map

$$K \setminus J = (K^+ \cap J^-) \setminus (J^+ \cap J^-) = \mathrm{Int}(K^+) \cap J^-,$$

so we deduce that $\mathrm{Int}(K^+(f_{\lambda_1}))$ is non-empty. By [BS1], $\mathrm{Int}(K^+(f_{\lambda_1}))$ is a finite union of attracting basins of periodic sinks, therefore f_{λ_1} admits an attracting periodic point. On the other hand by [DL, Thm. 4.2], periodic points stay of constant type in a J^*-stable family (this holds even in the presence of conservative maps, provided the family is substantial), so f_0

must have an attracting orbit, which is contradictory since it is conservative. The proof is complete. ☐

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