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On Hawking mass and Bartnik mass of CMC surfaces

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Given a constant mean curvature surface that bounds a compact manifold with nonnegative scalar curvature, we obtain intrinsic conditions on the surface that guarantee the positivity of its Hawking mass. We also obtain estimates of the Bartnik mass of such surfaces, without assumptions on the integral of the squared mean curvature. If the ambient manifold has negative scalar curvature, our method also applies and yields estimates on the hyperbolic Bartnik mass of these surfaces.

1. Introduction

Given a Riemannian 3-manifold M, let $\Sigma \subset M$ be a closed 2-surface with a unit normal vector filed ν. Σ is called a CMC surface if its mean curvature with respect to ν is a constant. Throughout this paper, we assume Σ is a CMC surface that is topologically a sphere.

When the ambient manifold M has nonnegative scalar curvature, a classic result of Christodoulou and Yau [11] is the following:

Theorem 1.1 ([11]). *Suppose Σ is a stable, CMC sphere in a 3-manifold M with nonnegative scalar curvature, then $\mathfrak{m}_{_H}(\Sigma) \geq 0$.*

Here $\mathfrak{m}_{_H}(\Sigma)$ is the Hawking quasi-local mass [13] of Σ in M, given by

$$\mathfrak{m}_{_H}(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi}\int_\Sigma H^2 \, d\sigma\right), \tag{1.1}$$

where $|\Sigma|$ is the area and H is the mean curvature of Σ, respectively, and $d\sigma$ denotes the area form on Σ. A CMC surface Σ is called stable if

$$\int_\Sigma |\nabla f|^2 - (|A|^2 + \mathrm{Ric}(\nu,\nu)) f^2 \, d\sigma \geq 0 \tag{1.2}$$

for any function f on Σ with $\int_\Sigma f \, d\sigma = 0$, where ∇ denotes the gradient on Σ, A is the second fundamental form of Σ and $\mathrm{Ric}(\nu,\nu)$ is the Ricci curvature of M along ν.

The stability condition (1.2) is a natural geometric condition and it plays a key role in the estimate of $\mathfrak{m}_{_H}(\Sigma)$ in [11].

In this paper, one of the main questions that we consider is the nonnegativity of $\mathfrak{m}_{_H}(\Sigma)$ without imposing the stability condition on Σ. Instead, we assume Σ bounds a finite region Ω with nonnegative scalar curvature. There are two reasons for making such a consideration:

i) First, from a quasi-local mass point of view, it is desirable to draw information on the quasi-local mass of Σ purely from knowledge on the geometric data (g, H), where g is the intrinsic metric on Σ and H is the mean curvature;

ii) Second, in the special case when g is a round metric on Σ, one indeed knows

$$\mathfrak{m}_{_H}(\Sigma) \geq 0$$

for any CMC surface Σ with positive constant mean curvature H_o. This is a consequence of the Riemannian positive mass theorem [22, 29]. To see this, suppose $\Sigma = \partial\Omega$ where Ω is compact and has nonnegative scalar curvature. Gluing Ω with an exterior Euclidean region $\mathbb{R}^3 \setminus B$, where B is a round ball with boundary ∂B isometric to Σ, one concludes $H_o \leq H_{_E}$, where $H_{_E}$ is the constant mean curvature of ∂B in $\mathbb{R}^3$ (see [18, 24]). As a result, $\mathfrak{m}_{_H}(\Sigma) \geq 0$.

In relation to ii) above, it is natural to ask if $\mathfrak{m}_{_H}(\Sigma)$ has positivity property when the intrinsic metric on Σ is not far from being round. As an application of our main result, Theorem 1.3 stated in a moment, we establish positivity of $\mathfrak{m}_{_H}(\Sigma)$ for these surfaces.

To formulate our theorems, we make use of a scaling invariant number ζ_g that measures how far a metric g is from a round metric. This ζ_g was introduced in [20] and we recall it here. Given any metric g with positive Gauss curvature K_g on the sphere S^2, let r_o be the area radius of (S^2, g), i.e., $|S^2|_g = 4\pi r_o^2$. Let $\{g(t)\}_{0\le t\le 1}$ be a smooth path of metrics on S^2 such that $g(0) = g$, $g(1)$ is round, $g(t)$ has positive Gauss curvature $K_{g(t)}$ and $\mathrm{tr}_{g(t)} g'(t) = 0$ for all t. (Existence of such a path, for instance, follows from Mantoulidis and Schoen's proof of [17, Lemma 1.2].) Associated to this path $\{g(t)\}_{0\le t\le 1}$, let α and β be two constants given by

$$\alpha = \frac{1}{4} \max_{t\in[0,1]} \max_{S^2} |g'(t)|^2_{g(t)}, \quad \beta = r_o^2 \min_{t\in[0,1]} \min_{S^2} K_{g(t)}. \tag{1.3}$$

It is clear $\beta \in (0, 1]$ by the Gauss-Bonnet Theorem, and $\alpha > 0$ if g is not a round metric. With these notations, we let

$$\zeta_g = \inf_{\{g(t)\}} \left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}}, \tag{1.4}$$

where the infimum is taken over all such paths $\{g(t)\}_{0\le t\le 1}$. We point out that ζ_g in (1.4) satisfies $2\zeta_g^2 = \eta(g)^{-1}$, where $\eta(g)$ was defined in [20, Section 4].

Evidently, $\zeta_g = 0$ if g is a round metric; moreover, ζ_g is invariant under constant scaling of g. For any $\gamma \in (0, 1)$, it was shown in [20, Proposition 4.1] that, if g is $C^{2,\gamma}$-close to a round metric g_*, normalized with area 4π, then $\zeta_g \le C||g - g_*||_{C^{0,\gamma}(\Sigma)}$ where C is an absolute constant.

The following theorem gives a sufficient condition on the intrinsic metric on Σ that guarantees the positivity of $\mathfrak{m}_{H}(\Sigma)$.

Theorem 1.2. *Let M be a Riemannian 3-manifold with nonnegative scalar curvature, with boundary ∂M, which is a minimal surface (possibly disconnected) minimizing area among all closed surfaces which bound a domain with ∂M. Suppose $\Sigma \subset M$ is a CMC surface bounding a domain Ω with ∂M and Σ has positive mean curvature with respect to the unit normal pointing out of Ω. Let g be the intrinsic metric on Σ. Suppose g has positive Gauss curvature. If*

$$\zeta_g < C\sqrt{\frac{|\partial M|}{|\Sigma|}},$$

then $\mathfrak{m}_{H}(\Sigma) > 0$. Here C is some absolute constant (for instance C can be $\frac{\sqrt{2}}{3}$).

Remark 1.1. A manifold M in Theorem 1.2 can be taken as an asymptotically flat 3-manifold for which the Riemannian Penrose inequality [5, 14] applies.

We will deduce Theorem 1.2 from a general result which holds without assumptions on ζ_g.

Theorem 1.3. *Suppose Σ is a CMC surface that bounds a compact 3-manifold Ω with nonnegative scalar curvature, which may have nonempty interior horizon. Precisely, this means that Σ is a boundary component of $\partial\Omega$ and $\Sigma_h := \partial\Omega \setminus \Sigma$, if nonempty, is a minimal surface that minimizes area among surfaces enclosing Σ_h. Suppose the intrinsic metric g on Σ has positive Gauss curvature and the mean curvature of Σ with respect to the outward normal ν is a positive constant H_o. Let $r_o = \sqrt{\frac{|\Sigma|}{4\pi}}$ and define $\tau = \frac{1}{2} r_o H_o$. Let θ be the unique root to*

$$\theta^3 - \frac{3\zeta_g \tau}{2}\theta^2 - 1 = 0. \tag{1.5}$$

Then the following holds:

a) If $\Sigma_h = \emptyset$, i.e. $\Sigma = \partial\Omega$, then

$$\tau \leq \theta.$$

b) If $\Sigma_h \neq \emptyset$, then

$$\tau^2 + \frac{r_h}{r_o} \leq \theta^2.$$

Here $r_h = \sqrt{\frac{|\Sigma_h|}{4\pi}}$.

c) Let $\mathfrak{m}_{B}(\Sigma)$ denote the Bartnik quasi-local mass of Σ, then

$$\begin{aligned} \mathfrak{m}_{B}(\Sigma) &\leq \sqrt{\frac{|\Sigma|}{16\pi}}\left(\theta^2 - 1\right) + \mathfrak{m}_{H}(\Sigma) \\ &= \sqrt{\frac{|\Sigma|}{16\pi}}\left(\theta^2 - \tau^2\right). \end{aligned}$$

In particular, this shows $\mathfrak{m}_{B}(\Sigma) \leq C r_o \left(1 + \zeta_g \tau\right) \zeta_g \tau + \mathfrak{m}_{H}(\Sigma)$, where C is an absolute constant.

We defer the definition of the Bartnik mass $\mathfrak{m}_{B}(\cdot)$ to the next section. For the moment, we give a few remarks about Theorem 1.3.

Remark 1.2. The constant τ satisfies $\tau^2 = \frac{1}{16\pi}\int_\Sigma H_o^2\, d\sigma$. Thus, $\mathfrak{m}_{_H}(\Sigma) > 0 \Leftrightarrow \tau < 1$. In terms of $\mathfrak{m}_{_H}(\Sigma)$, a) and b) of Theorem 1.3 can be rewritten as

$$\mathfrak{m}_{_H}(\Sigma) \geq \begin{cases} \frac{r_o}{2}(1-\theta^2), & \text{if } \Sigma_h = \emptyset; \\ \frac{r_o}{2}\left(1 + \frac{r_h}{r_o} - \theta^2\right), & \text{if } \Sigma_h \neq \emptyset. \end{cases} \tag{1.6}$$

Similarly, c) of Theorem 1.3 can be rewritten as

$$\mathfrak{m}_{_B}(\Sigma) \leq \begin{cases} \frac{r_o}{2}\left(\theta^2 - 1\right), & \text{if } \mathfrak{m}_{_H}(\Sigma) = 0; \\ \frac{\theta^2-\tau^2}{1-\tau^2}\, \mathfrak{m}_{_H}(\Sigma), & \text{if } \mathfrak{m}_{_H}(\Sigma) \neq 0. \end{cases} \tag{1.7}$$

Remark 1.3. If g is a round metric, then $\zeta_g = 0$ and hence $\theta = 1$. In this case, it is easily seen Theorem 1.3 is true. For instance, a) follows from ii) above; b) is a special case of the result in [19]; and c) follows from the fact that one can attach a spatial Schwarzschild manifold with mass $\mathfrak{m} = \mathfrak{m}_{_H}(\Sigma)$ to Ω at Σ.

Remark 1.4. Conclusions in a) and b) of Theorem 1.3 concern how nonnegative scalar curvature and interior horizon affect $\mathfrak{m}_{_H}(\Sigma)$ for a CMC surface. This question was studied by the first and the third authors in [20]. Under smallness assumptions on τ, results weaker than a) and b) were derived in [20].

An upper bound of $\mathfrak{m}_{_B}(\Sigma)$ for CMC surfaces was first derived by Lin and Sormani [15] for an arbitrary metric g on Σ. If $H_o = 0$ and the first eigenvalue of $-\Delta_g + K_g$ is positive, Mantoulidis and Schoen [17] proved $\mathfrak{m}_{_B}(\Sigma) = \mathfrak{m}_{_H}(\Sigma)$. Assuming $K_g > 0$ and imposing the smallness assumption on τ used in [20], an upper bound of $\mathfrak{m}_{_B}(\Sigma)$ was derived by Cabrera Pacheco, Cederbaum, McCormick and the first author [10]. A comparison of the estimates in [15] and [10] can be found in [10, Remark 1.2]. Our estimate of $\mathfrak{m}_{_B}(\Sigma)$ in c) of Theorem 1.3 shares the same feature as that in [10], but holds without assumptions on τ.

Remark 1.5. If one does not assume Σ bounds a manifold with nonnegative scalar curvature, the estimate of $\mathfrak{m}_{_B}(\Sigma)$ in c) of Theorem 1.3 is still valid provided the pair (g, H_o) satisfies $\mathfrak{m}_{_H}(\Sigma) \geq 0$. See Remark 2.4 for detailed reasons.

As a corollary of Remark 1.5 and the theorem of Christodoulou and Yau, we have

Corollary 1.1. *The Bartnik mass of any stable CMC surface* Σ *with positive Gauss curvature in a* 3*-manifold with nonnegative scalar curvature satisfies the estimate in c) of Theorem 1.3.*

We have an analogue of Theorem 1.2 with $\sqrt{\frac{|\partial M|}{|\Sigma|}}$ replaced by $\frac{2\mathfrak{m}_{B}(\Sigma)}{r_o}$.

Theorem 1.4. *Let* Σ *be a CMC surface, with positive mean curvature* H_o*, bounding a compact* 3*-manifold* Ω *with nonnegative scalar curvature. Suppose* $\mathfrak{m}_{B}(\Sigma) > 0$ *and the intrinsic metric* g *on* Σ *has positive Gauss curvature. If*

$$\zeta_g < C\left(1 + \frac{2\mathfrak{m}_{B}(\Sigma)}{r_o}\right)^{-1} \min\left\{\frac{2\mathfrak{m}_{B}(\Sigma)}{r_o}, 1\right\},$$

then $\mathfrak{m}_{H}(\Sigma) > 0$*. Here* $r_o = \sqrt{\frac{|\Sigma|}{4\pi}}$ *and* C *is some absolute constant (for instance* C *can be* $\frac{\sqrt{2}}{3}$*).*

Remark 1.6. In the setting of Theorem 1.4, one may also consider the Brown-York mass of Σ [7, 8], given by $\mathfrak{m}_{BY}(\Sigma) = \frac{1}{8\pi}\int_\Sigma (H_{E} - H_o)\, d\sigma$, where H_{E} is the mean curvature of the isometric embedding of (Σ, g) in $\mathbb{R}^3$. As H_o is a constant, one has

$$\mathfrak{m}_{BY}(\Sigma) = \mathfrak{m}_{H}(\Sigma) + \left(\frac{1}{8\pi}\int_\Sigma H_{E}\, d\sigma - r_o\right) + \frac{r_o}{2}(1-\tau)^2, \tag{1.8}$$

where the second term in the bracket is nonnegative by the Minkowski inequality. In [24], Shi and Tam proved $\mathfrak{m}_{BY}(\Sigma) \geq 0$. It would be interesting to know if the positivity of $\mathfrak{m}_{BY}(\Sigma)$ can be used in the study of $\mathfrak{m}_{H}(\Sigma)$.

Remark 1.7. In relation to the positivity of $\mathfrak{m}_{H}(\Sigma)$, a natural question is its rigidity. Under the assumption Σ is stable, recent results concerning $\mathfrak{m}_{H}(\Sigma) = 0$ were given by Sun [26] and by Shi, Sun, Tian and Wei [23].

Our proof of Theorem 1.3 is built on the previous work of the first and the third authors [20]. The techniques we use to prove Theorem 1.3 here can also be applied to the setting of manifolds with a negative scalar curvature lower bound. It is known in the literature the Hawking mass $\mathfrak{m}_{H}(\Sigma)$ has a hyperbolic analogue, $\mathfrak{m}_{H}^{\mathbb{H}}(\Sigma)$ (see (4.1)). Recently, Cabrera Pacheco, Cederbaum and McCormick [9] formulated a hyperbolic analogue of the Bartnik mass and derived results analogous to those in [17] and [10]. Combining the techniques in proving Theorem 1.3 and a gluing tool from [9], we obtain estimates of the hyperbolic Bartnik mass, which we denote by $\mathfrak{m}_{_B}^{\mathbb{H}}(\Sigma)$, for the boundary of a compact manifold with negative scalar curvature.

Theorem 1.5. *Suppose Σ is a CMC surface bounding a compact 3-manifold Ω with scalar curvature $R \geq -6\kappa^2$ for some constant $\kappa > 0$. Let g be the intrinsic metric on Σ and suppose its Gauss curvature satisfies $K_g > -3\kappa^2$. Let $\tau = \frac{1}{2}H_o r_o$, where r_o is the area radius of Σ and H_o is the positive constant mean curvature of Σ in Ω. Then the hyperbolic Bartnik mass $\mathfrak{m}_{_B}^{\mathbb{H}}(\Sigma)$ satisfies*

$$\begin{aligned} &\mathfrak{m}_{_B}^{\mathbb{H}}(\Sigma) - \mathfrak{m}_{_H}^{\mathbb{H}}(\Sigma) \\ &\leq \frac{r_o}{2}\left[\kappa^2 r_o^2\left(1+\frac{3}{2}\tau\xi\right)^2 + \left(1+\frac{3}{2}\tau\xi\right)^{\frac{2}{3}} - \kappa^2 r_o^2 - 1\right] \\ &\leq \frac{r_o}{2}\left(3\kappa^2 r_o^2 + 1\right)\left(1+\frac{3}{4}\tau\xi\right)\tau\xi. \end{aligned} \tag{1.9}$$

Here $\xi \geq 0$ is a constant that is specified as follows.

(i) When $\inf_\Sigma K_g \leq 0$, $\xi = \zeta_{g,\kappa}$, where $\zeta_{g,\kappa}$ is a constant determined by g, given by $\zeta_{g,\kappa} = \inf_{\{g(t)\}}\left(\frac{\alpha}{2\beta+6\kappa^2 r_o^2}\right)^{\frac{1}{2}}$. Here the infimum is taken over all paths of metrics $\{g(t)\}_{0\leq t\leq 1}$ with $g(0) = g$, $g(1)$ is round, $K_{g(t)} > -3\kappa^2$, and $tr_{g(t)} g'(t) = 0$, and α, β are two constants defined in (1.3).

(ii) When $\inf_\Sigma K_g > 0$, ξ is a constant given in (4.38). In particular, ξ satisfies $\xi \leq \zeta_g\theta^2 \leq \zeta_g\left(1+\frac{3}{2}\tau\zeta_g\right)^2$. Here ζ_g is given in (1.4) and θ is the unique root to $\theta^3 - \frac{3}{2}\tau\zeta_g\theta^2 - 1 = 0$.

The remainder of this paper is organized as follows. In Section 2, we consider manifolds with nonnegative scalar curvature and prove Theorem 1.3. In Section 3, we apply Theorem 1.3 to prove Theorems 1.2 and 1.4. In Section 4, we consider manifolds with negative scalar curvature and prove Theorem 1.5.

2. Manifolds with nonnegative scalar curvature

Let Ω, Σ, r_o, H_o and τ be given in Theorem 1.3. By Remark 1.3, it suffices to assume that the intrinsic metric g on Σ is not round. We divide the proof of Theorem 1.3 into a few steps:

Step 1. We review the construction of a suitable metric on $N = [0,1] \times \Sigma$ from [20]. Let $\{g(t)\}_{t\in[0,1]}$ be any given smooth path of metrics on Σ, satisfying $g(0) = g$, $g(1)$ is round, $K_{g(t)} > 0$ and $\mathrm{tr}_{g(t)} g'(t) = 0$, $\forall\, t$. Given any parameter $m \in (-\infty, \frac{1}{2}r_o)$, consider part of a spatial Schwarzschild metric

$$\gamma_m = \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 g_*, \;\; r \geq r_o,$$

where g_* is the standard metric with area 4π on the sphere S^2. Rewriting γ_m as $\gamma_m = ds^2 + u_m^2(s) g_*, \;\; s \geq 0$, one has $u_m(0) = r_0$ and

$$u_m'(s) = \left(1 - \frac{2m}{u_m(s)}\right)^{\frac{1}{2}}. \tag{2.1}$$

Let $k > 0$ be a constant given by

$$k = \tau \left(1 - \frac{2m}{r_o}\right)^{-\frac{1}{2}}. \tag{2.2}$$

Define a metric

$$\gamma^{(m)} = A^2 dt^2 + r_o^{-2} u_m^2(Akt) g(t).$$

Here $A > 0$ is some constant which will be chosen later. The following properties of $(N, \gamma^{(m)})$ follow from direct calculation (see (2.1)–(2.16) in [20]):

- each $\Sigma_t := \{t\} \times \Sigma$ has positive constant mean curvature w.r.t ∂_t;
- the induced metric on Σ_0 is g, and the mean curvature of Σ_0 w.r.t ∂_t is H_o;
- the Hawking mass of each Σ_t is

$$\mathfrak{m}_{_H}(\Sigma_t) = \frac{1}{2}\left(u_m(Akt) - r_o\right)(1 - k^2) + \mathfrak{m}_{_H}(\Sigma); \tag{2.3}$$

- the scalar curvature $R(\gamma^{(m)})$ of $\gamma^{(m)}$ satisfies

$$\begin{aligned} R(\gamma^{(m)}) &= 2u_m^{-2}\left[r_o^2 K_{g(t)} - k^2 - \frac{1}{8}|g'(t)|^2_{g(t)} A^{-2} u_m^2\right] \\ &\geq 2u_m^{-2}\left[\beta - k^2 - \frac{1}{2}\alpha A^{-2} u_m^2(Ak)\right]. \end{aligned} \tag{2.4}$$

By (2.4), a sufficient condition to have $R(\gamma^{(m)}) \geq 0$ is that there exists an $A > 0$ such that $\beta - k^2 - \frac{1}{2}\alpha A^{-2}u_m^2(Ak) \geq 0$. If this is the case, then necessarily $k^2 < \beta \leq 1$. As $k^2 < 1$ is equivalent to $m < m_o$, where $m_o = \frac{r_o}{2}(1-\tau^2)$ is the Hawking mass of Σ, such an A exists only if $m < m_o$.

Step 2. For any suitably given $m < m_o$, we choose an optimal $A = A_o$ such that $\gamma^{(m)}$ has nonnegative scalar curvature.

Lemma 2.1. *For each $m \in (-\infty, m_o)$ satisfying*

$$\beta > \left(1 + \frac{\alpha}{2}\right) k^2, \tag{2.5}$$

there exists a constant $A_o > 0$ such that

$$\beta - k^2 - \frac{\alpha}{2} A_o^{-2} u_m^2(A_o k) = 0. \tag{2.6}$$

Moreover, the set of all such A_o is bounded from above and away from zero as m tends to $-\infty$. That is, there are constants $B_2 > B_1 > 0$ and $\tilde{m} < 0$ such that $B_1 < A_o < B_2$ whenever $m < \tilde{m}$.

Proof. Since $\alpha > 0$, (2.6) is equivalent to

$$k^{-2} 2\alpha^{-1}(\beta - k^2) = (A_o k)^{-2} u_m^2(A_o k). \tag{2.7}$$

Consider the function $f_m(s) = s^{-1}u_m(s)$. One has $\lim_{s\to 0+} f_m(s) = \infty$ and

$$\lim_{s\to\infty} f_m(s) = \lim_{s\to\infty} u_m'(s) = \lim_{s\to\infty} \left(1 - \frac{2m}{u_m(s)}\right)^{\frac{1}{2}} = 1. \tag{2.8}$$

Thus, the range of f_m includes $(1, \infty)$. Since (2.5) implies

$$k^{-2} 2\alpha^{-1}(\beta - k^2) > 1,$$

the existence of such an A_o follows.

Now, by (2.6) and the fact $u_m(s) \geq r_o$, one has

$$\beta - k^2 = \frac{\alpha}{2} A_o^{-2} u_m^2(A_o k) \geq \frac{\alpha}{2} A_o^{-2} r_o^2, \tag{2.9}$$

which gives

$$A_o^2 \geq \frac{\alpha}{2} r_o^2 \left(\beta - k^2\right)^{-1}. \tag{2.10}$$

As $\lim_{m\to-\infty} k = 0$, this shows A_o is bounded away from 0 as $m \to -\infty$.

Next, suppose $m < 0$. By (2.1), $u_m'(s) \leq \left(1 - \frac{2m}{r_o}\right)^{\frac{1}{2}} = \tau k^{-1}$, which implies

$$u_m(s) \leq \; r_0 + \tau k^{-1} s.$$

Thus, for $0 \leq s \leq A_o k$,

$$u_m'(s) \leq \sqrt{1 - \frac{2m(r_0 + A_o \tau)}{u_m^2(s)}}, \tag{2.11}$$

or equivalently

$$u_m(s) u_m'(s) \leq \sqrt{u_m^2(s) - 2m(r_0 + A_o \tau)}. \tag{2.12}$$

Upon integration, (2.12) shows

$$u_m^2(A_o k) \leq r_o^2 + A_o^2 k^2 + 2A_o k \sqrt{r_o^2 - 2m(r_o + A_o \tau)}.$$

Combined with (2.6) and (2.2), this implies

$$\beta - k^2 \leq \frac{\alpha}{2} A_o^{-2} \left(r_o^2 + A_o^2 k^2 + 2A_o k \sqrt{r_o^2 - 2m(r_o + A_o \tau)} \right),$$

i.e.

$$\beta - k^2 - \frac{\alpha}{2} k^2 \leq \frac{\alpha}{2} \left(r_o^2 A_o^{-2} + 2A_o^{-1} \sqrt{\tau^2 r_o^2 + (\tau^2 - k^2) r_o A_o \tau} \right). \tag{2.13}$$

Since $\beta > 0$ and $\lim_{m \to -\infty} k = 0$, it follows from (2.13) that A_o is bounded from above as $m \to -\infty$. □

In what follows, for each m satisfying (2.5), we choose A to be the smallest root A_o to equation (2.6). By (2.4), the metric

$$\gamma^{(m)} = A_o^2 dt^2 + r_o^{-2} u_m^2(A_o k t) g(t)$$

has nonnegative scalar curvature. For each m, we glue $(N, \gamma^{(m)})$ to Ω by identifying Σ_0 with Σ. The argument in [20, Section 3] leading to (3.9) therein then gives

$$\mathfrak{m}_{_H}(\Sigma_1) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}}, \quad \text{if } \Sigma_h \neq \emptyset, \tag{2.14}$$

and

$$\mathfrak{m}_{_H}(\Sigma_1) \geq 0, \quad \text{if } \Sigma_h = \emptyset. \tag{2.15}$$

Here, by (2.3),

$$\mathfrak{m}_{_H}(\Sigma_1) = \frac{1}{2}\left(u_m(A_o k) - r_o\right)(1-k^2) + \mathfrak{m}_{_H}(\Sigma). \tag{2.16}$$

Step 3. We follow the idea in [20] by letting $m \to -\infty$ in (2.14) and (2.15). Since $\lim_{m\to-\infty} k = 0$, (2.5) is satisfied for every sufficiently negative m. By Lemma 2.1, there exists a sequence $\{m_i\}$ with $\lim_{i\to\infty} m_i = -\infty$ such that the corresponding sequence $\{A_o^{(i)}\}$, where $A_o^{(i)}$ is the A_o associated with m_i, has a finite limit. Consequently, by (2.6), the sequence $\{u_{m_i}(A_o^{(i)} k^{(i)})\}$ has a finite limit as well. Here $k^{(i)}$ is the k associated with m_i.

We evaluate $\lim_{i\to\infty} u_{m_i}(A_o^{(i)} k^{(i)})$. One way to achieve this is to implicitly solve (2.1). Suppose $m < 0$. Let $v_m(s) > 0$ be the function such that

$$\frac{-2m}{u_m(s)} = \sinh^{-2}(v_m(s)). \tag{2.17}$$

In term of $v_m(s)$, (2.1) becomes

$$-4m \sinh^2(v_m(s)) v_m'(s) = 1,$$

or equivalently

$$(-m)[\sinh(2v_m(s)) - 2v_m(s)]' = 1. \tag{2.18}$$

Plugging in

$$\sinh(2v_m(s)) = 2\left(\frac{-u_m(s)}{2m}\right)^{\frac{1}{2}}\left(1 - \frac{u_m(s)}{2m}\right)^{\frac{1}{2}}$$

and

$$v_m(s) = \ln\left(\left(\frac{-u_m(s)}{2m}\right)^{\frac{1}{2}} + \left(1 - \frac{u_m(s)}{2m}\right)^{\frac{1}{2}}\right),$$

we get

$$\begin{aligned} &2m\left[\ln\left(\left(\frac{-u_m(s)}{2m}\right)^{\frac{1}{2}}+\left(1-\frac{u_m(s)}{2m}\right)^{\frac{1}{2}}\right)-\left(\frac{-u_m(s)}{2m}\right)^{\frac{1}{2}}\left(1-\frac{u_m(s)}{2m}\right)^{\frac{1}{2}}\right] \\ &-\ 2m\left[\ln\left(\left(-\frac{r_o}{2m}\right)^{\frac{1}{2}}+\left(1-\frac{r_o}{2m}\right)^{\frac{1}{2}}\right)-\left(\frac{-r_o}{2m}\right)^{\frac{1}{2}}\left(1-\frac{r_o}{2m}\right)^{\frac{1}{2}}\right]=s. \end{aligned}$$

Taking $m = m_i$, $k = k^{(i)}$, $A_o = A_o^{(i)}$, $s = A_o^{(i)}k^{(i)}$, and let $u_{m_i}^{(i)} := u_{m_i}(A_o^{(i)}k^{(i)})$, we have

(2.19)

$$\begin{aligned} &2m_i\left[\ln\left(\left(-\frac{u_{m_i}^{(i)}}{2m_i}\right)^{\frac{1}{2}}+\left(1-\frac{u_{m_i}^{(i)}}{2m_i}\right)^{\frac{1}{2}}\right)-\left(\frac{-u_{m_i}^{(i)}}{2m_i}\right)^{\frac{1}{2}}\left(1-\frac{u_{m_i}^{(i)}}{2m_i}\right)^{\frac{1}{2}}\right] \\ &-2m_i\left[\ln\left(\left(-\frac{r_o}{2m_i}\right)^{\frac{1}{2}}+\left(1-\frac{r_o}{2m_i}\right)^{\frac{1}{2}}\right)-\left(\frac{-r_o}{2m_i}\right)^{\frac{1}{2}}\left(1-\frac{r_o}{2m_i}\right)^{\frac{1}{2}}\right] \\ &= A_o^{(i)}k^{(i)}. \end{aligned}$$

By Lemma 2.1, $\frac{u_{m_i}^{(i)}}{2m_i} = O(|m_i|^{-1})$ as $i \to \infty$. Hence,

$$\begin{aligned} &\ln\left(\left(-\frac{u_{m_i}^{(i)}}{2m_i}\right)^{\frac{1}{2}}+\left(1-\frac{u_{m_i}^{(i)}}{2m_i}\right)^{\frac{1}{2}}\right)-\left(\frac{-u_{m_i}^{(i)}}{2m_i}\right)^{\frac{1}{2}}\left(1-\frac{u_{m_i}^{(i)}}{2m_i}\right)^{\frac{1}{2}} \\ &= -\frac{2}{3}\left(-\frac{u_{m_i}^{(i)}}{2m_i}\right)^{\frac{3}{2}} + O(|m_i|^{-2}). \end{aligned}$$

Combined with (2.2), this gives

$$\begin{aligned} &\lim_{i\to\infty}\frac{2m_i}{k^{(i)}}\left[\ln\left(\left(-\frac{u_{m_i}^{(i)}}{2m_i}\right)^{\frac{1}{2}}+\left(1-\frac{u_{m_i}^{(i)}}{2m_i}\right)^{\frac{1}{2}}\right)-\left(\frac{-u_{m_i}^{(i)}}{2m_i}\right)^{\frac{1}{2}}\left(1-\frac{u_{m_i}^{(i)}}{2m_i}\right)^{\frac{1}{2}}\right] \\ &= \frac{2}{3}r_o^{-\frac{1}{2}}\tau^{-1}\lim_{i\to\infty} {u_{m_i}^{(i)}}^{\frac{3}{2}}. \end{aligned}$$

Similarly,

$$\lim_{i\to\infty}\frac{2m_i}{k^{(i)}}\left[\ln\left(\left(-\frac{r_o}{2m_i}\right)^{\frac{1}{2}}+\left(1-\frac{r_o}{2m_i}\right)^{\frac{1}{2}}\right)-\left(\frac{-r_o}{2m_i}\right)^{\frac{1}{2}}\left(1-\frac{r_o}{2m_i}\right)^{\frac{1}{2}}\right]$$
$$=\frac{2}{3}r_o\tau^{-1}.$$

Hence, by (2.19), we have

$$\lim_{i\to\infty}u^{(i)}_{m_i}=r_o\left(1+\frac{3}{2}\tau r_o^{-1}\lim_{i\to\infty}A^{(i)}_o\right)^{\frac{2}{3}}. \tag{2.20}$$

Now let $\bar{A}_o:=\lim_{i\to\infty}A^{(i)}_o$. By (2.10),

$$\bar{A}_o\geq\left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}}r_o>0.$$

Taking limit in (2.6), we have

$$\beta=\frac{\alpha}{2}\bar{A}_o^{-2}\left(\lim_{i\to\infty}u^{(i)}_{m_i}\right)^2. \tag{2.21}$$

Therefore, it follows from (2.20) and (2.21) that

$$\left(\frac{r_o}{\bar{A}_o}\right)^{\frac{3}{2}}+\frac{3\tau}{2}\left(\frac{r_o}{\bar{A}_o}\right)^{\frac{1}{2}}=\left(\frac{2\beta}{\alpha}\right)^{\frac{3}{4}}. \tag{2.22}$$

We now define $\theta>0$ such that

$$\frac{\bar{A}_o}{r_o}=\theta^2\left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}}. \tag{2.23}$$

Then (2.22) shows

$$\theta^3-\frac{3\tau}{2}\left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}}\theta^2-1=0. \tag{2.24}$$

By (2.20), (2.23) and (2.24), we have

$$\lim_{i\to\infty}u^{(i)}_{m_i}=r_o\left(1+\frac{3\tau}{2}\theta^2\left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}}\right)^{\frac{2}{3}}=r_o\theta^2. \tag{2.25}$$

From this and (2.16), we conclude

$$\begin{aligned}\lim_{i\to\infty} \mathfrak{m}_{_H}(\Sigma_1) &= \lim_{i\to\infty} \frac{1}{2}\left(u^{(i)}_{m_i} - r_o\right)(1 - {k^{(i)}}^2) + \mathfrak{m}_{_H}(\Sigma) \\ &= \frac{r_o}{2}\left(\theta^2 - 1\right) + \mathfrak{m}_{_H}(\Sigma_o) \\ &= \frac{r_o}{2}\left(\theta^2 - \tau^2\right).\end{aligned} \tag{2.26}$$

Here $\mathfrak{m}_{_H}(\Sigma_1)$ denotes the Hawking mass of Σ_1 in $(N, \gamma^{(m_i)})$.

Remark 2.1. Since $\{A_o^{(i)}\}$ can be taken to be any converging sequence, the argument above indeed shows

$$\lim_{m\to-\infty} u_m(A_o k) = r_o\theta^2 \quad \text{and} \quad \lim_{m\to-\infty} A_o = r_o\theta^2 \left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}}.$$

The following theorem follows directly from (2.14), (2.15) and (2.26).

Theorem 2.1. *Let Ω, Σ, g, r_o, H_o and τ be given in Theorem 1.3. Let $\{g(t)\}_{t\in[0,1]}$ be a smooth path of metrics on Σ satisfying $g(0) = g$, $g(1)$ is round, $K_{g(t)} > 0$ and $tr_{g(t)} g'(t) = 0$. Let α and β be the constants associated to $\{g(t)\}_{t\in[0,1]}$, given by (1.3). Let $\theta > 0$ be the number that is the unique root to*

$$\theta^3 - \frac{3\tau}{2}\left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}}\theta^2 - 1 = 0.$$

Then

$$\tau \le \theta \quad \text{if } \Sigma_h = \emptyset,$$

and

$$\tau^2 + \frac{r_h}{r_o} \le \theta^2 \quad \text{if } \Sigma_h \ne \emptyset.$$

Here $r_h = \sqrt{\frac{|\Sigma_h|}{4\pi}}$ is the area radius of Σ_h.

Remark 2.2. Let $f(x) = x^3 - \frac{3\tau}{2}\left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}} x^2 - 1$. As

$$f'(x) = 3x\left[x - \tau\left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}}\right],$$

it is easily seen that, given a number x,

$$x \le \theta \Longleftrightarrow f(x) \le 0.$$

Thus, the conclusion in Theorem 2.1 can be equivalently stated as

$$\tau^3\left[1 - \frac{3}{2}\left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}}\right] \le 1 \quad \text{if } \Sigma_h = \emptyset, \tag{2.27}$$

and

$$\left(\tau^2 + \frac{r_h}{r_o}\right)^{\frac{3}{2}} - \frac{3\tau}{2}\left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}}\left(\tau^2 + \frac{r_h}{r_o}\right) \le 1 \quad \text{if } \Sigma_h \ne \emptyset. \tag{2.28}$$

Part a) and b) of Theorem 1.3 now follow from Theorem 2.1 by considering sequences of paths of metrics $\{g(t)\}_{t\in[0,1]}$ with $\left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}} \to \zeta_g$.

Remark 2.3. Because we have chosen $A_o > 0$ to be the smallest root to (2.6), we have $\beta - k^2 - \frac{\alpha}{2}A^{-2}u_m^2(Ak) < 0$, $\forall A \in (0, A_o)$. Thus, if $\tilde{A}_o > 0$ is any number such that $\beta - k^2 - \frac{\alpha}{2}\tilde{A}_o^{-2}u_m^2(\tilde{A}_o k) \ge 0$, we must have $A_o \le \tilde{A}_o$, and hence $u_m(A_o k) \le u_m(\tilde{A}_o k)$. Thus, besides requiring no assumptions on τ, inequalities in a) and b) of Theorem 1.3 are stronger than those of Theorems 1.1 and 1.2 in [20].

In the remaining part of this section, we prove part c) of Theorem 1.3. First, we review the definition of $\mathfrak{m}_{_B}(\cdot)$. Given a metric g and a function H on a surface Σ that is topologically a sphere, the Bartnik mass $\mathfrak{m}_{_B}(\Sigma)$ associated to the triple (Σ, g, H) [2, 3] can be defined as

$$\inf\left\{m_{_{ADM}}(M,\gamma)\,|\,(M,\gamma) \text{ is an admissible extension of } (\Sigma, g, H)\right\}.$$

Here $m_{_{ADM}}(\cdot)$ denotes the ADM mass [1], and an asymptotically flat 3-manifold (M, γ) with boundary is an admissible extension of (Σ, g, H) if: ∂M is isometric to (Σ, g); the mean curvature of ∂M in (M, γ) equals H;

(M, g) has nonnegative scalar curvature; and either (M, γ) contains no closed minimal surfaces (except possibly ∂M), or ∂M is outer-minimizing in (M, γ) (see [5, 6, 14, 30] for instance).

Working with this definition, one sees that part c) of Theorem 1.3 would be a natural consequence of the previous three steps. The reason is, because Σ_1 has a round intrinsic metric and constant mean curvature in $(N, \gamma^{(m)})$, one can attach part of a spatial Schwarzschild manifold with mass $\mathfrak{m}_{_H}(\Sigma_1)$, outside a rotationally symmetric sphere isometric to Σ_1, to $(N, \gamma^{(m)})$ at Σ_1. The resulting manifold would be an admissible extension of (Σ, g, H_o), except it may not be smooth across Σ_1. If it were smooth across Σ_1, then $\mathfrak{m}_{_B}(\Sigma) \le \mathfrak{m}_{_H}(\Sigma_1)$ by definition. Passing to the limit in Step 3, one would obtain the estimate in c).

To give a precise proof of c), we can make use of a gluing result in [10]. For this purpose, we return to the end of Step 2 to point out a few additional feature of $(N, \gamma^{(m)})$. By (2.14) and (2.15), the Hawing mass of Σ_1 in $(N, \gamma^{(m)})$ satisfies

$$\mathfrak{m}_{_H}(\Sigma_1) \ge 0. \tag{2.29}$$

By (2.4) and (2.6), the scalar curvature of $\gamma^{(m)}$ at any $(x, t) \in \Sigma \times [0, 1) \subset N$ satisfies

$$\begin{aligned} R(\gamma^{(m)})(x,t) &= 2u_m^{-2}(A_okt)\left[r_o^2 K_{g(t)}(x) - k^2 - \frac{1}{8}|g'(t)|^2_{g(t)}(x)A_o^{-2}u_m^2(A_okt)\right] \\ &> 2u_m^{-2}(A_ok)\left[\beta - k^2 - \frac{1}{2}\alpha A_o^{-2}u_m^2(Ak)\right] = 0. \end{aligned} \tag{2.30}$$

At $t = 1$, we also have

$$\begin{aligned} R(\gamma^{(m)})(x,1) &= 2u_m^{-2}(A_ok)\left[1 - k^2 - \frac{1}{8}|g'(1)|^2_{g(1)}(x)A_o^{-2}u_m^2(A_ok)\right] \\ &> 2u_m^{-2}(A_ok)\left[\beta - k^2 - \frac{1}{2}\alpha A_o^{-2}u_m^2(Ak)\right] = 0, \end{aligned} \tag{2.31}$$

because $\beta < 1$ (since $g(1)$ is round while $g(0) = g$ is not round). Thus, $R(\gamma^{(m)}) > 0$ everywhere on N.

Now we can apply [10, Proposition 2.1] to $(N, \gamma^{(m)})$. We may first assume the path $\{g(t)\}_{t\in[0,1]}$ has a property $g(t) = g(1)$ for t in $(1-\delta, 1]$ for some $\delta > 0$. In this case, a direct application of [10, Proposition 2.1] gives

$$\mathfrak{m}_{_B}(\Sigma) \le \mathfrak{m}_{_H}(\Sigma_1). \tag{2.32}$$

In general, by approximating $\{g(t)\}_{t\in[0,1]}$ with paths satisfying such a property (see (3.9)–(3.13) in [10]), one knows (2.32) still holds.

Combining (2.26) and (2.32), we obtain

$$\begin{aligned} \mathfrak{m}_{_B}(\Sigma) &\le \lim_{i\to\infty} \mathfrak{m}_{_H}(\Sigma_1) \\ &= \frac{r_o}{2}\left(\theta^2 - 1\right) + \mathfrak{m}_{_H}(\Sigma_o). \end{aligned} \tag{2.33}$$

Elementary estimates show that the root θ to (1.5) satisfies $1 \le \theta \le 1 + \frac{3}{2}\tau\zeta_g$. Thus,

$$\mathfrak{m}_{_B}(\Sigma) \le \frac{3}{2} r_o \left(1 + \frac{3}{4}\tau\zeta_g\right)\tau\zeta_g + \mathfrak{m}_{_H}(\Sigma). \tag{2.34}$$

This completes the proof of part c) of Theorem 1.3.

Remark 2.4. In Theorem 1.3, we assume Σ bounds a compact 3-manifold with nonnegative scalar curvature. If this assumption is dropped, the above proof is still valid to show (2.33), provided a sufficient condition $\mathfrak{m}_{_H}(\Sigma) \ge 0$ is assumed on (g, H_o). This is because, by (2.16), $\mathfrak{m}_{_H}(\Sigma_1) > \mathfrak{m}_{_H}(\Sigma)$ for each $(N, \gamma^{(m)})$ used in the proof.

Remark 2.5. In [10], it was shown if (g, H_o) on Σ satisfies $\tau^2 < \frac{\beta}{1+\alpha}$, then

$$\mathfrak{m}_{_B}(\Sigma) \le \left[\frac{\alpha}{\beta - (1+\alpha)\tau^2}\right]^{\frac{1}{2}} \tau\mathfrak{m}_{_H}(\Sigma) + \mathfrak{m}_{_H}(\Sigma). \tag{2.35}$$

Comparing (2.33) and (2.35), we see (2.33) requires no assumptions on τ and it improves (2.35) when τ is small. For instance, as $\tau \to 0$,

$$\frac{\theta^2 - 1}{1 - \tau^2} = \left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}} \tau + O(\tau^2) \quad \text{and} \quad \left[\frac{\alpha}{\beta - (1+\alpha)\tau^2}\right]^{\frac{1}{2}} \tau = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}} \tau + O(\tau^2).$$

3. Applications of Theorem 1.3

We apply Theorem 1.3 to prove Theorems 1.2 and 1.4.

Lemma 3.1. *Given two constants $b > 0$ and $\lambda > 0$, consider the function*

$$\Phi(\tau) = \left(\tau^2 + \lambda\right)^{\frac{3}{2}} - b\tau\left(\tau^2 + \lambda\right) - 1, \quad \tau \in (0, \infty). \tag{3.1}$$

If $b < \min\{\frac{\lambda}{\sqrt{1+\lambda}}, \frac{1}{\sqrt{1+\lambda}}\}$, then $\Phi(\tau) > 0$ for any $\tau \ge 1$.

Proof. One has

$$(3.2)\qquad \begin{aligned}(1+\lambda)^{-1}\Phi(1) &= \sqrt{1+\lambda} - \frac{1}{1+\lambda} - b\\ &= \frac{\lambda}{(1+\lambda)\left(\sqrt{1+\lambda}+1\right)} + \frac{\lambda}{\sqrt{1+\lambda}} - b > 0,\end{aligned}$$

and

$$(3.3)\qquad \begin{aligned}(3\tau^2+\lambda)^{-1}\Phi'(\tau) &= \frac{3\left(\tau^2+\lambda\right)^{\frac{1}{2}}\tau}{3\tau^2+\lambda} - b\\ &\geq \frac{\tau}{\sqrt{\tau^2+\lambda}} - b > 0\end{aligned}$$

for $\tau \geq 1$. The lemma follows. □

Proof of Theorem 1.2. We take the constant $C = \frac{\sqrt{2}}{3}$. Suppose

$$(3.4)\qquad \zeta_g < \frac{\sqrt{2}}{3}\sqrt{\frac{\partial M}{|\Sigma|}}.$$

Applying b) of Theorem 1.3 to the domain Ω, bounded by Σ and ∂M, in M, we have

$$(3.5)\qquad \tau^2 + \frac{r_h}{r_o} \leq \theta^2,$$

where $r_o = \sqrt{\frac{|\Sigma|}{4\pi}}$, $r_h = \sqrt{\frac{|\partial M|}{4\pi}}$, $\tau = \frac{1}{2}r_o H_o$, H_o is the positive constant mean curvature of Σ, and $\theta > 0$ is the unique root to (1.5). Similarly to Remark 2.2, we know (3.5) is equivalent to

$$(3.6)\qquad \left(\tau^2 + \frac{r_h}{r_o}\right)^{\frac{3}{2}} - \frac{3\tau\zeta_g}{2}\left(\tau^2 + \frac{r_h}{r_o}\right) - 1 \leq 0.$$

Let $b = \frac{3\zeta_g}{2}$ and $\lambda = \frac{r_h}{r_o}$. Condition (3.4) becomes $b < \frac{1}{\sqrt{2}}\lambda$. Since $|\partial M| \leq |\Sigma|$, $\lambda \leq 1$. Thus, by (3.6) and Lemma 3.1, we have $\tau < 1$, i.e $\mathfrak{m}_{_H}(\Sigma) > 0$. □

Theorem 1.4 is proved in a similar way.

Proof of Theorem 1.4. Since Σ bounds a compact Ω with nonnegative scalar curvature, $\mathfrak{m}_{_B}(\Sigma)$ satisfies the estimate in c) of Theorem 1.3, i.e.

$$\tau^2 + \frac{2\mathfrak{m}_{_B}(\Sigma)}{r_o} \leq \theta^2. \tag{3.7}$$

Therefore,

$$\left(\tau^2 + \frac{2\mathfrak{m}_{_B}(\Sigma)}{r_o}\right)^{\frac{3}{2}} - \frac{3\tau\zeta_g}{2}\left(\tau^2 + \frac{2\mathfrak{m}_{_B}(\Sigma)}{r_o}\right) - 1 \leq 0. \tag{3.8}$$

Now suppose

$$\zeta_g < \frac{\sqrt{2}}{3}\left(1 + \frac{2\mathfrak{m}_{_B}(\Sigma)}{r_o}\right)^{-\frac{1}{2}} \min\left\{\frac{2\mathfrak{m}_{_B}(\Sigma)}{r_o}, 1\right\}. \tag{3.9}$$

Let $b = \frac{3\zeta_g}{2}$ and $\lambda = \frac{2\mathfrak{m}_{_B}(\Sigma)}{r_o}$, (3.9) shows

$$b < (1+\lambda)^{-\frac{1}{2}} \min\{\lambda, 1\}.$$

By (3.8) and Lemma 3.1, we conclude $\tau < 1$, i.e $\mathfrak{m}_{_H}(\Sigma) > 0$. □

4. Manifolds with negative scalar curvature

In the remaining of this paper, we turn attention to CMC surfaces in manifolds with a negative lower bound on the scalar curvature. Let M denote a Riemannian 3-manifold with scalar curvature $R \geq -6\kappa^2$, where $\kappa > 0$ is a constant. Let $\Sigma \subset M$ be a closed surface. In this context, the hyperbolic Hawking mass of Σ is given by

$$\mathfrak{m}^{\mathbb{H}}_{_H}(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma H^2\, d\sigma + \frac{1}{4\pi}\kappa^2|\Sigma|\right). \tag{4.1}$$

A natural analogue of the Bartnik mass is

$$\mathfrak{m}^{\mathbb{H}}_{_B}(\Sigma) = \inf\left\{\mathfrak{m}(M^{\mathbb{H}}, \gamma^{\mathbb{H}})\right\},$$

where $\mathfrak{m}(\cdot)$ is the mass of an asymptotically hyperbolic manifold and the infimum is taken over a space of "admissible asymptotically hyperbolic extensions" $(M^{\mathbb{H}}, \gamma^{\mathbb{H}})$ of (Σ, g, H). We refer readers to the recent work of Cabrera Pacheco, Cederbaum and McCormick [9] for a detailed discussion of this definition.

We now let Ω, Σ, g, H_o, r_o and τ be given in Theorem 1.5. If g is a round metric, then $\zeta_g = 0$ and $\xi = 0$. In this case, (1.9) reduces to $\mathfrak{m}^{\mathbb{H}}_{B}(\Sigma) \leq \mathfrak{m}^{\mathbb{H}}_{H}(\Sigma)$. This is true because a spatial AdS-Schwarzschild manifold with mass $\mathfrak{m} = \mathfrak{m}^{\mathbb{H}}_{H}(\Sigma)$, lying outside a rotationally symmetric sphere isometric to Σ, can be attached to Ω at Σ.

In what follows, we assume that g is not a round metric. Under the assumption $K_g > -3\kappa^2$, there exists a smooth path of metrics $\{g(t)\}_{0\leq t\leq 1}$ on Σ with $g(0) = g$, $g(1)$ is round, $K_{g(t)} > -3\kappa^2$, and $\mathrm{tr}_{g(t)} g'(t) = 0$. (Existence of such a path can be provided by the solution to the normalized Ricci flow on Σ starting at g. See [16, Lemma 4.2] and [9, Lemma 5.1] for instance.) We fix such a path $\{g(t)\}_{0\leq t\leq 1}$ and let α, β be the constants given in (1.3). Then $\alpha > 0$ and $1 > \beta > -3\kappa^2 r_o^2$.

Similar to Step 1 in the proof of Theorem 1.3, now one can consider a spatial AdS-Schwarzschild metric $\gamma_m = \left(1 - \frac{2m}{r} + \kappa^2 r^2\right)^{-1} dr^2 + r^2 g_*,\ r \geq r_o$, where m is any parameter such that $1 - \frac{2m}{r_o} + \kappa^2 r_o^2 > 0$. Rewriting γ_m as $\gamma_m = ds^2 + u_m^2(s) g_*,\ s \geq 0$, one has $u_m(0) = r_o$ and

$$u_m'(s) = \left(1 - \frac{2m}{u_m(s)} + \kappa^2 u_m^2(s)\right)^{\frac{1}{2}}. \tag{4.2}$$

Define a constant

$$k = \tau \left(1 - \frac{2m}{r_o} + \kappa^2 r_o^2\right)^{-\frac{1}{2}} \tag{4.3}$$

and a metric

$$\gamma^{(m)} = A^2 dt^2 + r_o^{-2} u_m^2(Akt) g(t)$$

on $N = [0,1] \times \Sigma$, where $A > 0$ is a constant to be chosen. Direct calculation shows

- each $\Sigma_t := \{t\} \times \Sigma$ has positive constant mean curvature w.r.t ∂_t;
- the induced metric on Σ_0 is g, and the mean curvature of Σ_0 w.r.t ∂_t is H_o;
- the hyperbolic Hawking mass of each Σ_t is

$$\begin{aligned} \mathfrak{m}^{\mathbb{H}}_{H}(\Sigma_t) = {} & \frac{1}{2}\left(u_m(Akt) - r_o\right)\left(1 - k^2\right) \\ & + \frac{1}{2}\kappa^2 (1 - k^2)(u_m^3(Akt) - r_o^3) + \mathfrak{m}^{\mathbb{H}}_{H}(\Sigma); \end{aligned} \tag{4.4}$$

- the scalar curvature $R(\gamma^{(m)})$ of $\gamma^{(m)}$ satisfies

$$R(\gamma^{(m)}) = 2u_m^{-2}(r_o^2 K_{g(t)} - k^2) - \frac{1}{4}|g'(t)|^2_{g(t)} A^{-2} - 6k^2\kappa^2 \tag{4.5}$$
$$\geq 2u_m^{-2}(\beta - k^2) - \alpha A^{-2} - 6k^2\kappa^2.$$

Remark 4.1. The manifold $(N, \gamma^{(m)})$, constructed above via the warping function of a AdS-Schwarzschild metric, was also used in [9]. Estimates on $\mathfrak{m}_{_B}^{\mathbb{H}}(\cdot)$ for a pair (g, H_o) were derived in [9] under suitable smallness conditions on H_o. In this section, by assuming (g, H_o) arises from the boundary of Ω and by making an optimal choice of A, we obtain estimates on $\mathfrak{m}_{_B}^{\mathbb{H}}(\cdot)$ that require no assumption on H_o.

By (4.5), a sufficient condition to guarantee $R(\gamma^{(m)}) \geq -6\kappa^2$ on N is

$$u_m^{-2}(Akt)(\beta - k^2) - \frac{1}{2}\alpha A^{-2} + 3\kappa^2(1 - k^2) \geq 0, \quad \forall\, t \in [0, 1]. \tag{4.6}$$

As $u_m(s)$ is monotone, (4.6) is equivalent to

$$u_m^{-2}(Ak)(\beta - k^2) - \frac{1}{2}\alpha A^{-2} + 3\kappa^2(1 - k^2) \geq 0, \quad \text{if } \beta - k^2 > 0, \tag{4.7}$$

or

$$r_o^{-2}(\beta - k^2) - \frac{1}{2}\alpha A^{-2} + 3\kappa^2(1 - k^2) \geq 0, \quad \text{if } \beta - k^2 \leq 0. \tag{4.8}$$

Next, as in Step 2 in the proof of Theorem 1.3, we choose an optimal $A = A_o$ so that (4.7) or (4.8) are met. If $\beta \leq 0$, using the fact $\beta + 3\kappa^2 r_o^2 > 0$, one easily sees an optimal A satisfying (4.8) is

$$A_o = r_o \left(\frac{\frac{1}{2}\alpha}{\beta + 3\kappa^2 r_o^2 - (1 + 3\kappa^2 r_o^2)\, k^2} \right)^{\frac{1}{2}}, \tag{4.9}$$

provided k is small.

If $\beta > 0$ (which occurs only if $\inf_\Sigma K_g > 0$), we choose an optimal A_o satisfying (4.7) according to the following lemma.

Lemma 4.1. *Suppose $\alpha > 0$ and $\beta > 0$. For every $m \in (-\infty, 0)$ satisfying $k^2 < \beta$, there exists a positive constant A_o such that*

$$(\beta - k^2) + \left[3\kappa^2(1-k^2) - \frac{1}{2}\alpha A^{-2}\right] u_m^2(A_o k) = 0. \tag{4.10}$$

Moreover, the set of all such A_o is bounded from above and away from zero as m tends to $-\infty$.

Proof. For each fixed m, consider the function

$$f_m(A) = (\beta - k^2) + \left[3\kappa^2(1-k^2) - \frac{1}{2}\alpha A^{-2}\right] u_m^2(Ak), \;\; A \in (0, \infty).$$

One has $\lim_{A \to 0+} f_m(A) = -\infty$ since $\alpha > 0$, and $\lim_{A\to\infty} f_m(A) = \infty$ because $\lim_{s\to\infty} u_m(s) = \infty$ and $k^2 < \beta < 1$. Moreover, $f_m(A)$ is strictly increasing in A. Hence, there exists a unique root $A_o > 0$ to (4.10). For this A_o, one has

$$3\kappa^2(1-k^2) - \frac{1}{2}\alpha A_o^{-2} \le 0, \tag{4.11}$$

for otherwise the left side of (4.10) would be positive. Thus,

$$A_o^2 \le \frac{1}{6}\alpha\kappa^{-2}(1-k^2)^{-1}. \tag{4.12}$$

As $\lim_{m\to-\infty} k = 0$, this shows that A_o is bounded from above as m tends to $-\infty$. On the other hand, by (4.10), (4.11) and the fact $u_m(s) \ge r_o$, one has

$$0 \le (\beta - k^2) + \left[3\kappa^2(1-k^2) - \frac{1}{2}\alpha A_o^{-2}\right] r_o^2,$$

i.e.

$$\alpha r_o^2 \left[2(\beta - k^2) + 6\kappa^2(1-k^2) r_o^2\right]^{-1} \le A_o^2. \tag{4.13}$$

This shows A_o is bounded away from 0 as m tends to $-\infty$. □

In what follows, we assume m is sufficiently negatively large so that k^2 is small. We choose $A = A_o > 0$ so that A_o is the unique root to (4.10) if $\beta > 0$; and A_o is given by (4.9) if $\beta \le 0$. In either case, $A_o = O(1)$, as $m \to -\infty$.

Before we compute $\lim_{m\to-\infty} A_o$ and $\lim_{m\to-\infty} u_m(A_o k)$, we point out the non-negativity of $\mathfrak{m}_{_H}^{\mathbb{H}}(\Sigma_1)$ in our setting.

Proposition 4.1. *Let* Ω, Σ, $(N, \gamma^{(m)})$, A_o, *be given above. Then* $\mathfrak{m}_{_H}^{\mathbb{H}}(\Sigma_1) \geq 0$.

Proof. This is essentially a consequence of the positive mass theorem on asymptotically hyperbolic manifolds (see [12, 28] for instance). More precisely, this follows from such a theorem on manifolds with corners along a hypersurface (see [4] and also [25, 27]). Consider three manifolds

$$\Omega, (N, \gamma^{(m)}), \text{ and } M_{\mathfrak{m}},$$

where $M_{\mathfrak{m}}$ is part of the spatial AdS-Schwarzschild manifold, with mass $\mathfrak{m} = \mathfrak{m}_{_H}^{\mathbb{H}}(\Sigma_1)$, lying outside a rotationally symmetric sphere S isometric to Σ_1. One can glue $M_{\mathfrak{m}}$ to $(N, \gamma^{(m)})$ by identifying S with Σ_1 and glue Ω to $(N, \gamma^{(m)})$ by identifying Σ with Σ_0. Applying [4, Theorem 1.1] to the resulting manifold, one concludes $\mathfrak{m} \geq 0$. □

The above proof of Proposition 4.1 indeed indicates $\mathfrak{m}_{_B}^{\mathbb{H}}(\Sigma) \leq \mathfrak{m}_{_H}^{\mathbb{H}}(\Sigma_1)$ if the manifold obtained by gluing $M_{\mathfrak{m}}$ and $(N, \gamma^{(m)})$ along Σ_1 is smooth. By invoking a gluing result in [9, Proposition 3.3], one can verify this assertion.

Proposition 4.2. *Let* Ω, Σ, $(N, \gamma^{(m)})$, A_o, *be given above. Then* $\mathfrak{m}_{_B}^{\mathbb{H}}(\Sigma) \leq \mathfrak{m}_{_H}^{\mathbb{H}}(\Sigma_1)$.

Proof. Since $\beta < 1$ and Σ_1 is round in $(N, \gamma^{(m)})$, an examination of (4.7) and (4.8) shows $R(\gamma^{(m)}) > -6\kappa^2$ near Σ_1 in N. The claim nows follows from Proposition 4.1 and [9, Proposition 3.3] in the same way that (2.32) follows from (2.29) and [10, Proposition 2.1]. □

Next, we proceed to evaluate $\lim_{m\to-\infty} A_o$ and $\lim_{m\to-\infty} u_m(A_o k)$. First, as $m < 0$, (4.2) implies

$$u_m'(s) \leq \left(1 - \frac{2m}{r_o} + \kappa^2 u_m^2(s)\right)^{\frac{1}{2}},$$

which, upon integration, gives

$$\kappa u_m(A_o k) + \sqrt{1 - \frac{2m}{r_o} + \kappa^2 u_m^2(A_o k)} \leq e^{\kappa A_o k}\left[\kappa r_o + \sqrt{1 - \frac{2m}{r_o} + \kappa^2 r_o^2}\right]. \tag{4.14}$$

This yields

$$u_m(A_o k) \leq u_m^*, \tag{4.15}$$

where

$$u_m^* = \frac{e^{\kappa A_o k}\left(\kappa r_o + \sqrt{1 - \frac{2m}{r_o} + \kappa^2 r_o^2}\right)^2 - e^{-\kappa A_o k}\left(1 - \frac{2m}{r_o}\right)}{2\kappa\left(\kappa r_o + \sqrt{1 - \frac{2m}{r_o} + \kappa^2 r_o^2}\right)} \\ = r_o\left[\frac{1}{2}\left(e^{\kappa A_o k} + e^{-\kappa A_o k}\right) + \frac{1}{2\kappa k}\left(e^{\kappa A_o k} - e^{-\kappa A_o k}\right)\frac{H_o}{2}\right]. \tag{4.16}$$

For $0 \leq s \leq A_o k$, by (4.2), we have

$$u'_m(s) \leq \left(\frac{u_m^* - 2m + \kappa^2 {u_m^*}^3}{u_m(s)}\right)^{\frac{1}{2}}, \tag{4.17}$$

which implies

$$u_m^{\frac{3}{2}}(A_o k) \leq \frac{3}{2} A_o k \sqrt{u_m^* - 2m + \kappa^2 {u_m^*}^3} + r_o^{\frac{3}{2}}. \tag{4.18}$$

On the other hand, by (4.2),

$$u'_m(s) \geq \left(-\frac{2m}{u_m(s)}\right)^{\frac{1}{2}}, \tag{4.19}$$

which implies

$$u_m^{\frac{3}{2}}(A_o k) \geq \frac{3}{2} A_o k \sqrt{-2m} + r_o^{\frac{3}{2}}. \tag{4.20}$$

Now, let $\{m_i\}$ denote any sequence that tends to $-\infty$ so that the corresponding sequence $\{A_o^{(i)}\}$ has a finite limit, where $A_o^{(i)}$ is the A_o associated with m_i. Let $\bar{A}_o := \lim_{m \to -\infty} A_o^{(i)}$. As $\lim_{i \to \infty} k = 0$, by (4.3) and (4.16),

$$\lim_{i \to \infty} u_{m_i}^* = r_o\left(1 + \frac{1}{2}\bar{A}_o H_o\right) \tag{4.21}$$

and

$$\lim_{i \to \infty} k\sqrt{-2m_i} = \tau r_o^{\frac{1}{2}} = \lim_{i \to \infty} k\sqrt{u_{m_i}^* - 2m_i + \kappa^2 {u_{m_i}^*}^3}. \tag{4.22}$$

Hence, if we let

$$\xi = \bar{A}_o r_o^{-1},$$

then, by (4.18) and (4.20),

$$\bar{u}_o := \lim_{i\to\infty} u_{m_i}(A_o^{(i)}k) = r_o\left(1+\frac{3}{2}\tau\xi\right)^{\frac{2}{3}}. \tag{4.23}$$

As a result of (4.23) and (4.4), we see that the limit of the hyperbolic Hawking mass of Σ_1 in $(N, \gamma^{(m_i)})$ is given by

$$\begin{aligned}\lim_{i\to\infty} \mathfrak{m}_{_H}^{\mathbb{H}}(\Sigma_1) &= \frac{1}{2}(\bar{u}_o - r_o) + \frac{1}{2}\kappa^2(\bar{u}_o^3 - r_o^3) + \mathfrak{m}_{_H}^{\mathbb{H}}(\Sigma)\\ &= \frac{r_o}{2}\left[\left(1+\frac{3}{2}\tau\xi\right)^{\frac{2}{3}} + \kappa^2 r_o^2\left(1+\frac{3}{2}\tau\xi\right)^2 - 1 - \kappa^2 r_o^2\right] + \mathfrak{m}_{_H}^{\mathbb{H}}(\Sigma). \end{aligned}\tag{4.24}$$

Here, by (4.9),

$$\xi = \left(\frac{\frac{1}{2}\alpha}{\beta + 3\kappa^2 r_o^2}\right)^{\frac{1}{2}}, \quad \text{if } \beta \le 0. \tag{4.25}$$

When $\beta > 0$, by (4.13),

$$\xi \ge \left(\frac{\frac{1}{2}\alpha}{\beta + 3\kappa^2 r_o^2}\right)^{\frac{1}{2}} > 0. \tag{4.26}$$

Hence, by (4.10) and (4.23),

$$\beta + \left(3\kappa^2 r_o^2 - \frac{\alpha}{2}\xi^{-2}\right)\left(1+\frac{3}{2}\tau\xi\right)^{\frac{4}{3}} = 0, \tag{4.27}$$

or equivalently

$$\left[\beta + 3\kappa^2 r_o^2\left(1+\frac{3}{2}\tau\xi\right)^{\frac{4}{3}}\right]\xi^2 - \frac{\alpha}{2}\left(1+\frac{3}{2}\tau\xi\right)^{\frac{4}{3}} = 0. \tag{4.28}$$

Remark 4.2. Consider

$$\Psi(x) = \beta + \left(3\kappa^2 r_o^2 - \frac{\alpha}{2}x^{-2}\right)\left(1+\frac{3}{2}\tau x\right)^{\frac{4}{3}}, \ x \in (0,\infty).$$

Then

$$\Psi'(x) = \left(6\kappa^2 r_o^2\tau + \alpha x^{-3} + \frac{1}{2}\alpha\tau x^{-2}\right)\left(1+\frac{3}{2}\tau x\right)^{\frac{1}{3}} > 0.$$

As $\lim_{x\to 0+}\Psi(x)=-\infty$ and $\lim_{x\to\infty}\Psi(x)=\infty$, $\Psi(x)$ has a unique root $\xi>0$.

Remark 4.3. Since $\{A_o^{(i)}\}$ can be any converging sequence, the argument above shows

$$\lim_{m\to-\infty} A_o = r_o\xi \quad \text{and} \quad \lim_{m\to-\infty} u_m(A_o k) = r_o\left(1+\frac{3}{2}\tau\xi\right)^{\frac{2}{3}}.$$

Suppose $\beta>0$, we want to estimate $\xi>0$ which is the solution to (4.28). Similar to (2.23), we make a change of variable by letting

$$\xi = \left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}}\theta_\kappa^2$$

for $\theta_\kappa>0$. Then (4.28) becomes

$$\left[1+3\kappa^2 r_o^2\beta^{-1}\left(1+\frac{3}{2}\left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}}\tau\theta_\kappa^2\right)^{\frac{4}{3}}\right]^{\frac{3}{4}}\theta_\kappa^3-\frac{3}{2}\left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}}\tau\theta_\kappa^2-1=0. \tag{4.29}$$

For $x\in(0,\infty)$, consider the function

$$f(x)=\left[1+3\kappa^2 r_o^2\beta^{-1}\left(1+\frac{3}{2}\left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}}\tau x^2\right)^{\frac{4}{3}}\right]^{\frac{3}{4}}x^3-\frac{3}{2}\left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}}\tau x^2-1. \tag{4.30}$$

By Remark 4.2, $f(x)$ has a unique positive root θ_κ. As in Theorem 2.1, we let $\theta>0$ be the unique root to

$$\theta^3-\frac{3}{2}\left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}}\tau\theta^2-1=0. \tag{4.31}$$

Then $f(\theta)\geq 0$. Therefore, we conclude

$$\theta_\kappa\leq\theta. \tag{4.32}$$

In particular, using the fact $\theta\leq 1+\frac{3}{2}\left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}}\tau$, we have

$$\theta_\kappa\leq 1+\frac{3}{2}\left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}}\tau. \tag{4.33}$$

Our above discussion has established the following theorem.

Theorem 4.1. *Let Ω, Σ, g, H_o, r_o and τ be given in Theorem 1.5. Suppose g is not a round metric and its Gauss curvature satisfies $K_g > -3\kappa^2$. Let $\{g(t)\}_{0\leq t\leq 1}$ be a given smooth path of metrics on Σ satisfying $g(0) = g$, $g(1)$ is round, $K_{g(t)} > -3\kappa^2$, and $tr_{g(t)} g'(t) = 0$. Then the hyperbolic Hawking mass $\mathfrak{m}^{\mathbb{H}}_{_B}(\Sigma)$ satisfies*

$$\begin{aligned} &\mathfrak{m}^{\mathbb{H}}_{_B}(\Sigma) - \mathfrak{m}^{\mathbb{H}}_{_H}(\Sigma) \\ &\leq \frac{r_o}{2}\left[\kappa^2 r_o^2 \left(1+\frac{3}{2}\tau\xi\right)^2 + \left(1+\frac{3}{2}\tau\xi\right)^{\frac{2}{3}} - \kappa^2 r_o^2 - 1\right]. \end{aligned} \tag{4.34}$$

Here $\xi > 0$ is a constant given by

$$\xi = \left(\frac{\frac{1}{2}\alpha}{\beta + 3\kappa^2 r_o^2}\right)^{\frac{1}{2}}, \quad \text{if } \beta \leq 0; \tag{4.35}$$

and ξ is the unique positive root to

$$\left[\beta + 3\kappa^2 r_o^2\left(1+\frac{3}{2}\tau\xi\right)^{\frac{4}{3}}\right]\xi^2 - \frac{\alpha}{2}\left(1+\frac{3}{2}\tau\xi\right)^{\frac{4}{3}} = 0, \quad \text{if } \beta > 0. \tag{4.36}$$

In the latter case, if one writes $\xi = \left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}}\theta_\kappa^2$ for a positive θ_κ, then $\theta_\kappa \leq \theta$ where $\theta > 0$ is the unique root to

$$\theta^3 - \frac{3}{2}\left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}}\tau\theta^2 - 1 = 0.$$

In particular, this shows

$$\xi \leq \left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}}\left[1 + \frac{3}{2}\left(\frac{\alpha}{2\beta}\right)^{\frac{1}{2}}\tau\right]^2. \tag{4.37}$$

Theorem 1.5 is a corollary of Theorem 4.1.

Proof of Theorem 1.5. Note that the second inequality in (1.9) simply follows from

$$\begin{aligned} &\kappa^2 r_o^2\left(1+\frac{3}{2}\tau x\right)^2 + \left(1+\frac{3}{2}\tau x\right)^{\frac{2}{3}} - \kappa^2 r_o^2 - 1 \\ &= \tau x\left(1+\frac{3}{4}\tau x\right)\left[3\kappa^2 r_o^2 + \frac{3}{\left(1+\frac{3}{2}\tau x\right)^{\frac{4}{3}} + \left(1+\frac{3}{2}\tau x\right)^{\frac{2}{3}} + 1}\right], \ x \geq 0. \end{aligned}$$

If $\inf_\Sigma K_g \le 0$, the pair (α, β) associated to any path $\{g(t)\}_{0\le t\le 1}$ with $g(0) = g$, $g(1)$ is round, $K_{g(t)} > -3\kappa^2$, and $\mathrm{tr}_{g(t)} g'(t) = 0$, necessarily has $\beta \le 0$. Thus, (i) follows from taking the infimum over such paths in (4.35) of Theorem 4.1.

Suppose $\inf_\Sigma K_g > 0$, moreover we assume g is not a round metric. In this case, we can restrict the attention to the paths $\{g(t)\}_{0\le t\le 1}$ with $g(0) = g$, $g(1)$ is round, $K_{g(t)} > 0$, and $\mathrm{tr}_{g(t)} g'(t) = 0$. A pair (α, β) associated to such a path has $\beta > 0$. Applying Theorem 4.1, by (4.36), we see (1.9) holds for (4.38)

$$\xi = \inf_{\{g(t)\}} \left\{ \text{the root of } \left[\beta + 3\kappa^2 r_o^2 \left(1 + \frac{3}{2}\tau x\right)^{\frac{4}{3}}\right] x^2 - \frac{\alpha}{2}\left(1 + \frac{3}{2}\tau x\right)^{\frac{4}{3}} = 0 \right\}.$$

Furthermore, such an ξ satisfies

$$\xi \le \zeta_g \theta^2,$$

where θ is the unique root to $\theta^3 - \frac{3}{2}\zeta_g \tau \theta^2 - 1 = 0$. Since $\theta \le 1 + \frac{3}{2}\zeta_g\tau$, we have $\xi \le \zeta_g \left(1 + \frac{3}{2}\zeta_g\tau\right)^2$. This completes the proof. $\square$

We end this paper with a remark that discusses the analogues of (1.6).

Remark 4.4. In the context of Theorem 1.5, one indeed has

$$\mathfrak{m}^{\mathbb{H}}_{H}(\Sigma) + \frac{r_o}{2}\left[\kappa^2 r_o^2 \left(1 + \frac{3}{2}\tau\xi\right)^2 + \left(1 + \frac{3}{2}\tau\xi\right)^{\frac{2}{3}} - \kappa^2 r_o^2 - 1\right] \ge 0. \tag{4.39}$$

This follows from Proposition 4.1 and (4.24). Recall that Proposition 4.1 is a consequence of the positive mass theorem on asymptotically hyperbolic manifolds.

Next suppose the compact manifold Ω in Theorem 1.5 has an additional CMC boundary component $\Sigma_h := \partial\Omega \setminus \Sigma$ whose mean curvature equals 2κ (with respect to the inward normal). Suppose Σ_h minimizes area among surfaces enclosing Σ_h in Ω. Assuming the Penrose inequality on asymptotically hyperbolic manifolds holds valid (see [21, 28] for a statement of this conjecture), one would have $\mathfrak{m}^{\mathbb{H}}_{H}(\Sigma_1) \ge \frac{1}{2} r_h$ as a replacement of Proposition 4.1. Here r_h is the area radius of Σ_h. This combined with (4.24) then implies

$$\mathfrak{m}^{\mathbb{H}}_{H}(\Sigma) + \frac{r_o}{2}\left[\kappa^2 r_o^2 \left(1 + \frac{3}{2}\tau\xi\right)^2 + \left(1 + \frac{3}{2}\tau\xi\right)^{\frac{2}{3}} - \kappa^2 r_o^2 - 1\right] \ge \frac{r_h}{2}. \tag{4.40}$$

Since the hyperbolic Penrose inequality is still open, the above inequality on compact manifolds Ω may serve as a test of the hyperbolic Penrose inequality.

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References

[1] R. Arnowitt, S. Deser, and C. W. Misner, *Coordinate invariance and energy expressions in general relativity*, Phys. Rev. (2) **122** (1961), 997–1006.

[2] R. Bartnik, *New definition of quasilocal mass*, Phys. Rev. Lett. **62** (1989), 2346–2348.

[3] R. Bartnik, *Energy in general relativity*, in: Tsing Hua Lectures on Geometry and Analysis, pp. 5–27, International Press, (1997).

[4] V. Bonini and J. Qing, *A positive mass theorem on asymptotically hyperbolic manifolds with corners along a hypersurface*, Ann. Henri Poincaré **9** (2008), no. 2, 347–372.

[5] H. L. Bray, *Proof of the Riemannian Penrose inequality using the positive mass theorem*, J. Differential Geom. **59** (2001), no. 2, 177–267.

[6] H. L. Bray and P. T. Chruściel, *The Penrose inequality*, in: P. T. Chruściel and H. Friedrich, editors, The Einstein Equations and the Large Scale Behavior of Gravitational Fields, pp. 39–70, Birkhäuser Verlag, Basel, (2004).

[7] J. D. Brown and J. J. W. York, *Quasilocal energy in general relativity*, in: Mathematical Aspects of Classical Field Theory (Seattle, WA, 1991), Vol. 132, pp. 129–142, Amer. Math. Soc., Providence, (1992).

[8] J. D. Brown and J. J. W. York, *Quasilocal energy and conserved charges derived from the gravitational action*, Phys. Rev. D (3) **47** (1993), no. 4, 1407–1419.

[9] A. J. Cabrera Pacheco, C. Cederbaum, and S. McCormick, *Asymptotically hyperbolic extensions and an analogue of the Bartnik mass*, J. Geom. Phys. **132** (2018), 338–357.

[10] A. J. Cabrera Pacheco, C. Cederbaum, S. McCormick, and P. Miao, *Asymptotically flat extensions of CMC Bartnik data*, Class. Quantum Grav. **34** (2017), 105001, 15pp.

[11] D. Christodoulou and S.-T. Yau, *Some remarks on the quasi-local mass*, in: Mathematics and General Relativity (Santa Cruz, CA, 1986), Vol. 71, pp. 9–14, Amer. Math. Soc., Providence, (1988).

[12] P. T. Chruściel and M. Herzlich, *The mass of asymptotically hyperbolic Riemannian manifolds*, Pacific J. Math. **212** (2003), no. 2, 231–264.

[13] S. W. Hawking, *Black holes in general relativity*, Commun. Math. Phys. **25** (1972), no. 2, 152–166.

[14] G. Huisken and T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differential Geom. **59** (2001), no. 3, 353–437.

[15] C.-Y. Lin and C. Sormani, *Bartnik's mass and Hamilton's modified Ricci flow*, Ann. Henri Poincaré **17** (2016), no. 10, 2783–2800.

[16] C.-Y. Lin and Y.-K. Wang, *On isometric embeddings into anti-de Sitter spacetimes*, Int. Math. Res. Not. IMRN (2015), no. 16, 7130–7161.

[17] C. Mantoulidis and R. Schoen, *On the Bartnik mass of apparent horizons*, Class. Quantum Grav. **32** (2015), no. 20, 205002, 16pp.

[18] P. Miao, *Positive mass theorem on manifolds admitting corners along a hypersurface*, Adv. Theor. Math. Phys. **6** (2002), no. 6, 1163–1182.

[19] P. Miao, *On a localized Riemannian Penrose inequality*, Commun. Math. Phys. **292** (2009), no. 1, 271–284.

[20] P. Miao and N. Q. Xie, *On compact 3-manifolds with nonnegative scalar curvature with a CMC boundary component*, Trans. Amer. Math. Soc. **370** (2018), no. 8, 5887–5906.

[21] A. Neves, *Insufficient convergence of inverse mean curvature flow on asymptotically hyperbolic manifolds*, J. Differential Geom. **84** (2010), no. 1, 191–229.

[22] R. Schoen and S.-T. Yau, *On the proof of the positive mass conjecture in general relativity*, Commun. Math. Phys. **65** (1979), no. 1, 45–76.

[23] Y.-G. Shi, J. Sun, G. Tian, and D. Wei, *Uniqueness of the mean field equation and rigidity of Hawking mass*, Calc. Var. Partial Diff. Equations **58** (2019), no. 2, 41, 16pp.

[24] Y.-G. Shi and L.-F. Tam, *Positive mass theorem and the boundary behaviors of compact manifolds with nonnegative scalar curvature*, J. Differential Geom. **62** (2002), no. 1, 79–125.

[25] Y.-G. Shi and L.-F. Tam, *Rigidity of compact manifolds and positivity of quasi-local mass*, Class. Quantum Grav. **24** (2007), 2357–2366.

[26] J. Sun, *Rigidity of Hawking mass for surfaces in three manifolds*, Pacific J. Math. **292** (2018), no. 2, 479–504.

[27] M.-T. Wang and S.-T. Yau, *A generalization of Liu-Yau's quasi-local mass*, Comm. Anal. Geom. **15** (2007), no. 2, 249–282.

[28] X. Wang, *The mass of asymptotically hyperbolic manifolds*, J. Differential Geom. **57** (2001), no. 2, 273–299.

[29] E. Witten, *A new proof of the positive energy theorem*, Commun. Math. Phys. **80** (1981), no. 3, 381–402.

[30] D. Wiygul, *The Bartnik-Bray outer mass of small metric spheres in time-symmetric* 3*-slices*, Commun. Math. Phys. **358** (2018), 269–293.

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