An action of $\mathfrak{gl}(1|1)$ on odd annular Khovanov homology

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We define an annular version of odd Khovanov homology and prove that it carries an action of the Lie superalgebra $\mathfrak{gl}(1|1)$ which is preserved under annular Reidemeister moves.

1. Introduction

In [14], Khovanov defined a link invariant *categorifying* the Jones polynomial. That is, he constructed a bigraded homology theory of links whose graded Euler characteristic is the Jones polynomial. Recalling [26, 32] that the Jones polynomial has an interpretation involving representations of the quantum group $U_q(\mathfrak{sl}(2))$, it is perhaps not surprising that an *annular* version of Khovanov homology defined in [1] and further studied in [27] (see also [11]) carries an action of the Lie algebra $\mathfrak{sl}(2)$ [10, 24]. Moreover, although annular Khovanov homology is *not* a link invariant (it is well-defined only up to isotopy in the complement of a standardly-imbedded unknot in S^3), the algebraic features of the Khovanov complex that have yielded the most geometric/topological information (e.g. [22, 25]) have coincided with

key $\mathfrak{sl}(2)$–representation-theoretic features of the annular Khovanov complex (cf. [9, Prop. 1]).

In [17], Ozsváth-Szabó-Rasmussen defined an *odd* version of Khovanov homology. When taken with $\mathbb{Z}_2$ coefficients, their construction agrees with Khovanov's original construction. Hence, odd Khovanov homology can be viewed as an alternative integral lift of Khovanov homology.

Our aim in the present work is to define a natural *annular* version of *odd* Khovanov homology (Subsection 3.1) and show that it carries a well-defined action, not of the Lie algebra $\mathfrak{sl}(2)$, but of the Lie superalgebra $\mathfrak{gl}(1|1)$ (Theorems 1 and 2). We will define this $\mathfrak{gl}(1|1)$ action explicitly on chain level, using two different descriptions of the odd annular Khovanov complex (Subsections 3.2 and 3.3).

In a follow-up paper with Casey Necheles [16], the second author extends results of Russell [28] to the odd setting by relating an annular version of Putyra's chronological cobordism category [23] to a dotted version of the odd Temperley-Lieb category (defined as in [7]) at $\delta = 0$. After setting dots equal to zero, the latter category becomes equivalent to a (non-full) subcategory of the category of $\mathfrak{gl}(1|1)$ representations. As a consequence, one obtains a natural interpretation of the $\mathfrak{gl}(1|1)$ action on odd annular Khovanov homology.

In a different direction, noting that:

- the bordered Heegaard-Floer tangle invariant defined by Petkova-Vértesi [20] carries a categorical action of $U_q(\mathfrak{gl}(1|1))$ [8],

- on a decategorified level, the bordered theory for knot Floer homology defined by Ozsváth-Szabó [19] carries an action of $U_q(\mathfrak{gl}(1|1))$ [15],

- conjecturally, there is an Ozsváth-Szabó spectral sequence relating odd Khovanov homology of (the mirror of) a link to the Heegaard-Floer homology of the manifold obtained as the connected sum of the double-branched cover of L with $S^1 \times S^2$ (cf. [4, 31]),

it is natural to ask the following:

Question 1. *Let $L \subseteq Y$ be a link in a 3–manifold satisfying either:*

1) *$Y = S^3$ and $L = L_0 \cup L'$ where L_0 is an unknot and L' is non-empty, or*

2) *Y is the double-branched cover of a knot $K \subseteq S^3$, and $L = p^{-1}(U)$ is the preimage of an unknot U in $S^3 - K$.*

Does $\widehat{HFK}(Y, L)$*, the knot Floer homology of* L *in* Y*, carry an action of* $\mathfrak{gl}(1|1)$*? In the latter case, how does this* $\mathfrak{gl}(1|1)$ *action relate to the* $\mathfrak{gl}(1|1)$ *action on* $AKh_{odd}(K \subseteq S^3 - N(U))$*?*

For a link $L = L_0 \cup L'$ which is realized as the closure, L', of a tangle T, linked by the tangle axis, L_0, Petkova-Vértesi [21] showed that the knot Floer homology of L can be identified with the Hochschild homology of the tangle Floer homology of T. In the case where T is a tangle in $\mathbb{R}^2 \times I$, the existence of a $\mathfrak{gl}(1|1)$ action on $\widehat{HFK}(L)$ could therefore be established by showing that the categorical $\mathfrak{gl}(1|1)$ action described in [8] induces an action on Hochschild homology.

In this context, it is worth noting that the even version of annular Khovanov homology has been identified (see [2] for a special case and [5] for the general case) with the Hochschild homology of the Chen-Khovanov tangle invariant, which categorifies the Reshetikhin-Turaev tangle invariant associated to the fundamental representation of $U_q(\mathfrak{sl}(2))$.

2. Preliminaries on $\mathfrak{gl}(1|1)$ representations

In this section, we will review basic facts about the representation theory of the Lie superalgebra $\mathfrak{gl}(1|1)$. For more details and background material on Lie superalgebras, we refer the reader to [29, 30, 33].

We will assume throughout this section that we are working over $\mathbb{C}$. By a vector superspace, we will mean a vector space V endowed with a $\mathbb{Z}_2$-grading $V = V_{\bar{0}} \oplus V_{\bar{1}}$. We will refer to this grading as the *supergrading* on V, and we will use the notation $|v|$ to denote the superdegree of a homogeneous element $v \in V$.

For $n \in \mathbb{Z}$, we will denote by $\langle n \rangle$ the shift functor which shifts the superdegree on a vector superspace by the image $\bar{n}$ of n under the quotient map $\mathbb{Z} \to \mathbb{Z}_2$. Thus if V is a vector superspace, then $V\langle n \rangle$ is the vector superspace with $(V\langle n \rangle)_{\bar{i}} = V_{\bar{i}+\bar{n}}$ for $\bar{i} \in \mathbb{Z}_2$.

2.1. Representations of Lie superalgebras

Recall that a *Lie superalgebra* is a vector superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ endowed with a bilinear *Lie superbracket* $[-, -]_s \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

1) $|[x, y]_s| = |x| + |y|$,
2) $[x, y]_s = -(-1)^{|x||y|}[y, x]_s$,

3) $[x, [y, z]_s]_s = [[x, y]_s, z]_s + (-1)^{|x||y|}[y, [x, z]_s]_s,$

for all homogeneous $x, y, z \in \mathfrak{g}$.

If V is a vector superspace, then $\mathfrak{gl}(V)$ denotes the Lie superalgebra whose underlying vector superspace is the space $\text{End}(V)$ of all linear endomorphisms of V, and whose Lie superbracket is given by the supercommutator $[x, y]_s := x \circ y - (-1)^{|x||y|} y \circ x$ for all homogeneous $x, y \in \text{End}(V)$. Here, it is understood that an endomorphism of V has superdegree $\bar{0}$ if it preserves the supergrading on V, and superdegree $\bar{1}$ if it reverses the supergrading on V.

A *homomorphism* between two Lie superalgebras $\mathfrak{g}$ and $\mathfrak{g}'$ is a linear map from $\mathfrak{g}$ to $\mathfrak{g}'$ which preserves both the supergrading and the Lie superbracket. A *representation* of a Lie superalgebra $\mathfrak{g}$ is a vector superspace V together with a homomorphism $\rho_V \colon \mathfrak{g} \to \mathfrak{gl}(V)$. As with ordinary Lie algebras, the map ρ_V is sometimes called the *action* of $\mathfrak{g}$ on V.

Let V and W be two representations of a Lie superalgebra $\mathfrak{g}$. Then $V \langle n \rangle$ is a representation of $\mathfrak{g}$ with $\rho_{V \langle n \rangle} = \rho_V$, and the dual space $V^* = \text{Hom}(V, \mathbb{C})$ is a representation of $\mathfrak{g}$ with $(V^*)_{\bar{i}} = \text{Hom}(V_{\bar{i}}, \mathbb{C})$ for $\bar{i} \in \mathbb{Z}_2$ and

$$\rho_{V^*}(x)(\varphi) = -(-1)^{|x||\varphi|}(\rho_V(x))^*(\varphi)$$

for all homogeneous $x \in \mathfrak{g}$ and $\varphi \in V^*$. Moreover, the tensor product $V \otimes W$ is a representation of $\mathfrak{g}$ with $|v \otimes w| = |v| + |w|$ and

$$\rho_{V \otimes W}(x)(v \otimes w) = (-1)^{|w||x|}(\rho_V(x)(v)) \otimes w + v \otimes (\rho_W(x)(w))$$

for all homogeneous $x \in \mathfrak{g}$, $v \in V$, and $w \in W$.

Remark 1. In the literature, the action of $\mathfrak{g}$ on $V \otimes W$ is more commonly defined by

$$\rho_{V \otimes W}(x)(v \otimes w) = (\rho_V(x)(v)) \otimes w + (-1)^{|x||v|}v \otimes (\rho_W(x)(w))$$

for all homogeneous $x \in \mathfrak{g}$, $v \in V$, and $w \in W$. It is easy to check that our definition yields an isomorphic representation, where the isomorphism is given by the endomorphism of $V \otimes W$ which sends a homogeneous element $v \otimes w$ to the element $(-1)^{|v||w|}v \otimes w$. In fact, one can use the standard definition of the tensor product to obtain an equivalent action of $\mathfrak{gl}(1|1)$ on odd annular Khovanov homology. This will be discussed in more detail in [16].

Remark 2. The representations $V \otimes W$ and $W \otimes V$ are isomorphic where the isomorphism is given by the linear map $\tau \colon V \otimes W \to W \otimes V$ which sends

a homogeneous element $v \otimes w$ to the element $(-1)^{|v||w|} w \otimes v$. We will henceforth call this map the *twist map*.

Given two homogeneous linear maps $f \colon V \to V'$ and $g \colon W \to W'$ between vector superspaces, let $f \otimes g \colon V \otimes W \to V' \otimes W'$ denote the homogeneous linear map defined by

$$(f \otimes g)(v \otimes w) := (-1)^{|g||v|} f(v) \otimes g(w)$$

for all homogeneous $v \in V$ and $w \in W$. Using this definition, we have:

Lemma 1. *If $\mathfrak{g}$ is a Lie superalgebra and $f \colon V \to V'$ and $g \colon W \to W'$ are homogeneous linear maps between $\mathfrak{g}$ representations which intertwine the actions of $\mathfrak{g}$, then the tensor product $f \otimes g \colon V \otimes W \to V' \otimes W'$ defined as above also intertwines the actions of $\mathfrak{g}$.*

Proof. Let $x \in \mathfrak{g}$, $v \in V$, and $w \in W$ be homogeneous. Then

$$\rho_{V' \otimes W'}(x)\Big[(f \otimes g)(v \otimes w)\Big] = (-1)^{|g||v|} \rho_{V' \otimes W'}(x)\Big[f(v) \otimes g(w)\Big]$$
$$= (-1)^{|g||v|}\Big[(-1)^{(|g|+|w|)|x|}\rho_{V'}(x)(f(v)) \otimes g(w) + f(v) \otimes \rho_{W'}(x)(g(w))\Big]$$
$$= (-1)^{|g||v|}\Big[(-1)^{(|g|+|w|)|x|}f(\rho_V(x)(v)) \otimes g(w) + f(v) \otimes g(\rho_W(x)(w))\Big]$$
$$= (-1)^{|g||v|+|g||x|+|w||x|}f(\rho_V(x)(v)) \otimes g(w) + (-1)^{|g||v|}f(v) \otimes g(\rho_W(x)(w))$$
$$= (-1)^{|w||x|}(-1)^{|g|(|x|+|v|)}f(\rho_V(x)(v)) \otimes g(w) + (-1)^{|g||v|}f(v) \otimes g(\rho_W(x)(w))$$
$$= (-1)^{|w||x|}(f \otimes g)(\rho_V(x)(v) \otimes w) + (f \otimes g)(v \otimes \rho_W(x)(w))$$
$$= (f \otimes g)\Big[(-1)^{|w||x|}\rho_V(x)(v) \otimes w + v \otimes \rho_W(x)(w)\Big]$$
$$= (f \otimes g)\Big[\rho_{V \otimes W}(x)(v \otimes w)\Big].$$

Hence $f \otimes g$ intertwines the maps $\rho_{V \otimes W}(x)$ and $\rho_{V' \otimes W'}(x)$, which proves the lemma. $\square$

For a vector superspace V, let Φ_V denote the linear involution $\Phi_V \colon V \to V$ defined by $\Phi_V(v) = (-1)^{|v|}v$ for every homogeneous element $v \in V$. The following lemma describes how the grading shift functor $\langle 1 \rangle$ interacts with duals and tensor products of representations.

Lemma 2. *Let V and W be two representations of a Lie superalgebra $\mathfrak{g}$. Then*

$$(V\langle 1\rangle)^* \cong V^*\langle 1\rangle,$$

where the isomorphism is given by the map $\Phi_{V^*}\colon V^* \to V^*$, and

$$V\langle 1\rangle \otimes W \cong (V \otimes W)\langle 1\rangle \cong V \otimes W\langle 1\rangle,$$

where the first isomorphism is given by the identity map of $V \otimes W$ and the second isomorphism is given by the map $\Phi_V \otimes \mathrm{id}_W$.

Proof. Let $x \in \mathfrak{g}$ and $\varphi \in V^*$ be homogeneous. Then the definition of Φ_{V^*} implies

$$(\Phi_{V^*} \circ \rho_{V^*}(x))(\varphi) = (-1)^{|x|+|\varphi|}(\rho_{V^*}(x))(\varphi) = (-1)^{|x|}(\rho_{V^*}(x) \circ \Phi_{V^*})(\varphi),$$

where the first equation follows because $\rho_{V^*}(x)(\varphi)$ has superdegree $|x| + |\varphi|$. Because $\rho_{V^*} = \rho_{V^*\langle 1\rangle}$, the left-most term in the above sequence of equations can be identified with $(\Phi_{V^*} \circ \rho_{V^*\langle 1\rangle}(x))(\varphi)$, and because

$$\begin{aligned}
(-1)^{|x|}\rho_{V^*}(x)(\varphi) &= -(-1)^{|x|}(-1)^{|x||\varphi|}(\rho_V(x))^*(\varphi)\\
&= -(-1)^{|x|(|\varphi|+1)}(\rho_{V\langle 1\rangle}(x))^*(\varphi)\\
&= \rho_{(V\langle 1\rangle)^*}(x)(\varphi),
\end{aligned}$$

the right-most term can be identified with $(\rho_{(V\langle 1\rangle)^*}(x) \circ \Phi_{V^*})(\varphi)$. Thus we have

$$\Phi_{V^*} \circ \rho_{V^*\langle 1\rangle}(x) = \rho_{(V\langle 1\rangle)^*}(x) \circ \Phi_{V^*},$$

and hence Φ_{V^*} is an isomorphism between $V^*\langle 1\rangle$ and $(V\langle 1\rangle)^*$.

The claim that the identity map of $V \otimes W$ is an isomorphism between $V\langle 1\rangle \otimes W$ and $(V \otimes W)\langle 1\rangle$ follows because $\rho_{V\langle 1\rangle \otimes W} = \rho_{V \otimes W} = \rho_{(V \otimes W)\langle 1\rangle}$, by our definition of the tensor product of two representations.

Finally, by Remark 2, we have a sequence of isomorphisms

$$V \otimes W\langle 1\rangle \xrightarrow{\ \tau\ } W\langle 1\rangle \otimes V \xrightarrow{\ id\ } (W \otimes V)\langle 1\rangle \xrightarrow{\ \tau\ } (V \otimes W)\langle 1\rangle.$$

This sequence takes a homogeneous element $v \otimes w \in V \otimes W$ to

$$(-1)^{|v|(|w|+1)}(-1)^{|w||v|}v \otimes w = (-1)^{|v|}v \otimes w = (\Phi_V \otimes \mathrm{id}_W)(v \otimes w),$$

and hence $\Phi_V \otimes \mathrm{id}_W$ is an isomorphism between $V \otimes W\langle 1\rangle$ and $(V \otimes W)\langle 1\rangle$. $\square$

2.2. The Lie superalgebra $\mathfrak{gl}(1|1)$ and some of its representations

Let $\mathbb{C}^{1|1}$ denote the vector superspace

$$\mathbb{C}^{1|1} := \mathbb{C}v_0 \oplus \mathbb{C}v_1$$

spanned by two homogeneous elements v_0 and v_1 of superdegrees $\bar{0}$ and $\bar{1}$, respectively. The Lie superalgebra $\mathfrak{gl}(1|1)$ is defined as the space of linear endomorphisms $\mathfrak{gl}(1|1) = \mathfrak{gl}(\mathbb{C}^{1|1}) = \mathrm{End}(\mathbb{C}^{1|1})$ with Lie superbracket given by the supercommutator, as described in the previous subsection. Explicitly, $\mathfrak{gl}(1|1)$ is spanned by the following elements

$$h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

where h_1 and h_2 have superdegree $\bar{0}$ and e and f have superdegree $\bar{1}$. The Lie superbracket on $\mathfrak{gl}(1|1)$ is given by

$$\begin{aligned}
[e, f]_s &= h_1 + h_2, & [e, e]_s &= [f, f]_s = [h_i, h_j]_s = 0, \\
[e, h_1]_s &= -e, & [f, h_1]_s &= f, \\
[e, h_2]_s &= e, & [f, h_2]_s &= -f,
\end{aligned}$$

where $i, j \in \{1, 2\}$.

Let $h_+ := h_1 + h_2$ and $h_- := h_1 - h_2$. Then h_+ is central in $\mathfrak{gl}(1|1)$, in the sense that $[h_+, x]_s = 0$ for all $x \in \mathfrak{gl}(1|1)$, and h_- satisfies

$$[e, h_-]_s = -2e, \quad [f, h_-]_s = 2f, \quad [h_-, h_-]_s = 0.$$

Note that the elements h_+, h_-, e, and f form a basis for $\mathfrak{gl}(1|1)$ and that the Lie superbracket on $\mathfrak{gl}(1|1)$ is completely characterized by the aforementioned properties of h_+ and h_- and by the relations $[e, f]_s = h_+$ and $[e, e]_s = [f, f]_s = 0$.

We will now describe a family of irreducible $\mathfrak{gl}(1|1)$ representations $L_{(m,n)}$ parameterized by pairs of integers $(m, n) \in \mathbb{Z}^2$. It is not hard to see that every finite-dimensional irreducible $\mathfrak{gl}(1|1)$ representation on which h_1 and h_2 act with integer eigenvalues is isomorphic to one of the representations in this family, up to a possible shift of the supergrading. See [6] and [29].

If $(m, n) \in \mathbb{Z}^2$ satisfies $m + n = 0$, then $L_{(m,n)}$ is 1-dimensional and supported in superdegree $\bar{0}$. The elements (h_1, h_2) act by scalar multiplication by (m, n) on this representation, and the elements e and f act by zero. Note that $L_{(0,0)}$ is a trivial representation, where trivial means that all generators of $\mathfrak{gl}(1|1)$ act by zero.

If $(m, n) \in \mathbb{Z}^2$ satisfies $m + n \neq 0$, then $L_{(m,n)}$ is 2-dimensional and spanned by two homogeneous vectors v_+ and v_- of superdegrees $\bar{n}$ and $\bar{n} + \bar{1}$, respectively. In this case, the action of $\mathfrak{gl}(1|1)$ relative to the basis $\{v_+, v_-\}$ is given by the following matrices:

$$\rho_{L_{(m,n)}}(h_1) = \begin{pmatrix} m & 0 \\ 0 & m-1 \end{pmatrix}, \quad \rho_{L_{(m,n)}}(h_2) = \begin{pmatrix} n & 0 \\ 0 & n+1 \end{pmatrix},$$

$$\rho_{L_{(m,n)}}(e) = \begin{pmatrix} 0 & m+n \\ 0 & 0 \end{pmatrix}, \quad \rho_{L_{(m,n)}}(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Note that for $m = 1$ and $n = 0$, the above matrices coincide with the matrices h_1, h_2, e, and f. Thus, the representation $L_{(1,0)}$ is equal to the fundamental representation $\mathbb{C}^{1|1}$ of $\mathfrak{gl}(1|1)$, on which $\mathfrak{gl}(1|1)$ acts by $\rho_{\mathbb{C}^{1|1}} = \mathrm{id}_{\mathfrak{gl}(1|1)}$. In the remainder of this section, we will denote the fundamental representation $L_{(1,0)} = \mathbb{C}^{1|1}$ by V. The dual representation, V^*, can be seen to be isomorphic to the representation $L_{(0,-1)}$, where an isomorphism is given by the linear map which sends the basis $\{v_+^*, v_-^*\}$ of $V^* = L_{(1,0)}^*$ to the basis $\{v_-, v_+\}$ of $L_{(0,-1)}$.

In the next subsection, we will see that the representations $V^* \otimes V$ and $V \otimes V^*$ each contain a trivial 1-dimensional subrepresentation and a trivial 1-dimensional quotient representation, but no trivial direct summand. In particular, these representations are indecomposable but not irreducible. We will actually study the isomorphic (up to a grading shift) representations $V^*\langle 1 \rangle \otimes V$ and $V \otimes (V^*\langle 1 \rangle)$ since these representations will be needed later in this paper.

2.3. $\mathfrak{gl}(1|1)$ action on the representations $V^*\langle 1 \rangle \otimes V$ and $V \otimes (V^*\langle 1 \rangle)$

As before, let V denote the fundamental representation of $\mathfrak{gl}(1|1)$. For reasons that will become clear later, we will identify the vector superspaces underlying the representations $V^*\langle 1 \rangle$ and V by using the identifications $v_+^* = -v_-$ and $v_-^* = v_+$. It is not hard to see that, under these identifications, the action of $\mathfrak{gl}(1|1)$ on $V^*\langle 1 \rangle \otimes V$ is given as follows:
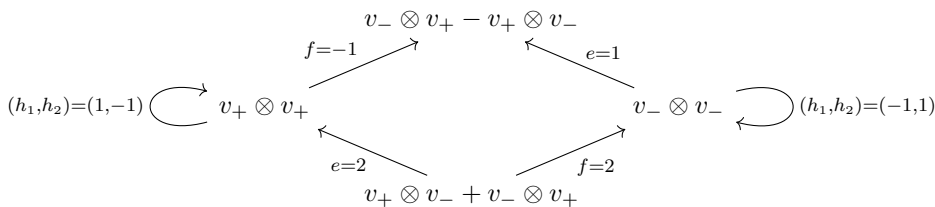
For example, e sends the vector $v_+ \otimes v_- + v_- \otimes v_+$ to $2v_+ \otimes v_+$ and f sends this vector to $2v_- \otimes v_-$. Likewise, h_1 and h_2 annihilate the vectors $v_+ \otimes v_-$ and $v_- \otimes v_+$ and act on the vectors $v_+ \otimes v_+$ and $v_- \otimes v_-$ by scalar multiplication by $(h_1, h_2) = (1, -1)$ and $(h_1, h_2) = (-1, 1)$, respectively. Since there are no arrows ending at the vector $v_+ \otimes v_- + v_- \otimes v_+$, this vector spans a trivial 1-dimensional quotient representation, and since there are no arrows emanating from $v_- \otimes v_+ - v_+ \otimes v_-$, this vector spans a trivial 1-dimensional subrepresentation. Let $p \colon V^* \langle 1 \rangle \otimes V \to \mathbb{C} \langle 1 \rangle$ and $i \colon \mathbb{C} \langle 1 \rangle \to V^* \langle 1 \rangle \otimes V$ denote the associated projection and inclusion maps, where $\mathbb{C}$ denotes the trivial 1-dimensional representation $\mathbb{C} = L_{(0,0)}$. Explicitly, p sends $v_+ \otimes v_-$ and $v_- \otimes v_+$ to 1 and $v_+ \otimes v_+$ and $v_- \otimes v_-$ to zero, and i sends 1 to $v_- \otimes v_+ - v_+ \otimes v_-$. For later use, we also introduce the maps

$$\tilde{p} \colon V^* \langle 1 \rangle \otimes V \longrightarrow \mathbb{C} \oplus (\mathbb{C} \langle 1 \rangle), \qquad \tilde{i} \colon \mathbb{C} \oplus (\mathbb{C} \langle 1 \rangle) \longrightarrow V^* \langle 1 \rangle \otimes V,$$

defined as the compositions

$$V^* \langle 1 \rangle \otimes V \xrightarrow{\ p\ } \mathbb{C} \langle 1 \rangle \xrightarrow{\ i_2\ } \mathbb{C} \oplus (\mathbb{C} \langle 1 \rangle),$$
$$\mathbb{C} \oplus (\mathbb{C} \langle 1 \rangle) \xrightarrow{\ p_1\ } \mathbb{C} \langle 1 \rangle \xrightarrow{\ i\ } V^* \langle 1 \rangle \otimes V,$$

respectively, where $i_2 \colon \mathbb{C} \langle 1 \rangle \to \mathbb{C} \oplus (\mathbb{C} \langle 1 \rangle)$ is the inclusion of $\mathbb{C} \langle 1 \rangle$ into the second summand, and $p_1 \colon \mathbb{C} \oplus (\mathbb{C} \langle 1 \rangle) \to \mathbb{C} \langle 1 \rangle$ is the projection onto the first summand, up to a shift of the supergrading. By construction, $\tilde{p}$ and $\tilde{i}$ intertwine the action of $\mathfrak{gl}(1|1)$ and have superdegrees $\bar{0}$ and $\bar{1}$, respectively.

The action of $\mathfrak{gl}(1|1)$ on $V \otimes (V^* \langle 1 \rangle)$ is given by almost the same diagram as the one above, with the only difference being that the arrow labeled $f = -1$ is replaced by an arrow labeled $f = 1$ and the arrow labeled $f = 2$ is replaced by an arrow labeled $f = -2$. In particular, there are maps $\tilde{p} \colon V \otimes (V^* \langle 1 \rangle) \to \mathbb{C} \oplus (\mathbb{C} \langle 1 \rangle)$ and $\tilde{i} \colon \mathbb{C} \oplus (\mathbb{C} \langle 1 \rangle) \to V \otimes (V^* \langle 1 \rangle)$ which are defined by the same formulas as before and which intertwine the action of $\mathfrak{gl}(1|1)$.

2.4. $\mathfrak{gl}(1|1)$ actions on the exterior algebra of a vector space

We end this section with a natural construction of $\mathfrak{gl}(1|1)$ actions on the exterior algebra of a vector space.

Let U be a vector space equipped with a symmetric bilinear inner product $\langle -, - \rangle$ and let $U' := \Lambda^*(U)$ be its exterior algebra. We may regard U' as a vector superspace by collapsing the natural $\mathbb{Z}_{\geq 0}$-grading on the exterior algebra to a $\mathbb{Z}_2$-grading. Choose two vectors $a, b \in U$ and a constant $N \in \mathbb{C}$,

and define a linear map $\rho_{U'} \colon \mathfrak{gl}(1|1) \to \mathfrak{gl}(U')$ by

$$\rho_{U'}(h_+)(v) = \langle a, b \rangle v,$$
$$\rho_{U'}(h_-)(v) = (N - 2\ell)v,$$
$$\rho_{U'}(e)(v) = a \lrcorner v,$$
$$\rho_{U'}(f)(v) = b \wedge v,$$

where $v \in U'$ is an element of the form $v = u_1 \wedge \cdots \wedge u_\ell$ for $u_j \in U$ and $\ell \geq 0$, and $\lrcorner$ is defined by

$$u \lrcorner v = \sum_{j=1}^{\ell} (-1)^{j-1} \langle u, u_j \rangle u_1 \wedge \cdots \wedge \widehat{u_j} \wedge \cdots \wedge u_\ell$$

for $u \in U$ and $v \in U'$ as before. Using this definition of $\lrcorner$, one can see that

$$a \lrcorner (b \wedge v) + b \wedge (a \lrcorner v) = \langle a, b \rangle v$$

for all $v \in U'$. In particular, this implies that the map $\rho_{U'}$ is compatible with the $\mathfrak{gl}(1|1)$ relation $[e, f]_s = h_+$. Since $\rho_{U'}(h_+)$ is given by multiplication by the constant $\langle a, b \rangle$, it is further clear that $\rho_{U'}(h_+)$ is central in $\mathfrak{gl}(U')$. Moreover, the map $v \mapsto a \lrcorner v$ (resp., $v \mapsto b \wedge v$) lowers (resp., raises) the number of factors in a wedge product by one, and together with the definition of $\rho_{U'}(h_-)$, this implies that $\rho_{U'}$ respects the $\mathfrak{gl}(1|1)$ relations $[e, h_-]_s = -2e$ and $[f, h_-]_s = 2f$. It is easy to see that $\rho_{U'}$ is also compatible with the $\mathfrak{gl}(1|1)$ relations $[e, e]_s = [f, f]_s = [h_-, h_-]_s = 0$, and hence $\rho_{U'}$ endows U' with a well-defined action of $\mathfrak{gl}(1|1)$.

One can slightly modify the above definition of $\rho_{U'}$ by replacing $b \wedge v$ by $v \wedge b$ and $a \lrcorner v$ by $v \llcorner a$, where $v \llcorner a := (-1)^{\ell-1} a \lrcorner v$ for $v \in U'$ as before. This modified definition also yields a well-defined action of $\mathfrak{gl}(1|1)$ on U'.

3. Odd annular Khovanov homology as a $\mathfrak{gl}(1|1)$ module

3.1. Odd annular Khovanov homology

Let A be a closed, oriented annulus, $I = [0, 1]$ the closed, oriented unit interval. Via the identification

$$A \times I = \{(r, \theta, z) \,|\, r \in [1, 2], \theta \in [0, 2\pi), z \in [0, 1]\} \subset (S^3 = \mathbb{R}^3 \cup \infty),$$
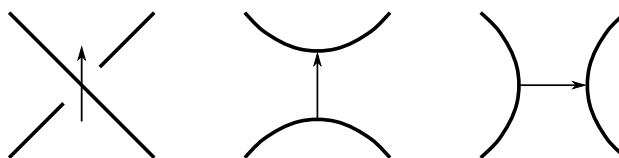
Figure 1: Direction of the arrows in the "0" resolution and the "1" resolution.

any link, $L \subset A \times I$, may naturally be viewed as a link in the complement of a standardly imbedded unknot, $(U = z\text{–axis} \cup \infty) \subset S^3$. Such an *annular link $L \subset A \times I$* admits a diagram, $\mathcal{P}(L) \subset A$, obtained by projecting a generic isotopy class representative of L onto $A \times \{1/2\}$. From this diagram one can construct a triply-graded chain complex called the *annular* Khovanov complex associated to the annular link L, by using a version of Khovanov's original construction [14] due to Asaeda-Przytycki-Sikora [1] and L. Roberts [27] (see also [11]). We now proceed to describe an odd version of the annular Khovanov complex, using the construction of Ozsváth-Rasmussen-Szabó in [17].

Begin by decorating the diagram with an arrow (called an *orientation* in [17]) at each crossing, as follows. Position the crossing so its overstrand connects the upper left (NW) to the lower right (SE) corner and draw an arrow on the crossing pointing either up or down. This choice will specify arrows for the two resolutions of the crossing, as follows. The arrow for the "0" resolution will agree with the arrow at the crossing, and the arrow for the "1" resolution will be rotated 90° clockwise. See Figure 1.

Now view the decorated diagram $\mathcal{P}(L) \subset A$ instead as a diagram on $S^2 - \{\mathbb{X}, \mathbb{O}\}$, where $\mathbb{X}$ (resp., $\mathbb{O}$) are basepoints on S^2 corresponding to the inner (resp., outer) boundary circles of A. If we temporarily forget the data of $\mathbb{X}$, we may view $\mathcal{P}(L)$ as a diagram on $\mathbb{R}^2 = S^2 - \{\mathbb{O}\}$ and form the ordinary bigraded odd Khovanov complex

$$\text{CKh}_{odd}(\mathcal{P}(L)) = \bigoplus_{(i,j) \in \mathbb{Z}^2} \text{CKh}_{odd}^i(\mathcal{P}(L); j)$$

as described in [17] and briefly recalled below.

Let $\mathcal{X}$ denote the set of crossings of $\mathcal{P}(L)$. For each map $I : \mathcal{X} \to \{0, 1\}$ one obtains an associated decorated imbedded 1–manifold, $\mathcal{P}^I(L) \subseteq S^2 - \{\mathbb{O}\}$ obtained by resolving and decorating each crossing as specified by I. Choosing an ordering of the c crossings identifies these decorated complete

resolutions with the vertices of a c–dimensional hypercube whose edges correspond to saddle cobordisms between decorated complete resolutions.

Remembering the data of $\mathbb{X}$, we now associate to each vertex of this *hypercube of decorated resolutions* a chain complex whose underlying $\mathbb{Z}^3$–graded vector space is defined as follows. For each $I : \mathcal{X} \to \{0,1\}$ let

$$V(I) := \mathrm{Span}_{\mathbb{C}}\{a_1, \ldots, a_n\}$$

be the formal span of the components, $a_1, \ldots, a_n$, of $\mathcal{P}^I(L)$. Then the $\mathbb{C}$ vector space we assign to the vertex I is

$$F(I) := \Lambda^*(V(I)).$$

It will be convenient to note that $F(I)$ has a distinguished basis indexed by subsets $S \subseteq \{1, \ldots, n\}$. Given such a subset $S = \{i_1, \ldots, i_\ell\} \subseteq \{1, \ldots, n\}$ whose elements have been arranged in order ($i_1 < \cdots < i_\ell$), we will denote the associated basis element of $F(I)$ by

$$a_S := a_{i_1} \wedge \cdots \wedge a_{i_\ell}.$$

Now each vector space $F(I)$ is endowed with a homological (i) and quantum (j) grading exactly as in [14, 17], and these gradings do not depend on the data of $\mathbb{X}$. The odd Khovanov differential,

$$\partial : \mathrm{CKh}_{odd}(\mathcal{P}(L)) \to \mathrm{CKh}_{odd}(\mathcal{P}(L)),$$

which also does not depend on the data of $\mathbb{X}$, is defined exactly as in [14, 17] as a signed sum (specified by an *edge assignment* as in [17, Defn. 1.1]) of elementary merge maps, $F_M : F(I) \to F(I')$ and split maps $F_\Delta : F(I) \to F(I')$ associated to edges of the hypercube.

For completeness, the definitions of F_M and F_Δ are also briefly recalled below.

Let $I_0, I_1 : \mathcal{X} \to \{0,1\}$ be two vertices for which there is an oriented edge from I_0 to I_1, as in [17, p.3].[1] If two components, a_1 and a_2, of $\mathcal{P}^{I_0}(L)$ merge to a single component, a, of $\mathcal{P}^{I_1}(L)$, there is a natural identification $V(I_1) \cong V(I_0)/(a_1 - a_2)$ coming from identifying a with $[a_1] = [a_2]$. The merge map

$$F_M : \Lambda^* V(I_0) \to \Lambda^* V(I_1)$$

is the map on the exterior algebra induced by the projection followed by this natural identification: $V(I_0) \to V(I_0)/(a_1 - a_2) \cong V(I_1)$.

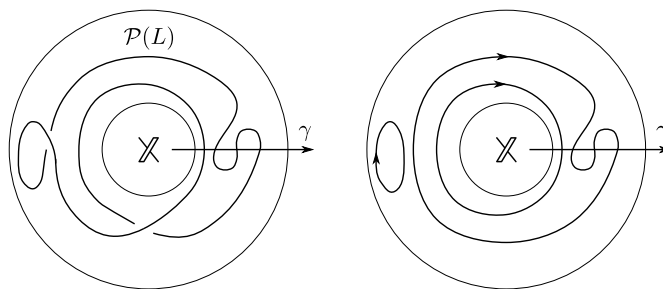[1] I_1 is sometimes called an *immediate successor* of I_0.

Figure 2: Annular link diagram $\mathcal{P}(L)$ and an oriented Kauffman state of $\mathcal{P}(L)$ of k-degree -2. For simplicity, the arrows at the crossings are not shown in the picture.

If a single component, a, of $\mathcal{P}^{I_0}(L)$ splits into two components, a_1 and a_2, of $\mathcal{P}^{I_1}(L)$ and the local arrow decorating the split region points from a_1 to a_2, then the split map is defined by

$$F_\Delta : \Lambda^*(V(I_0)) \longrightarrow \Lambda^* \left(\frac{V(I_1)}{(a_1 - a_2)} \right) \longrightarrow (a_1 - a_2) \wedge \Lambda^* V(I_1) \longrightarrow \Lambda^* V(I_1),$$

where the first map is the inverse of the natural identification described in the definition of the merge map, the final map is the inclusion, and the middle map is an explicit identification of the exterior algebra of the quotient, $V(I_1)/(a_1 - a_2)$, as

$$(a_1 - a_2) \wedge \Lambda^* V(I_1) \subset \Lambda^* V(I_1).$$

To obtain the annular (k) grading, begin by choosing an oriented arc γ from $\mathbb{X}$ to $\mathbb{O}$ that misses all crossings of $\mathcal{P}(L)$. It will be clear from the construction that the k-grading is independent of this choice.

Exactly as in the even case (cf. [12, Sec. 4.2]), we have a one-to-one correspondence between distinguished basis elements a_S of $F(I) \subseteq \mathrm{CKh}_{odd}(\mathcal{P}(L))$ and *orientations* of the Kauffman state $\mathcal{P}^I(L)$, defined as follows. Choose the *clockwise* (CW) orientation on a component a_i of the Kauffman state if $i \in S$ and the *counterclockwise* (CCW) orientation on a_i if $i \notin S$. The k-grading of a distinguished basis element is now defined to be the algebraic intersection number of the corresponding oriented Kauffman state with a fixed oriented arc γ from $\mathbb{X}$ to $\mathbb{O}$ that misses all crossings of $\mathcal{P}(L)$. See Figure 2.

Lemma 3. *The odd Khovanov differential*

$$\partial : CKh_{odd}(\mathcal{P}(L)) \to CKh_{odd}(\mathcal{P}(L))$$

is non-increasing in the k-grading. Indeed, it can be decomposed as $\partial = \partial_0 + \partial_-$, where ∂_0 has k-degree 0, and ∂_- has k-degree -2.

Proof. Roberts proves ([27, Lem. 1]) that the *even* Khovanov differential is non-increasing in this extra grading and decomposes according to k-degree as described in the statement of the lemma. As noted in [18] and [17], the odd merge and split maps F_M and F_Δ agree modulo 2 with the merge and split maps in even Khovanov homology, so the odd Khovanov differential, ∂, has precisely the same decomposition according to k-degree.   □

Decomposing $\partial^2 = 0$ into its k-homogeneous pieces, we see that ∂_0 and ∂_- are two anticommuting differentials on $\mathrm{CKh}_{odd}(\mathcal{P}(L))$. We can therefore define an odd version of annular Khovanov homology by taking the homology with respect to ∂_0:

Definition 1. The *odd annular Khovanov homology* of L, denoted $\mathrm{AKh}_{odd}(L)$, is the homology of the chain complex $(\mathrm{CKh}_{odd}(\mathcal{P}(L)), \partial_0)$.

Note that $\mathrm{AKh}_{odd}(L)$ is triply-graded since it carries well-defined i-, j-, and k-gradings. We will refer to the underlying chain complex, $(\mathrm{CKh}_{odd}(\mathcal{P}(L)), \partial_0)$, as the *odd annular Khovanov complex*.

Remark 3. Since ∂ is non-increasing with respect to the k-grading, the odd Khovanov complex can be viewed as a filtered chain complex, and the odd annular Khovanov complex can be identified with the associated graded complex. In particular, there is a spectral sequence converging from the odd annular Khovanov homology of L to the odd Khovanov homology of L, viewed as a link in S^3. Over $\mathbb{Z}_2$, this spectral sequence agrees with the corresponding spectral sequence between the even theories, and it was shown in [13, Prop. 18] that, over $\mathbb{Z}_2$, the latter spectral sequence does not always converge at the E^3 page. Indeed, the closed 4–braid counterexample

$$A(0,0) = \sigma_3 \sigma_2^{-2} \sigma_3^2 \sigma_2 \sigma_3^{-1} \sigma_1^{-1} \sigma_2 \sigma_1^2$$

the authors provide there would be a natural candidate for an annular link whose odd annular Khovanov homology does not converge at the E^3 page.

Remark 4. Note that, in general, a spectral sequence associated to a bi-complex over $\mathbb{Z}$ coefficients can converge earlier than the spectral sequence associated to the corresponding bicomplex over $\mathbb{Z}_2$ coefficients. For example, consider the bicomplex

$$
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\;2\;} & \mathbb{Z} \\
{\scriptstyle 1}\big\downarrow & & \\
\mathbb{Z} & \xrightarrow{\;2\;} & \mathbb{Z}
\end{array}
$$

where the horizontal arrows represent ∂_0 and the vertical arrow represents ∂_-. Then the first page of the associated spectral sequence is

$$
\begin{array}{ccc}
0 & \longrightarrow & \mathbb{Z}_2 \\
\big\downarrow & & \\
0 & \longrightarrow & \mathbb{Z}_2
\end{array}
$$

and all higher differentials in the spectral sequence vanish for grading reasons. Thus, the spectral sequence over $\mathbb{Z}$ coefficients collapses immediately. On the other hand, the spectral sequence associated to the same bicomplex over $\mathbb{Z}_2$ coefficients collapses one page later.

In the next subsection, we will define a $\mathfrak{gl}(1|1)$ action on $\mathrm{AKh}_{odd}(L)$ and show that this action is invariant under annular Reidemeister moves, hence yields an invariant of the isotopy class of $L \subseteq A \times I$. We will also discuss the interaction of the $\mathfrak{gl}(1|1)$ action with the i-, j-, and k-gradings on the complex and conclude that, when regarded simply as a triply-graded vector space without a Lie superalgebra action, $\mathrm{AKh}_{odd}(L)$ is an invariant of the isotopy class of $L \subseteq A \times I$.

3.2. Definition and invariance of the $\mathfrak{gl}(1|1)$ action on $\mathrm{AKh}_{odd}(L)$

As before, we will denote by $V(I)$ the formal span of the components $a_1, \ldots, a_n$ of $\mathcal{P}^I(L)$, and by $F(I) = \Lambda^*(V(I))$ the exterior algebra of $V(I)$.

We will further use the following notations: $|L|$ will denote the number of link components, n_+ (resp., n_-) will denote the number of positive (resp., negative) crossings in the link projection, and $|I|$ will denote the number of crossings $c \in \mathcal{X}$ such that $I(c) = 1$. Moreover, n_t (resp., n_e) will denote the number of trivial (resp., essential) components of $\mathcal{P}^I(L)$, where a component a_i is called trivial (resp., essential) if it is zero (resp., nonzero) in the first homology of $S^2 - \{\mathbb{X}, \mathbb{O}\}$.

Using these notations, we can define the tri-degree of an element $a_{i_1} \wedge \cdots \wedge a_{i_\ell} \in F(I)$ by $i = |I| - n_-$, $j = n - 2\ell + |I| + n_+ - 2n_-$, and $k = n_e - 2\ell_e$, where $n = n_t + n_e$ denotes the number of components of $\mathcal{P}^I(L)$ and ℓ_e denotes the number of indices $r \in \{1, \ldots, \ell\}$ for which the component a_{i_r} is essential.

We will now regard $F(I)$ as a vector superspace with the supergrading given by the modulo 2 reduction of $(j - |L|)/2$, where j denotes the quantum degree just defined. It is known that j always has the same parity as $|L|$, and hence the modulo 2 reduction of $(j - |L|)/2$ is well-defined.

In what follows, we will assume that the components $a_1, \ldots, a_n$ are ordered so that $a_1, \ldots, a_{n_t}$ are trivial and $a_{n_t+1}, \ldots, a_n$ are essential, and that the essential components are ordered according to their proximity to the basepoint $\mathbb{X}$, so that a_{n_t+1} is the essential component which is closest to $\mathbb{X}$.

We can then write $F(I)$ as a tensor product of two vector superspaces

$$F(I) = \Lambda^*(\mathrm{Span}_{\mathbb{C}}\{a_1, \ldots, a_{n_t}\}) \otimes \Lambda^*(\mathrm{Span}_{\mathbb{C}}\{a_{n_t+1}, \ldots, a_n\}),$$

where we define the supergrading on the two tensor factors as follows. If ℓ denotes the natural $\mathbb{Z}_{\geq 0}$ degree on the exterior algebra, then the superdegree on the first tensor factor is given by the modulo 2 reduction of $\ell + (n + |I| + n_+ - 2n_- - |L|)/2$, and the superdegree on the second tensor factor as the modulo 2 reduction of ℓ.

Now equip $\mathrm{Span}_{\mathbb{C}}\{a_{n_t+1}, \ldots, a_n\}$ with the unique symmetric bilinear form $\langle -, - \rangle$ for which the vectors $a_{n_t+1}, \ldots, a_n$ are orthonormal, and let a_I and b_I denote the vectors in $\mathrm{Span}_{\mathbb{C}}\{a_{n_t+1}, \ldots, a_n\}$ defined by

$$a_I := a_{n_t+1} + a_{n_t+2} + \cdots + a_n$$
$$\text{and} \quad b_I := a_{n_t+1} - a_{n_t+2} + \cdots + (-1)^{n_e-1}a_n.$$

Observe that $\langle a_I, b_I \rangle = m$, where

$$m = \begin{cases} 0 & \text{if } n_e \text{ is even,} \\ 1 & \text{if } n_e \text{ is odd.} \end{cases}$$

Since n_e has the same parity as the winding number of L around $\mathbb{X}$, m only depends on the parity of this winding number and not on the particular choice of I. Define a linear map $\rho_{U'}: \mathfrak{gl}(1|1) \to \mathfrak{gl}(U')$ for $U' :=$

$\Lambda^*(\mathrm{Span}_{\mathbb{C}}\{a_{n_t+1}, \ldots, a_n\})$ by

$$\rho_{U'}(h_+)(a_{i_1} \wedge \cdots \wedge a_{i_\ell}) = m a_{i_1} \wedge \cdots \wedge a_{i_\ell},$$
$$\rho_{U'}(h_-)(a_{i_1} \wedge \cdots \wedge a_{i_\ell}) = (n_e - 2\ell) a_{i_1} \wedge \cdots \wedge a_{i_\ell},$$
$$\rho_{U'}(e)(a_{i_1} \wedge \cdots \wedge a_{i_\ell}) = (a_{i_1} \wedge \cdots \wedge a_{i_\ell}) \llcorner a_I,$$
$$\rho_{U'}(f)(a_{i_1} \wedge \cdots \wedge a_{i_\ell}) = a_{i_1} \wedge \cdots \wedge a_{i_\ell} \wedge b_I$$

for all $n_t + 1 \leq i_1 < \cdots < i_\ell \leq n$.

Comparing with Subsection 2.4, we see that $\rho_{U'}$ endows the vector superspace U' with a well-defined action of $\mathfrak{gl}(1|1)$. We can extend this action to an action on $F(I)$ by regarding $\Lambda^*(\mathrm{Span}_{\mathbb{C}}\{a_1, \ldots, n_t\})$ as a trivial $\mathfrak{gl}(1|1)$ representation. In other words, we can define a $\mathfrak{gl}(1|1)$ action on $F(I)$ by setting

$$\rho_{F(I)}(x) := \mathrm{id} \otimes \rho_{U'}(x)$$

for all $x \in \mathfrak{gl}(1|1)$, where id denotes the identity map of $\Lambda^*(\mathrm{Span}_{\mathbb{C}}\{a_1, \ldots, n_t\})$, and $\otimes$ denotes the ordinary (ungraded) tensor product of linear maps.

Remark 5. The reader should note that h_- acts on a vector $a_{i_1} \wedge \cdots \wedge a_{i_\ell} \in F(I)$ by scalar multiplication by $k = n_e - 2\ell_e$, where ℓ_e denotes the number of essential components in the wedge product $a_{i_1} \wedge \cdots \wedge a_{i_\ell}$. Therefore, the k-grading on $F(I)$ can be viewed as the "weight space grading" with respect to the action of h_-. It is further clear that the $\mathfrak{gl}(1|1)$ action preserves the i-grading because the i-grading is constant on $F(I)$. Moreover, the $\mathfrak{gl}(1|1)$ action preserves the $(j - k)$-grading because the $\mathfrak{gl}(1|1)$ action preserves the number ℓ_t of trivial components in a wedge product $a_{i_1} \wedge \cdots \wedge a_{i_\ell} \in F(I)$, and $j - k = n - 2\ell + |I| + n_+ - 2n_- - (n_e - 2\ell_e) = n_t - 2\ell_t + |I| + n_+ - 2n_-$.

Lemma 4. *Let $I_0, I_1 \colon \mathcal{X} \to \{0, 1\}$ be two vertices of the resolution hypercube for which there is an oriented edge from I_0 to I_1 and let $F_W \colon F(I) \to F(I')$ for $W = M$ or $W = \Delta$ be the associated merge or split map. Then the k-degree preserving part of F_W intertwines the actions of $\mathfrak{gl}(1|1)$ on $F(I)$ and $F(I')$.*

Although it is possible to prove Lemma 4 directly, we will defer the proof to the next subsection, where we will give an alternative description of the $\mathfrak{gl}(1|1)$ action on $F(I)$.

Lemma 4 tells us that the boundary maps in the odd annular Khovanov complex intertwine the $\mathfrak{gl}(1|1)$ action, and this in turn implies that there is an induced $\mathfrak{gl}(1|1)$ action on the odd annular Khovanov homology of an
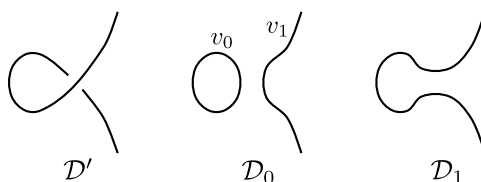
Figure 3: Reidemeister I. The diagram $\mathcal{D}'$ and the resolutions $\mathcal{D}_0$ and $\mathcal{D}_1$.

annular link diagram. By the remark preceding the lemma, it is further clear that this action preserves two gradings, namely the i-grading and the $(j - k)$-grading. In this sense, odd annular Khovanov homology becomes a bigraded $\mathfrak{gl}(1|1)$ representation.

Theorem 1. *If two annular link diagrams $\mathcal{P}(L)$ and $\mathcal{P}(L')$ differ by an annular Reidemeister move, then $AKh_{odd}(L)$ and $AKh_{odd}(L')$ are isomorphic as bigraded $\mathfrak{gl}(1|1)$ representations.*

Proof. Our proof of this theorem will closely follow the original proof of invariance of odd Khovanov homology given in [17]. In each step of the proof, we will verify that the relevant complexes and chain maps defined in [17] are compatible with the $\mathfrak{gl}(1|1)$ actions on the $F(I)$ when reinterpreted in the annular setting.

   Invariance under annular Reidemeister moves of type I. Consider an annular link diagram $\mathcal{D}'$ which is obtained from an annular link diagram $\mathcal{D}$ by performing a left-twist Reidemeister I move, so that $\mathcal{D}'$ contains a single positive crossing which is not already present in $\mathcal{D}$. Let $\mathcal{D}_0$ and $\mathcal{D}_1$ denote the two diagrams obtained from $\mathcal{D}'$ by resolving this crossing in the two possible ways. See Figure 3. Then $\mathcal{D}_1$ is isotopic to $\mathcal{D}$, and $\mathcal{D}_0$ is isotopic to a union of $\mathcal{D}$ with a small unknotted circle.

   Arguing as in the proof of [17, Prop.3.1], we can identify the odd annular Khovanov complex of $\mathcal{D}'$ with a mapping cone of a chain map

$$D \colon \mathrm{ACKh}_{odd}(\mathcal{D}_0) \longrightarrow \mathrm{ACKh}_{odd}(\mathcal{D}_1),$$

where $\mathrm{ACKh}_{odd}(\mathcal{D}_i)$ denotes the odd annular Khovanov complex of $\mathcal{D}_i$. The map D is surjective, and hence its mapping cone is quasi-isomorphic to $\ker(D)$. Moreover, since D is given by boundary maps, it intertwines the action of $\mathfrak{gl}(1|1)$, and hence $\ker(D)$ is itself a $\mathfrak{gl}(1|1)$ representation.

   Now let v_0 denote the small circular component of $\mathcal{D}_0$, and let v_1 denote the component that connects to v_0. Note that while v_0 is trivial in

all resolutions of $\mathcal{D}_0$, the component v_1 can be trivial in some resolutions of $\mathcal{D}_0$ and essential in others. Let $\mathrm{ACKh}_{odd}(\mathcal{D}_0, t)$ denote the subrepresentation of $\mathrm{ACKh}_{odd}(\mathcal{D}_0)$ corresponding to resolutions in which v_1 is trivial, and $\mathrm{ACKh}_{odd}(\mathcal{D}_0, e)$ the subrepresentation corresponding to resolutions in which v_1 is essntial. The kernel of D is then equal to the direct sum of $(v_0 - v_1) \wedge \mathrm{ACKh}_{odd}(\mathcal{D}_0, t)$ and $v_0 \wedge \mathrm{ACKh}_{odd}(\mathcal{D}_0, e)$, where in the first summand $\mathfrak{gl}(1|1)$ acts trivially on v_0 and v_1, and in the second summand $\mathfrak{gl}(1|1)$ acts trivially on v_0. It is now easy to see that the kernel of D is isomorphic as a complex of $\mathfrak{gl}(1|1)$ representations to $\mathrm{ACKh}_{odd}(\mathcal{D})$. Indeed, for each vertex I of the resolution cube of $\mathcal{D}_0$, define a sign $\epsilon(I)$ by $\epsilon(I) = (-1)^{\frac{1}{2}(n(I) - n(I_0) + |I|)}$ where $n(I)$ denotes the number of components in the resolution of $\mathcal{D}_0$ corresponding to I, and $n(I_0)$ denotes the number of components in the all-zero resolution of $\mathcal{D}_0$. Then contraction from the left with $\epsilon(I)v_0$ defines a chain map $\mathrm{ACKh}_{odd}(\mathcal{D}_0) \to \mathrm{ACKh}_{odd}(\mathcal{D})$ which intertwines the $\mathfrak{gl}(1|1)$ actions, and which restricts to an isomorphism on the kernel of D. In summary, we thus see that $\mathrm{ACKh}_{odd}(\mathcal{D}')$ and $\mathrm{ACKh}_{odd}(\mathcal{D})$ are quasi-isomorphic as complexes of $\mathfrak{gl}(1|1)$ representations, and this proves invariance under annular left-twist Reidemeister I moves.

Now suppose $\mathcal{D}'$ is obtained from $\mathcal{D}$ by an annular right-twist Reidemeister I move. We can then identify $\mathrm{ACKh}_{odd}(\mathcal{D}')$ with the mapping cone of an injective chain map

$$D \colon \mathrm{ACKh}_{odd}(\mathcal{D}_1) \longrightarrow \mathrm{ACKh}_{odd}(\mathcal{D}_0)$$

which is defined in terms of boundary maps and which thus intertwines the $\mathfrak{gl}(1|1)$ actions. The complex $\mathrm{ACKh}_{odd}(\mathcal{D}')$ is therefore quasi-isomorphic to the cokernel of D, and as a $\mathfrak{gl}(1|1)$ representation, the cokernel of D can be described explicitly as the direct sum of $\mathrm{ACKh}_{odd}(\mathcal{D}_0, t)/(v_0 - v_1)$ and $\mathrm{ACKh}_{odd}(\mathcal{D}_0, e)/(v_0)$. It is now easy to see that the inclusion $\mathrm{ACKh}_{odd}(\mathcal{D}) \to \mathrm{ACKh}_{odd}(\mathcal{D}_0)$ induces an isomorphism between $\mathrm{ACKh}_{odd}(\mathcal{D})$ and the cokernel of D, and this proves invariance under annular right-twist Reidemeister I moves.

Invariance under annular Reidemeister moves of type II. Next, assume $\mathcal{D}'$ is obtained from $\mathcal{D}$ by performing an annular Reidemeister move of type II. For $i, j \in \{0, 1\}$, let $\mathcal{D}_{ij}$ denote the diagram obtained from $\mathcal{D}'$ by choosing the i- and the j-resolution at the two crossings of $\mathcal{D}'$ which are not present in $\mathcal{D}$. Assume the numbering of the crossings is such that $\mathcal{D}_{01}$ is isotopic to $\mathcal{D}$, and $\mathcal{D}_{10}$ is obatined from $\mathcal{D}_{00}$ by adding a small unknotted circle. See Figure 4. Following the proof of [17, Prop. 3.2], we can write the complex
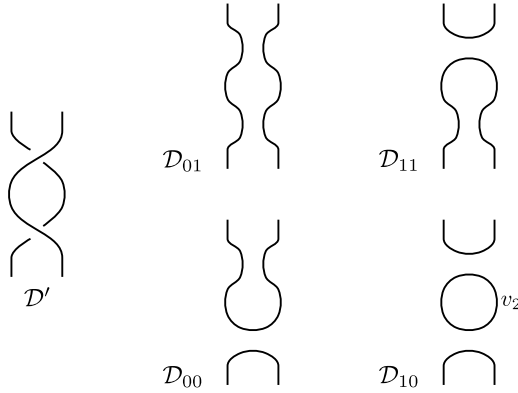
Figure 4: Reidemeister II. The diagram $\mathcal{D}'$ and the resolutions $\mathcal{D}_{ij}$.

$\mathrm{ACKh}_{odd}(\mathcal{D}')$ in the following form:

$$
\begin{array}{ccc}
\mathrm{ACKh}_{odd}(\mathcal{D}_{01}) & \longrightarrow & \mathrm{ACKh}_{odd}(\mathcal{D}_{11}) \\
\uparrow & & \uparrow \\
\mathrm{ACKh}_{odd}(\mathcal{D}_{00}) & \longrightarrow & \mathrm{ACKh}_{odd}(\mathcal{D}_{10})
\end{array}
$$

In this diagram, all arrows represent maps of $\mathfrak{gl}(1|1)$ representations. Let v_2 denote the small circular component in $\mathcal{D}_{10}$ and let $X \subset \mathrm{ACKh}_{odd}(\mathcal{D}_{10})$ be the subcomplex spanned by all elements of the form $a_{i_1} \wedge \cdots \wedge a_{i_\ell}$ which don't contain v_2 as a factor. Since v_2 is a trivial component, $\mathfrak{gl}(1|1)$ acts trivially on v_2, and hence the $\mathfrak{gl}(1|1)$ action preserves the subcomplex X. Moreover, the restriction $X \to \mathrm{ACKh}_{odd}(\mathcal{D}_{11})$ of the right vertical arrow to X is an isomorphism, and thus the above complex is quasi-isomorphic to a complex of the form

$$
\begin{array}{c}
\mathrm{ACKh}_{odd}(\mathcal{D}_{01}) \\
\uparrow \\
\mathrm{ACKh}_{odd}(\mathcal{D}_{00}) \longrightarrow \mathrm{ACKh}_{odd}(\mathcal{D}_{10})/X.
\end{array}
$$

In this complex, the horizontal arrow is an isomorphism, and so the above complex is quasi-isomorphic to $\mathrm{ACKh}_{odd}(\mathcal{D}_{01})$, which is in turn isomorphic to $\mathrm{ACKh}_{odd}(\mathcal{D})$.

Invariance under annular Reidemeister moves of type III. Suppose $\mathcal{D}'$ is obtained from $\mathcal{D}$ by performing an annular Reidemeister III move. By

repeating the arguments used in the proof of [17, Prop. 3.3], one can show that $\mathrm{ACKh}_{odd}(\mathcal{D})$ is quasi-isomorphic to a complex L, which fits into a short exact sequence

$$0 \longrightarrow R \xrightarrow{\Psi} L \xrightarrow{\Phi} P \longrightarrow 0.$$

In [17], the complexes L and R are depicted on the left and on the right of Figure 11, and the complex P is depicted in Figure 10. Using similar arguments as above, one can see that the complexes R, L, and P are complexes of $\mathfrak{gl}(1|1)$ representations, and that L is quasi-isomorphic to $\mathrm{ACKh}_{odd}(\mathcal{D})$ in the category of $\mathfrak{gl}(1|1)$ representations. Moreover, the maps Ψ and Φ are given by scalar multiplication on the spaces associated to the vertices of (partial) resolution cubes, and since $\mathfrak{gl}(1|1)$ acts on these spaces by linear endomorphisms, it is clear that Ψ and Φ intertwine the $\mathfrak{gl}(1|1)$ actions. As in the proof of [17, Prop. 3.3], the complex R is acyclic, and so L and hence $\mathrm{ACKh}_{odd}(\mathcal{D})$ is quasi-isomorphic in the category of $\mathfrak{gl}(1|1)$ representations to P.

By repeating the above arguments, one can show that $\mathrm{ACKh}_{odd}(\mathcal{D}')$ is quasi-isomorphic to an analogous complex P', and arguing as in the proof of [17, Prop. 3.3], one can see that this complex agrees with the complex P up to possible signs. Following [17], one can then show that P is isomorphic to P', via an isomorphism which is given by scalar multiplication on the spaces associated to vertices of (partial) resolution cubes. Since $\mathfrak{gl}(1|1)$ acts linearly on these spaces, it follows that this isomorphism intertwines the $\mathfrak{gl}(1|1)$ actions, and so the complexes $\mathrm{ACKh}_{odd}(\mathcal{D})$ and $\mathrm{ACKh}_{odd}(\mathcal{D}')$ are quasi-isomorphic in the category of $\mathfrak{gl}(1|1)$ representations. $\qquad\square$

Remark 6. The choice of the supergrading on odd annular Khovanov homology is not unique. In fact, by the definitions and by Remark 5, the boundary maps and the $\mathfrak{gl}(1|1)$ action preserve the $(j-k)$-grading. Therefore, any shift of the supergrading by a function of $j-k$ yields a new supergrading, which is also compatible with the $\mathfrak{gl}(1|1)$ action. For example, shifting the superdgree by the modulo 2 reduction of $(j-k+|L|+m)/2$, where m is as in the definition of the $\mathfrak{gl}(1|1)$ action, yields a new superdegree, which is given explicitly by the modulo 2 reduction of $(k-m)/2$.

3.3. An alternative description of the $\mathfrak{gl}(1|1)$ action

In this subsection, we will identify the vector superspace $F(I)$ with an isomorphic vector superspace $\tilde{F}(I)$. Using this identification, we will give a new description of the $\mathfrak{gl}(1|1)$ action on $F(I)$ and of the boundary maps in the

odd annular Khovanov complex. We will then use this description to prove a stronger version of Theorem 1.

As before, we will denote by $a_1, \ldots, a_n$ the components of the resolution $\mathcal{P}^I(L)$. Moreover, we will denote by V a 2-dimensional vector superspace spanned by two homogeneous elements v_+ and v_- of superdegrees $\bar{0}$ and $\bar{1}$, respectively. Let $\tilde{F}(I)$ denote the vector superspace

$$\tilde{F}(I) := (V \otimes \cdots \otimes V)\langle (n + |I| + n_+ - 2n_- - |L|)/2 \rangle,$$

where there are n tensor factors on the right-hand side, and where it is understood that the ith tensor factor corresponds to the component a_i. We can define an isomorphism of vector superspaces

$$\alpha_I \colon F(I) \longrightarrow \tilde{F}(I)$$

by sending the element $a_{i_1} \wedge \cdots \wedge a_{i_\ell} \in F(I)$ for $i_1 < \cdots < i_\ell$ to the element $v_{\epsilon_1} \otimes \cdots \otimes v_{\epsilon_n} \in \tilde{F}(I)$ where

$$\epsilon_i = \begin{cases} + & \text{if } i \notin \{i_1, \ldots, i_\ell\}, \\ - & \text{if } i \in \{i_1, \ldots, i_\ell\}. \end{cases}$$

Remark 7. Under this isomorphism, the (i, j, k)-trigrading on $F(I)$ corresponds to an (i, j, k)-trigrading on $\tilde{F}(I)$. Explicitly, the i-grading is constant on $\tilde{F}(I)$ and given by $i = |I| - n_-$. The j-grading on $\tilde{F}(I)$ is given by

$$j(v_{\epsilon_1} \otimes \cdots \otimes v_{\epsilon_n}) = j(v_{\epsilon_1}) + \cdots + j(v_{\epsilon_n}) + |I| + n_+ - 2n_-,$$

where the j-grading on the ith tensor factor is defined by $j(v_\pm) := \pm 1$. The k-grading on $\tilde{F}(I)$ is given by

$$k(v_{\epsilon_1} \otimes \cdots \otimes v_{\epsilon_n}) = k(v_{\epsilon_1}) + \cdots + k(v_{\epsilon_n}),$$

where the k-grading on the ith tensor factor is defined by $k(v_\pm) := 0$ if a_i is trivial, and $k(v_\pm) := \pm 1$ if a_i is essential.

Remark 8. It should be noted that the map α_I depends nontrivially on the ordering of the components $a_1, \ldots, a_n$ of the resolution $\mathcal{P}^I(L)$. However, different orderings lead to coherent maps α_I, in the following sense. If two orderings differ by exchanging the a_i and a_{i+1}, then the maps α_I and α'_I

associated to these two orderings fit into a commutative diagram

$$
\begin{array}{ccc}
 & & \tilde{F}(I) \\
 & \nearrow^{\alpha_I} & \uparrow \text{id}^{\otimes(i-1)}\otimes\tau\otimes\text{id}^{\otimes(n-i-1)} \\
F(I) & & \\
 & \searrow_{\alpha'_I} & \tilde{F}(I)
\end{array}
$$

where $\tau\colon V\otimes V\to V\otimes V$ denotes the twist map given by $\tau(v\otimes w) = (-1)^{|v||w|}w\otimes v$ for all homogeneous elements $v,w\in V$.

Remark 9. Let $\tilde{\alpha}_I$ denote the map α_I without the overall shifts of the supergrading on the domain and the codomain. That is, $\tilde{\alpha}_I$ is a map

$$
\tilde{\alpha}_I\colon \Lambda^*(\operatorname{Span}_{\mathbb{C}}\{a_1,\ldots,a_n\}) \longrightarrow V^{\otimes n},
$$

where the supergrading on the exterior algebra is defined by collapsing the natural $\mathbb{Z}_{\geq 0}$-grading on the exterior algebra to a $\mathbb{Z}_2$-grading. We can now identify the ith tensor factor of $V^{\otimes n}$ with $\Lambda^*(\operatorname{Span}_{\mathbb{C}}\{a_i\})$ by using the linear map given by $v_+\mapsto 1$ and $v_-\mapsto a_i$. Under this identification, the map $\tilde{\alpha}_I$ becomes the "obvious" algebra isomorphism

$$
\tilde{\alpha}_I\colon \Lambda^*(\operatorname{Span}_{\mathbb{C}}\{a_1,\ldots,a_n\}) \longrightarrow \Lambda^*(\operatorname{Span}_{\mathbb{C}}\{a_1\})\otimes\cdots\otimes\Lambda^*(\operatorname{Span}_{\mathbb{C}}\{a_n\})
$$

given by sending the generator a_i to the element $1\otimes\cdots\otimes 1\otimes a_i\otimes 1\otimes\cdots 1$. Here, $\otimes$ denotes the supergraded tensor product of supergraded algebras: it is the ordinary tensor product on the level of vector superspaces, but the algebra multiplication is given by $(a\otimes b)\cdot(c\otimes d) := (-1)^{|b||c|}(a\cdot c)\otimes(b\cdot d)$ for all homogeneous algebra elements a,b,c,d.

We will now define a $\mathfrak{gl}(1|1)$ action on the vector superspace $\tilde{F}(I)$. To define this action, we will first define a $\mathfrak{gl}(1|1)$ action on each tensor factor of

$$
\tilde{F}(I) = V^{\otimes n}\langle(n + |I| + n_+ - 2n_- - |L|)/2\rangle,
$$

and then regard $\tilde{F}(I)$ as the tensor product representation (with shifted supergrading).

As in the previous subsection, we will assume that the components of $\mathcal{P}^I(L)$ are ordered so that the trivial ones precede the essential ones, and that the essential components of $\mathcal{P}^I(L)$ are ordered according to their proximity to the basepoint $\mathbb{X}$.

If V is a tensor factor of $\tilde{F}(I)$ which corresponds to a trivial component of $\mathcal{P}^I(L)$, then we now define the $\mathfrak{gl}(1|1)$ action on V to be trivial. If V is a tensor factor of $\tilde{F}(I)$ which corresponds to an essential component a_i with $i - n_t - 1$ even, then we identify V with the fundamental representation $V = L_{(1,0)} = \mathbb{C}^{1|1}$ of $\mathfrak{gl}(1|1)$. Explicitly, the $\mathfrak{gl}(1|1)$ action on such a factor is given by:

$$(h_1,h_2)=(1,0) \quad v_+ \quad \overset{e=1}{\underset{f=1}{\rightleftharpoons}} \quad v_- \quad (h_1,h_2)=(0,1)$$
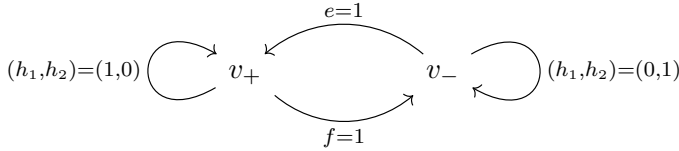
Finally, if V is a tensor factor of $\tilde{F}(I)$ which corresponds to an essential component a_i with $i - n_t - 1$ odd, then we identify V with the representation $V^*\langle 1 \rangle = L_{(1,0)}^*\langle 1 \rangle$ via the map which takes v_+ to v_-^* and v_- to $-v_+^*$. Explicitly, the $\mathfrak{gl}(1|1)$ action on such a factor is given by:

$$(h_1,h_2)=(0,-1) \quad v_+ \quad \overset{e=1}{\underset{f=-1}{\rightleftharpoons}} \quad v_- \quad (h_1,h_2)=(-1,0)$$
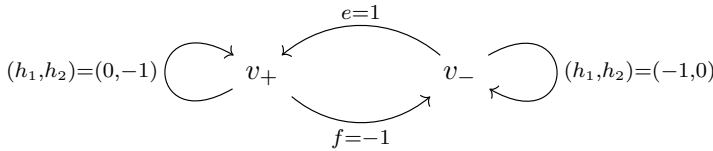
In summary, we obtain a $\mathfrak{gl}(1|1)$ action on $\tilde{F}(I)$, and we now claim that this action corresponds to the $\mathfrak{gl}(1|1)$ action on $F(I)$ defined in the previous subsection.

Lemma 5. *For each $I\colon \mathcal{X} \to \{0,1\}$, the isomorphism $\alpha_I\colon F(I) \to \tilde{F}(I)$ intertwines the $\mathfrak{gl}(1|1)$ actions on $F(I)$ and $\tilde{F}(I)$.*

Proof. On both $F(I)$ and $\tilde{F}(I)$, the generator $h_+ = h_1 + h_2$ acts by multiplication by m, where m is equal to 0 if n_e is even and equal to 1 if n_e is odd. Likewise, the generator $h_- = h_1 - h_2$ acts on both $F(I)$ and $\tilde{F}(I)$ by multiplication by k, where k denotes the annular grading. Thus it is clear that α_I intertwines the actions of h_+ and h_- and, therefore, also the actions of h_1 and h_2.

To see that α_I also intertwines the actions of e and f, we first note that e and f act trivially on $\Lambda^*\mathrm{Span}_{\mathbb{C}}\{a_1, \ldots, a_{n_t}\}$ and also on the first n_t tensor factors of $\tilde{F}(I)$, as these correspond to the trivial components $a_1, \ldots, a_{n_t}$.

Now suppose a_i is an essential component, and suppose that the essential components are ordered according to their proximity to the basepoint $\mathbb{X}$, as

before. If $i - n_t - 1$ is even, then the definitions of a_I and b_I imply that e and f act on $\Lambda^*\mathrm{Span}_{\mathbb{C}}\{a_i\}$ by the maps $v \mapsto v \llcorner a_i$ and $v \mapsto v \wedge a_i$, respectively. Explicitly,

$$
\begin{array}{ccc}
 & e=1 & \\
1 & \rightleftarrows & a_i \\
 & f=1 &
\end{array}
$$
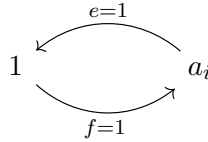
and this diagram is consistent with the diagram for the $\mathfrak{gl}(1|1)$ action on the ith tensor factor of $\tilde{F}(I)$.

Likewise, if $i - n_t - 1$ is odd, then e and f act on $\Lambda^*\mathrm{Span}_{\mathbb{C}}\{a_i\}$ by the maps $v \mapsto v \llcorner a_i$ and $v \mapsto v \wedge (-a_i)$, or more explicitly,

$$
\begin{array}{ccc}
 & e=1 & \\
1 & \rightleftarrows & a_i \\
 & f=-1 &
\end{array}
$$
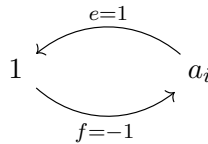
and again this diagram is consistent with the diagram for the $\mathfrak{gl}(1|1)$ action on the ith tensor factor of $\tilde{F}(I)$.

To complete the proof, we note that if the maps $v \mapsto v \llcorner a_i$ and $v \mapsto v \wedge (\pm a_i)$ act on a wedge product of the form

$$
a_{i_1} \wedge \cdots \wedge a_{i_r} \wedge \Lambda^*\mathrm{Span}_{\mathbb{C}}\{a_i\} \wedge a_{i_s} \wedge \cdots \wedge a_{i_\ell}
$$

for $i_1 < \cdots < i_r < i < i_s < \cdots < i_\ell$, then they pick up the sign $(-1)^{\ell-s+1}$, where the sign comes from permuting $\pm a_i$ across $a_{i_s} \wedge \cdots \wedge a_{i_\ell}$. Because of our definition of the tensor product of $\mathfrak{gl}(1|1)$ representations, the same sign occurs when e and f act on the tensor product

$$
v_{\epsilon_1} \otimes \cdots \otimes v_{\epsilon_{i-1}} \otimes V \otimes v_{\epsilon_{i+1}} \otimes \cdots \otimes v_{\epsilon_n},
$$

where here $\epsilon_1, \ldots, \epsilon_{i-1}, \epsilon_{i+1}, \ldots, \epsilon_n$ are related to $i_1, \ldots, i_r, i_s, \ldots, i_\ell$ as in the definition of the map α_I. $\qquad\square$

The next lemma will relate the merge and split maps F_M and F_Δ to the maps $m \colon V \otimes V \to V$ and $\Delta \colon V \to V \otimes V$ defined as follows:

$$
m = \begin{cases} v_+ \otimes v_+ \longmapsto v_+, & v_+ \otimes v_- \longmapsto v_-, \\ v_- \otimes v_- \longmapsto 0, & v_- \otimes v_+ \longmapsto v_-, \end{cases}
$$

$$
\Delta = \begin{cases} v_+ \longmapsto v_- \otimes v_+ - v_+ \otimes v_-, \\ v_- \longmapsto v_- \otimes v_-. \end{cases}
$$

Note that m is homogeneous of superdegree $\bar{0}$ and Δ is homogeneous of superdegree $\bar{1}$.

Lemma 6. *Let $I_0, I_1 \colon \mathcal{X} \to \{0,1\}$ be two vertices of the resolution hypercube for which there is an oriented edge from I_0 to I_1. Denote by $a_1, \ldots, a_n$ the components of $\mathcal{P}^{I_0}(L)$ and by $a'_1, \ldots, a'_{n \mp 1}$ the components of $\mathcal{P}^{I_1}(L)$, and let $\otimes$ denote the tensor product of homogeneous linear maps defined in Subsection 2.1.*

1) *If the components a_i and a_{i+1} merge into the component a'_i, and if $a_j = a'_j$ for $j < i$ and $a_j = a'_{j-1}$ for $j > i+1$, then*

$$\alpha_{I_1} \circ F_M \circ \alpha_{I_0}^{-1} = \mathrm{id}^{\otimes (i-1)} \otimes m \otimes \mathrm{id}^{\otimes (n-i-1)}.$$

2) *If the component a_i splits into the components a'_i and a'_{i+1}, and if $a_j = a'_j$ for $j < i$ and $a_j = a'_{j+1}$ for $j > i$, then*

$$\alpha_{I_1} \circ F_\Delta \circ \alpha_{I_0}^{-1} = \mathrm{id}^{\otimes (i-1)} \otimes \Delta \otimes \mathrm{id}^{\otimes (n-i)},$$

where it is assumed that the arrow decorating the split region points from a'_i to a'_{i+1}.

Proof. Recalling the definitions of F_M, F_Δ, and $\otimes$, it is easy to see that we can reduce to case where one of $\mathcal{P}^{I_0}(L)$ and $\mathcal{P}^{I_1}(L)$ has exactly two components and the other one has exactly one component.

Suppose first that $\mathcal{P}^{I_0}(L)$ has exactly two components a_1 and a_2, and that $\mathcal{P}^{I_1}(L)$ has a single component a'_1. Then the map $F_M \colon F(I_0) \to F(I_1)$ is given by sending each of the components a_1 and a_2 to a'_1. Explicitly,

$$F_M = \begin{cases} 1 \longmapsto 1, & a_1 \longmapsto a'_1, \\ a_1 \wedge a_2 \longmapsto 0, & a_2 \longmapsto a'_1, \end{cases}$$

and comparing with the definition of m, we see that $\alpha_{I_1} \circ F_M \circ \alpha_{I_0}^{-1} = m$.

Now suppose that $\mathcal{P}^{I_0}(L)$ has a single component a_1, and that $\mathcal{P}^{I_1}(L)$ has exactly two components a'_1 and a'_2. Then the map $F_\Delta \colon F(I_0) \to F(I_1)$ is given by first sending a_1 to either a'_1 or a'_2, and then wedge-multiplying the result from the left by $a'_1 - a'_2$ (where we assume that the arrow decorating

the split region points from a'_1 to a'_2). Explicitly,

$$F_\Delta = \begin{cases} 1 \longmapsto a'_1 - a'_2, \\ a_1 \longmapsto a'_1 \wedge a'_2, \end{cases}$$

and comparing with the definition of Δ, we see that $\alpha_{I_1} \circ F_\Delta \circ \alpha_{I_0}^{-1} = \Delta$. $\square$

We may now regard m and Δ as maps associated to merge and split operations between resolutions on an annulus. Since, up to a sign in the definition of $\Delta(v_+)$, m and Δ coincide with Khovanov's multiplication and comultiplication maps, it follows that they exhibit the same behavior with respect to the k-grading as the latter maps. In particular, it follows from [27] that there are decompositions

$$m = m_0 + m_- \quad \text{and} \quad \Delta = \Delta_0 + \Delta_-,$$

where m_0 and Δ_0 have k-degree 0 and m_- and Δ_- have k-degree -2. The explicit form of these decompositions depends on whether the components involved in the merge or split operation are trivial or essential.

If all involved components are trivial, then $m_0 = m$ and $\Delta_0 = \Delta$.

If a trivial component and an essential component are merged into a single essential component, then the map $v \mapsto m_0(v_+ \otimes v)$ is the identity map and the map $v \mapsto m_0(v_- \otimes v)$ is the zero map, where here it is assumed that the first factor in $V \otimes V$ corresponds to the trivial component. If the second factor corresponds to the trivial component, then the same holds true, but with m_0 replaced by $m_0 \circ \tau$.

If a single essential component is split into a trivial component and an essential component, then the map Δ_0 is given by $v \mapsto v_- \otimes v$, where here it is assumed that the first factor of $V \otimes V$ corresponds to the trivial component. If the second factor corresponds to the trivial component, then the same holds true, but with Δ_0 replaced by $-\tau \circ \Delta_0$.

Finally, if two essential components are merged into a single trivial component, or if a single trivial component is split into two essential components, then the corresponding maps m_0 and Δ_0 are given as follows:

$$m_0 = \begin{cases} v_+ \otimes v_+ \longmapsto 0, \quad v_+ \otimes v_- \longmapsto v_-, \\ v_- \otimes v_- \longmapsto 0, \quad v_- \otimes v_+ \longmapsto v_-, \end{cases}$$

$$\Delta_0 = \begin{cases} v_+ \longmapsto v_- \otimes v_+ - v_+ \otimes v_-, \\ v_- \longmapsto 0. \end{cases}$$

We can now prove Lemma 4 from the previous subsection.

Proof of Lemma 4. In view of Lemmas 5 and 6, it is enough to show that the maps

$$\mathrm{id}^{\otimes(i-1)} \otimes m_0 \otimes \mathrm{id}^{\otimes(n-i-1)} \quad \text{and} \quad \mathrm{id}^{\otimes(i-1)} \otimes \Delta_0 \otimes \mathrm{id}^{\otimes(n-i)}$$

intertwine the $\mathfrak{gl}(1|1)$ actions, where here the notation is to be understood as in the two parts of Lemma 5. Since id intertwines the $\mathfrak{gl}(1|1)$ actions and because of Lemma 1, it further suffices to show that m_0 and Δ_0 intertwine the $\mathfrak{gl}(1|1)$ actions when viewed as factors of these maps.

If m_0 and Δ_0 correspond to merges and splits which only involve trivial components, then this is obvious becasue in this case the $\mathfrak{gl}(1|1)$ actions are trivial on the domain and the codomain of m_0 and Δ_0.

If m_0 corresponds to a merge of a trivial component with an essential component, then m_0 (or $m_0 \circ \tau$) can be described in terms of the identity map or the zero map, depending on whether the trivial component is labeled by v_+ or v_-, and these maps, too, intertwine the $\mathfrak{gl}(1|1)$ action. (Note that if $m_0 \circ \tau$ intertwines the $\mathfrak{gl}(1|1)$ actions, then so does m_0, because τ is an isomorphism of $\mathfrak{gl}(1|1)$ representations).

If further Δ_0 corresponds to splitting an essential component into a trivial and an essential component, then Δ_0 (or $-\tau \circ \Delta_0$) is given by $v \mapsto v_- \otimes v$, and again this map intertwines the $\mathfrak{gl}(1|1)$ action.

Finally, if m_0 or Δ_0 corresponds to merging to essential components into a single trivial component, or to splitting a single trivial component into two essential components, then m_0 and Δ_0 agree with the maps $\tilde{p}$ and $\tilde{i}$ defined in Subsection 2.3, and we have already seen that these maps intertwine the $\mathfrak{gl}(1|1)$ action. $\square$

Using the alternative description of the $\mathfrak{gl}(1|1)$ action in terms of the spaces $\tilde{F}(I)$, we can now prove the following stronger version of Theorem 1.

Theorem 2. *If two annular link diagrams $\mathcal{P}(L)$ and $\mathcal{P}(L')$ differ by an annular Reidemeister move, then the odd annular Khovanov complexes $ACKh_{odd}(\mathcal{P}(L))$ and $ACKh_{odd}(\mathcal{P}(L'))$ are homotopy equivalent as complexes of $\mathfrak{gl}(1|1)$ representations.*

Proof. In [23], Putyra defines a category $\Bbbk\mathbf{ChCob}_{/\ell}(0)$, which generalizes Bar-Natan's category $\mathcal{C}ob_\ell^3(\emptyset)$ defined in [3]. To each decorated link diagram $\mathcal{P}(L)$, Putyra assigns a chain complex $Kh(\mathcal{P}(L))$, which lives in the additive closure of $\Bbbk\mathbf{ChCob}_{/\ell}(0)$, and which can be viewed as a generalization of Bar-Natan's formal Khovanov bracket of $\mathcal{P}(L)$.

Putyra shows that if two link diagrams $\mathcal{P}(L)$ and $\mathcal{P}(L')$ differ by a Reidemeister move, then there is a homotopy equivalence between the generalized Khovanov brackets $Kh(\mathcal{P}(L))$ and $Kh(\mathcal{P}(L'))$. Moreover, Putyra constructs a functor, called a chronological TQFT, which takes $Kh(\mathcal{P}(L))$ to the odd Khovanov complex of $\mathcal{P}(L)$.

Putyra's construction can be carried out equally well in the annular setting. In particular, if $\mathcal{P}(L)$ is an annular link diagram, then one can associate an annular version of the generalized Khovanov bracket, which lives in the additive closure of an annular version of the category $\Bbbk\mathbf{ChCob}_{/\ell}(0)$. Moreover, there is a chronological TQFT defined on this annular category, which takes the generalized annular Khovanov bracket of $\mathcal{P}(L)$ to the odd annular Khovanov complex $\mathrm{ACKh}_{odd}(\mathcal{P}(L))$. We will henceforth denote this chronological TQFT by $\mathcal{F}_o^{ann}$. Explicitly, $\mathcal{F}_o^{ann}$ is an additive functor given by assigning the maps $m_0\colon V \otimes V \to V$ and $\Delta_0\colon V \to V \otimes V$ to (suitably decorated) annular saddle cobordisms, and the maps $\iota\colon \mathbb{C} \to V$ and $\epsilon\colon V \to \mathbb{C}$ given by $\iota(1) = v_+$, $\epsilon(v_+) = 0$, and $\epsilon(v_-) = 1$ to (suitably decorated) annular cup and cap cobordisms.

We have already seen in the proof of Lemma 4 that the maps m_0 and Δ_0 intertwine the $\mathfrak{gl}(1|1)$ action. Since annular cup and cap cobordisms can only create or annihilate trivial components, and since the $\mathfrak{gl}(1|1)$ action is trivial on tensor factors corresponding to such components, it is further clear that the maps ι and ϵ also intertwine the $\mathfrak{gl}(1|1)$ action. We can thus view the functor $\mathcal{F}_o^{ann}$ as a functor with values in the representation category of $\mathfrak{gl}(1|1)$. Since this functor is also additive, it takes the homotopy equivalences that Putyra associates to Reidemeister moves to homotopy categories in the representation category of $\mathfrak{gl}(1|1)$, and this proves the theorem. $\qquad\square$

Remark 10. The category $\Bbbk\mathbf{ChCob}_{/\ell}(0)$ defined in [23] comes equipped with a $\mathbb{Z} \times \mathbb{Z}$-grading on its morphism sets. The modulo 2 reduction of the second $\mathbb{Z}$-factor in this $\mathbb{Z} \times \mathbb{Z}$-grading corresponds to the supergrading used in our definition of the $\mathfrak{gl}(1|1)$ action on odd annular Khovanov homology.

Acknowledgements

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