

Inverse Galois problem for del Pezzo surfaces over finite fields

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Dedicated to the memory of Sir Henry Peter Francis Swinnerton-Dyer

We completely solve the inverse Galois problem for del Pezzo surfaces of degree 2 and 3 over all finite fields.

1. Introduction

Recall that a *del Pezzo surface* is a smooth projective surface S over a field k with ample anticanonical class $-K_S$. Its *degree* is defined to be the integer $d = K_S^2$. One has $1 \leq d \leq 9$, with del Pezzo surfaces of lower degree generally having more complicated arithmetic and geometry. The case $d = 3$ is by far the most famous; such surfaces are exactly cubic surfaces. See [7, §IV] and [3, §8] for further background on del Pezzo surfaces.

Let S be a del Pezzo surface of degree $d \leq 6$ over a finite field \mathbb{F}_q . Over the algebraic closure S is the blow-up of \mathbb{P}^2 in $9 - d$ points in general position. But S can be non-rational over \mathbb{F}_q . To understand the geometry of S , one uses the action of the absolute Galois group $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ on the Picard group $\text{Pic } \overline{S}$. This action preserves the canonical class and the intersection pairing, and the group of such automorphisms of $\text{Pic } \overline{S}$ is isomorphic to the Weyl group $W(\mathbf{E}_{9-d})$ [7, Thm. 23.9] (we follow the conventions of [3, §8.2.2] regarding the root systems \mathbf{E}_{9-d}). We therefore obtain a well-defined conjugacy class $C(S) \subset W(\mathbf{E}_{9-d})$, induced by the image of the Frobenius element. The surfaces with trivial conjugacy class are exactly those that are blow ups of \mathbb{P}^2 in rational points. A natural problem is:

Question. Given d and q , which conjugacy classes of $W(\mathbf{E}_{9-d})$ can arise this way?

We call this the *inverse Galois problem for del Pezzo surfaces over finite fields*. An analogous problem for cubic surfaces over \mathbb{Q} has recently been

completely solved in [4, Theorem 0.1], where it was shown that every conjugacy class of subgroups of $W(\mathbf{E}_6)$ indeed arises for some cubic surface over \mathbb{Q} .

However *not* every conjugacy class of $W(\mathbf{E}_6)$ occurs over every finite field. For example, a simple point count shows that there is no split cubic surface over \mathbb{F}_2 (corresponding to the trivial conjugacy class), as such a surface would have more rational points than the ambient projective space $\mathbb{P}_{\mathbb{F}_2}^3$! (See [12, §2.3.3]).

In [1, Thm. 1.7] this question was completely answered for *all large* q . Specifically, the density of such surfaces was explicitly calculated for $d \leq 3$, showing that

$$(1.1) \quad \lim_{q \rightarrow \infty} \frac{\#\{S \in \mathcal{S}_d(\mathbb{F}_q) : C(S) = C\}}{\#\mathcal{S}_d(\mathbb{F}_q)} = \frac{\#C}{\#W(\mathbf{E}_{9-d})},$$

where $\mathcal{S}_d(\mathbb{F}_q)$ denotes the set of isomorphism classes of del Pezzo surfaces of degree d over \mathbb{F}_q . For $d > 3$ a similar formula holds, but one needs to weight each surface by the reciprocal of the size of its automorphism group.

Unfortunately the result (1.1) is not effective. One therefore still needs to determine the existence of each class, with particular attention to small finite fields.

There has been a long string of papers working on this problem solving various cases (see for example [1, 6, 8, 9, 14–16]), using a range of methods:

- blowing-up suitable collections of closed points;
- conic bundles;
- Galois twist by an automorphism.

The last method is very useful for del Pezzo surfaces of degrees 2 and 1, as there are always non-trivial automorphisms given by the Geiser and Bertini involution, respectively. (See [1, §4.1.2] and [1, §5.1.1] for explicit descriptions of these twists).

For $d = 5, 6$ there is a unique surface over the algebraic closure, and one constructs all classes over every \mathbb{F}_q by using twists. The case $d = 4$ was finished in [16]. The most general results for $d = 3, 2$ were also obtained in [16]. But there remains the incomplete case C_{14} for cubic surfaces (see [16, Thm. 1.3]), and four incomplete cases 28, 35, 47 and 56 for del Pezzo surfaces of degree 2 (see [16, Thm. 1.2]).

In this paper, we finish off these remaining cases and give a complete classification for cubic surfaces and del Pezzo surfaces of degree 2.

1.1. Cubic surfaces

The remaining case for cubic surfaces is conjugacy class C_{14} in Swinnerton-Dyer's notation [13, Tab. 1]. In [9, Lem. 3.5 and §5.4], this class was constructed for $q \equiv 1 \pmod{6}$. The construction was completely explicit; namely

$$x_0^3 = f(x_1, x_2, x_3)$$

such that the plane cubic $f(x_1, x_2, x_3) = 0$ is sufficiently general.

However this class has resisted for other q . The main difficulty is that such surfaces are minimal, and there is no non-trivial third root of unity in \mathbb{F}_q for $q \equiv 2 \pmod{3}$. Moreover, computational evidence suggests that for $q \equiv 2 \pmod{3}$ there is no such surface over \mathbb{F}_q with non-trivial automorphism group. So the aforementioned methods of blow-ups, twists, and conic bundles, yield nothing useful.

Despite being the most difficult case, it is one of the most common conjugacy classes; in the limit as $q \rightarrow \infty$, the density of such cubic surfaces in (1.1) is $1/9$, so that one expects to find such a surface very quickly. We show that this class in fact exists over all finite fields, which leads to the following complete classification.

We say that a class C_n of $W(\mathbf{E}_6)$ exists over \mathbb{F}_q if there is a smooth cubic surface over \mathbb{F}_q whose image of the Frobenius element in $W(\mathbf{E}_6)$ is an element of the conjugacy class C_n appearing in [13, Tab. 1].

Theorem 1.1. *Each class C_n of $W(\mathbf{E}_6)$ exists over \mathbb{F}_q , except in exactly the following cases.*

- 1) Class C_1 does not exist over $\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_5$.
- 2) Classes C_2, C_3 do not exist over $\mathbb{F}_2, \mathbb{F}_3$.
- 3) Classes C_4, C_5, C_9, C_{10} do not exist over \mathbb{F}_2 .

In particular, classes C_6 – C_8 and C_{11} – C_{25} exist over all finite fields.

An expanded version of Swinnerton-Dyer's table [13, Tab. 1] can be found in Manin's book [7, §31]. (Swinnerton-Dyer's numbering is column 0 of Manin's table.) Note however that Manin's table contains some mistakes; a corrected version can be found in [1, Tab. 7.1].

1.2. Del Pezzo surfaces of degree 2

The classes 47 and 56 can be obtained using blow-ups and twists of the last missing cubic surface class C_{14} . The existence of the classes 28 and 35 was only unknown over \mathbb{F}_3 ; we give an explicit construction of these, and thus obtain following complete classification.

We say that a class n of $W(\mathbf{E}_7)$ exists over \mathbb{F}_q if there is a smooth del Pezzo surface of degree 2 over \mathbb{F}_q whose image of the Frobenius element in $W(\mathbf{E}_7)$ is an element of the conjugacy class appearing in the n -th row of [15, Appendix, Tab. 1].

Theorem 1.2. *Each class n of $W(\mathbf{E}_7)$ exists over \mathbb{F}_q , except in exactly the following cases.*

- 1) Classes 1 and 49 do not exist over $\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_4, \mathbb{F}_5, \mathbb{F}_7, \mathbb{F}_8$.
- 2) Classes 5 and 10 do not exist over $\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_4, \mathbb{F}_5$.
- 3) Classes 2, 3, 18 and 31 do not exist over $\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_4$.
- 4) Classes 4, 6–9, 12–14, 17, 21, 22, 25, 28, 32, 33, 35, 38, 40, 50, 53, 55, 60 do not exist over \mathbb{F}_2 .

In particular, classes 11, 15, 16, 19, 20, 23, 24, 26, 27, 29, 30, 34, 36, 37, 39, 41–48, 51, 52, 54, 56–59 exist over all finite fields.

The table [15, Appendix, Tab. 1] is based upon Carter's [2, Tab. 10]. Urabe [17, Tab. 1] also compiled a table of the possibilities for del Pezzo surfaces of degree 2; but note that the rows in Urabe's table are ordered differently to Carter's table.

Knowledge about the possibilities for the Galois action of del Pezzo surfaces over small finite fields is often required in proofs of some results concerning properties of del Pezzo surfaces (see e.g. the proof of [10, Thm. 1]). We hope that the results in this paper should assist with such arguments in the future.

2. Cubic surfaces — Proof of Theorem 1.1

As explained in the introduction, by [16, Thm. 1.3] it suffices to construct the remaining case C_{14} over every finite field. To do so we require the following lemma.

Lemma 2.1. *Let k be an algebraically closed field and let Π_1, Π_2 and Π_3 be three planes in \mathbb{P}_k^3 which have exactly one common point Q . Let C_1, C_2 and C_3 be three cubic curves lying in Π_1, Π_2 and Π_3 respectively, such that $\Pi_i \cap C_j = \Pi_j \cap C_i$ and such that $Q \notin C_i$ for any i, j . Then there is one-dimensional family of cubic surfaces over k containing C_1, C_2 and C_3 .*

Proof. We can assume the planes are given by $\Pi_i : x_i = 0$, so $Q = (1 : 0 : 0 : 0)$. The cubic curves are given by $C_i : P_i(x_0, \dots, \widehat{x_i}, \dots, x_3) = 0$. Each polynomial P_i contains the monomial x_0^3 , since $Q \notin C_i$, so we may assume that the coefficient of x_0^3 is 1 for each polynomial P_i . Put

$$P(x_0, x_1, x_2, x_3) = P_1(x_0, x_2, x_3) + P_2(x_0, x_1, x_3) + P_3(x_0, x_1, x_2) - P_1(x_0, 0, x_3) - P_2(x_0, x_1, 0) - P_3(x_0, 0, x_2) + x_0^3.$$

Then one has

$$P(x_0, 0, x_2, x_3) = P_1(x_0, x_2, x_3), \quad P(x_0, x_1, 0, x_3) = P_2(x_0, x_1, x_3), \\ P(x_0, x_1, x_2, 0) = P_3(x_0, x_1, x_2),$$

since $P_1(x_0, 0, 0) = P_2(x_0, 0, 0) = P_3(x_0, 0, 0) = x_0^3$ and

$$P_1(x_0, 0, x_3) = P_2(x_0, 0, x_3), \quad P_1(x_0, x_2, 0) = P_3(x_0, 0, x_2), \\ P_2(x_0, x_1, 0) = P_3(x_0, x_1, 0).$$

Therefore for any $(\mu : \lambda) \in \mathbb{P}^1(k)$ the equation $\mu x_1 x_2 x_3 = \lambda P(x_0, x_1, x_2, x_3)$ gives a cubic surface passing through the curves C_1, C_2 and C_3 . \square

We apply this as follows. Let \mathbb{F}_q be a finite field and \mathbf{F} the Frobenius automorphism. Let L_1, L_2 and L_3 be a triple of conjugate non-coplanar lines in $\mathbb{P}_{\mathbb{F}_q}^3$ passing through a common point $Q \in \mathbb{P}^2(\mathbb{F}_q)$. Consider a point $p_1 \in L_1(\mathbb{F}_{q^9}) \setminus L_1(\mathbb{F}_{q^3})$, and its conjugates $p_{i+1} = \mathbf{F}^i p_1, i \in \{2, \dots, 9\}$. One has $p_i, p_{i+3}, p_{i+6} \in L_i$ (here and in what follows, subscripts are taken modulo 9). Consider the line E_1 passing through p_1 and p_2 , and its conjugates $E_{i+1} = \mathbf{F}^i E_1, i \in \{2, \dots, 9\}$. Denote this configuration of nine conjugate lines by \mathcal{E} . We want to show that there exists a smooth cubic surface X over \mathbb{F}_q containing \mathcal{E} .

The configuration \mathcal{E} is defined over \mathbb{F}_q , in particular the linear system of cubics containing it is defined over \mathbb{F}_q . For $\{i, j, k\} = \{1, 2, 3\}$, let Π_i be the plane spanned by L_j and L_k . Then $E_{i+1}, E_{i+4}, E_{i+7} \subset \Pi_i$ and $\Pi_i \cap \Pi_j \cap \mathcal{E} = \{p_k, p_{k+3}, p_{k+6}\}$. Therefore by Lemma 2.1 there is one-dimensional family \mathcal{X}

of cubic surfaces over \mathbb{F}_q passing through \mathcal{E} . We now show that there is a unique singular member.

Lemma 2.2. *There is a unique singular cubic surface over \mathbb{F}_q containing the configuration \mathcal{E} . This surface is the union of the planes Π_1 , Π_2 and Π_3 .*

Proof. Let X be a singular surface containing \mathcal{E} , and S a singular point of $X_{\overline{\mathbb{F}}_q}$. There is a line E in \mathcal{E} that does not pass through S . Without loss of generality, we may assume that $E = E_1$. Then E_1 meets the other lines in \mathcal{E} in at least 3 points, as it meets the lines E_2 , E_9 and E_4 in distinct points. Thus any line in \mathbb{P}^2 passing through one of these points and S meets both points in multiplicity at least 2, hence is contained in X . Thus the plane spanned by E and S contains at least 4 lines in X , therefore this plane lies in X . As the lines E_i are conjugate, the only possibility is that X is the union of planes of planes Π_1 , Π_2 and Π_3 . \square

So by Lemmas 2.1 and 2.2 there exists a smooth cubic surface X containing the configuration \mathcal{E} , as \mathcal{X} has $q + 1 \geq 3$ elements. Any such cubic surface contains nine conjugate lines. Therefore it splits after a field extension of degree at least 9, hence its conjugacy class in $W(\mathbf{E}_6)$ consists of elements of order divisible by 9. However from the classification [13, Tab. 1], one can see that C_{14} is the unique such class (in fact its elements have order exactly 9). Thus X has the claimed class. \square

3. Del Pezzo surfaces of degree 2 — Proof of Theorem 1.2

By Theorem 1.1, over every finite field \mathbb{F}_q there exists a cubic surface S with class C_{14} . But every cubic surface over a finite field has a rational point, by the Chevalley–Warning theorem [11, §2.2]. Moreover, by a consideration of the Galois action on the lines, no rational point lies on a line over $\overline{\mathbb{F}}_q$. Thus the blow-up of S in a rational point is a del Pezzo surface of degree 2. This constructs the missing class 47 over \mathbb{F}_q , since this class corresponds to the unique subgroup of order 9 in $W(\mathbf{E}_7)$. One then performs a Geiser twist to get the class 56 (see [16, Table 1]).

By [16, Thm. 1.2], this leaves open the existence of classes 28 and 35 over \mathbb{F}_3 . These surfaces are Geiser twists of each other (see [16, Table 1]), thus it suffices to construct the class 35. We claim that this is realised by the surface

$$X : (x^2 + xz - z^2)s^2 + (x^2 + y^2)st + (x^2 - xy - y^2 + xz - z^2)t^2 = 0 \subset \mathbb{P}_{s,t}^1 \times \mathbb{P}_{x,y,z}^2.$$

We first prove that X is a del Pezzo surface of degree 2. First, this surface is easily checked to be smooth over \mathbb{F}_3 . By the adjunction formula the anticanonical bundle is $\mathcal{O}_X(0, 1)$, thus the anticanonical map is given by the projection onto the second factor $X \rightarrow \mathbb{P}^2$. This map is proper and a simple calculation shows that it is quasi-finite, hence it is finite. Thus the anticanonical bundle, being the pull-back of an ample line bundle by a finite morphism, is ample. Thus X is del Pezzo, and from the equation it is obvious that it has degree 2.

Next, the projection onto the first factor equips X with the structure of a conic bundle. The singular fibres lie over the closed points of \mathbb{P}^1 given by $s = 0$, $t = 0$, $s^2 + t^2 = 0$ and $s^2 + st - t^2 = 0$, moreover one checks that these singular fibres are all irreducible, so that $X \rightarrow \mathbb{P}^1$ is relatively minimal. Therefore from [8, Thm. 2.6], one finds that the Frobenius element acts on the Picard group $\text{Pic } \bar{X}$ with eigenvalues $1, -1, -1, i, -i, i, -i$. However, an inspection of [15, Appendix, Tab. 1] reveals that class 35 is the only one with this property, hence the class is 35, as claimed. \square

Remark 3.1. In fact any del Pezzo surface of degree 2 with a conic bundle structure may be embedded as a surface of bidegree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^2$; see [5, Thms. 5.6, 5.7].

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