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Minimal model program for log canonical threefolds in positive characteristic

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Given a three-dimensional projective log canonical pair over a perfect field of characteristic larger than five, there exists a minimal model program that terminates after finitely many steps.

1. Introduction

Recently there have been large developments on the three-dimensional minimal model program of characteristic $p > 5$. The first remarkable achievement was made by Hacon and Xu. They proved the existence of minimal models for terminal threefolds over an algebraically closed field k of characteristic $p > 5$ [HX15]. Actually, they succeeded in establishing the so-called generalised minimal model program for terminal threefolds X over k such that K_X is pseudo-effective. Then Cascini, Xu and the third author dropped the assumption that K_X is pseudo-effective [CTX15]. As a consequence, it turned out that an arbitrary terminal threefold over k is birational to either a minimal model or a Mori fibre space. Finally, Birkar and Waldron established the minimal model program for klt threefolds over k [Bir16], [BW17].

The main objective of this paper is to give a generalisation of the three-dimensional minimal model program over k to the log canonical case. Furthermore, we treat perfect base fields, the relative setting and the non-$\mathbb{Q}$-factorial case. More specifically, our main theorem is as follows.

Theorem 1.1 (cf. Theorem 6.12). *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional log canonical pair over k, where Δ is an $\mathbb{R}$-divisor. Let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Then there exists a $(K_X + \Delta)$-MMP over Z that terminates.*

In particular, if X is $\mathbb{Q}$-factorial, then there is a sequence of birational maps of three-dimensional normal varieties:

$$X =: X_0 \overset{\varphi_0}{\dashrightarrow} X_1 \overset{\varphi_1}{\dashrightarrow} \cdots \overset{\varphi_{\ell-1}}{\dashrightarrow} X_\ell$$

such that if Δ_i denotes the proper transform of Δ on X_i, then the following properties hold:

(1) *For any $i \in \{0, \ldots, \ell\}$, (X_i, Δ_i) is a $\mathbb{Q}$-factorial log canonical pair which is projective over Z.*

(2) *For any $i \in \{0, \ldots, \ell - 1\}$, $\varphi_i : X_i \dashrightarrow X_{i+1}$ is either a $(K_{X_i} + \Delta_i)$-divisorial contraction over Z or a $(K_{X_i} + \Delta_i)$-flip over Z.*

(3) *If $K_X + \Delta$ is pseudo-effective over Z, then $K_{X_\ell} + \Delta_\ell$ is nef over Z.*

(4) *If $K_X + \Delta$ is not pseudo-effective over Z, then there exists a $(K_{X_\ell} + \Delta_\ell)$-Mori fibre space $X_\ell \to Y$ over Z.*

Remark 1.2. Theorem 1.1 is known for the following cases:

(1) (X, Δ) is $\mathbb{Q}$-factorial klt and Δ is a $\mathbb{Q}$-divisor [GNT, Theorem 2.13].

(2) k is algebraically closed, Z is projective over k, and $K_X + \Delta \equiv_f D$ for some effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D [Wal18, Corollary 1.8].

To prove the main theorem (Theorem 1.1), we also establish some fundamental results such as the cone theorem and the base point free theorem under the same generality.

Theorem 1.3 (cf. Theorem 4.6, Theorem 4.7). *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional log canonical pair over k, where Δ is an $\mathbb{R}$-divisor. Let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Then there exists a countable set $\{C_i\}_{i\in I}$ of curves on X which satisfies the following properties:*

(1) *$f(C_i)$ is a point for any $i \in I$.*

(2) $\overline{\mathrm{NE}}(X/Z) = \overline{\mathrm{NE}}(X/Z)_{K_X+\Delta\geq 0} + \sum_{i\in I} \mathbb{R}_{\geq 0}[C_i]$.

(3) *If A is an f-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, then there exists a finite subset J of I such that*

$$\overline{\mathrm{NE}}(X/Z) = \overline{\mathrm{NE}}(X/Z)_{K_X+\Delta+A\geq 0} + \sum_{j\in J} \mathbb{R}_{\geq 0}[C_j].$$

(4) *If k is algebraically closed, then the set $\{C_i\}_{i\in I}$ can be chosen so that C_i is a rational curve such that $0 < -(K_X + \Delta) \cdot C_i \leq 6$ for any $i \in I$.*

Theorem 1.4 (cf. Theorem 4.10, Theorem 5.2, Theorem 5.3). *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional log canonical pair over k, where Δ is an $\mathbb{R}$-divisor. Let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Let L be an f-nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Assume that one of the following conditions holds:*

(1) *$L - (K_X + \Delta)$ is f-semi-ample and f-big.*

(2) *$L = K_X + \Delta$ and L is f-big.*

(3) *$L = K_X + \Delta$ and Δ is an f-big $\mathbb{R}$-Cartier $\mathbb{R}$-divisor.*

Then L is f-semi-ample.

Remark 1.5. Theorem 1.3 and Theorem 1.4(1)(2) are known for the case when k is algebraically closed and Z is projective over k [Wal18, Theorem 1.1, Theorem 1.2].

Although the main theorem (Theorem 1.1) only asserts the existence of a minimal model program that terminates, we also obtain several results on termination of flips.

Theorem 1.6 (cf. Theorem 5.5, Theorem 6.11). *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional log canonical pair* *over k, where Δ is an $\mathbb{R}$-divisor. Let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Then the following hold.*

(1) *If $K_X + \Delta \equiv_f D$ for some effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on X, then there exists no infinite sequence that is a $(K_X + \Delta)$-MMP over Z.*

(2) *Let A be an ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. If $K_X+\Delta$ is f-pseudo-effective, then there exists no infinite sequence that is a $(K_X + \Delta)$-MMP over Z with scaling of A.*

We now summarise some known results related to this paper. Several results toward the abundance conjecture for threefolds has been established ([DW], [Wal17], [XZ], [Zha]). For threefolds of characteristic $p > 5$, the Iitaka conjecture has been proven to be true if the generic fibre is smooth [EZ18]. For three-dimensional log canonical pairs of characteristic $p > 5$, Das and Hacon prove that minimal lc centres are normal [DH16].

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2. Preliminaries

2.1. Notation

In this subsection, we summarise notation used in this paper.

- We will freely use the notation and terminology in [Har77] and [Kol13].
- For a scheme X, its *reduced structure* X_{red} is the reduced closed subscheme of X such that the induced morphism $X_{\mathrm{red}} \to X$ is surjective.
- For an integral scheme X, we define the *function field* $K(X)$ of X to be $\mathcal{O}_{X,\xi}$ for the generic point ξ of X.
- For a field k, we say that X is a *variety over k* or a *k-variety* if X is an integral scheme that is separated and of finite type over k. We

say that X is a *curve* over k or a k-*curve* (resp. a *surface* over k or a k-*surface*, resp. a *threefold* over k) if X is an k-variety of dimension one (resp. two, resp. three).

- Let $f : X \to Z$ be a projective morphism of noetherian separated schemes, where X is an integral normal scheme. For a Cartier divisor L on X, we say that L is f-*free* if the induced homomorphism $f^*f_*\mathcal{O}_X(L) \to \mathcal{O}_X(L)$ is surjective. Let M be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. We say that M is f-*ample* (resp. f-*semi-ample*) if we can write $M = \sum_{i=1}^{r} a_i M_i$ for some $r \geq 1$, positive real numbers a_i and f-ample (resp. f-free) Cartier divisors M_i. We say that M is f-*big* if we can write $M = A + E$ for some f-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor A and effective $\mathbb{R}$-divisor E. We define f-nef $\mathbb{R}$-divisors in the same way as in [Kol13, Definition 1.4]. We say that M is f-*numerically trivial*, denoted by $M \equiv_f 0$ or $M \equiv_Z 0$, if both M and $-M$ are f-nef.

  Given $\mathbb{R}$-Cartier $\mathbb{R}$-divisors D_1 and D_2 on X such that $D_1 + \lambda_0 D_2$ is f-nef for some $\lambda_0 \in \mathbb{R}_{>0}$, the f-*nef threshold* of (D_1, D_2) is the non-negative real number λ defined by

$$\lambda := \inf\{\mu \in \mathbb{R}_{\geq 0} \mid D_1 + \mu D_2 \text{ is } f\text{-nef}\}.$$

  It is well-known that inf can be replaced by min in this definition if Z is of finite type over a field.

- Let Δ be an $\mathbb{R}$-divisor on an integral normal separated noetherian scheme and let $\Delta = \sum_{i \in I} r_i D_i$ be its irreducible decomposition. We define $\Delta^{\geq 1} := \sum_{i \in I, r_i \geq 1} r_i D_i$ and $\Delta^{\wedge 1} := \sum_{i \in I} r'_i D_i$ where $r'_i := \min\{r_i, 1\}$. We also define $\Delta^{>1}$ and $\Delta^{<1}$ similarly. Furthermore, we set $\{\Delta\} := \Delta - \lfloor \Delta \rfloor$.

- Let X be a normal variety X over a field and let X' be a non-empty open subset X' of X. For a prime divisor F on X', the *closure* $\overline{F}$ of F in X is the prime divisor on X whose generic point is equal to the generic point of F. For an $\mathbb{R}$-divisor D on X', the *closure* $\overline{D}$ of D in X is defined as $\sum_{i \in I} r_i \overline{D}_i$, where $D = \sum_{i \in I} r_i D_i$ is the irreducible decomposition of D.

- Let k be a field and let $f : X \to Y$ be a birational k-morphism of normal k-varieties. For a prime divisor F on X, the *push-forward* f_*F of F by f (or to Y) is defined as follows: if $F \subset \mathrm{Ex}(f)$, then $f_*F = 0$, and if $F \not\subset \mathrm{Ex}(f)$, then f_*F is defined as the prime divisor whose generic point is equals to the generic point of $f(F)$. For an $\mathbb{R}$-divisor

D on X, the *push-forward* f_*D of D by f (or to Y) is defined as $\sum_{i\in I} r_i f_* D_i$, where $D = \sum_{i\in I} r_i D_i$ is the irreducible decomposition of D.

- Let k be a field and let $f : X \dashrightarrow Y$ be a birational map. Let D be an $\mathbb{R}$-divisor on X. The *proper transform* D' of D on Y is defined as $f'_*D|_{X'}$, where X' denotes the maximum open subset X' of X where f is defined, and $f' : X' \to Y$ is the induced birational morphism.
- Let D be a closed subset of a smooth scheme X over a perfect field k and let $D_1, \ldots, D_n$ be the irreducible components of D with the reduced scheme structures. We say that D is *simple normal crossing* if the scheme-theoretic intersection $\bigcap_{j\in J} D_j$ is smooth over k for every subset $J \subset \{1, \ldots, n\}$. For a variety X over k and a closed subset Z of X, we say that $f : Y \to X$ is a *log resolution* of (X, Z) if f is a projective birational k-morphism from a smooth variety Y over k such that $\mathrm{Ex}(f) \cup f^{-1}(Z)$ is purely codimension one and $\mathrm{Ex}(f) \cup f^{-1}(Z)$ is simple normal crossing. In dimension three, there exists a log resolution for such a pair (X, Z) by [CP08]. For a variety X over k and an $\mathbb{R}$-divisor Δ on X, a log resolution of (X, Δ) means a log resolution of $(X, \mathrm{Supp}\,\Delta)$.

2.2. Log pairs

We recall some notation in the theory of singularities in the minimal model program. For more details, we refer the reader to [KM98, Section 2.3] and [Kol13, Section 1].

We say that (X, Δ) is a *log pair* over a field k if X is a normal variety over k and Δ is an effective $\mathbb{R}$-divisor such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. For a proper birational morphism $f : X' \to X$ from a normal variety X' and a prime divisor E on X', the *log discrepancy* of (X, Δ) at E is defined as

$$a_E(X, \Delta) := \mathrm{coeff}_E(K_{X'} - f^*(K_X + \Delta)) + 1.$$

We say that a log pair (X, Δ) is *log canonical* if $a_E(X, \Delta) \geq 0$ for any prime divisor E over X. Moreover, we say that a log pair (X, Δ) is *klt*, if $a_E(X, \Delta) > 0$ for any prime divisor E over X.

We call a log pair (X, Δ) *dlt* when all coefficients of Δ belong to $[0, 1]$ and there exists a log resolution $f : Y \to X$ of (X, Δ) such that $a_E(X, \Delta) > 0$ for any f-exceptional divisor E on Y.

For a log pair (X,Δ), $\mathrm{Nklt}(X,\Delta)$ denotes the closed subset of X consisting of the non-klt points of (X,Δ). We equip $\mathrm{Nklt}(X,\Delta)$ with the reduced scheme structure.

Definition 2.1. Given a field k, a log pair (X,Δ) over k, and a projective k-morphisms $X \xrightarrow{f_1} Z_1 \to Z_2$ of quasi-projective k-schemes, we say that $f_1 : X \to Z_1$ is a $(K_X+\Delta)$-*Mori fibre space over* Z_2 if $\dim X > \dim Z_1$, $(f_1)_*\mathcal{O}_X = \mathcal{O}_{Z_1}$, $\rho(X/Z_1) = 1$ and $-(K_X+\Delta)$ is f_1-ample. If $Z_2 = \mathrm{Spec}\, k$, then $f_1 : X \to Z_1$ is simply called a $(K_X+\Delta)$-*Mori fibre space.*

2.3. Log minimal models

Definition 2.2. Let k be a field and let Z be a quasi-projective k-scheme. Let (X,Δ) and (Y,Δ_Y) be log pairs over k that are projective over Z.

(1) We say that (Y,Δ_Y) is a *log birational model over* Z of (X,Δ) if there exists a birational map $\varphi : X \dashrightarrow Y$ over Z such that $\Delta_Y = \widetilde{\Delta} + E$, where $\widetilde{\Delta}$ denotes the proper transform of Δ on Y and E is the sum of the prime divisors that are exceptional over X.

(2) We say that (Y,Δ_Y) is a *weak log canonical model over* Z of (X,Δ) if
  - (a) (Y,Δ_Y) is a log birational model over Z of (X,Δ),
  - (b) $K_Y+\Delta_Y$ is nef over Z, and
  - (c) for any prime divisor D on X that is exceptional over Y, it holds that
$$a_D(X,\Delta) \leq a_D(Y,\Delta_Y).$$

(3) We say that (Y,Δ_Y) is a *log canonical model over* Z of (X,Δ) if
  - (a) (Y,Δ_Y) is a weak log canonical model over Z of (X,Δ), and
  - (b) $K_Y+\Delta_Y$ is ample over Z.

(4) We say that (Y,Δ_Y) is a *log minimal model over* Z of (X,Δ) if
  - (i) (Y,Δ_Y) is a weak log canonical model over Z of (X,Δ),
  - (ii) (Y,Δ_Y) is a $\mathbb{Q}$-factorial dlt pair, and
  - (iii) for any prime divisor D on X that is exceptional over Y, it holds that
$$a_D(X,\Delta) < a_D(Y,\Delta_Y).$$

Remark 2.3. Let k be a field and let Z be a quasi-projective k-scheme. Fix a log canonical pair (X,Δ) over k that is projective over Z. Let (Y_1,Δ_{Y_1}) and (Y_2,Δ_{Y_2}) be weak log canonical models over Z of (X,Δ). If $g_1 : W \to Y_1$ and $g_2 : W \to Y_2$ are projective birational morphisms that commute with

$Y_1 \dashrightarrow Y_2$, then it follows from the same argument as in [Bir12a, Remark 2.6] that

$$g_1^*(K_{Y_1} + \Delta_{Y_1}) = g_2^*(K_{Y_2} + \Delta_{Y_2}).$$

Moreover, if (Y_2, Δ_{Y_2}) is a log canonical model of (X, Δ), then $K_{Y_1} + \Delta_{Y_1}$ is semi-ample over Z and the birational map $Y_1 \dashrightarrow Y_2$ is a morphism. In particular, if both (Y_1, Δ_{Y_1}) and (Y_2, Δ_{Y_2}) are log canonical models of (X, Δ), then the induced birational map $Y_1 \dashrightarrow Y_2$ is an isomorphism.

2.4. Minimal model program

Let us recall construction of a sequence of steps of log canonical minimal model program.

Definition 2.4 (cf. [Fuj17, 4.9.1]). Let k be a field. Let (X, Δ) be a log pair over k and let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Let $g : X \dashrightarrow Y$ be a birational map over Z to a normal k-variety Y projective over Z.

(1) We say that g is a $(K_X + \Delta)$-*divisorial contraction* over Z if
   - (a) g is a morphism,
   - (b) $\dim \operatorname{Ex}(g) = \dim X - 1$,
   - (c) $-(K_X + \Delta)$ is g-ample, and
   - (d) $\rho(X/Y) = 1$.

(2) We say that g is a $(K_X + \Delta)$-*flipping contraction* over Z if
   - (a) g is a morphism,
   - (b) $\dim \operatorname{Ex}(g) < \dim X - 1$,
   - (c) $-(K_X + \Delta)$ is g-ample, and
   - (d) $\rho(X/Y) = 1$.

(3) We say that g is a *step of a* $(K_X + \Delta)$-*MMP* over Z if there exist a $(K_X + \Delta)$-divisorial or $(K_X + \Delta)$-flipping contraction $h : X \to W$ over Z and a log canonical model $(Y = X^+, \Delta^+)$ of (X, Δ) over W such that $g = (h^+)^{-1} \circ h$, where $h^+ : X^+ \to W$ is the induced morphism. If h is a flipping contraction, then any of h^+ and g is called a $(K_X + \Delta)$-*flip* over Z.

Definition 2.5. Let k be a field. Let (X, Δ) be a log pair over k and let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z.

(1) A (possibly infinite) sequence

$$X = X_0 \overset{\varphi_0}{\dashrightarrow} X_1 \overset{\varphi_1}{\dashrightarrow} \cdots$$

is called a $(K_X + \Delta)$-*MMP over* Z if any φ_i is a step of a $(K_X + \Delta)$-MMP over Z. If $Z = \mathrm{Spec}\, k$, then a $(K_X + \Delta)$-MMP over Z is simply called a $(K_X + \Delta)$-*MMP*.

(2) For an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor C on X, a (possibly infinite) sequence

$$X = X_0 \overset{\varphi_0}{\dashrightarrow} X_1 \overset{\varphi_1}{\dashrightarrow} \cdots$$

is called a $(K_X + \Delta)$-*MMP over* Z *with scaling of* C if the sequence is a $(K_X + \Delta)$-MMP over Z and

$$\lambda_i := \min\{\lambda \in \mathbb{R}_{\geq 0} \mid K_{X_i} + \Delta_i + \lambda C_i \text{ is nef over } Z\}$$

exists and $K_{X_i} + \Delta_i + \lambda_i C_i$ is numerically trivial over Z_i, where $X_i \to Z_i$ is the associated divisorial or flipping contraction. In this case, $\lambda_0, \lambda_1, \cdots$ are called the *scaling coefficients*. Note that we do not assume that C is effective.

(3) We say that a $(K_X + \Delta)$-MMP over Z *terminates* if the sequence is a finite sequence:

$$X = X_0 \overset{\varphi_0}{\dashrightarrow} X_1 \overset{\varphi_1}{\dashrightarrow} \cdots \overset{\varphi_{\ell-1}}{\dashrightarrow} X_\ell$$

such that either $K_{X_\ell} + \Delta_\ell$ is nef over Z or there is a $(K_{X_\ell} + \Delta_\ell)$-Mori fibre space over Z.

For later use, we now summarise known results from [Wal18] on three-dimensional log canonical minimal model program in positive characteristic.

Theorem 2.6. *Let k be an algebraically closed field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional log canonical pair over k and let $f : X \to Z$ be a projective k-morphism to a projective k-scheme Z. If $K_X + \Delta$ is f-nef and f-big, then $K_X + \Delta$ is f-semi-ample.*

Proof. See [Wal18, Theorem 1.1]. □

Theorem 2.7. *Let k be an algebraically closed field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional log canonical pair over k and let $f : X \to$*

Z be a projective k-morphism to a projective k-scheme Z. If A is an f-semi-ample and f-big $\mathbb{R}$-Cartier $\mathbb{R}$-divisor such that $K_X + \Delta + A$ is f-nef, then $K_X + \Delta + A$ is f-semi-ample.

Proof. See [Wal18, Theorem 1.2]. $\square$

Theorem 2.8. *Let k be an algebraically closed field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional projective log canonical pair over k. Then there exists a countable set $\{C_i\}_{i \in I}$ of rational curves on X which satisfies the following conditions:*

(1) $\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X+\Delta \geq 0} + \sum_{i \in I} \mathbb{R}_{\geq 0}[C_i]$.

(2) $0 < -(K_X + \Delta) \cdot C_i \leq 6$ *for any* $i \in I$.

(3) *For any ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor A, there exists a finite subset J of I such that*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X+\Delta+A \geq 0} + \sum_{j \in J} \mathbb{R}_{\geq 0}[C_j].$$

Proof. See [Wal18, Theorem 1.7]. $\square$

Theorem 2.9. *Let k be an algebraically closed field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional $\mathbb{Q}$-factorial log canonical pair over k and let $f : X \to Z$ be a projective k-morphism to a projective k-scheme Z. Assume that there exists an effective $\mathbb{R}$-divisor M on X such that $K_X + \Delta \equiv_f M$. Then there exists a $(K_X + \Delta)$-MMP over Z that terminates.*

Proof. See [Wal18, Corollary 1.8]. $\square$

2.5. Compactifications of pairs

First we show the existence of a compactification of klt pairs.

Proposition 2.10. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional quasi-projective $\mathbb{Q}$-factorial klt pair over k and let $f : X \to Z$ be a k-morphism to a quasi-projective k-scheme Z. Then there exists an open immersion $j : X \to \overline{X}$ over Z to a normal $\mathbb{Q}$-factorial threefold $\overline{X}$ projective over Z such that $(\overline{X}, \overline{\Delta})$ is klt for the closure $\overline{\Delta}$ of Δ.*

Proof. Enlarging coefficients of Δ appropriately, we may assume that Δ is a $\mathbb{Q}$-divisor. Since X is quasi-projective over Z, there exists an open immersion $j_1 : X \to X_1$ over Z to an integral normal scheme X_1 projective over Z.

Let $\varphi : V \to X_1$ be a resolution of singularities of X_1 such that $\varphi^{-1}(\overline{\operatorname{Supp} \Delta}) \cup \operatorname{Ex}(\varphi)$ is a simple normal crossing divisor. Let E be the reduced divisor on V such that $\operatorname{Supp} E = \operatorname{Ex}(\varphi)$. Fix sufficiently small $\epsilon \in \mathbb{Q}_{>0}$. Then it holds that $(V, \varphi_*^{-1}\overline{\Delta} + (1-\epsilon)E)$ is klt. Moreover, it follows from [GNT, Theorem 2.13] that there exists a $(K_V + \varphi_*^{-1}\overline{\Delta} + (1-\epsilon)E)$-MMP over X_1 that terminates. Let $\overline{X}$ be the end result. Then the negativity lemma and the $\mathbb{Q}$-factoriality of X imply that $\overline{X} \to X_1$ is isomorphic over X. Moreover, $(\overline{X}, \overline{\Delta})$ is klt for the closure $\overline{\Delta}$ of Δ in $\overline{X}$. □

Using Proposition 2.10 and the ACC for log canonical thresholds, we prove the existence of a certain compactification of log canonical pairs (Proposition 2.11, Proposition 2.12).

Proposition 2.11. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional quasi-projective $\mathbb{Q}$-factorial log canonical pair over k such that $(X, \{\Delta\})$ is klt. Let $f : X \to Z$ be a k-morphism to a quasi-projective k-scheme Z. Then there exists an open immersion $j : X \to \overline{X}$ over Z to a normal $\mathbb{Q}$-factorial threefold $\overline{X}$ projective over Z such that $(\overline{X}, \overline{\Delta})$ is log canonical and $(\overline{X}, \{\overline{\Delta}\})$ is klt, where $\overline{\Delta}$ denotes the closure of Δ in $\overline{X}$.*

Proof. Since $(X, \{\Delta\})$ is klt, also $(X, \Delta - \epsilon\llcorner\Delta\lrcorner)$ is klt for any $\epsilon \in \mathbb{R}$ such that $0 < \epsilon \leq 1$. For a sufficiently small $\epsilon \in \mathbb{R}_{>0}$, we apply Proposition 2.10 to a $\mathbb{Q}$-factorial klt pair $(X, \Delta - \epsilon\llcorner\Delta\lrcorner)$ and a morphism $X \to Z$. Then we get an open immersion $j : X \to \overline{X}$ over Z to a normal $\mathbb{Q}$-factorial threefold $\overline{X}$ projective over Z such that $(\overline{X}, D)$ is klt for the closure D of $\Delta - \epsilon\llcorner\Delta\lrcorner$. Since $\epsilon \in \mathbb{R}_{>0}$ is sufficiently small, it follows from the ACC for log canonical thresholds [Bir16, Theorem 1.10] that $(\overline{X}, \overline{\Delta})$ is log canonical for the closure $\overline{\Delta}$ of Δ. □

Proposition 2.12. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional quasi-projective $\mathbb{Q}$-factorial log canonical pair over k such that X is klt. Let $f : X \to Z$ be a k-morphism to a quasi-projective k-scheme Z. Then there exists an open immersion $j : X \to \overline{X}$ over Z to a normal $\mathbb{Q}$-factorial threefold $\overline{X}$ projective over Z such that $(\overline{X}, \overline{\Delta})$ is log canonical and $\overline{X}$ is klt.*

Proof. Since $(X, 0)$ is klt, also $(X, (1-\epsilon)\Delta)$ is klt for any $\epsilon \in \mathbb{R}$ such that $0 < \epsilon \leq 1$. For a sufficiently small $\epsilon \in \mathbb{R}_{>0}$, we apply Proposition 2.10 to a $\mathbb{Q}$-factorial klt pair $(X, (1-\epsilon)\Delta)$ and a morphism $X \to Z$. Then we get an open immersion $j : X \to \overline{X}$ over Z to a normal $\mathbb{Q}$-factorial threefold $\overline{X}$ projective over Z such that $(\overline{X}, D)$ is klt for the closure D of $(1-\epsilon)\Delta$.

Since $\epsilon \in \mathbb{R}_{>0}$ is sufficiently small, it follows from the ACC for log canonical thresholds [Bir16, Theorem 1.10] that $(\overline{X}, \overline{\Delta})$ is log canonical for the closure $\overline{\Delta}$ of Δ. □

2.6. Perturbation of coefficients

Lemma 2.13. *Fix $\mathbb{K} \in \{\mathbb{Q}, \mathbb{R}\}$. Let k be an infinite perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional log pair over k and let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme. Let A be an f-semi-ample $\mathbb{K}$-Cartier $\mathbb{K}$-divisor on X. Then the following hold.*

(1) *If (X, Δ) is log canonical, then there exists an effective $\mathbb{K}$-Cartier $\mathbb{K}$-divisor A' such that $A \sim_{f,\mathbb{K}} A'$ and $(X, \Delta + A')$ is log canonical.*

(2) *If (X, Δ) is klt, then there exists an effective $\mathbb{K}$-Cartier $\mathbb{K}$-divisor A' such that $A \sim_{f,\mathbb{K}} A'$ and $(X, \Delta + A')$ is klt.*

Proof. First we prove (1). By standard argument, we may assume that A is an f-semi-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor (note that Δ could be still an $\mathbb{R}$-divisor). Taking a log resolution of (X, Δ), we may assume that X is smooth over k and Δ is a simple normal crossing $\mathbb{R}$-divisor. Hence, enlarging coefficients of Δ, the problem is reduced to the case when $\mathbb{K} = \mathbb{Q}$.

Since A is an f-semi-ample $\mathbb{Q}$-divisor, there exist projective morphisms of schemes

$$f : X \xrightarrow{g} Y \xrightarrow{h} Z$$

and an h-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor A_Y such that Y is a normal variety over k and $A \sim_{\mathbb{Q}} g^*A_Y$.

Fix an open immersion $j_Z : Z \hookrightarrow \overline{Z}$ to a projective k-scheme. We can find an open immersion $j_Y : Y \hookrightarrow \overline{Y}$ over $\overline{Z}$ to a normal k-variety $\overline{Y}$ and a globally ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $A_{\overline{Y}}$ such that $\overline{Y}$ is projective over $\overline{Z}$ and $A_{\overline{Y}}|_Y \sim_{Z,\mathbb{Q}} A_Y$. Applying Proposition 2.12 to a log canonical pair (X, Δ) and a morphism $X \to \overline{Y}$, there exists an open immersion $j_X : X \hookrightarrow \overline{X}$ over $\overline{Z}$ such that $\overline{X}$ is a $\mathbb{Q}$-factorial threefold over k and $(\overline{X}, \overline{\Delta})$ is log canonical, where $\overline{\Delta}$ denotes the closure of Δ in $\overline{X}$. To summarise, we have a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{j_X} & \overline{X} \\ \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle \overline{g}} \\ Y & \xrightarrow{j_Y} & \overline{Y} \\ \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle \overline{h}} \\ Z & \xrightarrow{j_Z} & \overline{Z}. \end{array}$$

We apply [Tan17, Theorem 1] to a log canonical pair $(\overline{X}, \overline{\Delta})$ and a semi-ample $\mathbb{Q}$-divisor $\overline{g}^* A_{\overline{Y}}$. Then there exists an effective $\mathbb{Q}$-divisor B on $\overline{X}$ such that $\overline{g}^* A_{\overline{Y}} \sim_{\mathbb{Q}} B$ and $(\overline{X}, \overline{\Delta} + B)$ is log canonical. Set $A' := B|_X$. It holds that $(X, \Delta + A')$ is log canonical and

$$A' = B|_X \sim_{\mathbb{Q}} (\overline{g}^* A_{\overline{Y}})|_X \sim_{Z,\mathbb{Q}} g^* A_Y \sim_{Z,\mathbb{Q}} A.$$

We have proved (1).

Next we prove (2). Suppose that (X, Δ) is klt. We apply (1) to (X, Δ) and $2A$. Then there exists an effective $\mathbb{K}$-Cartier $\mathbb{K}$-divisor B such that $2A \sim_{f,\mathbb{K}} B$ and $(X, \Delta + B)$ is log canonical. Set $A' := (1/2)B$. Then we have that $A \sim_{f,\mathbb{K}} A'$ and $(X, \Delta + A')$ is klt. $\square$

Lemma 2.14. *Let k be a perfect field. Let X be a quasi-projective $\mathbb{Q}$-factorial normal variety over k. Let Δ be an effective $\mathbb{Q}$-divisor such that all the coefficients of Δ are contained in $[0, 1]$. Let A be an ample $\mathbb{Q}$-divisor on X. Then there exists an effective $\mathbb{Q}$-divisor Δ' on X such that $\Delta' \sim_{\mathbb{Q}} \Delta + A$ and all the coefficients of Δ' are contained in $[0, 1]$.*

Proof. There exists a positive real number ϵ such that $0 < \epsilon < 1$ and $\epsilon\Delta + A$ is ample. After replacing Δ and A by $(1 - \epsilon)\Delta$ and ϵA respectively, we may assume that there exists a positive real number ϵ such that $0 < \epsilon < 1$ and all the coefficients of Δ are contained in $[0, 1 - \epsilon]$. If k is an infinite field (resp. a finite field), then [Sei50, Theorem 1] (resp. [Poo04, Theorem 1.1]) enables us to find a positive integer n and a smooth divisor H on the smooth locus X_{sm} of X such that

- $1/n < \epsilon$, and
- nA is a Cartier divisor such that $(nA)|_{X_{\mathrm{sm}}} \sim H$.

For the closure $\overline{H}$ of H in X, it holds that $nA \sim \overline{H}$ and $\overline{H}$ is a reduced divisor. Then it holds that

$$\Delta + A = \Delta + \frac{1}{n}(nA) \sim_{\mathbb{Q}} \Delta + \frac{1}{n}\overline{H} =: \Delta'$$

and all the coefficients of Δ' are contained in $[0, 1]$. $\square$

2.7. Non-vanishing theorem of relative dimension two

Lemma 2.15. *Let k be a field of characteristic $p > 0$. Let (X, Δ) be a log canonical pair over k and let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-variety such that $f_*\mathcal{O}_X = \mathcal{O}_Z$. Set X_K to be the generic fibre of f. Assume that*

(1) $\dim X_K \leq 2$, *and*

(2) $(K_X + \Delta)|_{X_K}$ *is pseudo-effective.*

Then there exists an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on X such that $K_X + \Delta \sim_{Z,\mathbb{R}} D$.

Proof. By the non-vanishing theorem for surfaces (cf. [Tan18b, Theorem 1.1] and [Tan, Theorem 1.1]), we obtain

$$(K_X + \Delta)|_{X_K} + \sum_{i=1}^{a} r_i \mathrm{div}(\varphi_i) = E$$

for some effective $\mathbb{R}$-divisor E on X_K, $r_1, \ldots, r_a \in \mathbb{R}$, and $\varphi_1, \ldots, \varphi_a \in K(X_K)$. By $K(X) = K(X_K)$, we get $\varphi_i \in K(X)$. We define an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor F by

$$K_X + \Delta + \sum_{i=1}^{a} r_i \mathrm{div}(\varphi_i) =: F.$$

We obtain $F|_{X_K} = E$. In particular, if $F = F_1 - F_2$ for effective $\mathbb{R}$-divisors F_1 and F_2 without any common irreducible components, then we get $f(\mathrm{Supp}\ F_2) \subsetneq Z$. Since Z is quasi-projective over k, there exists an effective Cartier divisor H_Z on Z such that $f(\mathrm{Supp}\ F_2) \subset \mathrm{Supp}\ H_Z$. Hence, for a sufficiently large positive integer m, it holds that $mf^*H_Z - F_2$ is an effective $\mathbb{R}$-divisor. We have that

$$K_X + \Delta + \sum_{i=1}^{a} r_i \mathrm{div}(\varphi_i) = F = F_1 - F_2 \sim_{Z,\mathbb{R}} F_1 + (mf^*H_Z - F_2) \geq 0,$$

as desired. □

3. MMP for the effective case

The purpose of this section is to establish the relative minimal model program for log canonical threefolds with effective log canonical divisors (Theorem 3.4). As a consequence, we obtain the existence of dlt modifications (Proposition 3.5). We start with the following lemma on dlt modification. Note that this result is known if the base field is algebraically closed [Bir16, Theorem 1.6].

Lemma 3.1. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional quasi-projective $\mathbb{Q}$-factorial log canonical pair over k, where Δ is a $\mathbb{Q}$-divisor. Then there exists a projective birational k-morphism $f : Y \to X$ from a normal threefold Y over k such that*

(1) *Y is $\mathbb{Q}$-factorial,*

(2) *$a_F(X, \Delta) = 0$ for any f-exceptional prime divisor F on Y, and*

(3) *if Δ_Y is the $\mathbb{R}$-divisor defined by $K_Y + \Delta_Y = f^*(K_X + \Delta)$, then Δ_Y is effective and $(Y, \{\Delta_Y\})$ is klt (see Subsection 2.1 for the definition of $\{\Delta_Y\}$).*

Proof. Let $\varphi : V \to X$ be a log resolution of (X, Δ). We can write

$$K_V + \varphi_*^{-1}\Delta + E = \varphi^*(K_X + \Delta) + \sum_i a_i E_i,$$

where E_i is a φ-exceptional prime divisor, $a_i := a_{E_i}(X, \Delta) \in \mathbb{Q}_{\geq 0}$ is the log discrepancy, and E is the reduced divisor on V such that $\operatorname{Supp} E = \operatorname{Ex}(\varphi)$. Fix a sufficiently small $\epsilon \in \mathbb{Q}_{>0}$ so that if $a_i > 0$, then $a_i - \epsilon > 0$. Since $\varphi_*^{-1}\Delta + (1 - \epsilon)E - \delta\varphi^*\Delta$ is still effective for sufficiently small $\delta > 0$, [GNT, Theorem 2.13] enables us to find a $(K_V + \varphi_*^{-1}\Delta + (1 - \epsilon)E)$-MMP over X that terminates. Let $g : Y \to X$ be the end result. Then (1) holds. The negativity lemma implies that this MMP contracts all the prime divisors E_i satisfying $a_i = a_{E_i}(X, \Delta) > 0$. In other words, any g-exceptional prime divisor F satisfies $a_F(X, \Delta) = 0$. Therefore, (2) holds. Furthermore, (3) holds because being dlt is preserved under the MMP. □

Theorem 3.2. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional $\mathbb{Q}$-factorial log canonical pair over k, where Δ is a*

$\mathbb{Q}$-divisor. Let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. If $K_X + \Delta$ is f-big, then the graded ring

$$\bigoplus_{m=0}^{\infty} f_*\mathcal{O}_X(\llcorner m(K_X + \Delta)\lrcorner)$$

is a finitely generated $\mathcal{O}_Z$-algebra.

Proof. By standard argument, we may assume that k is algebraically closed and $f_*\mathcal{O}_X = \mathcal{O}_Z$ (cf. [GNT, Remark 2.7]). By Lemma 3.1, the problem is reduced to the case when

(1) $(X, \{\Delta\})$ is klt.

Take an open immersion $j_Z : Z \to \overline{Z}$ such that $\overline{Z}$ is a projective normal threefold over k. Applying Proposition 2.11 to (X, Δ) and $X \to \overline{Z}$, we may assume that (1) and

(2) both X and Z are projective over k.

By Theorem 2.9, there exists a $(K_X + \Delta)$-MMP over Z that terminates. Replacing (X, Δ) by the end result, we may assume that (2) and

(3) $K_X + \Delta$ is f-nef.

By Theorem 2.7, the assertion in the statement holds. □

Proposition 3.3. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional $\mathbb{Q}$-factorial log canonical pair over k, where Δ is a $\mathbb{Q}$-divisor. Let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Let L be an f-nef and f-big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor such that $L - (K_X + \Delta)$ is f-semi-ample. Then L is f-semi-ample.*

Proof. We may assume that k is algebraically closed and $f_*\mathcal{O}_X = \mathcal{O}_Z$. By Lemma 2.13 (1), we are reduced to the case when $L = K_X + \Delta$. By Lemma 3.1, we may assume that $(X, \{\Delta\})$ is klt.

Fix an open immersion $j_Z : Z \hookrightarrow \overline{Z}$ such that $\overline{Z}$ is a normal variety projective over k. It follows from Proposition 2.11 that there exists an open immersion $j_1 : X \hookrightarrow X_1$ over $\overline{Z}$ to a $\mathbb{Q}$-factorial threefold X_1 projective over $\overline{Z}$ such that (X_1, Δ_{X_1}) is log canonical for the closure Δ_{X_1} of Δ in X_1. Let $f_1 : X_1 \to \overline{Z}$ be the induced morphism. Since $K_{X_1} + \Delta_{X_1} \sim_{f_1,\mathbb{Q}} D$ for some effective $\mathbb{Q}$-divisor D, Theorem 2.9 enables us to find a $(K_{X_1} + \Delta_{X_1})$-MMP over $\overline{Z}$ that terminates. Let X_2 be the end result and let Δ_{X_2} be the push-forward of Δ_{X_1}. Then $K_{X_2} + \Delta_{X_2}$ is f_2-nef, where $f_2 : X_2 \to \overline{Z}$ is the induced morphism. Since $K_X + \Delta$ is f-nef, the two morphisms f_1 and f_2 coincide over Z. In particular, there exists an open immersion $j_2 : X \hookrightarrow X_2$ such that $j_2(X) = f_2^{-1}(Z)$. It follows from Theorem 2.7 that $K_{X_2} + \Delta_{X_2}$ is f_2-semi-ample. Taking the restriction to X, we see that $K_X + \Delta$ is f-semi-ample, as desired. □

Theorem 3.4. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional $\mathbb{Q}$-factorial log canonical pair over k and let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Assume that $K_X + \Delta \equiv_{f,\mathbb{R}} D$ for some effective $\mathbb{R}$-divisor D on X. Then there exists a $(K_X + \Delta)$-MMP over Z that terminates.*

Proof. Assuming that $K_X + \Delta$ is not f-nef, let us find a $(K_X + \Delta)$-negative extremal ray and its contraction. There exists an f-ample $\mathbb{R}$-divisor A such that $K_X + \Delta + A$ is not f-nef.

Let us find an effective $\mathbb{Q}$-divisor Δ', an f-ample $\mathbb{Q}$-divisor A'_1 and an f-ample $\mathbb{R}$-divisor A'_2 such that

(1) $K_X + \Delta + A = K_X + \Delta' + A'_1 + A'_2$,

(2) all the coefficients of Δ' are contained in $[0, 1]$, and

(3) $K_X + \Delta' \equiv_{f,\mathbb{Q}} D'$ for some effective $\mathbb{Q}$-divisor D'.

We first take an ample $\mathbb{Q}$-divisor A'_1 on X such that $A - 3A'_1$ is f-ample. Then there exists a $\mathbb{Q}$-divisor Δ_1 on X such that $0 \leq \Delta_1 \leq \Delta$ and both $A'_1 + (\Delta - \Delta_1)$ and $A'_1 - (\Delta - \Delta_1)$ are f-ample. By Lemma 2.14, there exists an effective $\mathbb{Q}$-divisor Δ' such that $\Delta' \sim_{\mathbb{Q}} \Delta_1 + A'_1$ and all the coefficients of Δ' are contained in $[0, 1]$. Thus (2) holds. Then we obtain (3), since the $\mathbb{R}$-divisors appearing in the following:

$$(K_X + \Delta') - (K_X + \Delta) \sim_{f,\mathbb{R}} \Delta_1 + A'_1 - \Delta = A'_1 - (\Delta - \Delta_1)$$

are f-ample. Set $A'_2 := (K_X + \Delta + A) - (K_X + \Delta' + A'_1)$. By the equations:

$$\begin{aligned} A'_2 &= (K_X + \Delta + A) - (K_X + \Delta' + A'_1) \\ &= (A - 3A'_1) + (A'_1 + (\Delta - \Delta_1)) + (A'_1 + (\Delta_1 - \Delta')), \end{aligned}$$

it holds that A'_2 is f-ample, hence (1) holds.

Applying [GNT, Lemma 2.2] to (X, Δ') and A'_1, there exist finitely many curves $C_1, \ldots, C_r$ on X such that

$$\overline{\mathrm{NE}}(X/Z) = \overline{\mathrm{NE}}(X/Z)_{K_X+\Delta'+A'_1+A'_2\geq 0} + \sum_{i=1}^{r} \mathbb{R}_{\geq 0}[C_i].$$

Set $R := \mathbb{R}_{\geq 0}[C_1]$, which is a $(K_X+\Delta)$-negative extremal ray. By Proposition 3.3, there exists a contraction $\varphi : X \to Y$ of R. If φ is a flipping contraction, then Theorem 3.2 implies that a flip of φ exists.

Therefore, it suffices to prove that there exists no infinite sequence that is a $(K_X+\Delta)$-MMP consisting of flips:

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots .$$

This follows from the ACC for log canonical thresholds [Bir16, Theorem 1.10] and the assumption that $K_X + \Delta \equiv_{f,\mathbb{R}} D \geq 0$ ([Bir07, Lemma 3.2]). $\square$

Now we prove the existence of dlt modifications (Corollary 3.6). First we treat more general pairs in Proposition 3.5.

Proposition 3.5. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional quasi-projective log pair over k. Then there exists a projective birational k-morphism $f : Y \to X$ from a normal threefold over k that satisfies the following conditions:*

(1) $a_F(X, \Delta) \leq 0$ *holds for any f-exceptional prime divisor F on Y.*

(2) $(Y, \Delta_Y^{\wedge 1})$ *is a $\mathbb{Q}$-factorial dlt pair, where Δ_Y is the $\mathbb{R}$-divisor defined by $K_Y + \Delta_Y = f^*(K_X + \Delta)$ (see Subsection 2.1 for the definition of $\Delta_Y^{\wedge 1}$).*

(3) $\mathrm{Nklt}(Y, \Delta_Y) = f^{-1}(\mathrm{Nklt}(X, \Delta))$ *holds.*

Proof. The proof consists of two steps.

Step 1. *There exists a projective birational morphism $f_1 : Y_1 \to X$ that satisfies the conditions (1) and (2) in the statement.*

Proof of Step 1. Let $g : W \to X$ be a log resolution of (X, Δ). Write

$$g^*(K_X + \Delta) = K_W + \Delta_W = K_W + \widetilde{\Delta} + E,$$

where $\widetilde{\Delta}$ is the proper transform of Δ, and $E := \Delta_W - \widetilde{\Delta}$. Hence, E is a g-exceptional $\mathbb{R}$-divisor. Set F to be the reduced divisor such that Supp $F =$

$\mathrm{Ex}(g)$. There exists a $(K_W + \widetilde{\Delta}^{\wedge 1} + F)$-MMP over X that terminates (Theorem 3.4). Let $W \dashrightarrow Y_1 \to X$ be the end result. Since

$$K_W + \widetilde{\Delta}^{\wedge 1} + F \sim_{g,\mathbb{R}} -((\widetilde{\Delta} - \widetilde{\Delta}^{\wedge 1}) + E - F),$$

this MMP is also a $-((\widetilde{\Delta} - \widetilde{\Delta}^{\wedge 1}) + E - F)$-MMP over X. Hence by the negativity lemma, the push-forward of $(\widetilde{\Delta} - \widetilde{\Delta}^{\wedge 1}) + E - F$ on the end result Y_1 is effective. This implies that the g-exceptional divisors E_i with $\mathrm{coeff}_{E_i} E < 1$ are contracted in this MMP. Therefore the condition (1) holds. Since $\Delta_{Y_1}^{\wedge 1}$ is nothing but the push-forward of $\widetilde{\Delta}^{\wedge 1} + F$, the condition (2) follows because being dlt is preserved under an MMP. This completes the proof of Step 1. □

Step 2. *There exists a projective birational morphism $f : Y \to X$ that satisfies the conditions (1)–(3) in the statement.*

Proof of Step 2. The following argument is very similar to [dFKX17, Lemma 29]. By Step 1, we can find a projective birational morphism $f_1 : Y_1 \to X$ with the conditions (1) and (2). Define Δ_{Y_1} by $f_1^*(K_X + \Delta) = K_{Y_1} + \Delta_{Y_1}$.

There exists a $(K_{Y_1} + \Delta_{Y_1}^{<1})$-MMP over X that terminates (Theorem 3.4). Let $Y_1 \dashrightarrow Y_2 \xrightarrow{f_2} X$ be the end result, and let Δ_{Y_2} be the push-forward of Δ_{Y_1} on Y_2. We obtain

$$\mathrm{Supp}(\Delta_{Y_2}^{\geq 1}) = \mathrm{Nklt}(Y_2, \Delta_{Y_2}), \tag{3.5.1}$$

since $(Y_2, \Delta_{Y_2}^{<1})$ is klt.

Let us show the equation

$$\mathrm{Nklt}(Y_2, \Delta_{Y_2}) = f_2^{-1}(\mathrm{Nklt}(X, \Delta)). \tag{3.5.2}$$

By $f_2^*(K_X + \Delta) = K_{Y_2} + \Delta_{Y_2}$, f_2 induces a surjective morphism

$$\mathrm{Nklt}(Y_2, \Delta_{Y_2}) \to \mathrm{Nklt}(X, \Delta)$$

(cf. [KM98, Lemma 2.30]). In particular, the inclusion

$$\mathrm{Nklt}(Y_2, \Delta_{Y_2}) \subset f_2^{-1}(\mathrm{Nklt}(X, \Delta))$$

holds. Assuming that $\mathrm{Nklt}(Y_2, \Delta_{Y_2}) \not\supset f_2^{-1}(\mathrm{Nklt}(X, \Delta))$, let us derive a contradiction. Take a closed point $y \in f_2^{-1}(\mathrm{Nklt}(X, \Delta)) \setminus \mathrm{Nklt}(Y_2, \Delta_{Y_2})$. Set $x := f_2(y)$. Then it holds that $x \in \mathrm{Nklt}(X, \Delta)$ and $f_2^{-1}(x) \not\subset \mathrm{Nklt}(Y_2, \Delta_{Y_2})$. Since

$\mathrm{Nklt}(Y_2, \Delta_{Y_2}) \to \mathrm{Nklt}(X, \Delta)$ is surjective, there exists a curve C, contained in $f_2^{-1}(x)$, such that C intersects but is not contained in $\mathrm{Nklt}(Y_2, \Delta_{Y_2})$. By (3.5.1), it holds that $\Delta_{Y_2}^{\geq 1} \cdot C > 0$. On the other hand,

$$-\Delta_{Y_2}^{\geq 1} = -(\Delta_{Y_2} - \Delta_{Y_2}^{<1}) \equiv_{f_2} K_{Y_2} + \Delta_{Y_2}^{<1}$$

is f_2-nef. This is a contradiction. Hence, (3.5.2) holds.

Let $h : (Y, \Delta_Y) \to (Y_2, \Delta_{Y_2})$ be a projective birational morphism satisfying (1) and (2), whose existence is guaranteed by Step 1. Since Y_2 is $\mathbb{Q}$-factorial, $\mathrm{Ex}(h)$ is purely codimension one, and hence

$$\mathrm{Nklt}(Y, \Delta_Y) = h^{-1}(\mathrm{Nklt}(Y_2, \Delta_{Y_2}))$$

holds. Therefore the composition $f := f_2 \circ h : Y \to X$ satisfies the conditions (1)–(3). This completes the proof of Step 2. □

Step 2 completes the proof of Proposition 3.5. □

Corollary 3.6. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional quasi-projective log canonical pair over k. Then there exists a projective birational k-morphism $f : Y \to X$ from a normal threefold Y over k that satisfies the following conditions:*

(1) $a_F(X, \Delta) = 0$ *holds for any f-exceptional prime divisor F.*

(2) (Y, Δ_Y) *is a $\mathbb{Q}$-factorial dlt pair, where Δ_Y is the $\mathbb{R}$-divisor defined by* $K_Y + \Delta_Y = f^*(K_X + \Delta)$.

(3) $\mathrm{Nklt}(Y, \Delta_Y) = f^{-1}(\mathrm{Nklt}(X, \Delta))$ *holds.*

Proof. The claim directly follows from Proposition 3.5. □

4. Cone theorem

The purpose of this section is to prove the cone theorem for log canonical threefolds (Theorem 4.6, Theorem 4.7). To this end, we start with a projective case (Subsection 4.1). Then we treat the dlt case (Subsection 4.2). Finally, we will establish the cone theorem for log canonical threefolds (Subsection 4.3). As a consequence of the cone theorem, we obtain a result on the Shokurov polytope (Subsection 4.4).

4.1. Projective case

In this subsection, we prove Lemma 4.2. Let us start with a criterion to deduce the cone theorem.

Lemma 4.1. *Let k be a field. Let $f : X \to Z$ be a projective k-morphism from a normal k-variety X to a quasi-projective k-scheme Z. Let Δ be an $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let A be an f-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. For any ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor H, set a_H to be the f-nef threshold of $(K_X + \Delta + \frac{1}{2}A, H)$. Assume that there exist finitely many curves $C_1, \ldots, C_m$ on X such that*

(1) *$f(C_i)$ is a point for any $i \in \{1, \ldots, m\}$, and*
(2) *for any ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor H on X, it holds that $(K_X + \Delta + \frac{1}{2}A + a_H H) \cdot C_i = 0$ for some $i \in \{1, \ldots, m\}$.*

Then the following equation holds:

$$\overline{\mathrm{NE}}(X/Z) = \overline{\mathrm{NE}}(X/Z)_{K_X+\Delta+A\geq 0} + \sum_{i=1}^{m} \mathbb{R}_{\geq 0}[C_i].$$

Proof. We can apply the same argument as in [CTX15, Lemma 6.2]. □

Lemma 4.2. *Let k be an algebraically closed field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional log canonical pair over k and let $f : X \to Z$ be a projective k-morphism to a projective k-scheme Z. Let A be an f-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then there exist finitely many curves $C_1, \ldots, C_m$ on X such that*

(1) *$f(C_i)$ is a point for any $i \in \{1, \ldots, m\}$, and*
(2) *$\overline{\mathrm{NE}}(X/Z) = \overline{\mathrm{NE}}(X/Z)_{K_X+\Delta+A\geq 0} + \sum_{i=1}^{m} \mathbb{R}_{\geq 0}[C_i]$.*

Proof. Taking the Stein factorisation of f, we may assume that $f_*\mathcal{O}_X = \mathcal{O}_Z$. Replacing A by $A + f^*A_Z$ for some ample Cartier divisor A_Z on Z, the problem is reduced to the case when A is ample. Let H_Z be an ample Cartier divisor on Z. By [Tan17, Theorem 1], there exists an effective $\mathbb{R}$-divisor M such that $M \sim_{\mathbb{R}} 7f^*H_Z$ and $(X, \Delta' := \Delta + M)$ is log canonical. Applying Theorem 2.8 to (X, Δ) and $\frac{1}{2}A$, there exist finitely many curves $C_1, \ldots, C_n$

with the condition $0 > \left(K_X + \Delta + \frac{1}{2}A\right) \cdot C_i \geq -6$ and

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X+\Delta+\frac{1}{2}A\geq 0} + \sum_{i=1}^{n} \mathbb{R}_{\geq 0}[C_i]. \tag{4.2.1}$$

After permuting indices, we can find an integer m such that

- $0 \leq m \leq n$,
- $f(C_\alpha)$ is a point for any $\alpha \in \{1, \ldots, m\}$, and
- $f(C_\beta)$ is not a point for any $\beta \in \{m+1, \ldots, n\}$.

It is enough to prove that the curves $C_1, \ldots, C_m$ satisfy the condition (2). Take an ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor H on X and let a_H be the f-nef threshold of $(K_X + \Delta' + \frac{1}{2}A, H)$. Note that a_H is equal to the nef threshold of $(K_X + \Delta' + \frac{1}{2}A, H)$. Indeed, for any $\beta \in \{m+1, \ldots, n\}$, it holds that

$$\begin{aligned}&\left(K_X + \Delta' + \frac{1}{2}A + a_H H\right) \cdot C_\beta \\ \geq &\left(K_X + \Delta + 7f^*H_Z + \frac{1}{2}A\right) \cdot C_\beta \geq -6 + 7 = 1.\end{aligned}$$

Therefore, by (4.2.1), there exists $i \in \{1, \ldots, n\}$ such that $(K_X + \Delta' + \frac{1}{2}A + a_H H) \cdot C_i = 0$. By the inequality above, it holds that $1 \leq i \leq m$, as desired. □

4.2. Dlt case

The purpose of this subsection is to prove the cone theorem for the dlt case (Proposition 4.5). To this end, we first find an extremal ray that can be compactified (Lemma 4.3). We also need a basic fact on extremal rays (Lemma 4.4).

Lemma 4.3. *Let k be an algebraically closed field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional $\mathbb{Q}$-factorial log canonical pair over k such that $(X, \{\Delta\})$ is klt. Let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Let A be an f-ample $\mathbb{R}$-divisor on X such that $K_X + \Delta + A$ is f-nef but not f-ample. Then there exists a commutative* *diagram of quasi-projective k-schemes:*

$$\begin{array}{ccc} X & \xrightarrow{j_X} & \overline{X} \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle \overline{f}} \\ Z & \xrightarrow{j_Z} & \overline{Z}. \end{array}$$

such that

(a) *j_X and j_Z are open immersions to projective k-schemes $\overline{X}$ and $\overline{Z}$,*

(b) *$(\overline{X}, \overline{\Delta})$ is a $\mathbb{Q}$-factorial log canonical pair such that $(\overline{X}, \{\overline{\Delta}\})$ is klt, where $\overline{\Delta}$ denotes the closure of Δ in $\overline{X}$, and*

(c) *there is a $(K_{\overline{X}} + \overline{\Delta})$-negative extremal ray R of $\overline{\mathrm{NE}}(\overline{X}/\overline{Z})$ such that for the contraction $\overline{\psi} : \overline{X} \to \overline{Y}$ of R, its restriction $\psi : X \to Y$ over Z is not an isomorphism and $K_X + \Delta + A$ is ψ-numerically trivial.*

Proof. Fix an open immersion $j_Z : Z \to \overline{Z}$ to a projective k-scheme $\overline{Z}$. We divide the proof into three steps.

Step 1. *Assume that (X, Δ) is klt. Then there exist an open immersion $j^{(1)} : X \hookrightarrow X^{(1)}$ to a $\mathbb{Q}$-factorial projective threefold $X^{(1)}$, a projective morphism $f^{(1)} : X^{(1)} \to \overline{Z}$ and effective $\mathbb{R}$-divisors $\Delta^{(1)}$ and $A^{(1)}$ on $X^{(1)}$ such that*

(1) *$j_Z \circ f = f^{(1)} \circ j^{(1)}$,*

(2) *$\Delta^{(1)}$ is the closure of Δ,*

(3) *$A^{(1)}|_X \sim_{Z,\mathbb{R}} A$,*

(4) *$(X^{(1)}, \Delta^{(1)} + A^{(1)})$ is klt, and*

(5) *$K_{X^{(1)}} + \Delta^{(1)} + A^{(1)}$ is $f^{(1)}$-nef.*

Proof of Step 1. By Lemma 2.13 (2), we may assume that A is effective and $(X, \Delta + A)$ is klt. By Proposition 2.10, there is an open immersion $j : X \to X^{(0)}$ over $\overline{Z}$ to a normal $\mathbb{Q}$-factorial threefold $X^{(0)}$ projective over $\overline{Z}$ such that $(X^{(0)}, \Delta^{(0)} + A^{(0)})$ is klt, where $\Delta^{(0)}$ and $A^{(0)}$ denote the closures of Δ and A respectively.

It follows from [BW17, Theorem 1.6] that there exists a $(K_{X^{(0)}} + \Delta^{(0)} + A^{(0)})$-MMP over $\overline{Z}$ that terminates with a log minimal model $(X^{(1)}, \Delta^{(1)} +$

$A^{(1)}$). Since $(K_{X^{(0)}} + \Delta^{(0)} + A^{(0)})|_X = K_X + \Delta + A$ is nef over Z, the restriction of $X^{(0)} \dashrightarrow X^{(1)}$ over Z is an isomorphism. By construction, all the properties (1)–(5) hold. This completes the proof of Step 1. $\square$

Step 2. *The assertion of Lemma 4.3 holds if* (X, Δ) *is klt and* Δ *is big over* Z*.*

Proof of Step 2. Take $j^{(1)} : X \hookrightarrow X^{(1)}$, $f^{(1)} : X^{(1)} \to \overline{Z}$, $\Delta^{(1)}$ and $A^{(1)}$ as in Step 1. By [BW17, Theorem 1.5], there exists a $(K_{X^{(1)}} + \Delta^{(1)})$-MMP over $\overline{Z}$ with scaling of $A^{(1)}$ that terminates:

$$X^{(1)} =: X^{(1)}_0 \overset{\varphi_0}{\dashrightarrow} X^{(1)}_1 \overset{\varphi_1}{\dashrightarrow} \cdots \overset{\varphi_{\ell-1}}{\dashrightarrow} X^{(1)}_\ell.$$

For any i, we denote the associated morphism corresponding to the extremal ray by $\overline{\psi}_i : X^{(1)}_i \to \overline{Y}_i$. Then there are two possibilities as follows.

- (A) For any $i \in \{0, \ldots, \ell - 1\}$, the restriction of φ_i over Z is an isomorphism.
- (B) There exists $i \in \{0, \ldots, \ell - 1\}$ such that the restriction of φ_i over Z is not an isomorphism.

Assume that (A) occurs. Then we set $(\overline{X}, \overline{\Delta}) := (X^{(1)}_\ell, \Delta^{(1)}_\ell)$. It follows from the construction that (a) and (b) hold. In particular, $K_{\overline{X}} + \overline{\Delta}$ is not nef over $\overline{Z}$, since its restriction $(K_{\overline{X}} + \overline{\Delta})|_X = K_X + \Delta$ is not nef over Z. Hence, there is a $(K_{\overline{X}} + \overline{\Delta})$-negative extremal ray R of $\overline{\mathrm{NE}}(\overline{X}/\overline{Z})$ that corresponds to a $(K_{\overline{X}} + \Delta)$-Mori fibre space $\overline{\psi} : \overline{X} \to \overline{Y}$ over $\overline{Z}$. It is clear that the restriction of $\overline{\psi}$ over Z is not an isomorphism, hence (c) holds.

Assume that (B) occurs. Then there is an index $j \in \{0, \ldots, \ell\}$ such that the restriction of φ_j over Z is not an isomorphism and the restriction of $\varphi_{j'}$ over Z is an isomorphism over Z for any $j' < j$. In this case, we set $(\overline{X}, \overline{\Delta}) := (X^{(1)}_j, \Delta^{(1)}_j)$, $\overline{Y} := \overline{Y}_j$ and $\overline{\psi} := \overline{\psi}_j$. Then all the properties (a), (b) and (c) hold by construction. This completes the proof of Step 2. $\square$

Step 3. *The assertion of Lemma 4.3 holds without any additional assumptions.*

Proof of Step 3. Pick a sufficiently small positive real number ϵ such that $\frac{1}{2}A + \epsilon\lfloor\Delta\rfloor$ is f-ample. Since $(X, \Delta - \epsilon\lfloor\Delta\rfloor)$ is klt, there exists an effective $\mathbb{R}$-divisor $A' \sim_{f,\mathbb{R}} \frac{1}{2}A + \epsilon\lfloor\Delta\rfloor$ such that the pair $(X, \Delta' := \Delta - \epsilon\lfloor\Delta\rfloor + A')$ is klt by Lemma 2.13 (2). Again by Lemma 2.13 (2), there is an effective $\mathbb{R}$-divisor A'' on X such that $A'' \sim_{f,\mathbb{R}} \frac{1}{2}A$ and $(X, \Delta' + A'')$ is klt. By construction, Δ'

is f-big and $K_X + \Delta + A \sim_{f,\mathbb{R}} K_X + \Delta' + A''$. In particular, $K_X + \Delta' + A''$ is f-nef but not f-ample.

Applying Step 2 to $f : X \to Z$, (X, Δ') and A'', there exists a commutative diagram of quasi-projective k-schemes:

$$\begin{array}{ccc} X & \xrightarrow{j_X} & \overline{X} \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle \overline{f}} \\ Z & \xrightarrow{j_Z} & \overline{Z}. \end{array}$$

such that

(a) j_X and j_Z are open immersions to projective k-schemes $\overline{X}$ and $\overline{Z}$,

(b)' $(\overline{X}, \overline{\Delta'})$ is a $\mathbb{Q}$-factorial klt pair, where $\overline{\Delta'}$ denotes the closure of Δ' in $\overline{X}$, and

(c)' there is a $(K_{\overline{X}} + \overline{\Delta'})$-negative extremal ray R of $\overline{\mathrm{NE}}(\overline{X}/\overline{Z})$ such that for the contraction $\overline{\psi} : \overline{X} \to \overline{Y}$ of R, its restriction $\psi : X \to Y$ over Z is not an isomorphism and $K_X + \Delta' + A''$ is ψ-numerically trivial.

It suffices to prove that (b) and (c) hold. We first show (b). By $\Delta' = \Delta - \epsilon\lfloor\Delta\rfloor + A'$, we obtain $\overline{\Delta'} \geq \overline{\Delta} - \epsilon\lfloor\overline{\Delta}\rfloor$. Since $\overline{X}$ is $\mathbb{Q}$-factorial and the log pair $(\overline{X}, \overline{\Delta'})$ is klt, we see that $(\overline{X}, \overline{\Delta} - \epsilon\lfloor\overline{\Delta}\rfloor)$ is klt. Hence, also $(\overline{X}, \{\overline{\Delta}\})$ is klt. Furthermore, the log canonical threshold $\mathrm{lct}(\overline{X}, \{\overline{\Delta}\}; \lfloor\overline{\Delta}\rfloor)$ is greater than $1 - \epsilon$. As $\epsilon > 0$ was chosen to be sufficiently small, ACC for log canonical thresholds implies that $(\overline{X}, \overline{\Delta})$ is log canonical. Thus, (b) holds.

Let us show (c). Pick a curve C on X contracted by ψ, whose existence is guaranteed by (c)'. Then C generates R and therefore we have $(K_X + \Delta' + A'') \cdot C = 0$ and $(K_X + \Delta') \cdot C < 0$. Now recall that $K_X + \Delta + A \sim_{f,\mathbb{R}} K_X + \Delta' + A''$ and $K_X + \Delta + \frac{1}{2}A \sim_{f,\mathbb{R}} K_X + \Delta'$, which imply $(K_X + \Delta + A) \cdot C = 0$ and $(K_X + \Delta) \cdot C < 0$. Thus we see that $K_X + \Delta + A$ is ψ-numerically trivial and R is a $(K_{\overline{X}} + \overline{\Delta})$-negative extremal ray of $\overline{\mathrm{NE}}(\overline{X}/\overline{Z})$. Hence, (c) holds. This completes the proof of Step 3. $\square$

Step 3 completes the proof of Lemma 4.3. $\square$

Lemma 4.4. *Let k be an algebraically closed field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional $\mathbb{Q}$-factorial log canonical pair over k and let $f : X \to Z$ be a projective k-morphism to a projective k-scheme. Let R be a $(K_X + \Delta)$-negative extremal ray of $\overline{\mathrm{NE}}(X/Z)$ and let $\varphi : X \to Y$ be the contraction of R.*

(1) *If φ is birational, then there exists a non-empty open subset Y' of $\varphi(\mathrm{Ex}(\varphi))$ such that for any closed point $y \in Y'$, there is a rational curve C on X such that $\varphi(C) = \{y\}$ and $0 < -(K_X + \Delta) \cdot C \leq 6$.*

(2) *If φ is not birational, then for any closed point $y \in Y$, there is a rational curve C on X such that $\varphi(C) = \{y\}$ and $0 < -(K_X + \Delta) \cdot C \leq 6$.*

Proof. We may assume that $Z = \mathrm{Spec}\, k$. By the cone theorem, we can find an ample $\mathbb{R}$-divisor H such that $\overline{\mathrm{NE}}(X) \cap (K_X + \Delta + H)^{\perp} = R$. If $K_X + \Delta + H$ is not big i.e. φ is not birational, then the assertion follows from [CTX15, Corollary 1.5]. Hence, we may assume that φ is birational. If $\dim \mathrm{Ex}(\varphi) = 1$, then the assertion holds by Theorem 2.8. Therefore, the problem is reduced to the case when φ is a divisorial contraction. In this case, we can apply the same argument as in [BW17, the second paragraph of the proof of Lemma 3.2] without any changes. $\square$

Proposition 4.5. *Let k be an algebraically closed field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional $\mathbb{Q}$-factorial log canonical pair over k such that $(X, \{\Delta\})$ is klt. Let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Then the following hold.*

(1) *If H is an ample $\mathbb{R}$-divisor on X such that $K_X + \Delta + H$ is f-nef but not f-ample, then there exists a rational curve C such that $f(C)$ is a point, $(K_X + \Delta + H) \cdot C = 0$ and $0 < -(K_X + \Delta) \cdot C \leq 6$.*

(2) *If A is an f-ample $\mathbb{R}$-divisor on X, then there exist finitely many rational curves $C_1, \ldots, C_m$ such that*

$$\overline{\mathrm{NE}}(X/Z) = \overline{\mathrm{NE}}(X/Z)_{K_X+\Delta+A \geq 0} + \sum_{i=1}^{m} \mathbb{R}_{\geq 0}[C_i].$$

Proof. The assertion (1) follows from Lemma 4.3 and Lemma 4.4. We prove (2). Let H be an f-ample $\mathbb{R}$-divisor on X and let a_H be the f-nef threshold of $(K_X + \Delta + \frac{1}{2}A, H)$. By construction $K_X + \Delta + (\frac{1}{2}A + a_H H)$ is f-nef but not f-ample. By (1), there exists a rational curve C_H such that $f(C_H)$ is a point, $(K_X + \Delta + \frac{1}{2}A + a_H H) \cdot C_H = 0$ and $0 < -(K_X + \Delta) \cdot C_H \leq 6$. From this we have

$$0 < A \cdot C_H \leq (A + 2a_H H) \cdot C_H = -2(K_X + \Delta) \cdot C_H \leq 6 \cdot 2 = 12.$$

Therefore the subset $\{[C_H] \in \overline{\mathrm{NE}}(X/Z) |\, H \,\text{is}\, f\text{-ample}\}$ of $\overline{\mathrm{NE}}(X/Z)$ is a finite set. In this way we can find finitely many rational curves $C_{H_1}, \ldots, C_{H_m}$ satisfying the conditions of Lemma 4.1. Hence, (2) follows. $\square$

4.3. Log canonical case

Let us prove the main results in this section (Theorem 4.6, Theorem 4.7).

Theorem 4.6. *Let k be an algebraically closed field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional log canonical pair over k and let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Then there exists a countable set $\{C_i\}_{i \in I}$ of rational curves on X which satisfies the following conditions:*

(1) *$f(C_i)$ is a point for any $i \in I$.*

(2) *$\overline{\mathrm{NE}}(X/Z) = \overline{\mathrm{NE}}(X/Z)_{K_X+\Delta \geq 0} + \sum_{i \in I} \mathbb{R}_{\geq 0}[C_i]$.*

(3) *$0 < -(K_X + \Delta) \cdot C_i \leq 6$ for any $i \in I$.*

(4) *For any f-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor A, there exists a finite subset J of I such that*

$$\overline{\mathrm{NE}}(X/Z) = \overline{\mathrm{NE}}(X/Z)_{K_X+\Delta+A \geq 0} + \sum_{j \in J} \mathbb{R}_{\geq 0}[C_j].$$

Proof. The proof consists of two steps.

Step 1. *If X is $\mathbb{Q}$-factorial and $(X, \{\Delta\})$ is klt, then there exists a countable set $\{C_i\}_{i \in I}$ of rational curves on X that satisfies the conditions (1)–(3) of Theorem 4.6.*

Proof. Take any f-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor H. Then for any positive integer n, Proposition 4.5 enables us to find a finite set of rational curves $\{C_i\}_{i \in I_n}$ such that $f(C_i)$ is a point, $-(K_X + \Delta) \cdot C_i \leq 6$, and

$$\overline{\mathrm{NE}}(X/Z) = \overline{\mathrm{NE}}(X/Z)_{K_X+\Delta+\frac{1}{n}H \geq 0} + \sum_{i \in I_n} \mathbb{R}_{\geq 0}[C_i].$$

Therefore $I = \bigcup_{n \geq 1} I_n$ satisfies the conditions (1)–(3). This completes the proof of Step 1. □

Step 2. *The assertion of Theorem 4.6 holds without any additional assumptions.*

Proof. Let $g : Y \to X$ be a projective birational morphism with the conditions in Corollary 3.6. We apply Step 1 to (Y, Δ_Y) to conclude that there exists a countable set $\{C_i^Y\}_{i \in J}$ of rational curves on Y with the following conditions:

- $f(g(C_i^Y))$ is a point for any $i \in J$.
- $\overline{\mathrm{NE}}(Y/Z) = \overline{\mathrm{NE}}(Y/Z)_{K_Y+\Delta_Y \geq 0} + \sum_{i \in J} \mathbb{R}_{\geq 0}[C_i^Y]$.
- $0 < -(K_Y + \Delta_Y) \cdot C_i^Y \leq 6$ for each $i \in J$.

Set $C_i := g(C_i^Y)$ and $I := \{i \in J \mid \dim C_i = 1\}$. Then it is clear that (1) holds. We get (2) by $\overline{\mathrm{NE}}(X/Z) = g_*(\overline{\mathrm{NE}}(Y/Z))$. It follows from the projection formula that (3) holds.

We now show (4). Let R be an extremal ray of $\overline{\mathrm{NE}}(X/Z)$ which is $(K_X + \Delta + A)$-negative. By (1) and (2) we have already proved, we obtain $R = \mathbb{R}_{\geq 0}[C]$ for some curve C such that $f(C)$ is a point and $0 < -(K_X + \Delta) \cdot C \leq 6$. Then it holds that

$$A \cdot C < -(K_X + \Delta) \cdot C \leq 6.$$

Hence, there are only finitely many extremal rays satisfying this property (cf. [KM98, Corollary 1.19 (3)]). Thus (4) holds. This completes the proof of Step 2. $\square$

Step 2 completes the proof of Theorem 4.6. $\square$

Theorem 4.7. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional log canonical pair over k and let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Let A be an f-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then there exist finitely many curves $C_1, \ldots, C_m$ on X such that*

(1) *$f(C_i)$ is a point for any $i \in \{1, \ldots, m\}$, and*

(2) *$\overline{\mathrm{NE}}(X/Z) = \overline{\mathrm{NE}}(X/Z)_{K_X+\Delta+A \geq 0} + \sum_{i=1}^m \mathbb{R}_{\geq 0}[C_i]$.*

Proof. We may assume that

- $f_*\mathcal{O}_X = \mathcal{O}_Z$,
- k is algebraically closed in $K(Z)$, and
- both X and Z are geometrically integral and geometrically normal over k.

Indeed, we may assume the first condition by taking the Stein factorisation of f. Then, after replacing k by the algebraic closure of k in $K(Z)$, the second condition holds. The third condition automatically follows from the other two.

Let $\overline{k}$ be the algebraic closure of k. We set

$$f_{\overline{k}} : X_{\overline{k}} \to Z_{\overline{k}}$$

to be the base change $f \times_k \overline{k}$. Let $\Delta_{\overline{k}}$ and $A_{\overline{k}}$ be the $\mathbb{R}$-divisors defined as the pullbacks of Δ and A respectively. Then $(X_{\overline{k}}, \Delta_{\overline{k}}, f_{\overline{k}}, A_{\overline{k}})$ satisfies the assumptions listed in the statement. By Theorem 4.6, there exist finitely many curves $C'_1, \ldots, C'_m$ on $X_{\overline{k}}$ such that

(i) $f_{\overline{k}}(C'_i)$ is a point for any i, and

(ii) $\overline{\mathrm{NE}}(X_{\overline{k}}/Z_{\overline{k}}) = \overline{\mathrm{NE}}(X_{\overline{k}}/Z_{\overline{k}})_{K_{X_{\overline{k}}}+\Delta_{\overline{k}}+\frac{1}{2}A_{\overline{k}}\geq 0} + \sum_{i=1}^{m} \mathbb{R}_{\geq 0}[C'_i]$.

Let C_i be the image of C'_i for any $i \in \{1, \ldots, m\}$. It follows from (i) that the condition Lemma 4.1(1) holds. Therefore, it is enough to prove that also Lemma 4.1(2) holds. Let H be an ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X and let a_H be the f-nef threshold of $(K_X + \Delta + \frac{1}{2}A, H)$. We see that a_H is equal to the $f_{\overline{k}}$-nef threshold of $(K_{X_{\overline{k}}} + \Delta_{\overline{k}} + \frac{1}{2}A_{\overline{k}}, H_{\overline{k}})$ (cf. [Tan18a, Lemma 2.3]). Hence, by (ii), we get $(K_{X_{\overline{k}}} + \Delta_{\overline{k}} + \frac{1}{2}A_{\overline{k}} + a_H H_{\overline{k}}) \cdot C'_i = 0$ for some $i \in \{1, \ldots, m\}$, which in turn implies $(K_X + \Delta + \frac{1}{2}A + a_H H) \cdot C_i = 0$ (cf. [Tan18a, Lemma 2.3]). Therefore, Lemma 4.1(2) holds, as desired. □

4.4. Shokurov polytope

As a consequence of the cone theorem, we obtain a result on the Shokurov polytope. We first fix some terminologies.

Notation 4.8. Let k be an algebraically closed field of characteristic $p > 5$. Let X be a $\mathbb{Q}$-factorial klt threefold and let $f : X \to Z$ be a projective morphism to a quasi-projective k-scheme. Fix finitely many prime divisors $D_1, \ldots, D_n$ and set

$$V := \bigoplus_{i=1}^{n} \mathbb{R} \cdot D_i,$$

which is a subspace of the $\mathbb{R}$-vector space of the $\mathbb{R}$-divisors on X. For any $D = \sum d_i D_i \in V$, we set $||D|| := \max_{1\leq i\leq n}\{|d_i|\}$. We see that

$$\mathcal{L} := \{\Delta \in V \mid (X, \Delta) \text{ is log canonical}\}$$

is a rational polytope in V.

Proposition 4.9. *We use Notation 4.8. Fix $D \in \mathcal{L}$. Then there exist positive real numbers α and δ which satisfy the following properties.*

(1) *If Γ is an extremal curve of $\overline{\mathrm{NE}}(X/Z)$ and if $(K_X + D) \cdot \Gamma > 0$, then $(K_X + D) \cdot \Gamma > \alpha$.*

(2) *If $\Delta \in \mathcal{L}$, $||\Delta - D|| < \delta$, and $(K_X + \Delta) \cdot R \leq 0$ for an extremal ray R of $\overline{\mathrm{NE}}(X/Z)$, then $(K_X + D) \cdot R \leq 0$.*

(3) *Let $\{R_t\}_{t \in T}$ be a set of extremal rays of $\overline{\mathrm{NE}}(X/Z)$. Then the set*

$$\mathcal{N}_T := \{\Delta \in \mathcal{L} \,|\, (K_X + \Delta) \cdot R_t \geq 0 \textit{ for any } t \in T\}$$

is a rational polytope.

(4) *Assume that $K_X + D$ is f-nef, $\Delta \in \mathcal{L}$ and that*

$$X = X_1 \dashrightarrow X_2 \dashrightarrow \cdots$$

is a sequence of $(K_X + \Delta)$-MMP over Z that consists of flips which are $(K_X + D)$-numerically trivial. Then, for any i and any curve Γ on X_i whose image on Z is a point, if $(K_{X_i} + D_i) \cdot \Gamma > 0$, then $(K_{X_i} + D_i) \cdot \Gamma > \alpha$, where D_i denotes the push-forward of D on X_i.

(5) *In addition to the assumptions of (4), suppose that $||\Delta - D|| < \delta$. If $(K_{X_i} + \Delta_i) \cdot R \leq 0$ for an extremal ray R of $\overline{\mathrm{NE}}(X_i/Z)$, then $(K_{X_i} + D_i) \cdot R = 0$, where Δ_i denotes the push-forward of Δ on X_i.*

Proof. Since we have proved the cone theorem in the relative setting (Theorem 4.6), we can apply the same argument as in [BW17, Proposition 3.8] (cf. [Tan18b, Proposition A.3]). □

Theorem 4.10. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional log canonical pair over k and let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Let L be an f-nef and f-big $\mathbb{R}$-Cartier $\mathbb{R}$-divisor such that $L - (K_X + \Delta)$ is f-semi-ample. Then L is f-semi-ample.*

Proof. We may assume k is algebraically closed. By Lemma 2.13 (1), we may assume that $L = K_X + \Delta$. By Corollary 3.6, the problem is reduced to the case when (X, Δ) is a $\mathbb{Q}$-factorial dlt pair. Applying Proposition 4.9(3), we can find effective $\mathbb{Q}$-divisors $\Delta_1, \ldots, \Delta_n$ and positive real numbers $r_1, \ldots, r_n$

such that (X, Δ_i) is log canonical, $K_X + \Delta_i$ is f-nef for any $i \in \{1, \dots, n\}$,

$$\sum_{i=1}^{n} r_i = 1, \qquad \text{and} \qquad K_X + \Delta = \sum_{i=1}^{n} r_i(K_X + \Delta_i).$$

Replacing Δ_i and r_i appropriately, we may assume that each $K_X + \Delta_i$ is f-big. Hence, each $K_X + \Delta_i$ is f-semi-ample by Proposition 3.3. Therefore, also $L = K_X + \Delta$ is f-semi-ample. $\square$

Theorem 4.11. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional log canonical pair over k and let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. If $K_X + \Delta$ is f-big, then there exists a log canonical model of (X, Δ) over Z.*

Proof. By Corollary 3.6, we may assume that (X, Δ) is a $\mathbb{Q}$-factorial dlt pair. Furthermore, by running a $(K_X + \Delta)$-MMP over Z (Theorem 3.4), we may assume that $K_X + \Delta$ is f-nef. Then Theorem 4.10 implies that $K_X + \Delta$ is f-semi-ample. Let $g : X \to Y$ be the birational morphism $g : X \to Y$ over Z with $g_*\mathcal{O}_X = \mathcal{O}_Y$ induced by $K_X + \Delta$. Then $(Y, \Delta_Y := g_*\Delta)$ is a log canonical model of (X, Δ) over Z. $\square$

5. Base point free theorem

The purpose of this section is to prove the base point free theorem for log canonical threefolds (Theorem 5.3). As a consequence, we obtain the contraction theorem (Theorem 5.4). We also establish the minimal model program for effective log canonical pairs (Theorem 5.5), which is a generalisation of Theorem 3.4.

Lemma 5.1. *Let k be an algebraically closed field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional projective $\mathbb{Q}$-factorial dlt pair over k, where Δ is a $\mathbb{Q}$-divisor. Let A be an effective big $\mathbb{Q}$-divisor on X such that $(X, \Delta + A)$ is log canonical. If $K_X + \Delta + A$ is nef, then $K_X + \Delta + A$ is semi-ample.*

Proof. The same argument as in [Wal18, Step 2-4 of the proof of Theorem 1.2] works without any changes. $\square$

Theorem 5.2. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional log canonical pair over k and let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Assume that there* *exist effective* $\mathbb{R}$*-divisors* Δ_1 *and* Δ_2 *such that* $\Delta = \Delta_1 + \Delta_2$ *and* Δ_2 *is an* f*-big* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor. If* $K_X + \Delta$ *is* f*-nef, then* $K_X + \Delta$ *is* f*-semi-ample.*

Proof. By standard argument, the problem is reduced to the case when k is algebraically closed and $f_*\mathcal{O}_X = \mathcal{O}_Z$. The proof is divided into three steps.

Step 1. *The assertion of Theorem 5.2 holds, if* Z *is projective over* k *and both* Δ_1 *and* Δ_2 *are* $\mathbb{Q}$*-divisors.*

Proof of Step 1. By Theorem 4.6, we may assume that $Z = \operatorname{Spec} k$. Applying Corollary 3.6 to (X, Δ_1), there exists a projective birational morphism $g : Y \to X$ such that

- Y is $\mathbb{Q}$-factorial, and
- if B is defined by $K_Y + B = g^*(K_X + \Delta_1)$, then B is effective and $(Y, \{B\})$ is klt.

In particular, it holds that

- $(Y, 0)$ is klt, and
- if we define Δ_Y by $K_Y + \Delta_Y = g^*(K_X + \Delta)$, then Δ_Y is an effective big $\mathbb{Q}$-divisor such that (Y, Δ_Y) is log canonical.

Therefore, Lemma 5.1 implies that $K_Y + \Delta_Y$ is semi-ample, hence so is $K_X + \Delta$. This completes the proof of Step 1. $\square$

Step 2. *The assertion of Theorem 5.2 holds, if both* Δ_1 *and* Δ_2 *are* $\mathbb{Q}$*-divisors.*

Proof of Step 2. If $Z = \operatorname{Spec} k$, then the assertion follows from Step 1. Hence, we may assume that $\dim Z \geq 1$. In particular, $\dim X_K \leq 2$, where X_K denotes the generic fibre of f. Applying Corollary 3.6 to (X, Δ_1), we may assume that X is $\mathbb{Q}$-factorial and $(X, 0)$ is klt.

Take an open immersion $Z \hookrightarrow \overline{Z}$ to a scheme $\overline{Z}$ projective over k. By Proposition 2.12, there exists an open immersion $X \hookrightarrow X^{(1)}$ over $\overline{Z}$ to a $\mathbb{Q}$-factorial threefold projective $X^{(1)}$ over $\overline{Z}$ and an effective $\mathbb{Q}$-divisor $\Delta^{(1)} := \overline{\Delta}$ on $X^{(1)}$ such that $(X^{(1)}, \Delta^{(1)})$ is log canonical and $(X^{(1)}, 0)$ is klt.

Since $\dim X_K \leq 2$ and $K_{X^{(1)}} + \Delta^{(1)}$ is pseudo-effective over $\overline{Z}$, it follows from Lemma 2.15 that $K_{X^{(1)}} + \Delta^{(1)} \equiv_f D$ for some effective $\mathbb{R}$-divisor D on $X^{(1)}$. By Theorem 2.9, there exists a $(K_{X^{(1)}} + \Delta^{(1)})$-MMP over $\overline{Z}$ that terminates. Let $(X^{(2)}, \Delta^{(2)})$ be the end result. Since $K_X + \Delta$ is f-nef, there exists an open immersion $X \hookrightarrow X^{(2)}$ over $\overline{Z}$ such that $\Delta^{(2)}|_X = \Delta$. Since $(X^{(2)}, \Delta^{(2)})$ is log canonical, $K_{X^{(2)}} + \Delta^{(2)}$ is nef over $\overline{Z}$, and $\Delta^{(2)}$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor which is big over $\overline{Z}$, Step 1 implies that $K_{X^{(2)}} + \Delta^{(2)}$ is semi-ample over $\overline{Z}$. Restricting to X, it holds that $K_X + \Delta$ is semi-ample over Z. This completes the proof of Step 2. $\square$

Step 3. *The assertion of Theorem 5.2 holds without any additional assumptions.*

Proof of Step 3. We may assume that k is an algebraically closed field. Applying Corollary 3.6 to a log canonical pair (X, Δ_1), the problem is reduced to the case when X is $\mathbb{Q}$-factorial and klt. In particular, we may assume that $\Delta_1 = 0$ and $\Delta = \Delta_2$.

Then Proposition 4.9 implies that there exist positive real numbers $r_1, \ldots, r_m$ and effective $\mathbb{Q}$-divisors $\Delta_1, \ldots, \Delta_m$ on X such that

- $\sum_{i=1}^m r_i = 1$,
- (X, Δ_i) is log canonical, $K_X + \Delta_i$ is f-nef, and
- $K_X + \Delta = \sum_{i=1}^m r_i(K_X + \Delta_i)$.

Replacing r_i and Δ_i appropriately, we may assume that Δ_i is f-big for any $i \in \{1, \ldots, m\}$. Hence, Step 2 implies that $K_X + \Delta_i$ is f-semi-ample for any $i \in \{1, \ldots, m\}$. Therefore, also $K_X + \Delta$ is f-semi-ample. This completes the proof of Step 3. $\square$

Step 3 completes the proof of Theorem 5.2. $\square$

Theorem 5.3. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional log canonical pair over k and let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Let L be an f-nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $L - (K_X + \Delta)$ is f-semi-ample and f-big. Then L is f-semi-ample.*

Proof. We may assume that k is an algebraically closed field. Furthermore, Corollary 3.6 reduces the problem to the case when X is $\mathbb{Q}$-factorial. Lemma 2.13 (1) implies that there exists an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor A such that $A \sim_{f,\mathbb{R}} L - (K_X + \Delta)$ and $(X, \Delta + A)$ is log canonical. Since $\Delta + A$ is f-big, the assertion follows from Theorem 5.2. $\square$

Theorem 5.4. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional log canonical pair over k and let $f : X \to Z$ be a* *projective k-morphism to a quasi-projective k-scheme Z. Let R be a $(K_X + \Delta)$-negative extremal ray of $\overline{\mathrm{NE}}(X/Z)$. Then,*

(1) *there exists a projective Z-morphism $\varphi_R : X \to Y$ such that*
  (1.1) *Y is a normal variety over k projective over Z,*
  (1.2) *$(\varphi_R)_*\mathcal{O}_X = \mathcal{O}_Y$, and*
  (1.3) *if C is a curve on X such that $f(C)$ is a point, then $\varphi_R(C)$ is a point if and only if $[C] \in R$.*

Moreover, if $\varphi_R : X \to Y$ is a projective Z-morphism satisfying the properties (1.1)–(1.3), then the following hold.

(2) *Fix $\mathbb{K} \in \{\mathbb{Q}, \mathbb{R}\}$. If L is a $\mathbb{K}$-Cartier $\mathbb{K}$-divisor on X such that $L \equiv_{\varphi_R} 0$, then there exists a $\mathbb{K}$-Cartier $\mathbb{K}$-divisor L_Y on Y such that $L \sim_{f,\mathbb{K}} (\varphi_R)^* L_Y$.*

(3) $\rho(Y/Z) = \rho(X/Z) - 1$.

(4) *If X is $\mathbb{Q}$-factorial and if either*
  (4.1) $\dim X > \dim Y$, *or*
  (4.2) *φ_R is birational and* $\dim \mathrm{Ex}(\varphi_R) = 2$,

  then Y is $\mathbb{Q}$-factorial.

Proof. Let us show (1). By Theorem 4.7, there exists an f-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor A such that $L := K_X + \Delta + A$ is f-nef and $\overline{\mathrm{NE}}(X/Z) \cap L^{\perp} = R$. It follows from Theorem 5.3 that there is a projective Z-morphism $\varphi_R : X \to Y$ which satisfies (1.1)–(1.3). Hence, (1) holds.

Let us prove (2). We first treat the case when $\overline{\mathrm{NE}}(X/Z) \cap L^{\perp} = R$. In this case, L or $-L$ is f-nef, hence we may assume that L is f-nef. By Theorem 4.6, $L - \epsilon(K_X + \Delta)$ is f-ample for some $\epsilon \in \mathbb{Q}_{>0}$. Thus, L is f-semi-ample by Theorem 5.3. In particular, L is φ_R-semi-ample. Since $L \equiv_{\varphi_R} 0$, we can find L_Y as in the statement. By Theorem 4.7, the general case is reduced to the case when $\overline{\mathrm{NE}}(X/Z) \cap L^{\perp} = R$. Hence, (2) holds.

We now prove (3). Fix a curve Γ contracted by φ_R. By (2), we have an exact sequence:

$$0 \to N^1(Y/Z)_{\mathbb{Q}} \to N^1(X/Z)_{\mathbb{Q}} \xrightarrow{\cdot\Gamma} \mathbb{Q} \to 0,$$

where

$$N^1(X/Z)_{\mathbb{Q}} := \frac{\mathrm{Pic}\, X \otimes_{\mathbb{Z}} \mathbb{Q}}{\equiv_Z}, \quad N^1(Y/Z)_{\mathbb{Q}} := \frac{\mathrm{Pic}\, Y \otimes_{\mathbb{Z}} \mathbb{Q}}{\equiv_Z}.$$

Then (3) holds by $\rho(X/Z) = \dim_{\mathbb{Q}} N^1(X/Z)_{\mathbb{Q}}$ and $\rho(Y/Z) = \dim_{\mathbb{Q}} N^1(Y/Z)_{\mathbb{Q}}$.

The assertion (4) follows from the same argument as in [KM98, Corollary 3.18]. □

Theorem 5.5. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional log canonical pair over k and let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Assume that $K_X + \Delta \equiv_{f,\mathbb{R}} D$ for some effective $\mathbb{R}$-divisor D on X. Then we can run an arbitrary $(K_X + \Delta)$-MMP and it terminates.*

Proof. By Theorem 4.7, Theorem 4.11 and Theorem 5.4, we can run an arbitrary $(K_X + \Delta)$-MMP. If X is $\mathbb{Q}$-factorial, then any $(K_X + \Delta)$-MMP terminates by ACC for log canonical thresholds [Bir16, Theorem 1.10] and the assumption $K_X + \Delta \equiv_{f,\mathbb{R}} D \geq 0$ (cf. [Bir07, Lemma 3.2]). The general case can be reduced to this case by a standard argument (cf. [Bir12a, Remark 2.9]). □

6. MMP for log canonical pairs

The purpose of this section is to prove the main theorem of this paper (Theorem 6.12). We first check that we may run log minimal model programs with scaling under mild conditions (Subsection 6.1). Second, we show that the existence of log minimal models implies the termination for certain sequences of flips (Subsection 6.2). Third, we prove the existence of log minimal models (Subsection 6.3). Finally, we establish the main theorem of this paper (Subsection 6.4).

6.1. Existence of extremal rays for MMP with scaling

In this subsection, we check that we can run log minimal model programs with scaling for any three-dimensional log canonical pairs (Theorem 6.2).

Lemma 6.1. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional geometrically integral log pair over k. Let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Assume that there exist an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor C on X and an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor C' on $X_{\overline{k}} := X \times_k \overline{k}$ such that if $\Delta_{\overline{k}}$ and $C_{\overline{k}}$ denote the pullbacks of Δ and C to $X_{\overline{k}}$ respectively, then*

(a) *$(X_{\overline{k}}, \Delta_{\overline{k}} + C')$ is log canonical,*

(b) *$K_X + \Delta + C$ is f-nef, and*

(c) $C_{\overline{k}} \sim_{Z,\mathbb{R}} C'$.

Then $K_X + \Delta$ *is* f*-nef or there exists a* $(K_X + \Delta)$*-negative extremal ray* R *of* $\overline{\mathrm{NE}}(X/Z)$ *such that* $(K_X + \Delta + \lambda C) \cdot R = 0$*, where* λ *denotes the* f*-nef threshold of* $(K_X + \Delta, C)$*.*

Proof. By standard arguments, we may assume that $f_*\mathcal{O}_X = \mathcal{O}_Z$. We divide the proof into three steps.

Step 1. *If* k *is an algebraically closed field, then the assertion of Lemma 6.1 holds.*

Proof of Step 1. By Theorem 4.6, the same argument as in [Fuj17, Theorem 4.7.3] works without any changes. □

Step 2. *There exists a curve* G *on* X *such that* $f(G)$ *is a point,* $(K_X + \Delta) \cdot G < 0$ *and* $(K_X + \Delta + \lambda C) \cdot G = 0$*.*

Proof of Step 2. Let $f_{\overline{k}} : X_{\overline{k}} \to Z_{\overline{k}}$ be the base change $f \times_k \overline{k}$. Applying Step 1 to $f_{\overline{k}} : X_{\overline{k}} \to Z_{\overline{k}}$, we can find a curve G' on $X_{\overline{k}}$ such that $f_{\overline{k}}(G')$ is a point, $(K_{X_{\overline{k}}} + \Delta_{\overline{k}}) \cdot G' < 0$ and $(K_{X_{\overline{k}}} + \Delta_{\overline{k}} + \lambda C_{\overline{k}}) \cdot G' = 0$. Note that the $f_{\overline{k}}$-nef threshold of $(K_{X_{\overline{k}}} + \Delta_{\overline{k}}, C_{\overline{k}})$ is λ (cf. [GNT, Remark 2.7]). Let G be the image of G' on X. Then $f(G)$ is a point, $(K_X + \Delta) \cdot G < 0$ and $(K_X + \Delta + \lambda C) \cdot G = 0$ (cf. [Tan18a, Lemma 2.3]). Hence G is the required curve. This completes the proof of Step 2. □

Step 3. *The assertion of Lemma 6.1 holds without any additional assumptions.*

Proof of Step 3. Let G be a curve as in Step 2. Let A be an f-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $(K_X + \Delta + A) \cdot G < 0$. By Theorem 4.7, there exist $G_0 \in \overline{\mathrm{NE}}(X/Z)_{K_X+\Delta+A\geq 0}$, $r_1, \ldots, r_m \in \mathbb{R}_{\geq 0}$, and curves $G_1, \ldots, G_m$ generating $(K_X + \Delta + A)$-negative extremal rays of $\overline{\mathrm{NE}}(X/Z)$ such that the equation

$$[G] = [G_0] + \sum_{i=1}^{m} r_i [G_i]$$

holds in $\overline{\mathrm{NE}}(X/Z)$. Since $(K_X + \Delta + A) \cdot G < 0$, there is $j \in \{1, \ldots, m\}$ such that $r_j > 0$ and $(K_X + \Delta + A) \cdot G_j < 0$. In particular, we get $(K_X +$

$\Delta) \cdot G_j < 0$. On the other hand, since $K_X + \Delta + \lambda C$ is f-nef, the equation $(K_X + \Delta + \lambda C) \cdot G = 0$ implies that $(K_X + \Delta + \lambda C) \cdot G_j = 0$. Therefore G_j generates a $(K_X + \Delta)$-negative extremal ray R of $\overline{\mathrm{NE}}(X/Z)$ such that $(K_X + \Delta + \lambda C) \cdot R = 0$. This completes the proof of Step 3. $\square$

Step 3 completes the proof of Lemma 6.1. $\square$

By Lemma 6.1, we can run an MMP under the same assumption. However, for now, we do not know whether it terminates.

Theorem 6.2. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional geometrically integral log pair over k. Let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Assume that there exist an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor C on X and an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor C' on $X_{\overline{k}} := X \times_k \overline{k}$ such that if $\Delta_{\overline{k}}$ and $C_{\overline{k}}$ denote the pullbacks of Δ and C to $X_{\overline{k}}$ respectively, then*

(a) *$(X_{\overline{k}}, \Delta_{\overline{k}} + C')$ is log canonical,*

(b) *$K_X + \Delta + C$ is f-nef, and*

(c) *$C_{\overline{k}} \sim_{Z,\mathbb{R}} C'$.*

Then there exists a $(K_X + \Delta)$-MMP over Z with scaling of C

$$X =: X_0 \dashrightarrow X_1 \dashrightarrow \cdots \tag{6.2.1}$$

such that either

(1) *(6.2.1) terminates, or*

(2) *(6.2.1) is an infinite sequence.*

Proof. The assertion follows from Theorem 4.11, Theorem 5.4, and Lemma 6.1 (cf. [Fuj17, 4.9.1]). $\square$

Remark 6.3. Let k, (X, Δ) and $f : X \to Z$ be as in Theorem 6.2. If C is a sufficiently large multiple of an f-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, then there exists C' on $X_{\overline{k}}$ that satisfies all the conditions (a), (b) and (c) of Theorem 6.2.

6.2. Criterion for termination of flips

In this subsection, we prove that assuming the existence of log minimal models, the termination holds for pseudo-effective minimal model programs with scaling whose scaling coefficients are strictly decreasing. The idea of the proof can be found in the proof of [Bir12a, Theorem 4.1(iii)].

Proposition 6.4. *Let k be an algebraically closed field of characteristic $p > 5$. Let X be a $\mathbb{Q}$-factorial normal threefold over k and let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Let Δ and C be effective $\mathbb{R}$-divisors on X such that*

(a) *$(X, \Delta + C)$ is log canonical,*

(b) *$K_X + \Delta + C$ is f-nef, and*

(c) *C is f-big.*

Assume that the following holds:

(i) *If (V, Δ_V) is a three-dimensional $\mathbb{Q}$-factorial log canonical pair that is projective over Z and $K_V + \Delta_V$ is pseudo-effecitve over Z, then there exists a log minimal model of (V, Δ_V) over Z.*

Then there exists no infinite sequence

$$X = X_0 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow X_{i+1} \dashrightarrow \cdots \tag{6.4.1}$$

such that

(ii) *the sequence (6.4.1) is a $(K_X + \Delta)$-MMP over Z with scaling of C, and*

(iii) *if $\lambda_0, \lambda_1, \ldots$ are the real numbers defined by*

$$\lambda_i := \min\{\mu \in \mathbb{R}_{\geq 0} \,|\, K_{X_i} + \Delta_i + \mu C_i \text{ is nef}\},$$

then it holds that $\lim_{i\to\infty} \lambda_i \neq \lambda_i$ for any i.

Proof. Assume that there exists an infinite sequence (6.4.1) which satisfies (ii) and (iii). Let us derive a contradiction.

Set $\lambda_\infty := \lim_{i\to\infty} \lambda_i$. Replacing Δ and C by $\Delta + \lambda_\infty C$ and $(1 - \lambda_\infty)C$ respectively, we may assume that

(iii)' $\lambda_\infty = \lim_{i\to\infty} \lambda_i = 0$, and

(iv) $K_X + \Delta$ is f-pseudo-effective.

It is clear that $\lambda_0 > 0$. By (i), there is a log minimal model (X', Δ') of (X, Δ) over Z. Let $g : Y \to X$ be a log resolution of $(X, \Delta + C)$ such that the induced birational map $h : Y \dashrightarrow X'$ is a morphism. Set $\Gamma := g_*^{-1}\Delta + E_g$, where E_g denotes the reduced divisor on Y such that $\operatorname{Supp} E_g = \operatorname{Ex}(g)$. By construction, (Y, Γ) is a log birational model over Z of (X, Δ) (cf. Definition 2.2) and $(Y, \Gamma + g_*^{-1}C)$ is log canonical. Moreover, by a property of weak log canonical model, we can write

$$K_Y + \Gamma = h^*(K_{X'} + \Delta') + E$$

with an h-exceptional divisor $E \geq 0$. It follows from Theorem 3.4 that there exists a $(K_Y + \Gamma)$-MMP over X' that terminates with a log minimal model (Y', Γ') over X'. The negativity lemma implies that $K_{Y'} + \Gamma' = h'^*(K_{X'} + \Delta')$, where $h' : Y' \to X'$ denotes the induced birational morphism. In particular $K_{Y'} + \Gamma'$ is nef over Z.

Pick a sufficiently small $\epsilon > 0$ so that the MMP $Y \dashrightarrow Y'$ defined above is a $(K_Y + \Gamma + \epsilon g_*^{-1}C)$-MMP over Z. Let C' be the proper transform of $g_*^{-1}C$ on Y'. Then $(Y', \Gamma' + \epsilon C')$ is a $\mathbb{Q}$-factorial log canonical pair. Since it follows from (iv) that $K_Y + \Gamma + \epsilon g_*^{-1}C$ is big over Z, so is $K_{Y'} + \Gamma' + \epsilon C'$. Thus, again by Theorem 3.4, there is a $(K_{Y'} + \Gamma' + \epsilon C')$-MMP over Z that terminates with a log minimal model $(Y'', \Gamma'' + \epsilon C'')$ over Z. As ϵ is sufficiently small, it follows from Proposition 4.9 that the MMP $Y' \dashrightarrow Y''$ is $(K_{Y'} + \Gamma')$-numerically trivial. Therefore the $\mathbb{R}$-divisor $K_{Y''} + \Gamma''$ is also nef over Z. Hence, we see that $K_{Y''} + \Gamma'' + \epsilon' C''$ is nef over Z for any $\epsilon' \in [0, \epsilon]$.

By (iii)', there exists $i > 0$ such that $0 \leq \lambda_{i+1} < \lambda_i \leq \epsilon$. By the construction of (Y, Γ) and by the basic property of the log MMP, we see that $(X_{i+1}, \Delta_{i+1} + \lambda_i C_{i+1})$ and $(X_{i+1}, \Delta_{i+1} + \lambda_{i+1} C_{i+1})$ are weak log canonical models over Z of $(Y, \Gamma + \lambda_i g_*^{-1}C)$ and $(Y, \Gamma + \lambda_{i+1} g_*^{-1}C)$ respectively. On the other hand, by the construction of (Y'', Γ'') and by the choices of ϵ and i, we see that also $(Y'', \Gamma'' + \lambda_i C'')$ and $(Y'', \Gamma'' + \lambda_{i+1} C'')$ are weak log canonical models over Z of $(Y, \Gamma + \lambda_i g_*^{-1}C)$ and $(Y, \Gamma + \lambda_{i+1} g_*^{-1}C)$ respectively. Now let $\varphi : W \to X_{i+1}$ and $\psi : W \to Y''$ be a common resolution of $X_{i+1} \dashrightarrow Y''$. Then it follows from Remark 2.3 that

$$\varphi^*(K_{X_{i+1}} + \Delta_{i+1} + \lambda_i C_{i+1}) = \psi^*(K_{Y''} + \Gamma'' + \lambda_i C'') \qquad \text{and}$$
$$\varphi^*(K_{X_{i+1}} + \Delta_{i+1} + \lambda_{i+1} C_{i+1}) = \psi^*(K_{Y''} + \Gamma'' + \lambda_{i+1} C'').$$

By $\lambda_i \neq \lambda_{i+1}$, we get

$$\varphi^*(K_{X_{i+1}} + \Delta_{i+1}) = \psi^*(K_{Y''} + \Gamma'').$$

Therefore, $K_{X_{i+1}} + \Delta_{i+1}$ is nef over Z. Hence, the $(K_X + \Delta)$-MMP (6.4.1) terminates. This contradicts the fact that the sequence (6.4.1) was chosen to be an infinite sequence. □

6.3. Existence of log minimal models

In this subsection, we prove the existence of log minimal models (Theorem 6.9). To this end, we first treat the projective case (Theorem 6.6). The proof of Theorem 6.6 is similar to the one of [Bir12b, Corollary 1.7]. We start with an auxiliary result, which is known to experts.

Lemma 6.5. *Let k be an algebraically closed field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional $\mathbb{Q}$-factorial terminal pair over k and let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Then there exists no infinite sequence*

(6.5.1) $$X = X_0 \overset{\varphi_0}{\dashrightarrow} X_1 \overset{\varphi_1}{\dashrightarrow} \cdots$$

such that

(1) *the sequence* (6.5.1) *is a $(K_X + \Delta)$-MMP over Z, and*
(2) *for any i, $\varphi_i : X_i \dashrightarrow X_{i+1}$ is a $(K_{X_i} + \Delta_i)$-flip, where Δ_i denotes the proper transform of Δ to X_i.*

Proof. By the fact that the singular locus of X is zero-dimensional [Kol13, Corollary 2.30], we can apply the same argument as in [KM98, Theorem 6.17]. □

Theorem 6.6. *Let k be an algebraically closed field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional projective log canonical pair over k such that $K_X + \Delta$ is pseudo-effective. Then there exists a log minimal model of (X, Δ).*

Proof. Set $S := \lfloor \Delta \rfloor$ and $B := \{\Delta\}$. In particular, we have $\Delta = S + B$. Taking a log resolution of (X, Δ), we may assume that

(1) (X, Δ) is a $\mathbb{Q}$-factorial dlt pair.

If there exists a $(K_X + \Delta)$-MMP that terminates, then there is nothing to prove. Hence, the problem is reduced to the case when

(2) an arbitrary $(K_X + \Delta)$-MMP does not terminate.

We divide the proof into several steps.

Step 1. *There exist an effective big $\mathbb{R}$-divisor H on X and an infinite sequence*

$$X = X_0 \overset{f_0}{\dashrightarrow} X_1 \overset{f_1}{\dashrightarrow} \cdots \tag{6.6.1}$$

such that

(3) $K_X + \Delta + H$ *is nef,* $(X, \Delta + H)$ *is dlt,*

(4) *the sequence (6.6.1) is a* $(K_X + \Delta)$*-MMP with scaling of* H*, and*

(5) $\lim_{n\to\infty} \lambda_n = 0$*, where* λ_n *is the scaling coefficient.*

Proof of Step 1. Pick an ample $\mathbb{R}$-divisor H such that $K_X + \Delta + H$ is nef and $(X, \Delta + H)$ is dlt [Bir16, Lemma 9.2]. Hence, (3) holds. By Theorem 6.2 and (2), there exists an infinite sequence which is a $(K_X + \Delta)$-MMP with scaling of H:

$$(X, \Delta) = (X_0, \Delta_0) \overset{f_0}{\dashrightarrow} (X_1, \Delta_1) \overset{f_1}{\dashrightarrow} \cdots \dashrightarrow (X_i, \Delta_i) \dashrightarrow \cdots . \tag{6.6.2}$$

Clearly, (4) holds.

It suffices to prove (5). Assuming that $\lambda := \lim_{n\to\infty} \lambda_n > 0$, let us derive a contradiction. The sequence (6.6.2) is a $(K_X + \Delta + \frac{\lambda}{2}H)$-MMP, thus there exists a $(K_X + \Delta + \frac{\lambda}{2}H)$-MMP that does not terminate. On the other hand, $K_X + \Delta + \frac{\lambda}{2}H$ is big, hence any $(K_X + \Delta + \frac{\lambda}{2}H)$-MMP terminates by Theorem 5.5. This is a contradiction. Therefore, (5) holds. $\square$

We denote the proper transforms of Δ, S, B and H on X_i by Δ_i, S_i, B_i and H_i, respectively. Note that even if we replace (X, Δ) by (X_i, Δ_i), the properties (1)–(5) still hold. In particular, we may assume that

(6) $1 > \lambda_0 \geq \lambda_1 \geq \cdots$,

(7) for any i, $X_i \dashrightarrow X_{i+1}$ is a flip, and

(8) for any i, $\mathrm{Ex}(f_i)$ is disjoint from $\mathrm{Supp}\, S_i$ (cf. [Bir16, Proposition 5.5], [Wal18, Proposition 4.2]).

Hence, $(X_i, B_i + \lambda_i H_i)$ is klt, because $(X, B + \lambda_i H)$ is klt and the MMP

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_i$$

is $(K_X + B + \lambda_i H)$-non-positive. For any $i \geq 0$, let $\mu_i : (W_i, \Psi_i) \to (X_i, B_i + \lambda_i H_i)$ be a projective birational morphism such that (W_i, Ψ_i) is a $\mathbb{Q}$-factorial terminal pair and $K_{W_i} + \Psi_i = \mu_i^*(K_{X_i} + B_i + \lambda_i H_i)$. Let $g_i : W_i \dashrightarrow W_{i+1}$ be the induced birational map. Since $a_D(X_i, B_i + \lambda_i H_i) \leq a_D(X_{i+1}, B_{i+1} + \lambda_{i+1} H_{i+1})$ for any exceptional prime divisor D over X, the induced birational map $g_i^{-1} : W_{i+1} \dashrightarrow W_i$ does not contract any prime divisor. Replacing (X, Δ) by (X_i, Δ_i) for some $i \gg 0$, we may assume that

(9) $g_i : W_i \dashrightarrow W_{i+1}$ is isomorphic in codimension one for any i.

Set $h_i : W_0 \dashrightarrow W_i$ to be the induced birational map. Then we get

$$\Psi_0 \geq h_{1*}^{-1}\Psi_1 \geq \cdots \geq h_{i*}^{-1}\Psi_i \geq \cdots \geq 0.$$

Therefore there exists an $\mathbb{R}$-divisor Ψ_∞ on W_0 such that $\Psi_\infty = \lim_{i\to\infty} h_{i*}^{-1}\Psi_i$. Then (W_0, Ψ_∞) is a $\mathbb{Q}$-factorial terminal pair. Set

$$G_i = \mu_i^* S_i.$$

Step 2. *The following hold.*

(i) $(W_0, G_0 + \Psi_\infty)$ *is log canonical.*

(ii) *If there exists a log minimal model of* $(W_0, G_0 + \Psi_\infty)$*, then there exists a log minimal model of* (X, Δ).

Proof of Step 2. If we define Ξ_i on W_0 by $K_{W_0} + \Xi_i = \mu_0^*(K_X + B + \lambda_i H)$, then $\Xi_i \geq h_{i*}^{-1}\Psi_i$ by the negativity lemma. Since $\lim_{i\to\infty}\lambda_i = 0$, we have $K_{W_0} + G_0 + \Psi_\infty \leq \mu_0^*(K_X + \Delta)$. Therefore $(W_0, G_0 + \Psi_\infty)$ is log canonical.

If $(W_0, G_0 + \Psi_\infty)$ has a log minimal model, then $K_X + \Delta$ has a weak Zariski decomposition in the sense of [Bir16, Section 8.1]. So we see that (X, Δ) has a log minimal model by [Bir16, Proposition 8.3]. $\square$

Step 3. *The following hold.*

(i) $G_0 = (h_i)_*^{-1} G_i$ *for any* i.

(ii) $K_{W_0} + G_0 + \Psi_\infty = \lim_{i\to\infty}(h_i)_*^{-1}(K_{W_i} + G_i + \Psi_i)$.

(iii) $K_{W_0} + G_0 + \Psi_\infty \in \overline{\mathrm{Mov}}(X)$.

(iv) *The stable base locus of* $(h_i)_*^{-1}(K_{W_i} + G_i + \Psi_i)$ *is disjoint from* $\mathrm{Supp}\, G_0$.

Proof of Step 3. The assertion (i) follows from (8) and (9). Then (ii) holds by (i).

We now show (iii). Since $K_{X_i} + \Delta_i + \lambda_i H_i$ is nef and big, it follows from Theorem 2.6 that $K_{X_i} + \Delta_i + \lambda_i H_i$ is semi-ample. Since

$$K_{W_i} + G_i + \Psi_i = \mu_i^*(K_{X_i} + S_i + B_i + \lambda_i H_i) = \mu_i^*(K_{X_i} + \Delta_i + \lambda_i H_i),$$

also $K_{W_i} + G_i + \Psi_i$ is semi-ample. Then (ii) implies (iii).

Let us prove (iv). Let $\varphi_0 : Y \to W_0$ and $\varphi_i : Y \to W_i$ be a common resolution of W_0 and W_i. Since $K_{W_i} + G_i + \Psi_i$ is nef, the negativity lemma induces an equation

$$\varphi_0^*((h_i)_*^{-1}(K_{W_i} + G_i + \Psi_i)) = \varphi_i^*(K_{W_i} + G_i + \Psi_i) + F$$

for some φ_0-exceptional effective $\mathbb{R}$-divisor F. Since $K_{W_i} + G_i + \Psi_i$ is semi-ample, the stable base locus of $(h_i)_*^{-1}(K_{W_i} + G_i + \Psi_i)$ is $\varphi_0(\operatorname{Supp} F)$. By (8), there is an open set $U_i \subset X$ containing S such that the induced birational map $X \dashrightarrow X_i$ is an isomorphism on U_i. Restricting the above equation to $(\mu_0 \circ \varphi_0)^{-1}(U_i)$, the negativity lemma implies that $F|_{(\mu_0 \circ \varphi_0)^{-1}(U_i)} = 0$. Thus we have $\varphi_0(\operatorname{Supp} F) \cap \mu_0^{-1}(U_i) = \emptyset$. Since $\mu_0^{-1}(U_i)$ contains $\operatorname{Supp} G_0$, we see that the stable base locus of $(h_i)_*^{-1}(K_{W_i} + G_i + \Psi_i)$, which is equal to $\varphi_0(\operatorname{Supp} F)$, is disjoint from $\operatorname{Supp} G_0$. □

Step 4. *There exists a log minimal model of* $(W_0, G_0 + \Psi_\infty)$.

Proof of Step 4. Let

$$W_0 =: V_0 \dashrightarrow V_1 \dashrightarrow \cdots \qquad (6.6.3)$$

be an arbitrary $(K_{W_0} + G_0 + \Psi_\infty)$-MMP. It suffices to prove that the MMP (6.6.3) terminates. By Step 3(iii), the MMP (6.6.3) consists of flips. Furthermore, Step 3(ii)(iv) imply that the MMP (6.6.3) occurs disjointly from Supp G_0. In particular, the sequence (6.6.3) is a $(K_{W_0} + \Psi_\infty)$-MMP. Since (W_0, Ψ_∞) is a $\mathbb{Q}$-factorial terminal pair, the sequence (6.6.3) terminates by Lemma 6.5. □

Step 2 and Step 4 complete the proof of Theorem 6.6. □

Proposition 6.7. *Let k be an algebraically closed field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional $\mathbb{Q}$-factorial log canonical pair over k such that $(X, \{\Delta\})$ is klt. Let $f : X \to Z$ be a projective k-morphism to a projective k-scheme Z. Let C be an f-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor such that $K_X + \Delta + C$ is f-nef. Then the following hold.*

(1) *There exists no infinite sequence that is a* $(K_X + \Delta)$*-MMP over* Z *with scaling of* C.

(2) *There exists a* $(K_X + \Delta)$*-MMP over* Z *with scaling of* C *that terminates.*

Proof. By Theorem 6.2, (1) implies (2).

Let us prove (1). Assume that there exists an infinite sequence

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \tag{6.7.1}$$

that is a $(K_X + \Delta)$-MMP over Z with scaling of C. Let us derive a contradiction. If the scaling coefficients satisfy $\epsilon := \lim \lambda_i > 0$, then the sequence (6.7.1) is a $(K_X + \Delta + \epsilon C)$-MMP with scaling of C. Since there exists an effective $\mathbb{R}$-divisor Δ' such that $\Delta' \sim_{Z,\mathbb{R}} \Delta + \epsilon C$ and (X, Δ') is klt, it follows from [BW17, Theorem 1.5] that the sequence (6.7.1) terminates, which is a contradiction. Hence, we may assume that $\lim \lambda_i = 0$. By Theorem 6.6, the sequence (6.7.1) terminates by Proposition 6.4. Therefore, we get a contraction in any case. Hence, (1) holds. ☐

Corollary 6.8. *Let* k *be an algebraically closed field of characteristic* $p > 5$. *Let* (X, Δ) *be a three-dimensional* $\mathbb{Q}$*-factorial log canonical pair over* k *such that* $(X, \{\Delta\})$ *is klt. Let* $f : X \to Z$ *be a projective* k*-morphism to a quasi-projective* k*-scheme* Z. *Then there exists a* $(K_X + \Delta)$*-MMP over* Z *that terminates.*

Proof. The assertion follows from Proposition 2.11 and Proposition 6.7. ☐

Theorem 6.9. *Let* k *be an algebraically closed field of characteristic* $p > 5$. *Let* (X, Δ) *be a three-dimensional log canonical pair over* k *and let* $f : X \to Z$ *be a projective* k*-morphism to a quasi-projective* k*-scheme* Z. *If* $K_X + \Delta$ *is* f*-pseudo-effective, then there exists a log minimal model of* (X, Δ) *over* Z.

Proof. If (X, Δ) is $\mathbb{Q}$-factorial dlt pair, then the assertion follows from Corollary 6.8. The general case is reduced to this case by Corollary 3.6. ☐

Corollary 6.10. *Let* k *be a perfect field of characteristic* $p > 5$. *Let* (X, Δ) *be a three-dimensional geometrically integral log pair over* k. *Let* $f : X \to Z$ *be a projective* k*-morphism to a quasi-projective* k*-scheme* Z. *Assume that there exist an* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor* C *on* X *and an effective* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor* C' *on* $X_{\overline{k}} := X \times_k \overline{k}$ *such that if* $\Delta_{\overline{k}}$ *and* $C_{\overline{k}}$ *denote the pullbacks of* Δ *and* C *to* $X_{\overline{k}}$ *respectively, then*

(a) $(X_{\overline{k}}, \Delta_{\overline{k}} + C')$ *is log canonical,*

(b) $K_X + \Delta + C$ *is* f*-nef,*

(c) C *is* f*-big, and*

(d) $C_{\overline{k}} \sim_{Z,\mathbb{R}} C'$.

Then there exists no infinite sequence

$$X = X_0 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow X_{i+1} \dashrightarrow \cdots \tag{6.10.1}$$

such that

(1) *the sequence (6.10.1) is a* $(K_X + \Delta)$*-MMP over* Z *with scaling of* C*, and*

(2) *if* $\lambda_0, \lambda_1, \ldots$ *are the real numbers defined by*

$$\lambda_i := \min\{\mu \in \mathbb{R}_{\geq 0} \mid K_{X_i} + \Delta_i + \mu C_i \text{ is nef over } Z\},$$

then it holds that $\lim_{i\to\infty} \lambda_i \neq \lambda_i$ *for any* i.

Proof. Under the additional assumption that

- k is algebraically closed and X is $\mathbb{Q}$-factorial and klt,

the assertion immediately follows from Proposition 6.4 and Theorem 6.9.

Let us go back to the general situation. Let

$$X' = X'_0 \dashrightarrow X'_1 \dashrightarrow \cdots$$

be the sequence obtained by the base changes $X'_i := X_i \times_k \overline{k}$ to the algebraic closure $\overline{k}$ of k. Then each birational map $g'_i : X'_i \dashrightarrow X'_{i+1}$ has a decomposition

$$X'_i \xrightarrow{\varphi_i} Z'_i \xleftarrow{\psi_i} X'_{i+1},$$

such that

(i) Z'_i is a normal threefold projective over Z,

(ii) φ_i and ψ_i are birational,

(iii) $-(K_{X'_i} + \Delta'_i)$ is φ_i-ample, and

(iv) $(X'_{i+1}, \Delta'_{i+1})$ is a log canonical model of (X'_i, Δ'_i) over Z'_i.

We apply Corollary 3.6 to (X'_0, Δ'_0). Then we obtain a projective birational morphism $g_0 : Y_0 \to X'_0$ satisfying the properties listed in Corollary 3.6. In particular, if we define Δ_{Y_0} by $K_{Y_0} + \Delta_{Y_0} = g_0^*(K_{X'_0} + \Delta'_0)$, then (Y_0, Δ_{Y_0}) is a $\mathbb{Q}$-factorial dlt pair. Then there exists a $(K_{Y_0} + \Delta_{Y_0})$-MMP over Z'_0 that terminates. Let (Y_1, Δ_{Y_1}) be the end result, which is a log minimal model of (X'_0, Δ'_0) over Z'_0. By (iv) and Remark 2.3, the induced birational map $g_1 : Y_1 \dashrightarrow X'_1$ is a morphism and $K_{Y_1} + \Delta_{Y_1} = g_1^*(K_{X'_1} + \Delta'_1)$. Repeating the same procedure, we obtain a sequence

$$Y_0 \dashrightarrow Y_1 \dashrightarrow Y_2 \dashrightarrow \cdots ,$$

which is a $(K_{Y_0} + \Delta_{Y_0})$-MMP over $Z' := Z \times_k \overline{k}$. Furthermore, the MMP $Y_i \dashrightarrow Y_{i+1}$ is $(K_{Y_i} + \Delta_{Y_i} + \lambda_i g_i^* C'_i)$-numerically trivial, where C'_i is the pullback of C_i. Therefore this sequence is a $(K_{Y_0} + \Delta_{Y_0})$-MMP over Z' with scaling of $g_0^* C'_0$. Hence, this terminates by the case treated above. □

6.4. MMP

The purpose of this subsection is to show Theorem 6.12. We first treat the minimal model program for the pseudo-effective log canonical pairs (Theorem 6.11). In this case, any $(K_X + \Delta)$-MMP with scaling of an ample divisor terminates.

Theorem 6.11. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional geometrically integral log pair over k. Let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Assume that there exist an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor C on X and an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor C' on $X_{\overline{k}} := X \times_k \overline{k}$ such that if $\Delta_{\overline{k}}$ and $C_{\overline{k}}$ denote the pullbacks of Δ and C to $X_{\overline{k}}$ respectively, then*

(a) *$K_X + \Delta$ is f-pseudo-effective,*

(b) *$(X_{\overline{k}}, \Delta_{\overline{k}} + C')$ is log canonical,*

(c) *$K_X + \Delta + C$ is f-nef,*

(d) *C is f-big, and*

(e) *$C_{\overline{k}} \sim_{Z,\mathbb{R}} C'$.*

Then the following hold.

(1) *There exists no infinite sequence that is a $(K_X + \Delta)$-MMP over Z with scaling of C.*

(2) *There exists a $(K_X+\Delta)$-MMP over Z with scaling of C that terminates.*

Proof. By Theorem 6.2, (1) implies (2). Hence it suffices to prove (1).

Assume that there exists an infinite sequence that is a $(K_X+\Delta)$-MMP over Z with scaling of C:

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots . \tag{6.11.1}$$

Let us derive a contradiction. For the scaling coefficients $\lambda_0, \lambda_1, \ldots$, we set $\lambda := \lim_{i\to\infty} \lambda_i$. If $\lambda \neq \lambda_i$ for any i, then the sequence (6.11.1) terminates by Corollary 6.10. Hence, we may assume that $\lambda = \lambda_i$ for some i. Then the infinite sequence (6.11.1) is a $(K_X+\Delta+\frac{\lambda}{2}C)$-MMP over Z. Let

$$X_{\overline{k}} = X_{0,\overline{k}} \dashrightarrow X_{1,\overline{k}} \dashrightarrow \cdots$$

be the infinite sequence obtained by applying the base change $(-)\times_k \overline{k}$ to (6.11.1). Fix a projective birational morphism $g : Y^{(0)} \to X_{\overline{k}} = X_{0,\overline{k}}$ which satisfies the properties listed in Corollary 3.6. In particular, if $\Delta_{Y^{(0)}}$ is the $\mathbb{R}$-divisor defined by $K_{Y^{(0)}} + \Delta_{Y^{(0)}} = g^*(K_{X_{\overline{k}}} + \Delta_{\overline{k}})$, then $(Y^{(0)}, \Delta_{Y^{(0)}})$ is a $\mathbb{Q}$-factorial dlt pair. Set $C_{Y^{(0)}}$ to be the pullback of C' to $Y^{(0)}$. By standard argument, we can construct an infinite sequence that is a $(K_{Y^{(0)}} + \Delta_{Y^{(0)}} + \frac{1}{2}C_{Y^{(0)}})$-MMP over Z. As $K_X+\Delta+\frac{\lambda}{2}C$ is big over Z, $K_{Y^{(0)}} + \Delta_{Y^{(0)}} + \frac{1}{2}C_{Y^{(0)}}$ is big over $Z_{\overline{k}}$. This contradicts Theorem 5.5. Therefore, (1) holds. $\square$

Theorem 6.12. *Let k be a perfect field of characteristic $p>5$. Let (X,Δ) be a three-dimensional log canonical pair over k. Let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Then there exists a $(K_X+\Delta)$-MMP over Z that terminates.*

Proof. By standard argument, we may assume that $f_*\mathcal{O}_X = \mathcal{O}_Z$ and k is algebraically closed in $K(X)$. In particular, X is geometrically integral over k. Fix an algebraic closure $\overline{k}$ of k. Let $f_{\overline{k}} : X_{\overline{k}} \to Z_{\overline{k}}$ be the base change $f \times_k \overline{k}$.

If $K_X+\Delta$ is f-pseudo-effective, then the assertion follows from Theorem 6.11. Thus we may assume that $K_X+\Delta$ is not f-pseudo-effective. Fix a projective birational morphism $g : Y \to X$ which satisfies the properties (1) and (2) of Corollary 3.6. We prove the assertion of Theorem 6.12 by induction on $\rho(Y/Z)$.

If $\rho(X/Z) \leq 1$, then the assertion holds. Indeed, if $\rho(X/Z) = 0$, then there is nothing to show. If $\rho(X/Z) = 1$, then the Stein factorisation of $f : X \to Z$ is a $(K_X + \Delta)$-Mori fibre space over Z.

Therefore, we may assume that $\rho(X/Z) > 1$. Pick an f-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor C on X and an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $\overline{C}$ on $X_{\overline{k}}$ such that

(i) $(X_{\overline{k}}, \Delta_{\overline{k}} + \overline{C})$ is log canonical, $K_X + \Delta + C$ is f-nef, and

(ii) if we set

$$\nu = \inf\{\mu \in \mathbb{R}_{\geq 0} \mid K_X + \Delta + \mu C \text{ is } f\text{-pseudo-effevtive}\},$$

then $K_X + \Delta + \nu C \not\equiv_f 0$.

(iii) $\overline{C} \sim_{\mathbb{R}, Z_{\overline{k}}} C_{\overline{k}}$, where $C_{\overline{k}}$ denotes the pullback of C to $X_{\overline{k}}$.

Note that $0 < \nu \leq 1$. By Theorem 6.11, there exists a $(K_X + \Delta + \nu C)$-MMP over Z with scaling of C that terminates:

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_\ell =: X'. \tag{6.12.1}$$

If Δ' and C' denote the proper transforms of Δ and C on X' respectively, then $K_{X'} + \Delta' + \nu C'$ is nef over Z. Moreover, by Theorem 5.2, we see that $K_{X'} + \Delta' + \nu C'$ is semi-ample over Z. We obtain projective morphisms:

$$X' \xrightarrow{f'} Z' \xrightarrow{\psi} Z,$$

where f' is the projective morphism over Z with $f'_*\mathcal{O}_{X'} = \mathcal{O}_{Z'}$ that is induced by $K_{X'} + \Delta' + \nu C'$, and ψ is the induced morphism. By the choice of ν, it holds that $K_{X'} + \Delta' + \nu C'$ is not big over Z. Therefore we get $\dim X' > \dim Z'$, which implies that $K_{X'} + \Delta' \equiv_{f'} -\nu C'$ is not f'-pseudo-effective. Recall that $g : Y \to X$ is a projective morphism which satisfies the properties (1) and (2) of Corollary 3.6. Let Δ_Y be the $\mathbb{R}$-divisor defined by $K_Y + \Delta_Y = g^*(K_X + \Delta)$. Since the sequence (6.12.1) is a $(K_X + \Delta)$-MMP over Z, we can construct $g' : Y' \to X'$ and an $\mathbb{R}$-divisor $\Delta_{Y'}$ on Y' such that

(1) g' and $(Y', \Delta_{Y'})$ satisfies the properties (1) and (2) of Corollary 3.6, and

(2) the induced birational map $\varphi : Y \dashrightarrow Y'$ is decomposed as a $(K_Y + \Delta_Y)$-MMP over Z:

$$Y = Y^{(0)} \dashrightarrow \cdots \dashrightarrow Y^{(m)} = Y',$$

and $\varphi_*\Delta_Y = \Delta_{Y'}$.

In particular, we have $\rho(Y'/Z) \leq \rho(Y/Z)$.

We prove that $\rho(Y'/Z') < \rho(Y/Z)$. When $K_{X'} + \Delta' + \nu C' \not\equiv_{\psi \circ f'} 0$, there is a curve on Z' whose image on Z is a point. Therefore we get $\rho(Y'/Z') < \rho(Y'/Z) \leq \rho(Y/Z)$. When $K_{X'} + \Delta' + \nu C' \equiv_{\psi \circ f'} 0$, the birational map $X \dashrightarrow X'$ contract a prime divisor E because $K_X + \Delta + \nu C \not\equiv_f 0$ by the above property (ii). Then

$$a_E(Y', \Delta_{Y'}) = a_E(X', \Delta') > a_E(X, \Delta) = a_E(Y, \Delta_Y).$$

Since $Y = Y^{(0)} \dashrightarrow \cdots \dashrightarrow Y^{(m)} = Y'$ is a $(K_Y + \Delta_Y)$-MMP over Z, it contracts $g_*^{-1}E$. Therefore we have $\rho(Y'/Z') \leq \rho(Y'/Z) < \rho(Y/Z)$. In any case, we obtain $\rho(Y'/Z') < \rho(Y/Z)$.

We apply the induction hypothesis to (X', Δ'), $f' : X' \to Z'$ and $g' : Y' \to X'$. Since $K_{X'} + \Delta'$ is not f'-pseudo-effective, there is a $(K_{X'} + \Delta')$-MMP over Z' that terminates:

$$(X', \Delta') = (X_\ell, \Delta_\ell) \dashrightarrow (X_{\ell+1}, \Delta_{\ell+1}) \dashrightarrow \cdots \dashrightarrow (X_n, \Delta_n) = (X'', \Delta'').$$

Then there is a $(K_{X''} + \Delta'')$-Mori fibre space $g'' : X'' \to Z''$ over Z'. In particular, g'' is a $(K_{X''} + \Delta'')$-Mori fibre space over Z. Hence, the sequence

$$(X, \Delta) = (X_0, \Delta_0) \dashrightarrow \cdots \dashrightarrow (X_n, \Delta_n) = (X'', \Delta'')$$

is a $(K_X + \Delta)$-MMP over Z that terminates. □

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